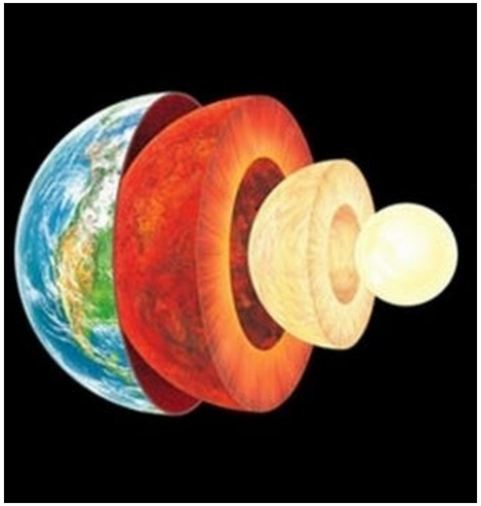
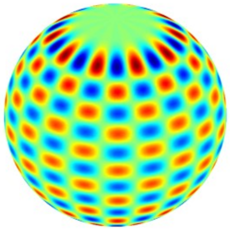
Solid Earth Lading: Theory



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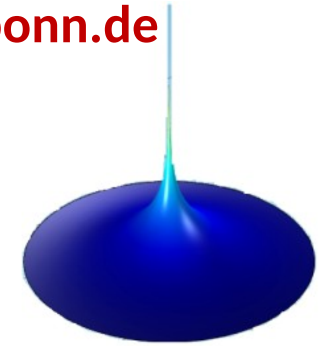

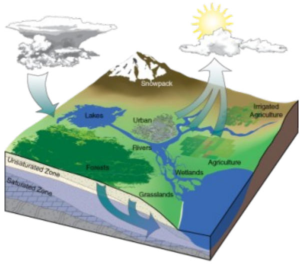

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* Green's function

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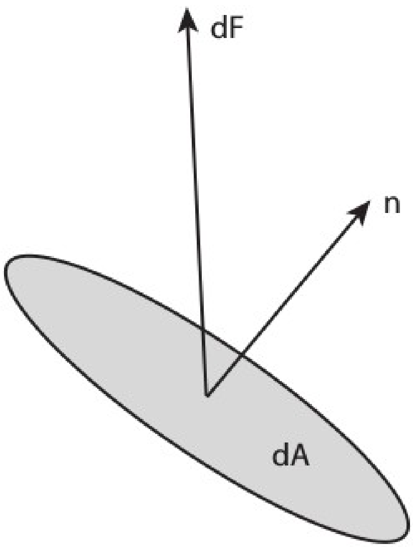
We distinguish between **body forces** that act at all points in the earth, such as gravity, from **surface forces**.

Surface forces are those that act either on actual surfaces within the earth, such as a fault or an igneous dike, or forces that one part of the earth exerts on an adjoining part.

The **traction (stress) vector T** is defined as the limit of the surface force per unit area **dF** acting on a surface element **dA**, with unit normal **n** as the size of the area element tends to zero



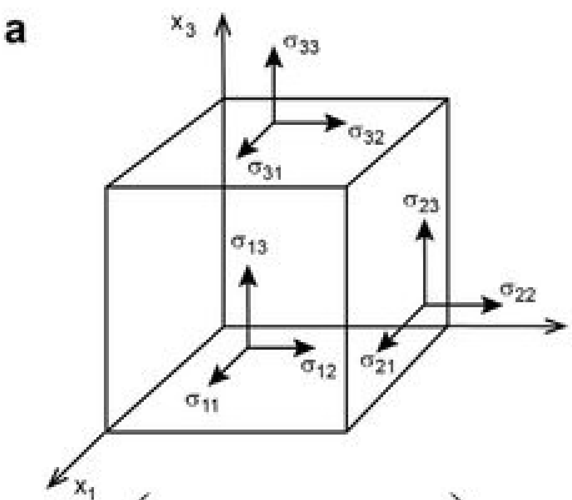
$$\mathbf{T} = \lim_{\Delta \mathbf{A} \to 0} \Delta\mathbf{F}/\Delta\mathbf{A}$$

The traction depends:

1. Forces acting on the body

2. Orientation of the surface elements

Let's formulate the traction components acting on three **mutually orthogonal surfaces** populate a second-rank stress tensor **σ**



$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

The mean normal stress is equal to minus the pressure
$\sigma_{kk}/3 = -p$ also called **volumetric stress**.

where
$\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$

Strain and rotation tensor

Displacement vector: **u** = (u_1, u_2, u_3) or **u** = (u_x, u_y, u_z)

Cartesian coordinates: **x** = (x_1, x_2, x_3) or **x** = (x, y, z)

Linearization of the displacement using Taylor's series expansion as:

$$u_i\left(x_0 + dx\right) = u_i\left(x_0\right) + \frac{\partial u_i}{\partial x_j}\, dx_j$$

∇.u

we can rewrite the second term of expansion (∇.u) as:

$$u_i\left(x_0 + dx\right) = u_i\left(x_0\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)dx_j + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)dx_j$$

where we define as symmetric tensor ε_{ij} as strain tensor and *anti symmetric* tensor w_{ij} as *rotation* tensor:

$$u_i\left(x_0 + dx\right) = u_i\left(x_0\right) + \varepsilon_{ij} dx_j + \omega_{ij} dx_j$$

$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) & 0 \end{bmatrix}$$

In vector form:

$$\boldsymbol{\epsilon} = \tfrac{1}{2}((\nabla \mathbf{u})^T + \nabla \mathbf{u})$$

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

*** The strain tensor is symmetric and only 6 components are independent:**

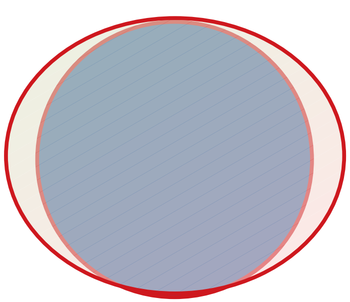
$\varepsilon_{21} = \varepsilon_{12}$, $\varepsilon_{31} = \varepsilon_{13}$, $\varepsilon_{23} = \varepsilon_{32}$

*** Strains are unitless but have directions, e.g. axial strain, shear strain...**

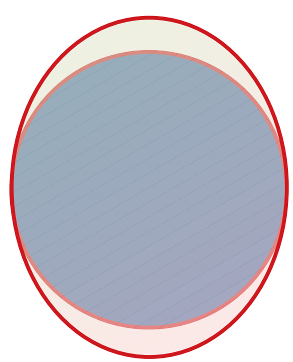
Diagonal elements: axial strains Off-diagonal: Shear Strains

$$\begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2}+\dfrac{\partial u_2}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_3}+\dfrac{\partial u_3}{\partial x_1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2}+\dfrac{\partial u_2}{\partial x_1}\right) & \dfrac{\partial u_2}{\partial x_2} & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_3}+\dfrac{\partial u_3}{\partial x_2}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_3}+\dfrac{\partial u_3}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_3}+\dfrac{\partial u_3}{\partial x_2}\right) & \dfrac{\partial u_3}{\partial x_3} \end{bmatrix}$$

For example a **dilatation** (**uniaxial strains**):
i.e. length changes in the x,y or z directions



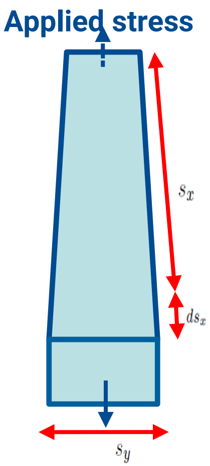
$dX/X = \varepsilon_{xx}$



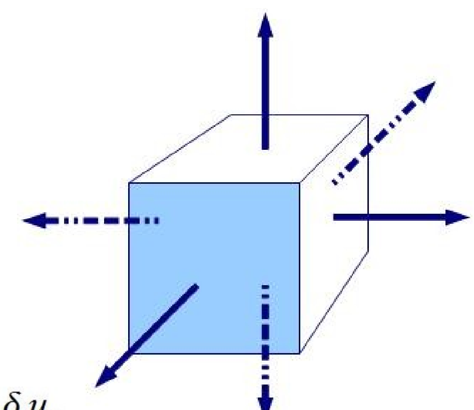
$dY/Y = \varepsilon_{yy}$

Strain in x direction

$$q_x = \frac{ds_x}{s_x}$$



Volume change: sum of strains in x,y,z directions



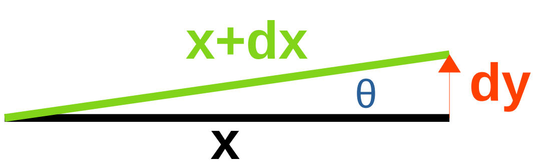
$$\frac{dV}{V} = \frac{\delta u_x}{\delta x} + \frac{\delta u_y}{\delta y} + \frac{\delta u_z}{\delta z} = (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$

Diagonal elements: axial strains **Off-diagonal: Shear Strains**

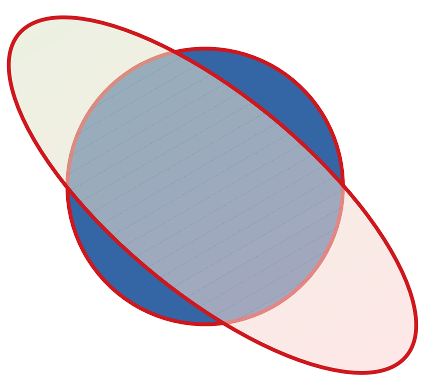
$$
\begin{bmatrix}
\frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) \\
\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) \\
\frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3}
\end{bmatrix}
$$

