CHAPTER 9

State feedback control

Objectives

After completing this chapter, the reader will be able to do the following:

1. Design state feedback control using pole placement.
2. Design servomechanisms using state−space models.
3. Analyze the behavior of multi variable zeros under state feedback.
4. Design state estimators (observers) for state−space models.
5. Design controllers using observer state feedback.
6. Assign system poles using transfer functions with output feedback.

State variable feedback allows the flexible selection of linear system dynamics. Often, not all state variables are available for feedback, and the remainder of the state vector must be estimated. This chapter includes an analysis of state feedback and its limitations. It also includes the design of state estimators for use when some state variables are not available and the use of state estimates in feedback control.

Throughout this chapter, we assume that the state vector $\mathbf{x}$ is $n\times1$, the control vector $\mathbf{u}$ is $m\times1$, and the output vector $\mathbf{y}$ is $l\times1$. We drop the subscript d for discrete system matrices. With minor changes, the design methodologies are applicable to continuous-time systems.

Chapter Outline

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9.1 State and output feedback

State feedback involves the use of the state vector to compute the control action for specified system dynamics. Fig. 9.1 shows a linear system (A, B, C) with constant state feedback gain matrix K. Using the rules for matrix multiplication, we deduce that the matrix K is $m \times n$ so that for a single-input (SI) system, K is a row vector.

The equations for the linear system and the feedback control law are, respectively, given by:

$$\begin{aligned} \mathbf{x}(k+1) &= A\mathbf{x}(k) + B\mathbf{u}(k) \\ \mathbf{y}(k) &= C\mathbf{x}(k) \end{aligned} \tag{9.1}$$

$$\mathbf{u}(k) = -K\mathbf{x}(k) + \mathbf{v}(k) \tag{9.2}$$

where $\mathbf{v}(k)$ is the reference input vector.

The two equations can be combined to yield the closed-loop state equation

$$\begin{aligned} \mathbf{x}(k+1) &= A\mathbf{x}(k) + B[-K\mathbf{x}(k) + \mathbf{v}(k)] \\ &= [A - BK]\mathbf{x}(k) + B\mathbf{v}(k) \end{aligned} \tag{9.3}$$

We define the closed-loop state matrix as

$$A_{cl} = A - BK \tag{9.4}$$

and rewrite the closed-loop system state−space equations in the form

$$\begin{aligned} \mathbf{x}(k+1) &= A_{cl}\mathbf{x}(k) + B\mathbf{v}(k) \\ \mathbf{y}(k) &= C\mathbf{x}(k) \end{aligned} \tag{9.5}$$

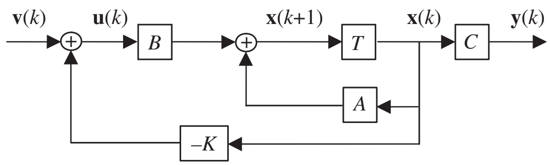

Figure 9.1
Block diagram of constant state feedback control.

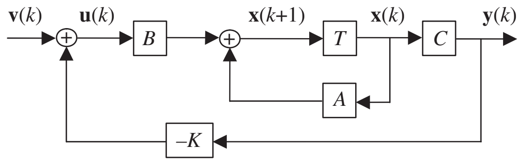

Figure 9.2
Block diagram of constant output feedback control.

The dynamics of the closed-loop system depend on the **eigenstructure** (eigenvalues and eigenvectors) of the matrix A_{cl}. Thus, the desired system dynamics can be chosen with appropriate choice of the gain matrix K. Limitations of this control scheme are addressed in the next section.

For many physical systems, it is either prohibitively costly or impossible to measure all the state variables. The output measurements $\mathbf{y}$ must then be used to obtain the control $\mathbf{u}$ as shown in Fig. 9.2.

The feedback control for output feedback is

$$\begin{aligned}\mathbf{u}(k) &= -K_y\mathbf{y}(k) + \mathbf{v}(k)\\ &= -K_yC\mathbf{x}(k) + \mathbf{v}(k)\end{aligned} \tag{9.6}$$

Substituting in the state equation gives the closed-loop system

$$\begin{aligned}\mathbf{x}(k+1) &= A\mathbf{x}(k) + B[-K_yC\mathbf{x}(k) + \mathbf{v}(k)]\\ &= [A - BK_yC]\mathbf{x}(k) + B\mathbf{v}(k)\end{aligned} \tag{9.7}$$

The corresponding state matrix is

$$A_y = A - BK_yC \tag{9.8}$$

Intuitively, less can be accomplished using output feedback than state feedback because less information is used in constituting the control law. In addition, the postmultiplication by the C matrix in Eq. (9.8) restricts the choice of closed-loop dynamics. However, output feedback is a more general design problem because state feedback is the special case where C is the identity matrix.

9.2 Pole placement

Using output or state feedback, the poles or eigenvalues of the system can be assigned subject to system-dependent limitations. This is known as **pole placement**, **pole assignment**, or **pole allocation**. We state the problem as follows.

Definition 9.1: Pole placement

Choose the gain matrix K or K_y to assign the system eigenvalues to an arbitrary set $\{\lambda_i, i = 1, \ldots, n\}$.

The following theorem gives conditions that guarantee a solution to the pole-placement problem with state feedback.

Theorem 9.1: State feedback

If the pair (A, B) is controllable, then there exists a feedback gain matrix K that arbitrarily assigns the system poles to any set $\{\lambda_i, i = 1, \ldots, n\}$. Furthermore, if the pair (A, B) is stabilizable, then the controllable modes can all be arbitrarily assigned.

Proof

Necessity

We first show that controllability is invariant under state feedback. If the system is not controllable, then by Theorem 8.4 we have $\mathbf{w}_i^T B = 0^T$ for some left eigenvector $\mathbf{w}_i^T$. Premultiplying the closed-loop state matrix by $\mathbf{w}_i^T$ gives

$$\mathbf{w}_i^T A_{cl} = \mathbf{w}_i^T (A - BK) = \lambda_i \mathbf{w}_i^T - \mathbf{w}_i^T BK = \lambda_i \mathbf{w}_i^T$$

Thus, λ_i is an eigenvalue and $\mathbf{w}_i^T$ is a left eigenvector of the closed-loop system for any state feedback gain matrix K. Hence, the i^{th} **eigenpair** (eigenvalue and eigenvector) of A is unchanged by state feedback and cannot be arbitrarily assigned. Therefore, controllability is a necessary condition for arbitrary pole assignment.

Sufficiency

We first give the proof for the SI case where **b** is a column matrix. For a controllable pair (A, B), we can assume without loss of generality that the pair is in controllable form. We rewrite Eq. (9.4) as

$$\mathbf{b}\mathbf{k}^T = A - A_{cl}$$

Proof—cont'd

Substituting the controllable form matrices gives

$$\begin{bmatrix} \mathbf{0}_{n-1\times 1} \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{0}_{n-1\times 1} & I_{n-1} \\ \hline -a_0 \quad -a_1 & \ldots \quad \ldots \quad -a_{n-1} \end{array}\right] - \left[\begin{array}{c|c} \mathbf{0}_{n-1\times 1} & I_{n-1} \\ \hline -a_0^d \quad -a_1^d & \ldots \quad \ldots \quad -a_{n-1}^d \end{array}\right]$$

That is,

$$\left[\begin{array}{cccc} & \mathbf{0}_{n-1\times n} & & \\ \hline k_1 & k_2 & \cdots & k_n \end{array}\right] = \left[\begin{array}{ccccc} & & \mathbf{0}_{n-1\times n} & & \\ \hline a_0^d - a_0 & a_1^d - a_1 & \ldots & \ldots & a_{n-1}^d - a_{n-1} \end{array}\right]$$

Equating the last rows of the matrices yields

$$\begin{bmatrix} k_1 & k_2 & \ldots & k_n \end{bmatrix} = \begin{bmatrix} a_0^d - a_0 & a_1^d - a_1 & \ldots & \ldots & a_{n-1}^d - a_{n-1} \end{bmatrix} \tag{9.9}$$

which is the control that yields the desired characteristic polynomial coefficients.

Sufficiency Proof 2

We now give a more general proof by contraposition—that is, we assume that the result is not true and prove that the assumption is not true. So we assume that the eigenstructure of the j^{th} mode is unaffected by any state feedback for any choice of the matrix K and prove that the system is uncontrollable. In other words, we assume that the j^{th} eigenpair is the same for the open-loop and closed-loop systems. Then we have

$$\mathbf{w}_j^T BK = \mathbf{w}_j^T \{A - A_{cl}\} = \left\{\lambda_j \mathbf{w}_j^T - \lambda_j \mathbf{w}_j^T\right\} = 0^T$$

Assuming K full rank, we have $\mathbf{w}_j^T B = \mathbf{0}$. By Theorem 8.4, the system is not controllable.

For a controllable SI system, the matrix (row vector) K has n entries with n eigenvalues to be assigned, and the pole placement problem has a unique solution. Clearly, with more inputs, the K matrix has more rows and consequently more unknowns than the n equations dictated by the n specified eigenvalues. This freedom can be exploited to obtain a solution that has desirable properties in addition to the specified eigenvalues. For example, the eigenvectors of the closed-loop state matrix can be selected subject to constraints. There is a rich literature that covers eigenstructure assignment, but it is beyond the scope of this text.

The sufficiency proof of Theorem 9.1 provides a method to obtain the feedback matrix K to assign the poles of a controllable system for the SI case with the system in controllable form. This approach is explored later. We first give a simple procedure applicable to low-order systems.