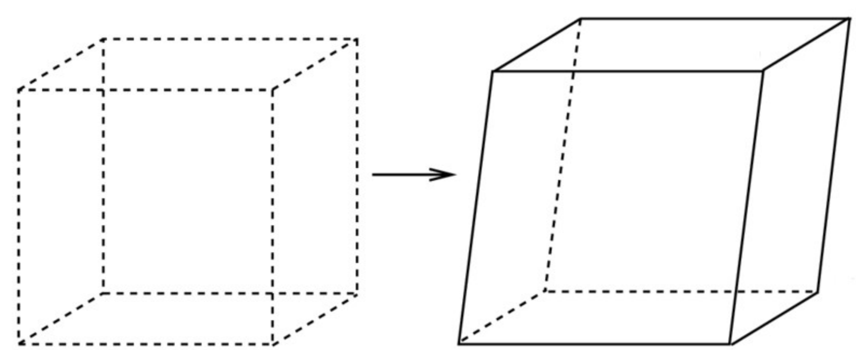
Shear strain examples: think of angle change of a side

A simple picture for shear strain



shear strain: (dy/x)= tanθ = θ
as **displacement in the orthogonal direction divided by the length**





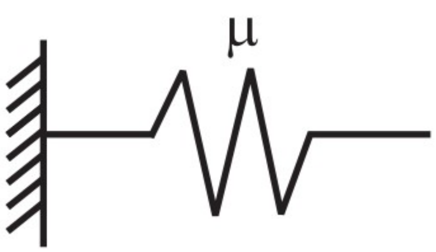
Constitutive equation: elastic rheology

The constitutive equation describes how material stress and strain (or strain rate) are related to each other which define a rheology.

On short timescales (seconds to years and several decades) the Earth deforms well with elasticity but on longer (geologically) large time scales (multi-decades to thousands and millions of years), the Earth mantle behaves like a very viscous fluid.

General linear elasticity is described by generalized Hooke's law:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$



σ_{ij} stress tensor

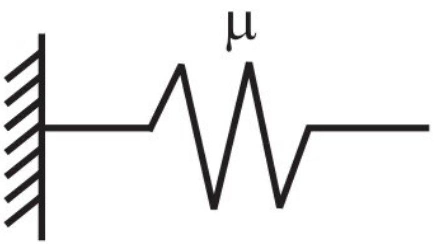
ε_{kl} strain tensor

C_{ijkl} elastic properties of a material

Constitutive equation: elastic rheology

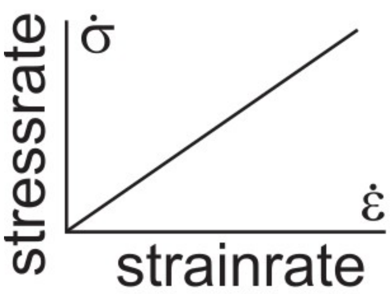
For an **isotropic body** (properties of materials are directional independent) the elastic parameters reduce to two scalar parameters λ and μ, called Lame's parameters. The result is Hooke's law (stress and strain or their rates are linearly dependent).

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$



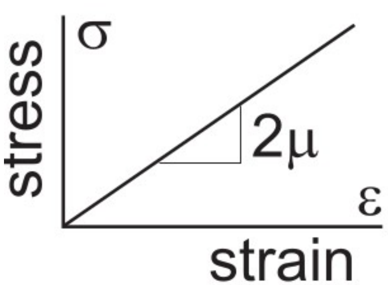
σ_{ij} stress tensor ε_{ij} strain tensor

$\varepsilon_{kk} = (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$ or volumetric strain



μ shear modulus (unit: Pa) relating shear stress to shear strain

λ is unitless (no name!) but both called Lame's parameters



Shear component ($i \neq j$) gives:
$\sigma_{ij} = 2\mu\varepsilon_{ij}$ e.g. i = 1 and j = 2 $\sigma_{12} = 2\mu\varepsilon_{12}$

Normal components (i = j = 1, 2, or 3):
$\sigma_{ii} = (3\lambda + 2\mu)\varepsilon_{ii} = 3K\varepsilon_{ii}$

Lame's parameters can define other elastic parameters which are convenient for particular applications:

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$

Young's modulus (Unit: Pa)

$$\nu = \frac{\lambda}{2(\lambda + \mu)}$$

Poisson number (unitless)

$$E\epsilon_{ij} = (1 + \nu)\sigma_{ij} - \nu \sigma_{kk} \delta_{ij}$$

$\sigma_{kk} = (\sigma_{11} + \sigma_{22} + \sigma_{33})$ or volumetric stress

Bulk or compression modulus (*K*) relates mean normal stress $\sigma_{kk}/3$ to volumetric strain ε_{kk}:

$$\sigma_{kk} = 3K\epsilon_{kk}$$

There are five widely used elastic constants (*μ, λ, K, ν, E*) only two are independent.

$$K = \frac{2\mu(1 + \nu)}{3(1 - 2\nu)} = \lambda + \frac{2}{3}\mu,$$

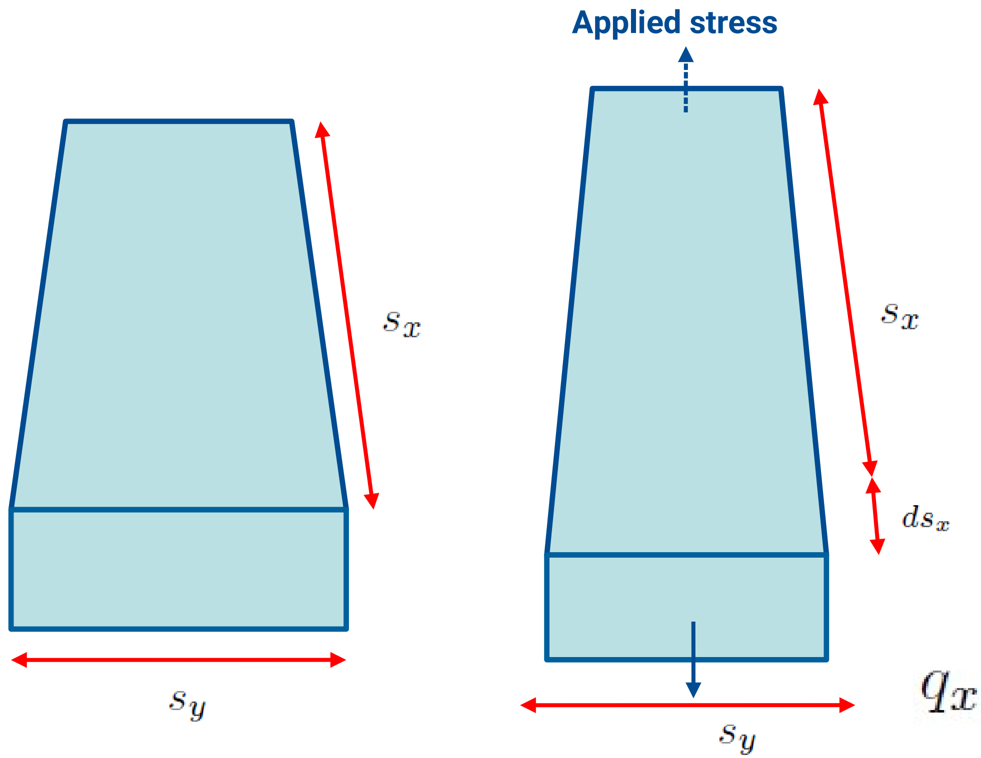
$$\lambda = \frac{2\mu\nu}{(1 - 2\nu)},$$

$$E = 2\mu(1 + \nu),$$

$$\frac{\mu}{\lambda + \mu} = 1 - 2\nu,$$

$$\frac{\lambda}{\lambda + 2\mu} = \frac{\nu}{1 - \nu}.$$

Young's modulus relates stress and strain for the special case of uniaxial stress:

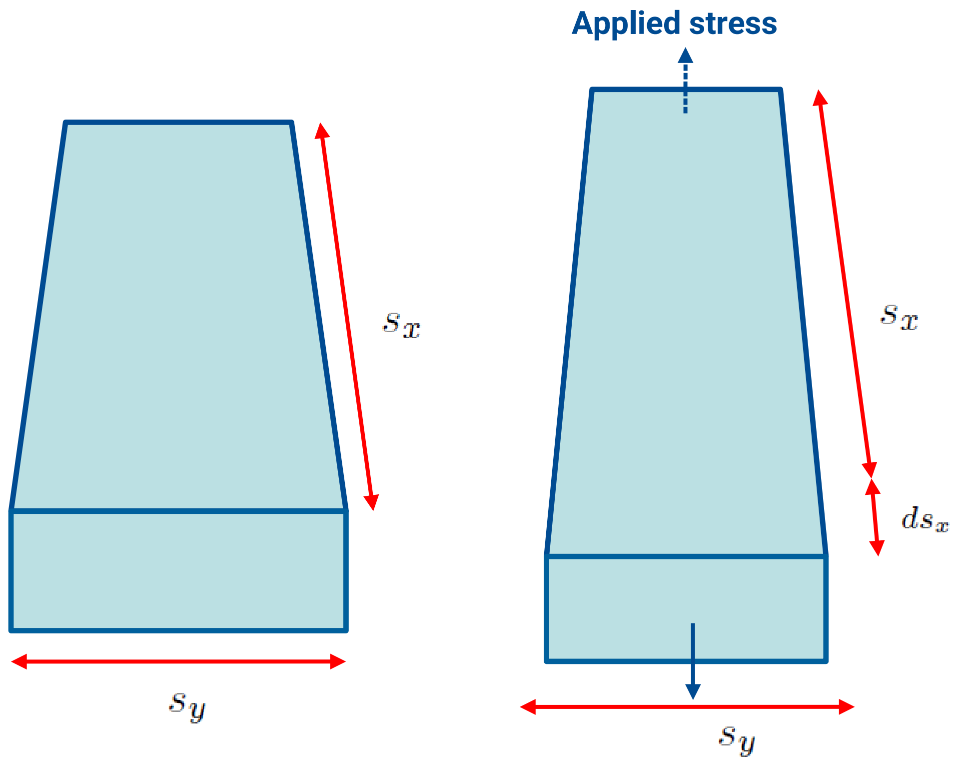


$$q_x = \frac{ds_x}{s_x} = \frac{\sigma}{E}$$

Strain in direction of applied stress is proportional to 1/*E*.