Procedure 9.1: Pole placement by equating coefficients

1. Evaluate the desired characteristic polynomial from the specified eigenvalues λ_i^d, $i = 1, \ldots, n$ using the expression

$$\Delta_c^d(\lambda) = \prod_{i=1}^{n} \left(\lambda - \lambda_i^d\right) \tag{9.10}$$

2. Evaluate the closed-loop characteristic polynomial using the expression

$$\det\{\lambda I_n - (A - BK)\} \tag{9.11}$$

3. Equate the coefficients of the two polynomials to obtain n equations to be solved for the entries of the matrix K.

Example 9.1: Pole assignment

Assign the eigenvalues $\{0.3 \pm j0.2\}$ to the pair

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution

For the given eigenvalues, the desired characteristic polynomial is

$$\Delta_c^d(\lambda) = (\lambda - 0.3 - j0.2)(\lambda - 0.3 + j0.2) = \lambda^2 - 0.6\lambda + 0.13$$

The closed-loop state matrix is

$$\begin{aligned} A - \mathbf{b}\mathbf{k}^T &= \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [\, k_1 \quad k_2 \,] \\ &= \begin{bmatrix} 0 & 1 \\ 3 - k_1 & 4 - k_2 \end{bmatrix} \end{aligned}$$

The closed-loop characteristic polynomial is

$$\begin{aligned} \det\left\{\lambda I_n - \left(A - \mathbf{b}\mathbf{k}^T\right)\right\} &= \det\begin{bmatrix} \lambda & -1 \\ -(3 - k_1) & \lambda - (4 - k_2) \end{bmatrix} \\ &= \lambda^2 - (4 - k_2)\lambda - (3 - k_1) \end{aligned}$$

Equating coefficients gives the two equations

Example 9.1: Pole assignment—cont'd

1. $4 - k_2 = 0.6 \Rightarrow k_2 = 3.4$
2. $-3 + k_1 = 0.13 \Rightarrow k_1 = 3.13$

that is,

$$\mathbf{k}^T = [3.13, 3.4]$$

Because the system is in controllable form, the same result can be obtained as the coefficients of the open-loop characteristic polynomial minus those of the desired characteristic polynomial using Eq. (9.9).

9.2.1 Pole placement by transformation to controllable form

Any controllable single-input—single-output (SISO) system can be transformed into controllable form using the transformation

$$T_c = \mathscr{C}\mathscr{C}_c^{-1} = \left[\mathbf{b} \mid A\mathbf{b} \mid \ldots \mid A^{n-1}\mathbf{b}\right] \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$T_c^{-1} = \mathscr{C}_c\mathscr{C}^{-1} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & t_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & \cdots & t_{n-2,n} \\ 0 & 1 & t_{2,n} & \cdots & t_{n-1,n} \\ 1 & t_{2,n} & t_{3,n} & \cdots & t_{n,n} \end{bmatrix} \left[\mathbf{b} \mid A\mathbf{b} \mid \ldots \mid A^{n-1}\mathbf{b}\right]^{-1} \tag{9.13}$$

where $\mathscr{C}$ is the controllability matrix, the subscript c denotes the controllable form, and the entries of the matrix T_c^{-1} are given by

$$t_{2,n} = a_{n-1}$$

$$t_{j+1,n} = -\sum_{i=0}^{j-1} a_{n-i-1} t_{j-i,n}, \qquad j = 2, \ldots, n-1$$

The forms of the matrix T_c and its inverse given above can be derived by induction and the proof is left as an exercise. The state feedback for a system in controllable form is

$$u = -\mathbf{k}_c^T \mathbf{x}_c = -\mathbf{k}_c^T \left(T_c^{-1} \mathbf{x} \right) \tag{9.14}$$

$$\mathbf{k}^T = \begin{bmatrix} a_0^d - a_0 & a_1^d - a_1 & \dots & \dots & a_{n-1}^d - a_{n-1} \end{bmatrix} T_c^{-1} \tag{9.15}$$

We now have the following pole placement procedure.

Procedure 9.2

1. Obtain the characteristic polynomial of the pair (A, B) using the Leverrier algorithm described in Section 7.4.1.
2. Obtain the transformation matrix T_c^{-1} using the coefficients of the polynomial from step 1.
3. Obtain the desired characteristic polynomial coefficients from the given eigenvalues using Eq. (9.10).
4. Compute the state feedback matrix using Eq. (9.15).

Procedure 9.2 requires the numerically troublesome inversion of the controllability matrix to obtain T. However, it does reveal an important characteristic of state feedback. From Eq. (9.15), we observe that the feedback gains tend to increase as the change from the open-loop to the closed-loop polynomial coefficients increases. Procedure 9.2, like Procedure 9.1, works well for low-order systems but can be implemented more easily using a computer-aided design (CAD) program.

Example 9.2

Design a feedback controller for the pair

$$A = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.5 & 0.2 \\ 0.2 & 0 & 0.4 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0.01 \\ 0 \\ 0.005 \end{bmatrix}$$

to obtain the eigenvalues $\{0.1, 0.4 \pm j\, 0.4\}$.

Solution

The characteristic polynomial of the state matrix is

$$\lambda^3 - \lambda^2 + 0.27\lambda - 0.01 \quad \text{i.e.,} \quad a_2 = -1, a_1 = 0.27, \quad a_0 = -0.01$$

The transformation matrix T_c^{-1} is

$$T_c^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0.73 \end{bmatrix} \times 10^3 \begin{bmatrix} 10 & 1.5 & 0.6 \\ 0 & 1 & 1.3 \\ 5 & 4 & 1.9 \end{bmatrix}^{-1} = 10^3 \begin{bmatrix} 0.1923 & 1.25 & -0.3846 \\ -0.0577 & 0.625 & 0.1154 \\ 0.0173 & 0.3125 & 0.1654 \end{bmatrix}$$

The desired characteristic polynomial is

$$\lambda^3 - 0.9\lambda^2 + 0.4\lambda - 0.032 \quad \text{i.e.,} \quad a_2^d = -0.9, a_1^d = 0.4, a_0^d = -0.032$$

Hence, we have the feedback gain vector

Example 9.2—cont'd

$$\mathbf{k}^T = \begin{bmatrix} a_0^d - a_0 & a_1^d - a_1 & a_2^d - a_2 \end{bmatrix} T_c^{-1}$$

$$= \begin{bmatrix} -0.032 + 0.01 & 0.4 - 0.27 & -0.9 + 1 \end{bmatrix} \times 10^3 \begin{bmatrix} 0.1923 & 1.25 & -0.3846 \\ -0.0577 & 0.625 & 0.1154 \\ 0.0173 & 0.3125 & 0.1654 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 85 & 40 \end{bmatrix}$$

9.2.2 Pole placement using a matrix polynomial

The gain vector for pole placement can be expressed in terms of the desired closed-loop characteristic polynomial. The expression, known as **Ackermann's formula**, is

$$\mathbf{k}^T = \mathbf{t}_1^T \Delta_c^d(A) \tag{9.16}$$

where $\mathbf{t}_1^T$ is the first row of the matrix T_c^{-1} of Eq. (9.13) and $\Delta_c^d(\lambda)$ is the desired closed-loop characteristic polynomial. From Theorem 9.1, we know that state feedback can arbitrarily place the eigenvalues of the closed-loop system for any controllable pair (A, $\mathbf{b}$). In addition, any controllable pair can be transformed into controllable form (A_c, $\mathbf{b}_c$). By the Cayley−Hamilton theorem, the state matrix satisfies its own characteristic polynomial $\Delta(\lambda)$ but not that corresponding to the desired pole locations. That is,

$$\Delta_c(A) = \sum_{i=0}^{n-1} a_i A^i = \sum_{i=0}^{n-1} a_i A_c^i = 0, \quad a_n = 1$$

$$\Delta_c^d(A) = \sum_{i=0}^{n} a_i^d A^i \neq 0, \quad a_n^d = 1$$

Subtracting and using the identity $A_c = T_c^{-1} A T_c$ gives

$$T_c^{-1} \Delta_c^d(A) T_c = \sum_{i=0}^{n-1} \left(a_i^d - a_i\right) A_c^i \tag{9.17}$$

The state matrix in controllable form possesses an interesting property, which we use in this proof. If the matrix raised to power i, with $i = 1, 2, \ldots, n{-}1$, is premultiplied by the first elementary vector

$$\mathbf{e}_1^T = \begin{bmatrix} 1 & 0_{n-1\times 1}^T \end{bmatrix}$$

the result is the $(i+1)$th elementary vector—that is,

$$\mathbf{e}_1^T A_c^i = \mathbf{e}_{i+1}^T, \qquad i = 0, \ldots, n-1 \tag{9.18}$$

Premultiplying Eq. (9.17) by the elementary vector $\boldsymbol{e}_1^T$, and then using Eq. (9.18), we obtain

$$\begin{aligned}
\mathbf{e}_1^T T_c^{-1} \Delta_c^d(A) T_c &= \sum_{i=0}^{n-1} \left(a_i^d - a_i\right) \mathbf{e}_1^T A_c^i \\
&= \sum_{i=0}^{n-1} \left(a_i^d - a_i\right) \mathbf{e}_{i+1}^T \\
&= \begin{bmatrix} a_0^d - a_0 & a_1^d - a_1 & \cdots & a_{n-1}^d - a_{n-1} \end{bmatrix}
\end{aligned}$$

Using Eq. (9.15), we obtain

$$\begin{aligned}
\mathbf{e}_1^T T_c^{-1} \Delta_c^d(A) T_c &= \begin{bmatrix} a_0^d - a_0 & a_1^d - a_1 & \cdots & a_{n-1}^d - a_{n-1} \end{bmatrix} \\
&= \mathbf{k}_c^T = \mathbf{k}^T T_c
\end{aligned}$$

Postmultiplying by T_c^{-1} and observing that the first row of the inverse is $\boldsymbol{t}_1^T = \boldsymbol{e}_1^T T_c^{-1}$, we obtain Ackermann's formula (9.16).

Minor modifications in Procedure 9.2 allow pole placement using Ackermann's formula. The formula requires the evaluation of the first row of the matrix T_c^{-1} rather than the entire matrix. However, for low-order systems, it is often simpler to evaluate the inverse and then use its first row. The following example demonstrates pole placement using Ackermann's formula.

Example 9.3

Obtain the solution described in Example 9.2 using Ackermann's formula.

Solution

The desired closed-loop characteristic polynomial is

$$\Delta_c^d(\lambda) = \lambda^3 - 0.9\lambda^2 + 0.4\lambda - 0.032 \quad \text{i.e.,} \quad a_2^d = -0.9, a_1^d = 0.4, a_0^d = -0.032$$

The first row of the inverse transformation matrix is

$$\begin{aligned}
\mathbf{t}_1^T = e_1^T T_c^{-1} &= 10^3 \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.1923 & 1.25 & -0.3846 \\ -0.0577 & 0.625 & 0.1154 \\ 0.0173 & 0.3125 & 0.1654 \end{bmatrix} \\
&= 10^3 \begin{bmatrix} 0.1923 & 1.25 & -0.3846 \end{bmatrix}
\end{aligned}$$

Example 9.3—cont'd

We use Ackermann's formula to compute the gain vector

$$\begin{aligned}\mathbf{k}^T &= \mathbf{t}_1^T \Delta_c^d(A) \\ &= \mathbf{t}_1^T \left\{ A^3 - 0.9A^2 + 0.4A - 0.032 I_3 \right\} \\ &= 10^3 [\,0.1923 \quad 1.25 \quad -0.3846\,] \times 10^{-3} \begin{bmatrix} 6 & 0 & 18 \\ 4 & 68 & 44 \\ 36 & 0 & 48 \end{bmatrix} \\ &= [\,-10 \quad 85 \quad 40\,]\end{aligned}$$

9.2.3 Choice of the closed-loop eigenvalues

Procedures 9.1 and 9.2 yield the feedback gain matrix once the closed-loop eigenvalues have been arbitrarily selected. The desired locations of the eigenvalues are directly related to the desired transient response of the system. In this context, considerations similar to those made in Section 6.6 can be applied to select the desired time response. However, the designer must take into account that poles associated with fast modes will lead to high gains for the state feedback matrix and consequently to a high **control effort**. High gains may also lead to performance degradation due to nonlinear behavior such as analog-to-digital converter (ADC) or actuator saturation. If all the desired closed-loop eigenvalues are selected at the origin of the complex plane, the deadbeat control strategy is implemented (see Section 6.7), and the closed-loop characteristic polynomial is chosen as

$$\Delta_c^d(\lambda) = \lambda^n \tag{9.19}$$

Substituting in Ackermann's formula (9.16) gives the feedback gain matrix

$$\mathbf{k}^T = \mathbf{t}_1^T A^n \tag{9.20}$$

The resulting control law will drive all the states to zero in at most n sampling intervals starting from any initial condition. However, the limitations of deadbeat control discussed in Section 6.7 apply—namely, the control variable can assume unacceptably high values, and undesirable intersample oscillations can occur.

Example 9.4

Determine the gain vector $\mathbf{k}$ using Ackermann's formula for the discretized state–space model of the armature-controlled DC motor (see Example 7.16) where

$$A_d = \begin{bmatrix} 1.0 & 0.01 & 0.0 \\ 0.0 & 0.9995 & 0.0095 \\ 0.0 & -0.09470 & 0.8954 \end{bmatrix}, \quad B_d = \begin{bmatrix} 1.622 \times 10^{-6} \\ 4.821 \times 10^{-4} \\ 9.468 \times 10^{-2} \end{bmatrix}$$

for the following choices of closed-loop eigenvalues:

1. $\{0.1, 0.4 \pm j0.4\}$
2. $\{0.4, 0.6 \pm j0.33\}$
3. $\{0, 0, 0\}$ (deadbeat control)

Simulate the system in each case to obtain the zero-input response starting from the initial condition $\mathbf{x}(0) = [1,1,1]^T$, and discuss the results.

Solution

The characteristic polynomial of the state matrix is

$$\Delta(\lambda) = \lambda^3 - 2.895\lambda^2 + 2.791\lambda - 0.896$$

That is,

$$a_2 = -2.895, \quad a_1 = 2.791, \quad a_0 = -0.896$$

The controllability matrix of the system is

$$\mathscr{C} = 10^{-3} \begin{bmatrix} 0.001622 & 0.049832 & 0.187964 \\ 0.482100 & 1.381319 & 2.185571 \\ 94.6800 & 84.37082 & 75.73716 \end{bmatrix}$$

Using Eq. (9.13) gives the transformation matrix

$$T_c^{-1} = 10^4 \begin{bmatrix} 1.0527 & -0.0536 & 0.000255 \\ -1.9948 & 0.20688 & -0.00102 \\ 0.94309 & -0.14225 & 0.00176 \end{bmatrix}$$

1. The desired closed-loop characteristic polynomial is

$$\Delta_c^d(\lambda) = \lambda^3 - 0.9\lambda^2 + 0.4\lambda - 0.032$$

That is,

$$a_2^d = -0.9, \quad a_1^d = 0.4, \quad a_0^d = -0.032$$

By Ackermann's formula, the gain vector is

$$\begin{aligned} \mathbf{k}^T &= \mathbf{t}_1^T \Delta_c^d(A) \\ &= \mathbf{t}_1^T \{A^3 - 0.9A^2 + 0.4A - 0.032I_3\} \\ &= 10^4 [1.0527 \quad -0.0536 \quad 0.000255] \times \{A^3 - 0.9A^2 + 0.4A - 0.032I_3\} \\ &= 10^3 [4.9268 \quad 1.4324 \quad 0.0137] \end{aligned}$$

Example 9.4—cont'd

The zero-input response for the three states and the corresponding control variable u are shown in Fig. 9.3.

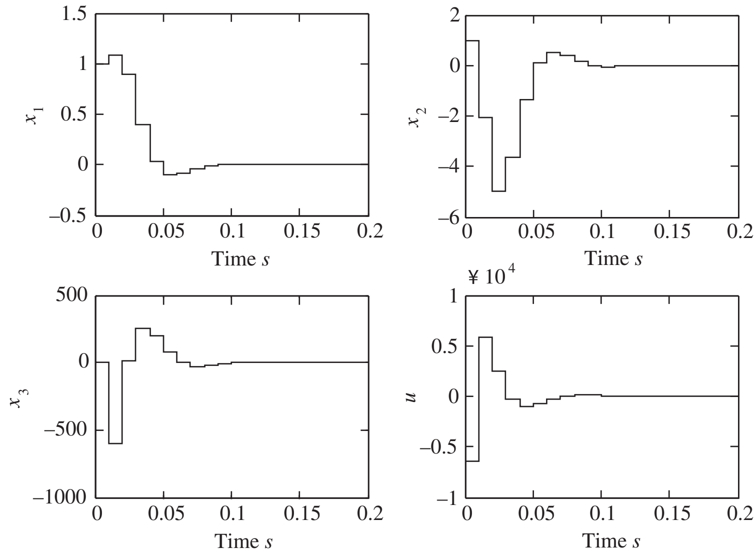

Figure 9.3

Zero-input state response and control variable for case 1 of Example 9.4.

2. The desired closed-loop characteristic polynomial is

$$\Delta_c^d(\lambda) = \lambda^3 - 1.6\lambda^2 + 0.9489\lambda - 0.18756$$

That is,

$$a_2^d = -1.6, \quad a_1^d = 0.9489, \quad a_0^d = -0.18756$$

And the gain vector is

$$\begin{aligned}
\mathbf{k}^T &= \mathbf{t}_1^T \Delta_c^d(A) \\
&= \mathbf{t}_1^T \{A^3 - 1.6A^2 + 0.9489A - 0.18756I_3\} \\
&= 10^4 [1.0527 \quad -0.0536 \quad 0.000255] \times \{A^3 - 1.6A^2 + 0.9489A - 0.18756I_3\} \\
&= 10^3 [1.6985 \quad 0.70088 \quad 0.01008]
\end{aligned}$$

The zero-input response for the three states and the corresponding control variable u are shown in Fig. 9.4.

Example 9.4—cont'd

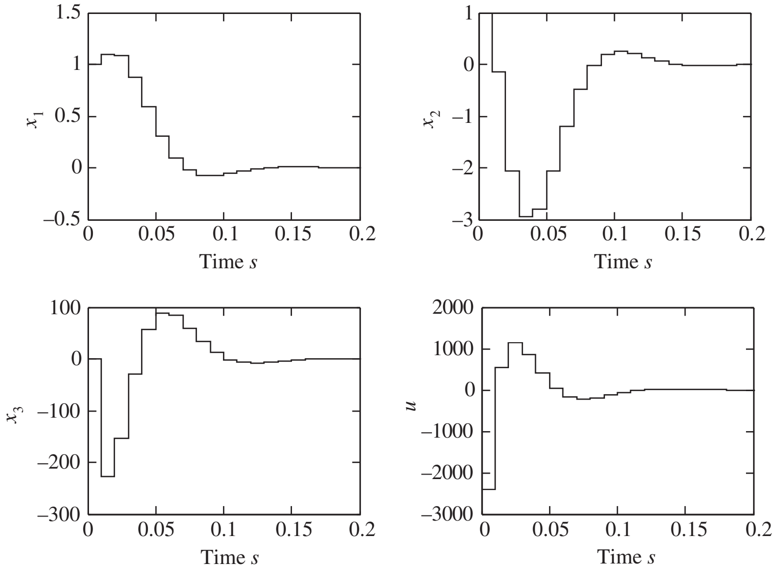

Figure 9.4

Zero-input state response and control variable for case 2 of Example 9.4.