***E* is always a positive number and it can be viewed as a measure of stiffness!**

Poisson's ratio measures the ratio of strain in the orthogonal direction to that in the direction the stress is applied.



$$q_y = q_z = -\frac{\nu}{E}\sigma$$

Strain in direction of normal to stress direction is proportional to *ν*

ν **is bounded by −1 ≤ ν ≤ 0.5. For ν = 0.5, 1/*K* = 0, and the material is incompressible**

Constitutive equation: elastic parameters
Some physical insights

Granite:
E = 50 GPa

Sandstone:
E = 20 GPa = $20 \cdot 10^9$N/m²

Normal stress 5 MPa

$$q_x = \frac{ds_x}{s_x} = \frac{\sigma}{E}$$

Normal stress 5 MPa

Strain: q=0.001

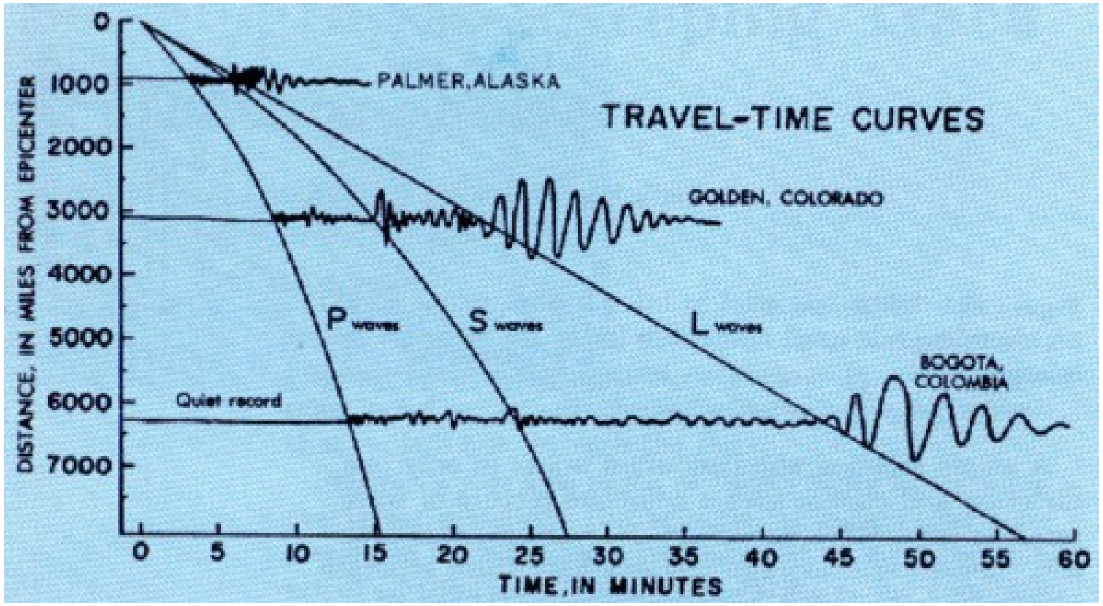
Strain: q=?

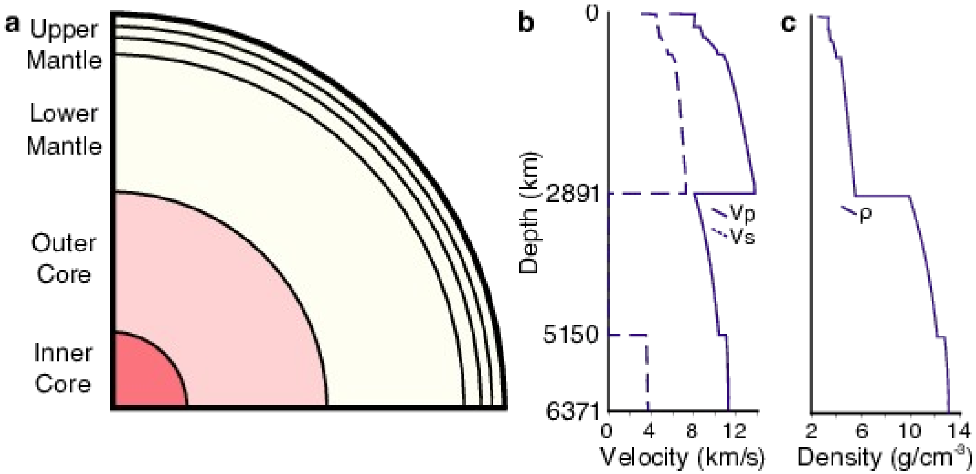
The elastic parameters of rocks are computed in labs be applying a known stress and measuring strain.

This is done in seismology:

1) **Measuring travel times of Earthquake-generated seismic waves (P wave and S wave)**
2) **Velocities v_p and v_s can be converted to elastic parameters using empirical relations such as**

$$\kappa = \rho\left(v_p^2 - \frac{4}{3}v_s^2\right)$$

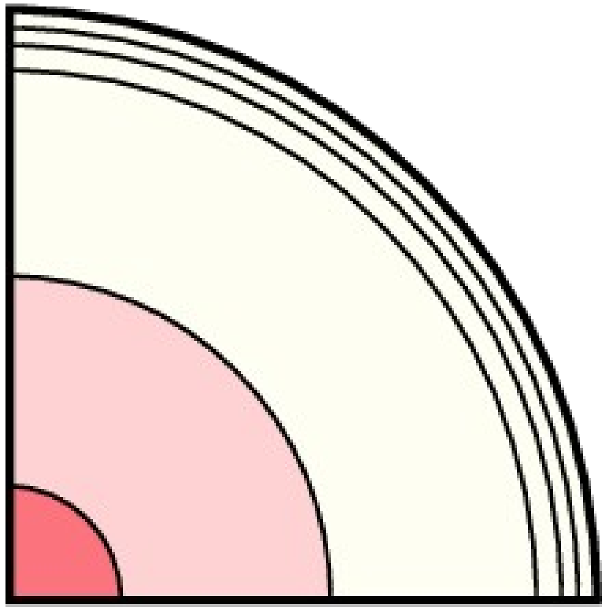




Primary (compressional) P-Wave

Secondary (shear) S-Wave

Constitutive equation: elastic parameters values within the Earth



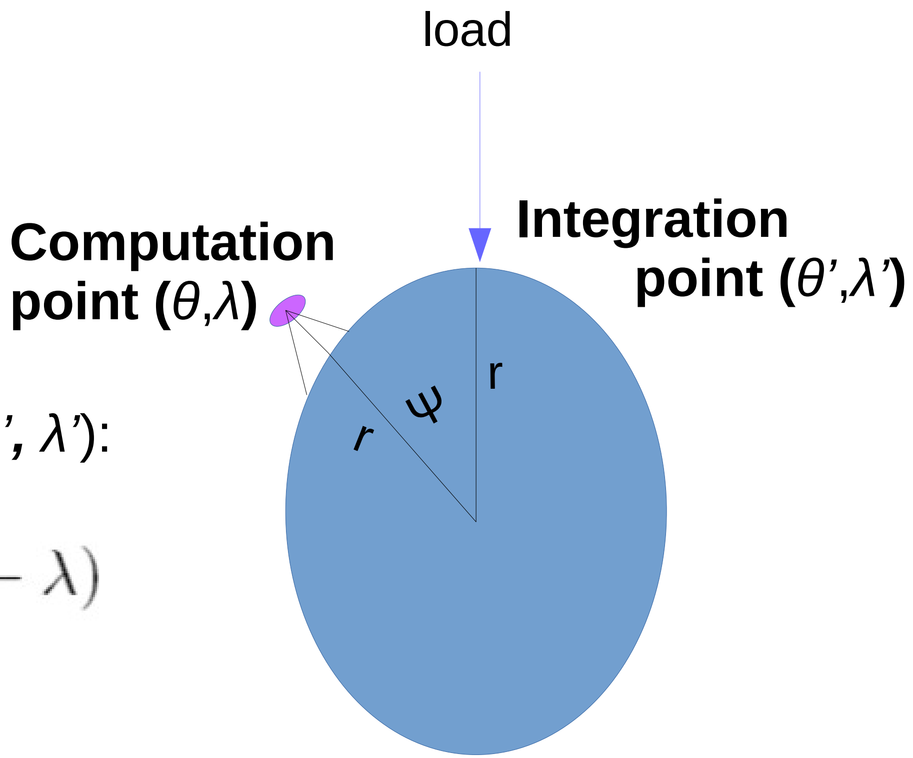
Region	$\frac{r}{r_E}$ [-]	ρ [g cm^{-3}]	λ [10^{12}g cm^{-1} s^{-2}]	μ [10^{12}g cm^{-1} s^{-2}]	ν [-]
Inner core	$0 \ldots 0.20$	12.3	13.4	1.1	0.46
Outer fluid core	$0.20 \ldots 0.55$	11.2	$12.7 \ldots 7.1$	0	0.50
Lower mantle	$0.55 \ldots 0.84$	5.1	$4.4 \ldots 2.4$	2.4	0.29
Upper mantle	$0.84 \ldots 0.99$	3.8	$2.2 \ldots 0.8$	1.1	0.27
Crust	$0.99 \ldots 1$	2.8	0.4	0.4	0.26

Spatial convolution of Green's function and load

We want evaluate the deformations at point of interest like at a GPS site with a coordinate (θ,λ) at time t due to surface density variations $\Delta\sigma(\theta', \lambda')$, that is at time t **over a certain integration domain** *dσ:*

$$v(\theta,\ \lambda,\ t) = a^2 \iint_{\sigma} \Delta\sigma(\theta',\ \lambda',\ t)\ \ G(\psi)\ \ d\sigma$$

a **is average radius of Earth.**



Ψ **is spherical distance between (θ,λ) and (θ', λ'):**

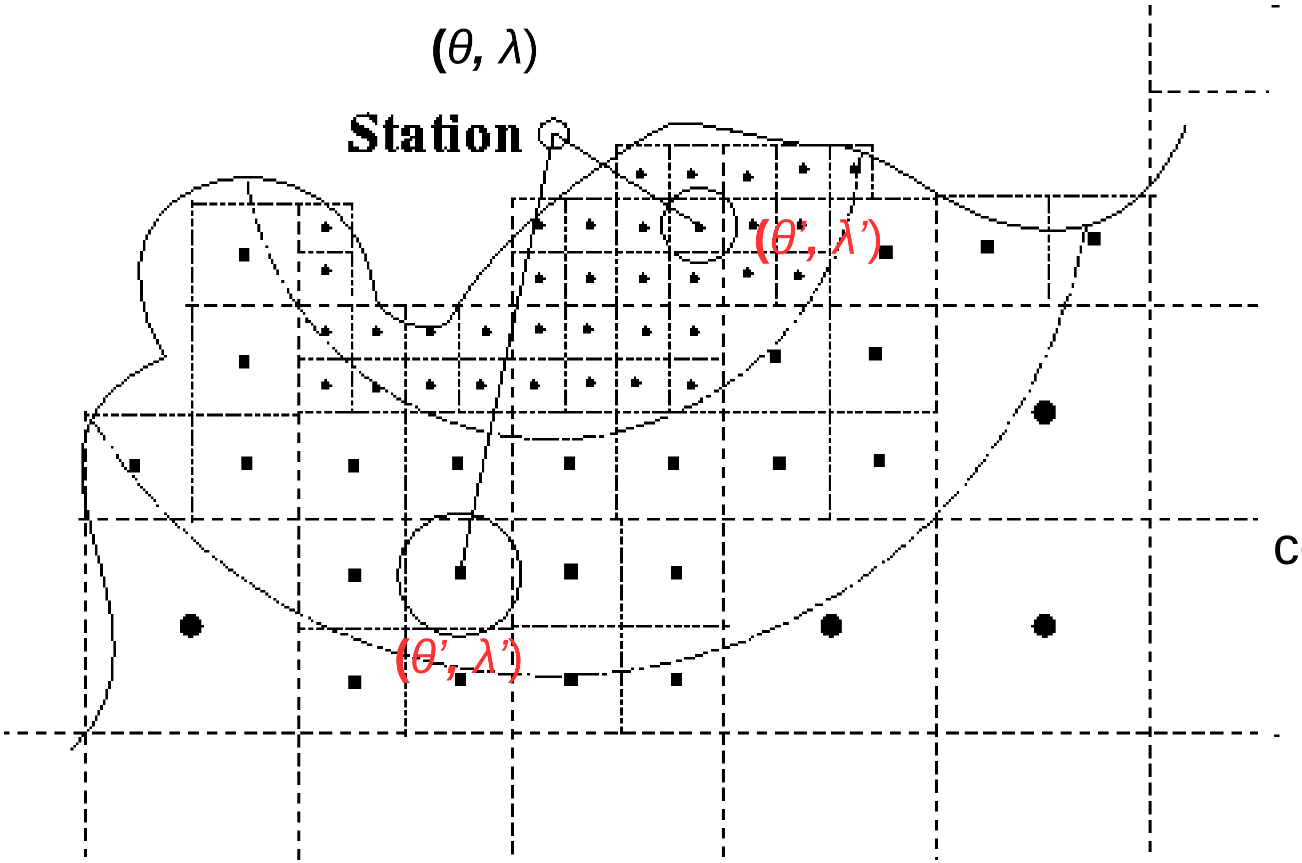
$$\cos\psi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\lambda' - \lambda)$$

G(*Ψ*) is a Green's function for vertical or horizontal displacement and is calculated based on elastic-half space model or spherical symmetric isotropic layered model.

In general the displacement Green's function is response of an Elastic earth to unit point load mass exerted at the pole.

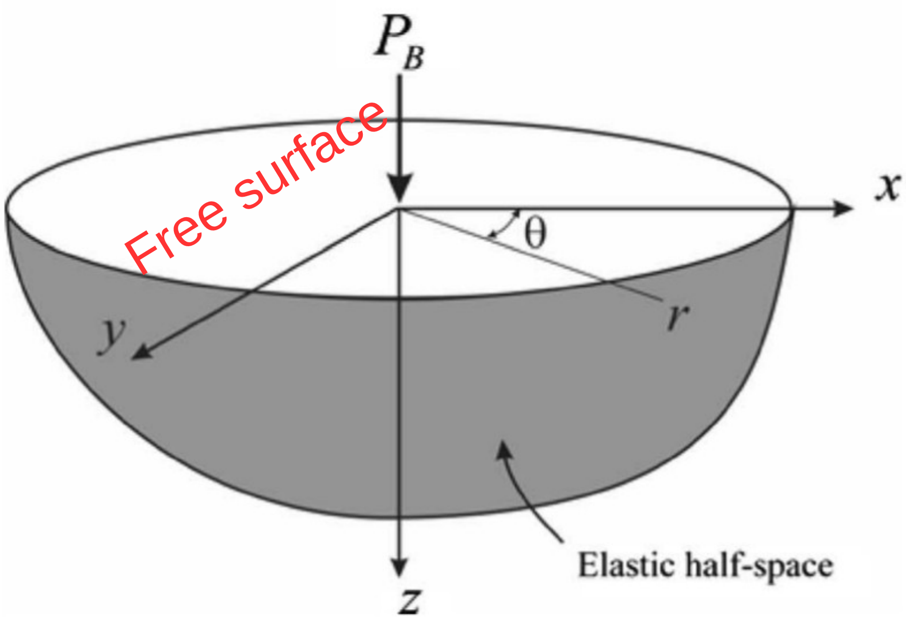
The convolution integral is generally discretized and numerically evaluated by means of summation of the products $\sigma(\theta', \lambda')G(\psi)$ **over** j **compartments within the entire globe or a specific domain.**

$$v\ (\theta, \lambda, t) = \sum_{k}^{j} \Delta\sigma_k G(\psi_k) A_k$$



Stokes integral is another convolution integral in geodesy which is used for geoid determination (next lectures)

Boussinesq problem



* A cylindrical coordinates system (r, θ, z), centered on the force application point.

* A point load P_B acts on the surface

* Stress-free surface
(no shear stress on surface)

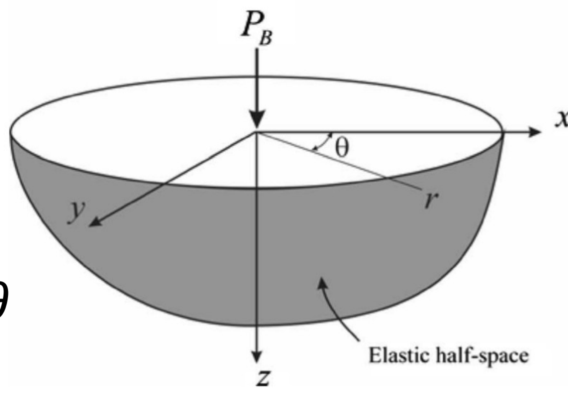
* Half-space elastic
(Lame parameters defined the elastic rheology)

* Homogeneous and isotopic Earth
(Elastic parameters are radially and laterally constants)
$\mu(\mathbf{r}) = \mu$ and $\lambda(\mathbf{r}) = \lambda$

* Non-gravitational Earth (the Earth gravity filed is ignored)

* Flat Earth assumption

Boussinesq, J (1885). Applications des Potentials a l' Equilibre et Mouvement des Solides Elastiques. p.231.