3. The desired closed-loop characteristic polynomial is

$$\Delta_c^d(\lambda) = \lambda^3 \quad \text{i.e.,} \quad a_2^d = 0, \quad a_1^d = 0, \quad a_0^d = 0$$

and the gain vector is

$$\begin{aligned}\mathbf{k}^T &= \mathbf{t}_1^T \Delta_c^d(A) \\ &= \mathbf{t}_1^T \{A^3\} \\ &= 10^4 [1.0527 \quad -0.0536 \quad 0.000255] \times \{A^3\} \\ &= 10^4 [1.0527 \quad 0.2621 \quad 0.0017]\end{aligned}$$

The zero-input response for the three states and the corresponding control variable u are shown in Fig. 9.5.

We observe that when eigenvalues associated with faster modes are selected, higher gains are required for the state feedback and the state variables have transient responses with larger oscillations. Specifically, for the deadbeat control of case 3, the gain values are one order of magnitude larger than those of cases 1 and 2, and the magnitude of its transient oscillations is much larger. Further, for deadbeat control, the zero state is reached in $n = 3$ sampling intervals, as predicted by the theory. However, unpredictable transient intersample behavior occurs between 0.01 and 0.02 in the motor velocity x_2. This is shown in Fig. 9.6, where the analog velocity and the sampled velocity of the motor are plotted.

Example 9.4—cont'd

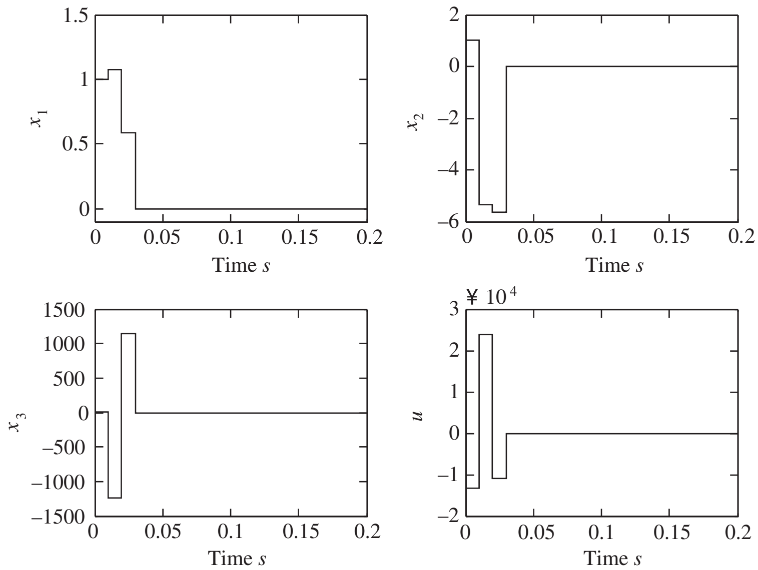

Figure 9.5
Zero-input state response and control variable for case 3 of Example 9.4.

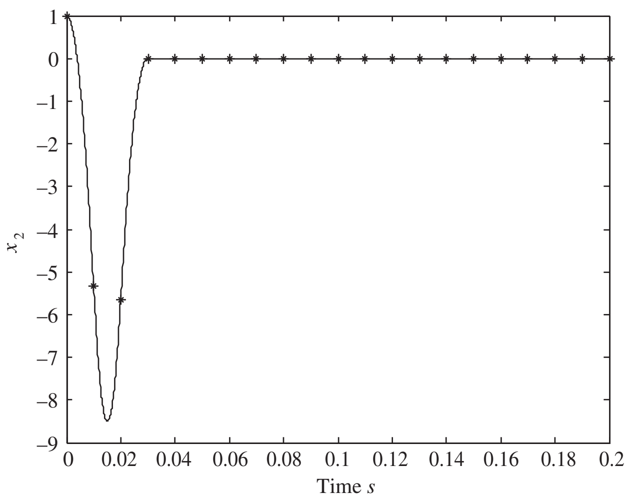

Figure 9.6
Sampled and analog velocity of the motor with deadbeat control.

Another consideration in selecting the desired closed-loop poles is the **robustness** of the system to modeling uncertainties. Although we have so far assumed that the state–space model of the system is known perfectly, this is never true in practice. Thus, it is desirable that control systems have poles with low sensitivity to perturbations in their state–space matrices. It is well known that low pole sensitivity is associated with designs that minimize the condition number of the modal matrix of eigenvectors of the state matrix; that is, the ratio of the maximum to the minimum singular value (see Appendix III for a discussion of singular values).

For a SISO system, the eigenvectors are fixed once the eigenvalues are chosen. Thus, the choice of eigenvalues determines the pole sensitivity of the system. For example, selecting the same eigenvalues for the closed-loop system leads to a high condition number of the eigenvector matrix and therefore to a closed-loop system that is very sensitive to coefficient perturbations. For multi-input–multi-output (MIMO) systems, the feedback gains for a given choice of eigenvalues are nonunique. This allows for more than one choice of eigenvectors and can be exploited to obtain robust designs. However, robust pole placement for MIMO systems is beyond the scope of this text and is not discussed further.

9.2.4 MATLAB commands for pole placement

The pole placement command is **place**. The following example illustrates the use of the command:

$$\gg \mathbf{A} = [\mathbf{0}, \mathbf{1}; \mathbf{3}, \mathbf{4}];$$
$$\gg \mathbf{B} = [\mathbf{0}; \mathbf{1}];$$
$$\gg \mathbf{poles} = [\mathbf{0.3}+\mathbf{j}*\mathbf{.2}, \mathbf{0.3}-\mathbf{j}*\mathbf{0.2}];$$
$$\gg \mathbf{K} = \mathbf{place}(\mathbf{A}, \mathbf{B}, \mathbf{poles})$$
$$\text{place: ndigits} = 16$$
$$\text{K} = 3.1300 \quad 3.4000$$

ndigits is a measure of the accuracy of pole placement.

Alternatively, we can obtain the same answer using Ackermann's formula and the command

$$\gg \mathbf{K} = \mathbf{acker(A, B, poles)}$$

It is also possible to compute the state feedback gain matrix of Eqs. (9.15) or (9.16) using basic MATLAB commands as follows:

1. Generate the characteristic polynomial of the state matrix to use Eq. (9.15).

$$\gg \mathbf{poly}(\mathbf{A})$$

2. Obtain the coefficients of the desired characteristic polynomial from a set of desired eigenvalues given as the entries of vector **poles.**

$$\gg \textbf{desired} = \textbf{poly}(\textbf{poles})$$

The vector **desired** contains the desired coefficients in descending order.

3. For Eq. (9.16), generate the polynomial matrix for a matrix **A** corresponding to the desired polynomial:

$$\gg \textbf{polyvalm}(\textbf{desired}, \textbf{A})$$

9.2.5 Pole placement for multi-input systems

For multi-input systems, the solution to the pole placement problem for a specified set of eigenvalues is not unique. However, there are constraints on the eigenvectors of the closed-loop matrix. Using the singular value decomposition of the input matrix (see Appendix III), we can obtain a complete characterization of the eigenstructure of the closed-loop matrix that is assignable by state feedback.

Theorem 9.2

For any controllable pair (A, B), there exists a state feedback matrix K that assigns the eigenpairs $\{(\lambda_i, \mathbf{v}_{di}), i = 1, \ldots, n$ if and only if

$$U_1^T [A - \lambda_i I_n] \mathbf{v}_{di} = \mathbf{0}_{n\times 1} \tag{9.21}$$

where

$$B = U \begin{bmatrix} Z \\ \mathbf{0}_{n-m\times m} \end{bmatrix} \tag{9.22}$$

with

$$U = [U_0 | U_1], U^{-1} = U^T \tag{9.23}$$

The state feedback gain is given by

$$K = -Z^{-1} U_0^T \left[V_d \Lambda V_d^{-1} - A \right] \tag{9.24}$$

Proof

The singular value decomposition of the full-rank input matrix is

$$B = U\begin{bmatrix} \Sigma_B \\ \mathbf{0}_{n-m\times m} \end{bmatrix} V^T = U\begin{bmatrix} Z \\ \mathbf{0}_{n-m\times m} \end{bmatrix}$$

$$Z = \Sigma_B V^T$$

where the matrices of singular vectors satisfy $U^{-1} = U^T$, $V^{-1} = V^T$ and the inverse of the matrix Z is

$$Z^{-1} = V\Sigma_B^{-1}$$

The closed-loop state matrix is

$$A_{cl} = A - BK = V_d \Lambda V_d^{-1}$$

$$\text{with } \Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_n\}, V_d = [\mathbf{v}_{d1}, \ldots, \mathbf{v}_{dn}]$$

We write the matrix of left singular vectors U as in Eq. (9.23), and then multiply by the closed-loop state matrix

$$U^T\left[A - V_d \Lambda V_d^{-1}\right] = \begin{bmatrix} U_0^T \\ U_1^T \end{bmatrix}\left[A - V_d \Lambda V_d^{-1}\right] = U^T BK$$

Using the singular value decomposition of B, we have

$$\begin{bmatrix} U_0^T\left(A - V_d \Lambda V_d^{-1}\right) \\ U_1^T\left(A - V_d \Lambda V_d^{-1}\right) \end{bmatrix} = \begin{bmatrix} ZK \\ \mathbf{0}_{n-m\times m} \end{bmatrix}$$

The lower part of the matrix equality is equivalent to Eq. (9.21), while the upper part gives Eq. (9.24).