* **Analytical solution**

* **Because the surface point load is axially symmetric, the displacement doesn't depend on the longitude and all components of surface displacements are independent of θ**

$$u_r = \frac{P}{4\pi\mu R}\left[\frac{rz}{R} - \frac{(1-2\nu)r}{R+z}\right]$$

$$u_\theta = 0$$

$$u_z = \frac{P}{4\pi\mu R}\left[2(1-\nu) + \frac{z^2}{R^2}\right]$$

P = **Load magnitude for example a column of water thickness *h* exerts a pressure *P* = *ρgh***

R = **horizontal distance between the observation point and the point load**

* **There are only two adjustable model parameters: the Young's modulus (μ) and the Poisson's ratio (ν).**

* **The sensitivity to the Poisson's ratio is actually negligible so that in practice only the Young's modulus is adjusted and ν is generally set to a standard value of 0.25**

* **The half-space model may be reasonable if the load extends over small area for example lakes, dams, etc.**

Displacement field in Boussinesq problem

$$u_r = \frac{P}{4\pi\mu R}\left[\frac{rz}{R} - \frac{(1-2\nu)r}{R+z}\right]$$

$$u_\theta = 0$$

$$u_z = \frac{P}{4\pi\mu R}\left[2(1-\nu) + \frac{z^2}{R^2}\right]$$

u_r, u_z are Green's function for vertical and horizontal displacement and should be convolved with distribution of loads (here surface density) within an area of interest to calculate the deformation at a point of desire:

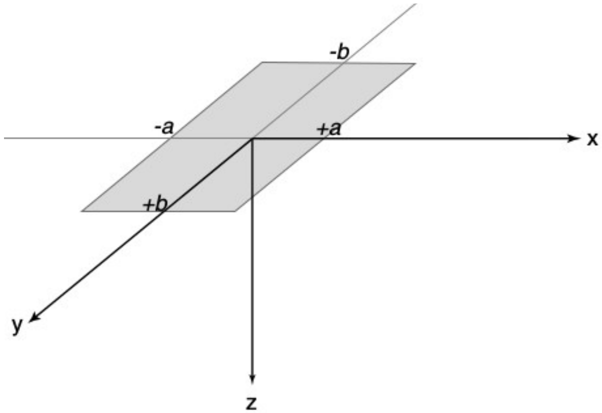
$$v(\theta,\ \lambda,\ t) = a^2 \iint_\sigma \Delta\sigma(\theta',\ \lambda',\ t)\ G(\psi)\ d\sigma$$

Because the extend of loads are small for the half-space model, the convolution integral is evaluated over a limited domain within which loading data are available (e.g. within a lakes or a dams and etc.)

The half space model can be further developed to take into account the complex shape of loads and the radial non-homogeneity of elastic parameters:

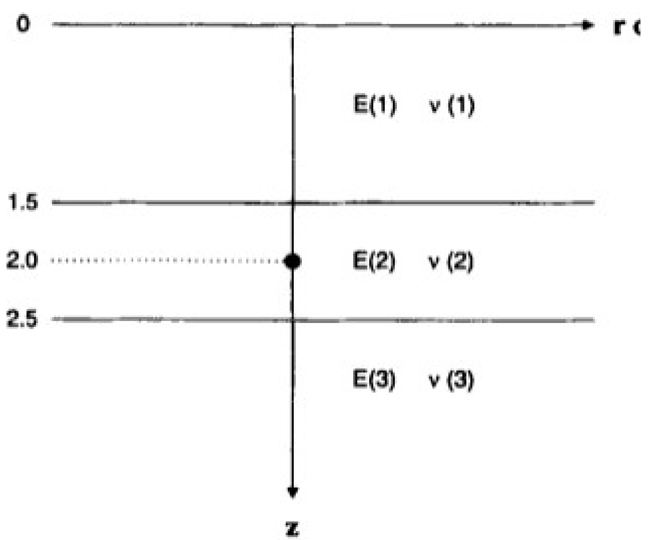
Some examples:

* A rectangular load has been solved by Becker & Bevis (2004) and obtained a semi-analytical solution.

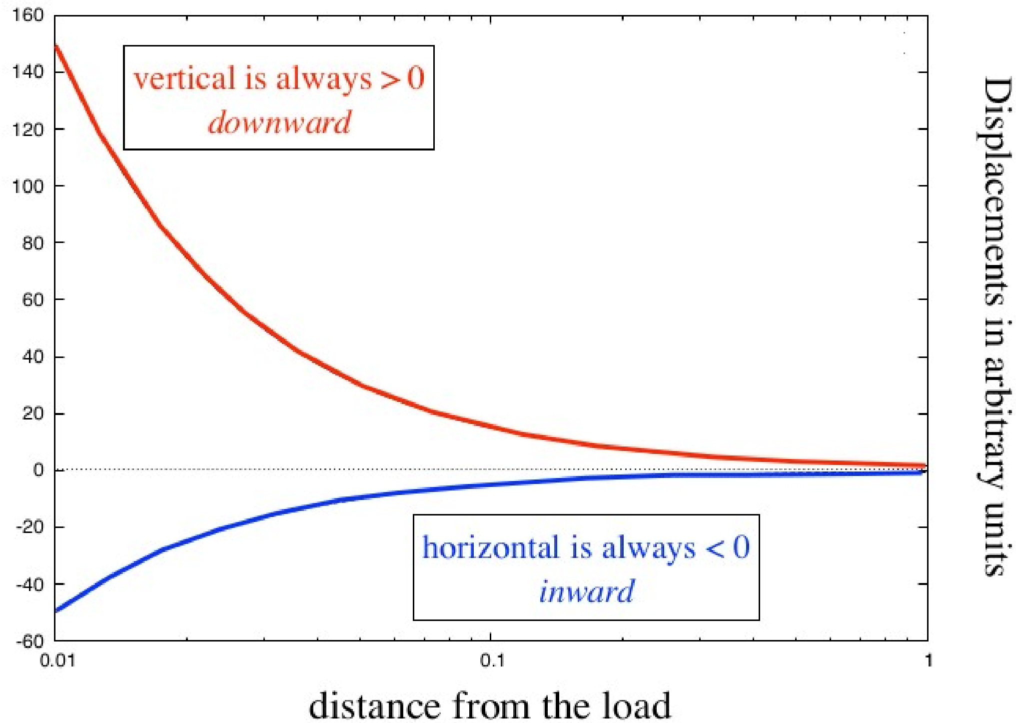


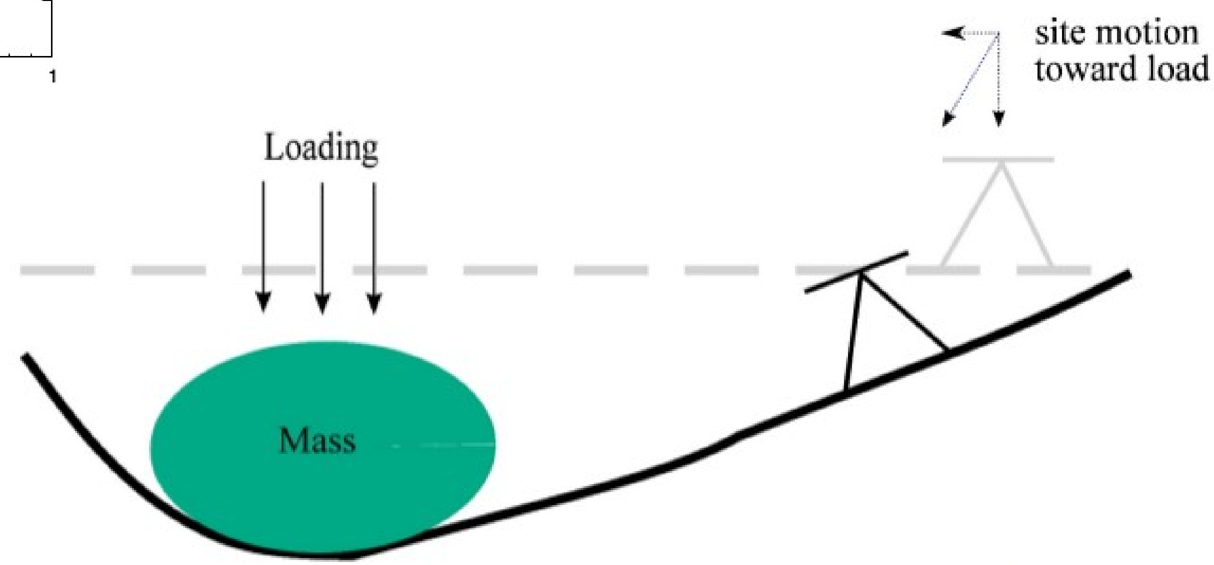
* A simple line model by Jaeger et al. (2007) and used first by Jiang et al (2010) to model uplift in Greenland due to mass loss.

* A multi-layer half pace model suggested by Pan (1997) to take into account the effect of strata

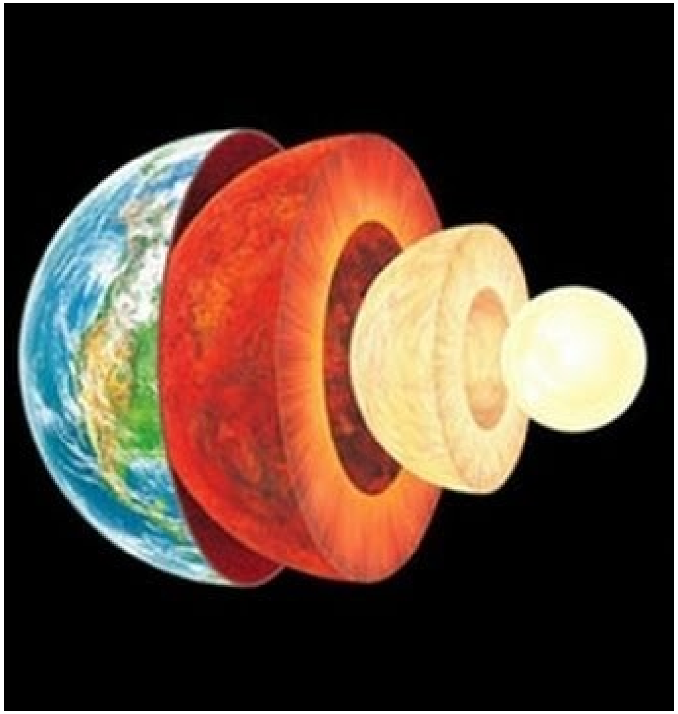


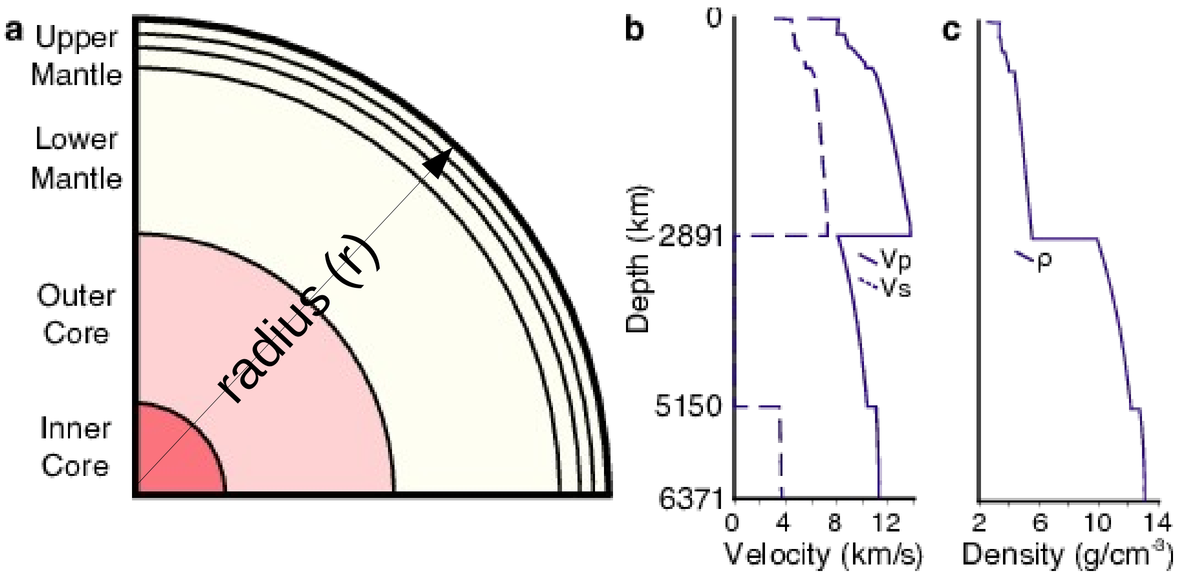
For elastic half-space model, load always cause downward and inward deformation





Spherical and layered Earth model





Elastic parameters and density vary only radially and are constant laterally. This is 1D or spherically symmetric Earth model.

Elastic and Self-gravitating

$$\mu = \mu(r)$$

$$\lambda = \lambda(r)$$

$$\rho = \rho(r)$$

PREM Earth model **(Dziewonski and Anderson, 1981)**
Iasp91 Earth model **(Kennett and Engdahl, 1991)**
ak135 model **(Kennett et al., 1995)**

Three equations for a spherical Earth model

1. Equations of motion or equation momentum conservation

$$\nabla \cdot \boldsymbol{\sigma}_1 - \nabla(\rho_0 g \mathbf{u} \cdot \mathbf{e}_r) - \rho_0 \nabla \phi_1 - \rho_1 g \mathbf{e}_r = 0$$

2. Poisson equation inside the Earth

$$\nabla^2 \phi_1 = 4\pi G \rho_1,$$

3. Continuity equation

$$\rho_1 = -\nabla \cdot (\rho_0 \mathbf{u}) = -\mathbf{u} \cdot \mathbf{e}_r \, \partial_r \rho_0 - \rho_0 \nabla \cdot \mathbf{u},$$

After integrating these three differential equations with some boundary condition for a spherical Earth model, the unitless elastic loading Love numbers are computed.

The Green's functions are then calculated using these Love numbers, allowing to calculate the displacement or stress for distributed loads using Green's integral

Equation of conservation of momentum

Law conservation of momentum requires that the body forces F (per unit mass) acting on the element of the body are balanced by the stresses that act on the surface of the element.

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{F} = 0$$

Body forces F include act at all points in the Earth and include:

1. **Gravity including gravitational & centrifugal forces**

2. **Loads including those in physical contact with Earth's surface such as hydrological loads, ice-water redistribution, atmospheric pressure, earthquake, etc. or gravitation pull from outside like tidal forces.**

σ: stress tensor

ρ: density

The stress tensor is the sum of the initial hydrostatic pressure (p_0) and a shear change (non-hydrostatic stress that later can be related to strain)

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 - p_0 \mathbf{I}.$$

The hydrostatic pressure p_0 has negative sign (compressive stress)
*** positive means stress act outward normal to the surface.**

so equation of conservation of momentum gets:

$$\nabla \cdot \boldsymbol{\sigma}_1 - \nabla p_0 + \rho \mathbf{F} = 0.$$

Let the body deform elastically u = (u_θ, u_λ, u_r) in t_0, then the pressure after a small time increment δt will be:

$$p_0(t_0 + \delta t) = p_0(t_0) - \mathbf{u} \cdot \nabla p_0.$$

Note: pressure increase mean negative displacement and vice versa

then equation of conservation of momentum reads:

$$\nabla \cdot \boldsymbol{\sigma}_1 - \nabla p_0(t_0) + \nabla(\mathbf{u} \cdot \nabla p_0) + \rho \mathbf{F} = 0.$$

Note the hydrostatic pressure depends only on radius (depth):

$p(\boldsymbol{r}) = \rho\, g\, \mathbf{r}$ **[N/m²] or [kg/s²m]**

ρ: **density**
g: **gravitational attraction**
both depends on depth

$$\frac{dp}{dr} = -\rho(r)g(r)$$

The gradient of initial pressure p_0 gets

$$\nabla p_0 = -\rho_0 g \mathbf{e}_r,$$

ρ_0 **Initial density**

where $\mathbf{e}_r$ is the unit vector, positive outward from the Earth center.

$$\nabla \cdot \boldsymbol{\sigma}_1 - \nabla p_0(t_0) + \nabla(\mathbf{u} \cdot \nabla p_0) + \rho \mathbf{F} = 0.$$

$$\downarrow$$

$$\nabla \cdot \boldsymbol{\sigma}_1 - \nabla p_0(t_0) - \nabla(\rho_0 g \mathbf{u} \cdot \mathbf{e}_r) + \rho \mathbf{F} = 0.$$

Let's assume that the body force F is only gravity - it can be expressed as the negative gradient of the potential:

$$\mathbf{F} = -\nabla\phi.$$

The potential is sum of initial state Φ_0 **and the small perturbation** Φ_1

$$\phi = \phi_0 + \phi_1,$$

$\rho \boldsymbol{F} = -\rho\,\nabla(\Phi_0 + \Phi_1) = -\rho\,\nabla\Phi_0 - \rho\,\nabla\Phi_1$

The initial density is also changes ρ_0**:** $\rho = \rho_0 + \rho_1$

$\rho \boldsymbol{F} = -\rho\,\nabla\Phi_0 - \rho\,\nabla\Phi_1 = -(\rho_0 + \rho_1)\,\nabla\Phi_0 - (\rho_0 + \rho_1)\,\nabla\Phi_1 = -\rho_0\nabla\Phi_0 - \rho_1\nabla\Phi_0 - \rho_0\nabla\Phi_1 - \rho_1\nabla\Phi_1$

where from previous slide $\rho_0\nabla\Phi_0 = \rho_0 g$ **is** $\nabla\rho_0$ **and let** $\rho_1\nabla\Phi_0 = \rho_1 \mathrm{g}$ **and** $\rho_1\nabla\Phi_1 \approx 0$

Now replace in momentum equation

$$\nabla\cdot\boldsymbol{\sigma}_1 - \nabla p_0(t_0) - \nabla(\rho_0 g\mathbf{u}\cdot\mathbf{e}_r) + \rho\mathbf{F} = 0.$$

and we get:

$$\nabla\cdot\boldsymbol{\sigma}_1 - \nabla(\rho_0 g\mathbf{u}\cdot\mathbf{e}_r) - \rho_0\nabla\phi_1 - \rho_1 g\mathbf{e}_r = 0$$

Equation of conservation of momentum

$$\nabla \cdot \boldsymbol{\sigma}_1 - \nabla(\rho_0 g \mathbf{u} \cdot \mathbf{e}_r) - \rho_0 \nabla \phi_1 - \rho_1 g \mathbf{e}_r = 0$$

Contribution due to shear stress change

Contribution of deformation

Effects of density changes (<u>compressibility</u>)

Contribution due to changes in gravitational potential (gravity), often called self-gravitational effect

For a incompressible Earth effects of pressure on the density are zero so the last term will be zero.