Condition (9.21) implies that the eigenvectors that can be assigned by state feedback must be in the null space of the matrix U_1^T; that is, they must be mapped to zero by the matrix. This shows the importance of the singular value decomposition of the input matrix and its influence on the choice of available eigenvectors. Recall that the input matrix must also make the pair (A, B) controllable. While the discussion of this section helps us characterize state feedback in the multivariable case, it does not really provide an easy recipe for the choice of state feedback control. In fact, no such recipe exists, although there are alternative methods to choose the eigenvectors to endow the closed-loop system with desirable properties, these methods are beyond the scope of this text.

Example 9.5

For the pair

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -.005 & -.11 & -0.7 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

and the eigenvalues {0.1, 0.2, 0.3},

1. Determine a basis set for the space from which each eigenvector can be chosen.
2. Select a set of eigenvectors and determine the state feedback gain matrix that assigns the desired eigenstructure. Verify that the closed-loop state matrix has the correct eigenstructure.

Solution

The singular value decomposition of the input matrix is obtained using MATLAB

$$>> [\mathrm{Ub}, \mathrm{sigb}, \mathrm{Vb}] = \mathrm{svd}(B)$$

$$\begin{array}{lll} \mathrm{Ub} = & & \\ -0.5000 & -0.5000 & -0.7071 \\ -0.5000 & -0.5000 & 0.7071 \\ -0.7071 & 0.7071 & 0 \end{array}$$

$$\begin{array}{ll} \mathrm{sigb} = & \\ 1.8478 & 0 \\ 0 & 0.7654 \\ 0 & 0 \end{array}$$

$$\begin{array}{ll} \mathrm{Vb} = & \\ -0.3827 & 0.9239 \\ -0.9239 & -0.38 \end{array}$$

The system is third order with two inputs—that is, $n = 3$, $m = 2$, $n-m = 1$—and the matrix of left singular vectors is partitioned as

$$U_b = [\,U_0 | U_1\,] = \left[\begin{array}{cc|c} -0.5 & -0.5 & -0.7071 \\ -0.5 & -0.5 & 0.7071 \\ -0.7071 & 0.7071 & 0 \end{array}\right]$$

For the eigenvalue $\lambda_1 = 0.1$, Eq. (9.21) gives

$$U_1^T[A - 0.1I_3]\mathbf{v}_{d1} = [\,0.0707 \quad -0.7778 \quad 0.7071\,]\mathbf{v}_{d1} = \mathbf{0}_{3\times 1}$$

Example 9.5—cont'd

The corresponding eigenvector is

$$\mathbf{v}_{d1} = \begin{bmatrix} 0.7383 & 0.4892 & 0.4643 \end{bmatrix}^T$$

Similarly, we determine the two remaining eigenvectors and obtain the modal matrix of eigenvectors

$$V_d = \begin{bmatrix} 0.7383 & 0.7620 & 0.7797 \\ 0.4892 & 0.4848 & 0.4848 \\ 0.4643 & 0.4293 & 0.3963 \end{bmatrix}$$

The closed-loop state matrix is

$$A_{cl} = V_d \Lambda V_d^{-1} = \begin{bmatrix} 0.2377 & 2.0136 & -2.3405 \\ 0.2377 & 1.0136 & -1.3405 \\ 0.6511 & -0.2695 & -0.6513 \end{bmatrix}$$

We also need the matrix

$$\Sigma_b V_b = \begin{bmatrix} -0.7071 & -1.7071 \\ 0.7071 & -0.2929 \\ 0 & 0 \end{bmatrix}$$

The state feedback gain matrix is

$$K = -Z^{-1} U_0^T [A_{cl} - A] = \begin{bmatrix} -0.4184 & 1.1730 & -2.3892 \\ -0.2377 & -1.0136 & 2.3405 \end{bmatrix}$$

The MATLAB command **place** gives the solution

$$K_m = \begin{bmatrix} -0.0346 & 0.3207 & -2.0996 \\ -0.0499 & 0.0852 & 0.7642 \end{bmatrix}$$

that also assigns the desired eigenvalues but with a different choice of eigenvectors.

9.2.6 Pole placement by output feedback

As one would expect, using output feedback limits our ability to assign the eigenvalues of the state system relative to what is achievable using state feedback. It is, in general, not possible to arbitrarily assign the system poles even if the system is completely controllable and completely observable. Only l poles can be arbitrarily assigned if the system has l linearly independent outputs. It is possible to arbitrarily assign the controllable dynamics of the system using dynamic output feedback, and a satisfactory solution can be obtained if the system is stabilizable and detectable. Several approaches are available for the design of such a dynamic controller. One solution is to obtain an estimate of the state using the output and input of the system and use it in state feedback as explained in Section 9.6.

9.3 Servo problem

The schemes shown in Figs. 9.1 and 9.2 are regulators that drive the system state to zero starting from any initial condition capable of rejecting impulse disturbances. In practice, it is often necessary to track a constant reference input $\mathbf{r}$ with zero steady-state error. For this purpose, a possible approach is to use the **two degree-of-freedom control scheme** in Fig. 9.7, so called because we now have two matrices to select: the feedback gain matrix K and the reference gain matrix F.

The reference input of Eq. (9.2) becomes $\mathbf{v}(k) = F\mathbf{r}(k)$, and the control law is chosen as

$$\mathbf{u}(k) = -K\mathbf{x}(k) + F\mathbf{r}(k) \tag{9.25}$$

with $\mathbf{r}(k)$ the reference input to be tracked. The corresponding closed-loop system equations are

$$\begin{aligned} \mathbf{x}(k+1) &= A_{cl}\mathbf{x}(k) + BF\mathbf{r}(k) \\ y(k) &= C\mathbf{x}(k) \end{aligned} \tag{9.26}$$

where the closed-loop state matrix is

$$A_{cl} = A - BK$$

Using the formula for the transfer function, the z-transform of the corresponding output is given by (see Section 7.8)

$$\mathbf{Y}(z) = C[zI_n - A_{cl}]^{-1}BF\mathbf{R}(z)$$

The steady-state tracking error for a unit step input is given by

$$\begin{aligned} \underset{z\to 1}{\mathscr{Lim}}(z-1)\{\mathbf{Y}(z) - \mathbf{R}(z)\} &= \underset{z\to 1}{\mathscr{Lim}}\left\{C[zI_n - A_{cl}]^{-1}BF - I\right\} \\ &= C[I_n - A_{cl}]^{-1}BF - I \end{aligned}$$

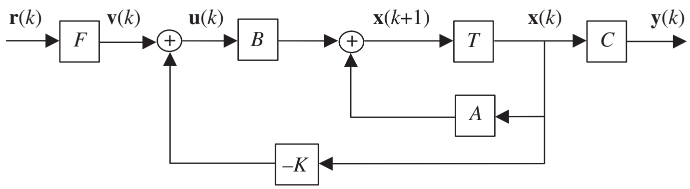

Figure 9.7
Block diagram of the two degree-of-freedom controller.

For zero steady-state error, we require the condition

$$C[I_n - A_{cl}]^{-1}BF = I_n \tag{9.27}$$

If the system is square ($m = l$) and A_{cl} is stable (no unity eigenvalues), we solve for the reference gain

$$F = \left[C(I_n - A_{cl})^{-1}B\right]^{-1} \tag{9.28}$$

Example 9.6

Design a state–space controller for the discretized state–space model of the DC motor speed control system described in Example 6.9 (with $T = 0.02$) to obtain zero steady-state error due to a unit step, a damping ratio of 0.7, and a settling time of about 1 s.

Solution

The discretized transfer function of the system with digital-to-analog converter (DAC) and ADC is

$$G_{ZAS}(z) = \left(1 - z^{-1}\right)\mathscr{Z}\left\{\frac{G(s)}{s}\right\} = 1.8604 \times 10^{-4}\frac{z + 0.9293}{(z - 0.8187)(z - 0.9802)}$$

The corresponding state–space model, computed with the MATLAB command **ss(G)**, is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1.799 & -0.8025 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.01563 \\ 0 \end{bmatrix}u(k)$$

$$y(k) = \begin{bmatrix} 0.01191 & 0.01107 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

The desired eigenvalues of the closed-loop system are selected as $\{0.9 \pm j0.09\}$, as in Example 6.9. This yields the feedback gain vector

$$K = \begin{bmatrix} -0.068517 & 0.997197 \end{bmatrix}$$

and the closed-loop state matrix

$$A_{cl} = \begin{bmatrix} 1.8 & -0.8181 \\ 1 & 0 \end{bmatrix}$$

The feedforward gain is

$$F = \left[C(I_n - A_{cl})^{-1}B\right]^{-1}$$

$$= \left[\begin{bmatrix} 0.01191 & 0.01107 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1.8 & -0.8181 \\ 1 & 0 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0.01563 \\ 0 \end{bmatrix}\right]^{-1} = 50.42666$$

The response of the system to a step reference input r is shown in Fig. 9.8. The system has a settling time of about 0.84 s and percentage overshoot of about 4%, with a peak time of about 1 s. All design specifications are met.

Example 9.6—cont'd

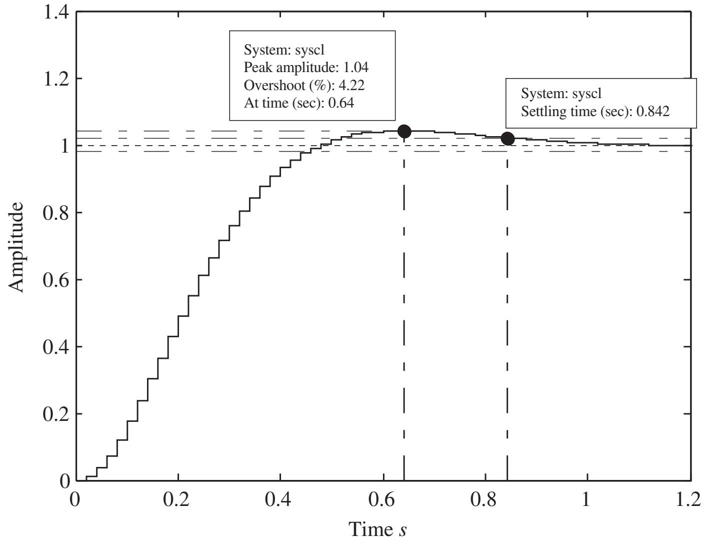

Figure 9.8
Step response of the closed-loop system of Example 9.6.

The control law (9.25) is equivalent to a **feedforward action** determined by F to yield zero steady-state error for a constant reference input $\mathbf{r}$. Because the forward action does not include any form of feedback, this approach is not robust to modeling uncertainties. Thus, modeling errors (which always occur in practice) will result in nonzero steady-state error. To eliminate such errors, we introduce the **integral control** shown in Fig. 9.9, with a new state added for each control error integrated.

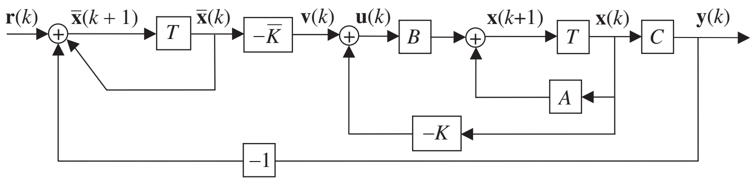

Figure 9.9
Control scheme with integral control.

The resulting state–space equations are

$$\begin{aligned} \mathbf{x}(k+1) &= A\mathbf{x}(k) + B\boldsymbol{u}(k) \\ \overline{\mathbf{x}}(k+1) &= \overline{\mathbf{x}}(k) + \mathbf{r}(k) - \mathbf{y}(k) \\ \mathbf{y}(k) &= C\mathbf{x}(k) \\ \mathbf{u}(k) &= K\mathbf{x}(k) - \overline{K}\overline{\mathbf{x}}(k) \end{aligned} \tag{9.29}$$

where $\overline{\mathbf{x}}$ is $l \times 1$. The state–space equations can be combined and rewritten in terms of an augmented state vector $\mathbf{x}_a(k) = [\mathbf{x}(k)\ \overline{\mathbf{x}}(k)]^T$ as

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \overline{\mathbf{x}}(k+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & I_l \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \overline{\mathbf{x}}(k) \end{bmatrix} - \begin{bmatrix} B \\ 0 \end{bmatrix} \begin{bmatrix} K & \overline{K} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \overline{\mathbf{x}}(k) \end{bmatrix} + \begin{bmatrix} 0 \\ I_l \end{bmatrix} \mathbf{r}(k)$$

$$\mathbf{y}(k) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k+1) \\ \overline{\mathbf{x}}(k+1) \end{bmatrix}$$

That is,

$$\begin{aligned} \mathbf{x}_a(k+1) &= \left(\widetilde{A} - \widetilde{B}\widetilde{K}\right)\mathbf{x}_a(k) + \begin{bmatrix} 0 \\ I_l \end{bmatrix} \mathbf{r}(k) \\ \mathbf{y}(k) &= \begin{bmatrix} C & 0 \end{bmatrix} \mathbf{x}_a(k) \end{aligned} \tag{9.30}$$

where

$$\widetilde{A} = \begin{bmatrix} A & 0 \\ -C & I_l \end{bmatrix} \quad \widetilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} \quad \widetilde{K} = \begin{bmatrix} K & \overline{K} \end{bmatrix} \tag{9.31}$$

It can be shown that the system of Eq. (9.31) is controllable if and only if the original system is controllable. The eigenvalues of the closed-loop system state matrix $A_{cl} = \left(\widetilde{A} - \widetilde{B}\widetilde{K}\right)$ can be arbitrarily assigned by computing the gain matrix $\widetilde{K}$ using any of the procedures for the regulator problem as described in Section 9.2.

Example 9.7

Solve the design problem presented in Example 9.6 using integral control.

Solution

The state–space matrices of the system are

$$A = \begin{bmatrix} 1.799 & -0.8025 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0.01563 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.01191 & 0.01107 \end{bmatrix}$$

Adding integral control, we obtain

$$\widetilde{A} = \begin{bmatrix} A & 0 \\ -C & 1 \end{bmatrix} = \begin{bmatrix} 1.799 & -0.8025 & 0 \\ 1 & 0 & 0 \\ -0.01191 & -0.01107 & 1 \end{bmatrix} \quad \widetilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} 0.01563 \\ 0 \\ 0 \end{bmatrix}$$

In Example 9.6, the eigenvalues were selected as $\{0.9 \pm j0.09\}$. Using integral control increases the order of the system by one, and an additional eigenvalue must be selected. The desired eigenvalues are selected as $\{0.9 \pm j0.09,\ 0.2\}$, and the additional eigenvalue at 0.2 is chosen for its negligible effect on the overall dynamics. This yields the feedback gain vector

Example 9.7—cont'd

$$\widetilde{K} = [\,51.1315 \quad -40.4431 \quad -40.3413\,]$$

The closed-loop system state matrix is

$$A_{cl} = \begin{bmatrix} 1 & -0.1706 & 0.6303 \\ 1 & 0 & 0 \\ -0.0119 & -0.0111 & 1 \end{bmatrix}$$

The response of the system to a unit step reference signal **r** is shown in Fig. 9.10. The figure shows that the control specifications are satisfied. The settling time of 0.87 is well below the specified value of 1 s, and the percentage overshoot is about 4.2%, which is less than the value corresponding to $\zeta = 0.7$ for the dominant pair.

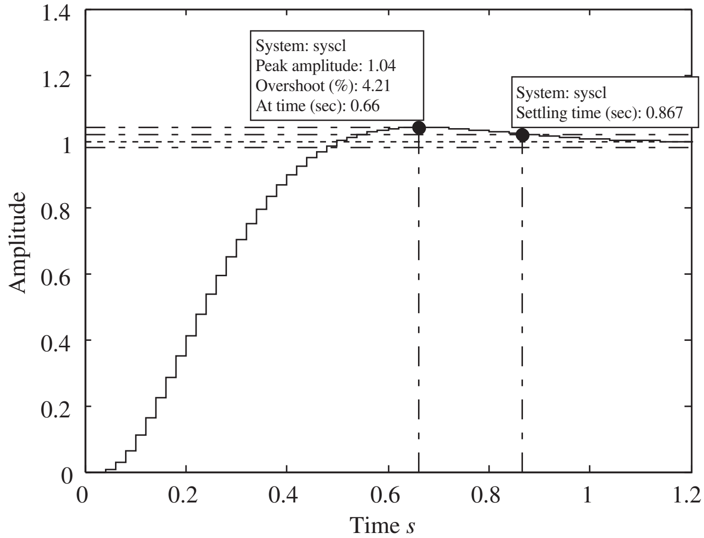

Figure 9.10
Step response of the closed-loop system of Example 9.7.

9.4 Invariance of system zeros

A severe limitation of the state-feedback control scheme is that it cannot change the location of the zeros of the system, which significantly affect the transient response. To show this, we consider the z-transform of the system including the direct transmission matrix D:

$$\mathbf{x}(k+1) = (A - BK)\mathbf{x}(k) + B\mathbf{v}(k)$$

$$\mathbf{y}(k) = C\mathbf{x}(k) - DK\mathbf{x}(k) - D\mathbf{v}(k)$$

The z-transform is given by

$$\begin{aligned} -(zI_n - A + BK)\mathbf{X}(z) + B\mathbf{V}(z) &= 0 \\ C\mathbf{X}(z) + D\mathbf{V}(z) &= \mathbf{Y}(z) \end{aligned} \tag{9.32}$$

If $z = z_0$ is a zero of the system, then $\mathbf{Y}(z_0)$ is zero with $\mathbf{V}(z_0)$ and $\mathbf{X}(z_0)$ nonzero. Thus, for $z = z_0$, the state−space Eq. (9.32) can be rewritten as

$$\begin{bmatrix} -(z_0 I_n - A + BK) & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathbf{X}(z_0) \\ \mathbf{V}(z_0) \end{bmatrix} = \mathbf{0} \tag{9.33}$$

Using the vector $\mathbf{V}(z_0) + K\mathbf{X}(z_0)$ in place of $\mathbf{V}(z_0)$, we rewrite Eq. (9.33) in terms of the state−space matrices of the open-loop system as

$$\begin{bmatrix} -(z_0 I_n - A + BK) & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathbf{X}(z_0) \\ \mathbf{V}(z_0) + K\mathbf{X}(z_0) \end{bmatrix} = \begin{bmatrix} -(z_0 I_n - A) & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathbf{X}(z_0) \\ \mathbf{V}(z_0) \end{bmatrix} = 0$$

We observe that with the state feedback $\mathbf{u}(k) = -K\mathbf{x}(k) + \mathbf{r}(k)$, the open-loop state−space quadruple (A, B, C, D) becomes

$$(A - BK, B, C - DK, D)$$

Thus, the zeros of the closed-loop system are the same as those of the plant and are invariant under state feedback. Similar reasoning can establish the same result for the system of Eq. (9.26).