PREM is constructed based on an incompressible model.

The perturbed gravitational potential field satisfies the Poisson equation within the volume of the Earth:

$$\nabla^2 \phi_1 = 4\pi G \rho_1$$

For an incompressible Earth the density change ρ_1 will be zero so equation above reduces to the Laplace equation:

$$\nabla^2 \phi_1 = 0.$$

Three equations for a spherical Earth model

Equations of motion: $$\nabla \cdot \boldsymbol{\sigma}_1 - \nabla(\rho_0 g \mathbf{u} \cdot \mathbf{e}_r) - \rho_0 \nabla \phi_1 - \rho_1 g \mathbf{e}_r = 0$$

Poisson equation inside the Earth: $$\nabla^2 \phi_1 = 4\pi G \rho_1,$$

Continuity equation: $$\rho_1 = -\nabla \cdot (\rho_0 \mathbf{u}) = -\mathbf{u} \cdot \mathbf{e}_r \, \partial_r \rho_0 - \rho_0 \nabla \cdot \mathbf{u},$$

For an incompressible Earth model the density perturbation ρ_1 will be zero so we will get two equations:

$$\nabla \cdot \boldsymbol{\sigma}_1 - \nabla(\rho_0 g \mathbf{u} \cdot \mathbf{e}_r) - \rho_0 \nabla \phi_1 = 0 \qquad \nabla^2 \phi_1 = 0$$

Spherical harmonic solution for displacement vector u = $u_r\mathbf{e}_\mathbf{r} + u_v\mathbf{e}_\mathbf{v}$ **and perturbed potential** Φ:

$$\mathbf{u} = \sum_{n=0}^{\infty} \left(U_n(r) P_n(\cos\psi) \mathbf{e}_\mathbf{r} + V_n(r) \frac{\partial P_n(\cos\psi)}{\partial \psi} \mathbf{e}_\mathbf{v} \right)$$

$$\phi = \sum_{n=0}^{\infty} \Phi_n(r) P_n(\cos\psi)$$

with radial-dependent spherical harmonic coefficients $U_n(r)$, $V_n(r)$ and $\Phi_n(r)$.

* $U_n(r)$, $V_n(r)$ have unit of meter and $\Phi_n(r)$ has potential unit.

Loading Love numbers

Since we want to compute deformation, stress and strain on the Earth's surface r = R

The $U_n(R)$, $V_n(R)$ and $\Phi_n(R)$ coefficients can be related to potential of point load mass W and some degree dependent coefficients, the so-called loading Love numbers as:

$$U_n = W\,h_n/g \qquad V_n = W\,l_n/g \qquad \Phi_n = W\,k_n$$

h_n actually converts the potential of point load mass (W) to coefficients of vertical displacement

l_n actually converts the potential of point load mass (W) to coefficients of horizontal displacement

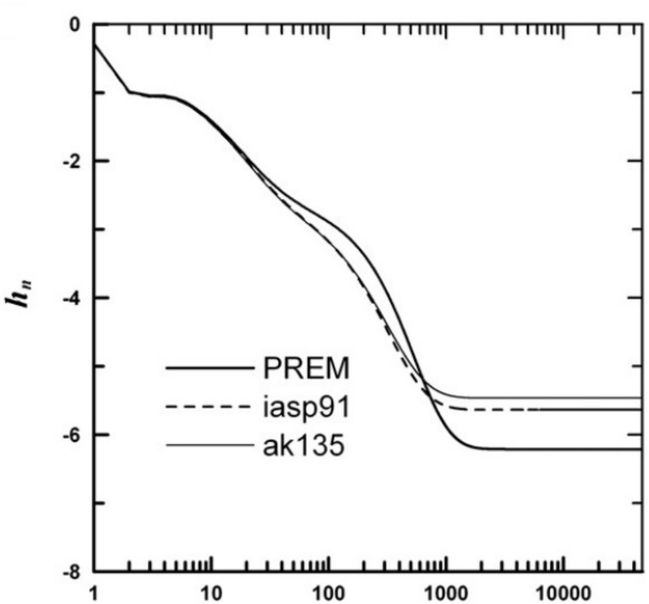
k_n actually converts the potential of point load mass (W) to coefficients of perturbed potential

g is gravitational acceleration at Earth's surface

Loading Love numbers

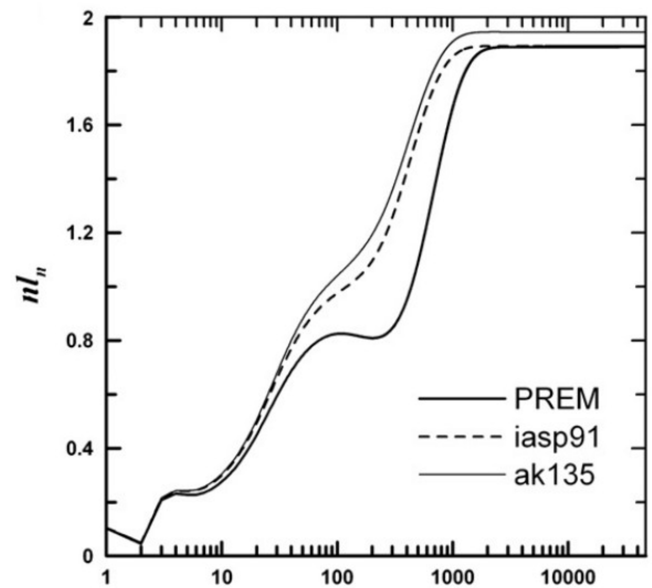
Putting back the $U_n(R)$, $V_n(R)$ and $\Phi_n(R)$ coefficients into differential equations and considering the elastic consecutive equation, with applying the boundary conditions corresponding to different layers of 1D Earth model, the Love number are estimated (Farrell, 1972).

h_n first Love number



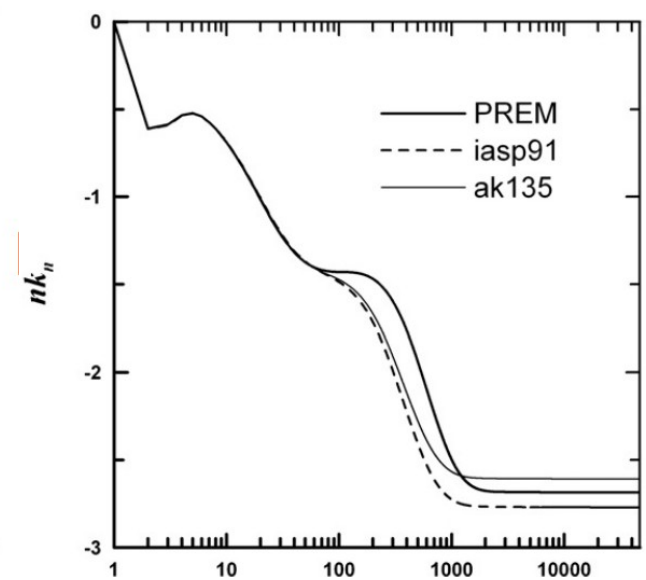
h_n **is used for calculating vertical deformation**

l_n second Love number



l_n **is used for calculating horizontal deformation**

k_n third Love number

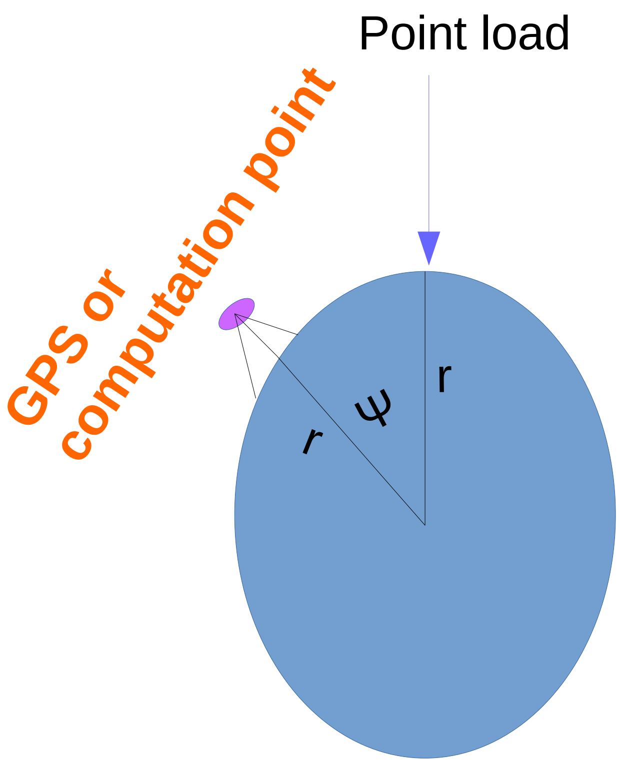


k_n **is used for calculating potential (mass) changes**

Use Love numbers to calculate displacement (Green's function)

$$\mathbf{u} = \sum_{n=0}^{\infty} \left(U_n(r) P_n(\cos\psi) \mathbf{e_r} + V_n(r) \frac{\partial P_n(\cos\psi)}{\partial\psi} \mathbf{e_v} \right)$$

U_r:vertical disp. U_v: horizontal disp.