Example 9.8

Consider the following continuous-time system:

$$G(s) = \frac{s+1}{(2s+1)(3s+1)}$$

Obtain a discrete model for the system with digital control and a sampling period $T = 0.02$, and then design a state−space controller with integral control and with the same closed-loop eigenvalues as in Example 9.7.

Solution

The analog system with DAC and ADC has the transfer function

$$G_{ZAS}(z) = \left(1 - z^{-1}\right)\mathscr{Z}\left\{\frac{G(s)}{s}\right\} = 33.338 \times 10^{-4} \frac{z - 0.9802}{(z - 0.9934)(z - 0.99)}$$

with an open-loop zero at 0.9802. The corresponding state−space model (computed with the MATLAB command **ss(G)**) is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1.983 & -0.983 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.0625 \\ 0 \end{bmatrix} u(k)$$

Example 9.8—cont'd

$$y(k) = [\,0.0534 \quad 0.0524\,]\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

The desired eigenvalues of the closed-loop system are selected as $\{0.9 \pm j0.09\}$ with an additional eigenvalue at 0.2, and this yields the feedback gain vector

$$\widetilde{K} = [\,15.7345 \quad -24.5860 \quad -2.19016\,]$$

The closed-loop system state matrix is

$$A_{cl} = \begin{bmatrix} 1 & -0.55316 & 13.6885 \\ 1 & 0 & 0 \\ 0.05342 & -0.05236 & 1 \end{bmatrix}$$

The response of the system to a unit step reference signal **r**, shown in Fig. 9.11, has a huge peak overshoot due to the closed-loop zero at 0.9802. The closed-loop control cannot change the location of the zero.

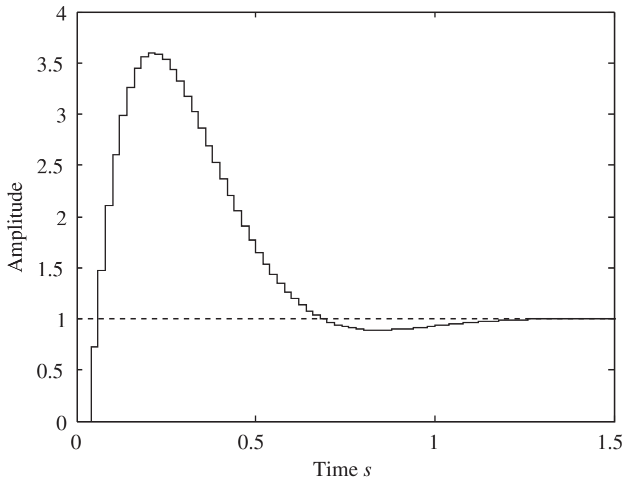

Figure 9.11
Step response of the closed-loop system of Example 9.8.

9.5 State estimation

In most applications, measuring the entire state vector is impossible or prohibitively expensive. To implement state feedback control, an estimate $\hat{\mathbf{x}}(k)$ of the state vector can be used. The state vector can be estimated from the input and output measurements by using a **state estimator** or **observer**.

9.5.1 Full-order observer

To estimate all the states of the system, one could in theory use a system with the same state equation as the plant to be observed. In other words, one could use the open-loop system

$$\widehat{\mathbf{x}}(k+1) = A\widehat{\mathbf{x}}(k) + B\mathbf{u}(k)$$

However, this open-loop estimator assumes perfect knowledge of the system dynamics and lacks the feedback needed to correct the errors that are inevitable in any implementation. The limitations of this observer become obvious on examining its error dynamics. We define the estimation error as $\widetilde{\mathbf{x}} = \mathbf{x} - \widehat{\mathbf{x}}$. We obtain the error dynamics by subtracting the open-loop observer dynamics from the system dynamics Eq. (9.1).

$$\widetilde{\mathbf{x}}(k+1) = A\widetilde{\mathbf{x}}(k)$$

The error dynamics are determined by the state matrix of the system and cannot be chosen arbitrarily. For an unstable system, the observer will be unstable and cannot track the state of the system.

A practical alternative is to feed back the difference between the measured and the estimated output of the system, as shown in Fig. 9.12. This yields to the following observer:

$$\widehat{\mathbf{x}}(k+1) = A\widehat{\mathbf{x}}(k) + B\mathbf{u}(k) + L[\mathbf{y}(k) - C\widehat{\mathbf{x}}(k)] \tag{9.34}$$

Subtracting the observer state equation from the system dynamics yields the estimation error dynamics

$$\widetilde{\mathbf{x}}(k+1) = (A - LC)\widetilde{\mathbf{x}}(k) \tag{9.35}$$

The error dynamics are governed by the eigenvalues of the observer matrix $A_0 = A - LC$. We transpose the matrix to obtain

$$A_0^T = A^T - C^T L^T \tag{9.36}$$

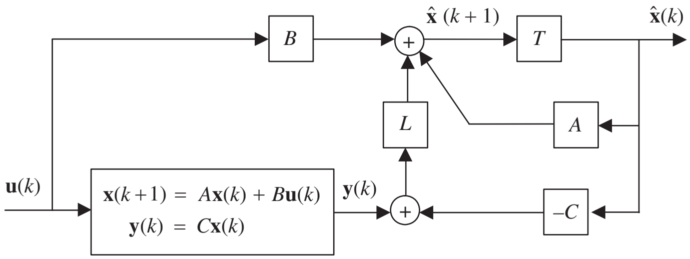

Figure 9.12
Block diagram of the full-order state estimator.

which has the same eigenvalues as the observer matrix. We observe that Eq. (9.36) is identical to the controller design Eq. (9.4) with the pair (A, B) replaced by the pair (A^T, C^T). We therefore have the following theorem.

Theorem 9.3: State estimation

If the pair (A, C) is observable, then there exists a feedback gain matrix L that arbitrarily assigns the observer poles to any set $\{\lambda_i, i = 1, \ldots, n\}$. Furthermore, if the pair (A, C) is detectable, then the observable modes can all be arbitrarily assigned.

Proof

Based on Theorem 8.12, system (A, C) is observable (detectable) if and only if (A^T, C^T) is controllable (stabilizable). Therefore, Theorem 9.3 follows from Theorem 9.1.

Based on Theorem 9.3, the matrix gain L can be determined from the desired observer poles, as discussed in Section 9.2. Hence, we can arbitrarily select the desired observer poles or the associated characteristic polynomial. From Eq. (9.36), it follows that the MATLAB command for the solution of the observer pole placement problem is

$$>> \mathrm{L} = \mathbf{place}(\mathbf{A}', \mathbf{C}', \mathbf{poles})'$$

Example 9.9

Determine the observer gain matrix L for the discretized state—space model of the armature-controlled DC motor of Example 9.4 with the observer eigenvalues selected as $\{0.1, 0.2 \pm j0.2\}$.

Solution

Recall that the system matrices are

$$A = \begin{bmatrix} 1.0 & 0.1 & 0.0 \\ 0.0 & 0.995 & 0.0095 \\ 0.0 & -0.0947 & 0.8954 \end{bmatrix} \quad B = \begin{bmatrix} 1.622 \times 10^{-6} \\ 4.821 \times 10^{-4} \\ 9.468 \times 10^{-2} \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

The MATLAB command **place** gives the observer gain

$$L = \begin{bmatrix} 2.3949 \\ 18.6734 \\ 4.3621 \end{bmatrix}$$

Expression (9.34) represents a **prediction observer**, because the estimated state vector (and any associated control action) at a given sampling instant does not depend on the current measured value of the system output. Alternatively, a **filtering observer** estimates the state vector based on the current output (assuming negligible computation time). The error correction term for the filtering observer uses the difference between the current output $\mathbf{y}(k+1)$ and its estimate and has the form

$$\begin{aligned}\widehat{\mathbf{x}}(k+1) &= A\widehat{\mathbf{x}}(k) + B\mathbf{u}(k) + L[\mathbf{y}(k+1) - C\widehat{\mathbf{x}}(k+1)] \\ &= A\widehat{\mathbf{x}}(k) + B\mathbf{u}(k) + L[\mathbf{y}(k+1) - C(A\widehat{\mathbf{x}}(k) + B\mathbf{u}(k))]\end{aligned} \tag{9.37}$$

The error dynamics are now represented by

$$\widetilde{\mathbf{x}}(k+1) = (A - LCA)\widetilde{\mathbf{x}}(k) \tag{9.38}$$

Expression (9.38) is the same as Expression (9.35) with the matrix product CA substituted for C. The observability matrix $\overline{\mathscr{O}}$ of system (A, CA) is

$$\overline{\mathscr{O}} = \begin{bmatrix} CA \\ --- \\ CA^2 \\ --- \\ \vdots \\ --- \\ CA^n \end{bmatrix} = \mathscr{O}A$$

where $\mathscr{O}$ is the observability matrix of the pair (A, C) (see Section 8.3). Thus, if the pair (A, C) is observable, the pair (A, CA) is observable unless A has one or more zero eigenvalues. If A has zero eigenvalues, the pair (A, CA) is detectable because the zero eigenvalues are associated with stable modes. Further, the zero eigenvalues are associated with the fastest modes, and the design of the observer can be completed by selecting a matrix L that assigns suitable values to the remaining eigenvalues of $A-LCA$.