For a point load with unit mass:

$$G_{u_r} = \frac{R}{M} \sum_{n=0}^{\infty} h_n P_n(\cos\Psi)$$

$$G_{u_v} = \frac{R}{M} \sum_{n=1}^{\infty} l_n \frac{\partial}{\partial\Psi} P_n(\cos\Psi)$$

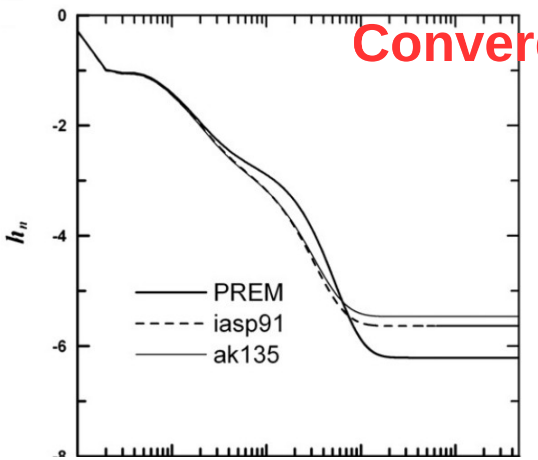
M **mean mass of Earth**
R **mean radius of Earth**

Similar Green's functions exist for indirect effect of gravity, effect of tilt and strain components at the surface (Farrell, 1972)

Use Kummar's transformation to converge the Green's functions

$$G_{u_r} = \frac{R}{M} \sum_{n=0}^{\infty} h_n P_n(\cos\Psi)$$

$$G_{u_v} = \frac{R}{M} \sum_{n=1}^{\infty} l_n \frac{\partial}{\partial\Psi} P_n(\cos\Psi)$$



Convergence problem?

$$h_\infty = \lim_{n\to\infty} h_n$$

$$nl_\infty = \lim_{n\to\infty} nl_n$$

$$nk_\infty = \lim_{n\to\infty} nk_n$$

$$G_{u_r} = \frac{R}{M} h_\infty \sum_{n=0}^{\infty} P_n(\cos\Psi) + \frac{R}{M} \sum_{n=0}^{\infty} (h_n - h_\infty) P_n(\cos\Psi)$$

$$G_{u_v} = \frac{R}{M} l_\infty \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial}{\partial\Psi} P_n(\cos\Psi) + \frac{R}{M} \sum_{n=1}^{\infty} (nl_n - l_\infty) \frac{1}{n} \frac{\partial}{\partial\Psi} P_n(\cos\Psi)$$

$$G_{u_r} = \frac{R}{2M} \frac{h_\infty}{\sin(\Psi/2)} + \frac{R}{M} \sum_{n=0}^{\infty} (h_n - h_\infty) P_n(\cos\Psi)$$

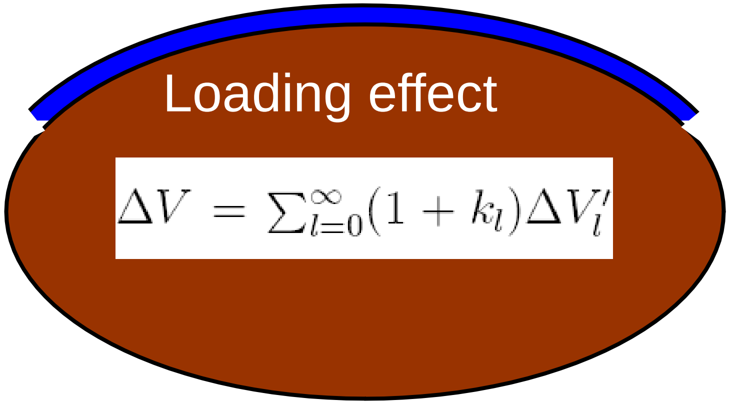
$$G_{u_v} = -\frac{R}{M} l_\infty \frac{\cos(\Psi/2)[1 + 2\sin(\Psi/2)]}{2\sin(\Psi/2)[1 + \sin(\Psi/2)]} + \frac{R}{M} \sum_{n=1}^{\infty} (nl_n - l_\infty) \frac{1}{n} \frac{\partial}{\partial\Psi} P_n(\cos\Psi)$$

Third Love number K_l

Total changes in potential is sum of potential load and indirect effect due to solid Earth deformation. The last term is represented by third Love number K_l.

$\Phi_n = W k_n$

$$\begin{Bmatrix} \Delta C_{lm}^{\text{solid Earth}} \\ \Delta S_{lm}^{\text{solid Earth}} \end{Bmatrix} = k_l \begin{Bmatrix} \Delta C_{lm}^{\text{surf mass}} \\ \Delta S_{lm}^{\text{surf mass}} \end{Bmatrix}$$



$$\Delta V(r,\lambda,\theta) = 4\pi G \frac{r_E^3 \rho_w}{r} \sum_{n=0}^{\infty} \left(\frac{r_E}{r}\right)^n \frac{1+k_l}{2n+1} \sum_{m=0}^{n} (\Delta c_{nm}^{\sigma} \cos m\lambda + \Delta s_{nm}^{\sigma} \sin m\lambda) P_{nm}(\cos\theta)$$

K_l is used to relate the potential and surface density SHCs

Potential (geoid height) SHCs ↔ Mass density SHCs

What GRACE delivers

$$\begin{Bmatrix} \Delta C_{lm}^{N} \\ \Delta S_{lm}^{N} \end{Bmatrix} = \frac{3\rho_w}{\rho_e} \frac{1+k_l}{2l+1} \begin{Bmatrix} \Delta C_{lm}^{\sigma} \\ \Delta S_{lm}^{\sigma} \end{Bmatrix}$$

The convolution integral for calculating deformation can be represent by spherical harmonic expression which is easier to use when dealing with global loading data such as GRACE and hydrological models:

$$v(\theta, \lambda, t) = a^2 \iint_\sigma \Delta\sigma(\theta', \lambda', t)\; G(\psi)\; d\sigma$$

Note the integration domain is entire Earth

$$G_{u_r} = \frac{R}{M} \sum_{n=0} h_n P_n(\cos \Psi)$$

$$P_n(\cos\psi) = \frac{1}{2n+1} \sum_{m=0}^{n} \bar{P}_{nm}(\cos\theta')\bar{P}_{nm}(\cos\theta)\cos m(\lambda - \lambda')$$

By putting P_n in G_u and then in convolution integral and using the spherical harmonic expression of surface load $\Delta\sigma$:

$$\Delta\sigma(\theta, \lambda, t) = a\rho_w \sum_{n=0}^{\infty} \sigma_n(\theta, \lambda, t) = a\rho_w \sum_{n=0}^{\infty} \sum_{m=0}^{n} (\Delta\bar{C}_{nm}^{\sigma} \cos m\lambda + \Delta\bar{S}_{nm}^{\sigma} \sin m\lambda)\bar{P}_{nm}(\cos\theta)$$

$$\begin{Bmatrix} \Delta\bar{C}_{nm}^{\sigma} \\ \Delta\bar{S}_{nm}^{\sigma} \end{Bmatrix} = \frac{\rho_e}{3\rho_w} \frac{2n+1}{1+k'_n} \begin{Bmatrix} \Delta\bar{C}_{nm} \\ \Delta\bar{S}_{nm} \end{Bmatrix}$$

Potential (geoid) SHCs or often called potential Stokes coefficients *(what GRACE delivers to us)*

after applying orthogonal propertiy of Legendre polynomial, after a few simplification we get (see Mitrovia et al. 1994, JGR**):**

Up component of displacement:

$$\Delta h = R\sum_{l=1}^{\infty}\sum_{m=0}^{l}\bar{P}_{lm}(\cos\theta)\cdot[\Delta\bar{C}_{lm}\cos(m\lambda)+\Delta\bar{S}_{lm}\sin(m\lambda)]\cdot\frac{h_l}{1+k_l}$$

East component of displacement:

$$\Delta e = \frac{R}{\sin\theta}\sum_{l=1}^{\infty}\sum_{m=0}^{l}\bar{P}_{lm}(\cos\theta)\cdot m\cdot[-\Delta\bar{C}_{lm}\sin(m\lambda)+\Delta\bar{S}_{lm}\cos(m\lambda)]\cdot\frac{l_l}{1+k_l}$$

North component of displacement:

$$\Delta n = -R\sum_{l=1}^{\infty}\sum_{m=0}^{l}\frac{\partial}{\partial\theta}\bar{P}_{lm}(\cos\theta)\cdot[\Delta\bar{C}_{lm}\cos(m\lambda)+\Delta\bar{S}_{lm}\sin(m\lambda)]\cdot\frac{l_l}{1+k_l}$$

The expansion begins from degree 1. Degree 0 corresponds to entire mass change of Earth and is set to zero due to conservation of mass.

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* Spherical and layered Earth model

* Spatial convolution of Green's function and load

* Boussinesq problem: half-space models

* Three equations for a spherical Earth model: equation of conservation of momentum, Poisson equation and equation of continuity

* Loading Love numbers

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* Spherical harmonic approach for computing displacement

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