Example 9.10

Determine the filtering observer gain matrix L for the system of Example 9.9.

Solution

Using the MATLAB command **place**,

$$\gg \mathrm{L} = \mathbf{place}(\mathbf{A}', (\mathbf{C} * \mathbf{A})', \mathbf{poles})'$$

we obtain the observer gain

L =

1.0e + 022*

0.009910699530206

0.140383004697920

4.886483453094875

9.5.2 Reduced-order observer

A full-order observer is designed so that the entire state vector is estimated from knowledge of the input and output of the system. The reader might well ask, why estimate n state variables when we already have l measurements that are linear functions of the same variables? Would it be possible to estimate $n-l$ variables only and use them with the measurements to estimate the entire state?

This is precisely what is done in the design of a **reduced-order observer**. A reduced-order observer is generally more efficient than a full-order observer. However, a full-order observer may be preferable in the presence of significant measurement noise. In addition, the design of the reduced-order observer is more complex.

We consider the linear time-invariant system Eq. (9.1) where the input matrix B and the output matrix C are assumed full rank. Then the entries of the output vector $\mathbf{y}(k)$ are linearly independent and form a partial state vector of length l, leaving $n-l$ variables to be determined. We thus have the state vectorc

$$\begin{bmatrix} \mathbf{y} \\ \hline \mathbf{z} \end{bmatrix} = Q_o^{-1}\mathbf{x} = \begin{bmatrix} C \\ \hline M \end{bmatrix}\mathbf{x} \tag{9.39}$$

where M is a full-rank $n-l \times n$ matrix with rows that are linearly independent of those of C, and $\mathbf{z}$ is the unknown partial state.

The state−space matrices for the transformed state variables are

$$C_t = CQ_o = \left[I_l \;\middle|\; \mathbf{0}_{l\times n-l} \right] \tag{9.40}$$

$$B_t = Q_o^{-1}B = \begin{array}{c} \begin{bmatrix} \overline{B}_1 \\ \hline \overline{B}_2 \end{bmatrix} \end{array} \begin{array}{l} \}\; l \\ \}\; n-l \end{array} \tag{9.41}$$

$$A_t = Q_o^{-1}AQ_o = \begin{array}{c} \begin{bmatrix} \overline{A}_1 & \overline{A}_2 \\ \overline{A}_3 & \overline{A}_4 \end{bmatrix} \\ \underbrace{\qquad}_{l}\;\underbrace{\qquad}_{n-l} \end{array} \begin{array}{l} \}\; l \\ \}\; n-l \\ \; \end{array} \tag{9.42}$$

The identity in the output matrix indicates that for the transformed system the measurement vector $\boldsymbol{y}(k)$ is not part of the state vector. The state equation for the unknown partial state is

$$\mathbf{z}(k+1) = \overline{A}_3\mathbf{y}(k) + \overline{A}_4\mathbf{z}(k) + \overline{B}_2\mathbf{u}(k) \tag{9.43}$$

Using the state equation for the known partial state, that is, the measurements, we define an output variable to form a state–space model with Eq. (9.43) as

$$\mathbf{y}_z(k) = \mathbf{y}(k+1) - \overline{A}_1\mathbf{y}(k) + \overline{B}_1\mathbf{u}(k) = \overline{A}_2\mathbf{z}(k) \tag{9.44}$$

This output represents the portion of the known partial state $\mathbf{y}(k+1)$ that is computed using the unknown partial state. The observer dynamics, including the error in computing $\mathbf{y}_z$, are assumed linear time invariant of the form

$$\begin{aligned}\widehat{\mathbf{z}}(k+1) &= \overline{A}_3\mathbf{y}(k) + \overline{A}_4\widehat{\mathbf{z}}(k) + \overline{B}_2\mathbf{u}(k) + L\left[\mathbf{y}_z(k) - \overline{A}_2\widehat{\mathbf{z}}(k)\right] \\ &= \left(\overline{A}_4 - L\overline{A}_2\right)\widehat{\mathbf{z}}(k) + \overline{A}_3\mathbf{y}(k) + L\left[\mathbf{y}(k+1) - \overline{A}_1\mathbf{y}(k) - \overline{B}_1\mathbf{u}(k)\right] + \overline{B}_2\mathbf{u}(k)\end{aligned} \tag{9.45}$$

where $\widehat{\mathbf{z}}$ denotes the estimate of the partial state vector $\mathbf{z}$. Unfortunately, the observer Eq. (9.45) includes the term $\mathbf{y}(k+1)$, which is not available at time k. Moving the term to the LHS reveals that its use can be avoided by estimating the variable

$$\overline{\mathbf{x}}(k) = \widehat{\mathbf{z}}(k) - L\mathbf{y}(k) \tag{9.46}$$

Using the observer dynamics Eq. (9.45) and the definition (9.46), we obtain the observer

$$\begin{aligned}\overline{\mathbf{x}}(k+1) &= \widehat{\mathbf{z}}(k+1) - L\mathbf{y}(k+1) \\ &= \left[\overline{A}_4 - L\overline{A}_2\right](\overline{\mathbf{x}}(k) + L\mathbf{y}(k)) + \left[\overline{A}_3 - L\overline{A}_1\right]\mathbf{y}(k) + \left[\overline{B}_2 - L\overline{B}_1\right]\mathbf{u}(k)\end{aligned}$$

The observer for the unknown partial state is given by

$$\overline{\mathbf{x}}(k+1) = A_o\overline{\mathbf{x}}(k) + A_y\mathbf{y}(k) + B_o\mathbf{u}(k) \tag{9.47}$$

where

$$\begin{aligned}A_o &= \overline{A}_4 - L\overline{A}_2 \\ A_y &= A_oL + \overline{A}_3 - L\overline{A}_1 \\ B_o &= \overline{B}_2 - L\overline{B}_1\end{aligned} \tag{9.48}$$

The block diagram of the reduced-order observer is shown in Fig. 9.13.

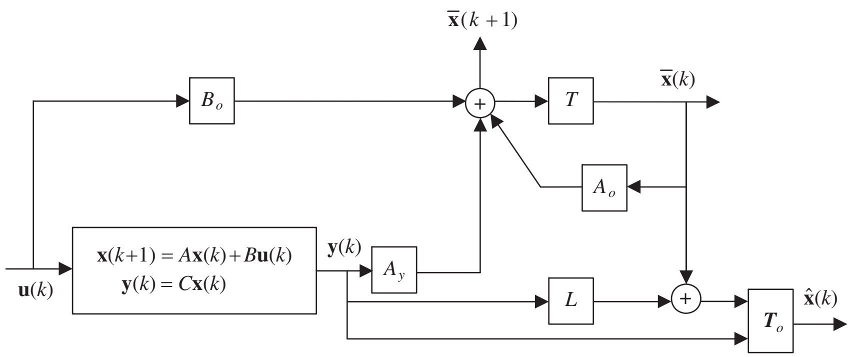

Figure 9.13
Block diagram of the reduced-order observer.

The dynamic of the reduced-order observer Eq. (9.47) is governed by the matrix A_o. The eigenvalues of A_o must be selected inside the unit circle and must be sufficiently fast to track the state of the observed system. This reduces observer design to the solution of Eq. (9.48) for the observer gain matrix L. Once L is obtained, the other matrices in Eq. (9.48) can be computed and the state vector $\hat{\mathbf{x}}$ can be obtained using the equation

$$\begin{aligned}\hat{\mathbf{x}}(k) &= Q_o\begin{bmatrix}\mathbf{y}(k)\\ \hat{\mathbf{z}}(k)\end{bmatrix}\\ &= Q_o\begin{bmatrix}I_l & 0_{l\times n-l}\\ L & I_{n-l}\end{bmatrix}\begin{bmatrix}\mathbf{y}(k)\\ \overline{\mathbf{x}}(k)\end{bmatrix} = T_o\begin{bmatrix}\mathbf{y}(k)\\ \overline{\mathbf{x}}(k)\end{bmatrix}\end{aligned} \tag{9.49}$$

where the transformation matrix Q_o is defined in Eq. (9.39).

Transposing the state matrix of Eq. (9.48) yields

$$A_o^T = \overline{A}_4^T - \overline{A}_2^T L^T \tag{9.50}$$

Because Eq. (9.50) is identical in form to the controller design Eq. (9.4), it can be solved as discussed in Section 9.2. We recall that the poles of the matrix A_o^T can be arbitrarily assigned provided that the pair $\left(\overline{A}_4^T, \overline{A}_2^T\right)$ is controllable. From the duality concept discussed in Section 8.6, this is equivalent to the observability of the pair $\left(\overline{A}_4, \overline{A}_2\right)$. Theorem 9.4 gives a necessary and sufficient condition for the observability of the pair.

Theorem 9.4

The pair $\left(\overline{A}_4, \overline{A}_2\right)$ is observable if and only if system (A, C) is observable.

Proof

The proof is left as an exercise.

Example 9.11

Design a reduced-order observer for the discretized state—space model of the armature-controlled DC motor of Example 9.4 with the observer eigenvalues selected as $\{0.2 \pm j0.2\}$.

Solution

Recall that the system matrices are

Example 9.11—cont'd

$$A = \begin{bmatrix} 1.0 & 0.1 & 0.0 \\ 0.0 & 0.995 & 0.0095 \\ 0.0 & -0.0947 & 0.8954 \end{bmatrix} \quad B = \begin{bmatrix} 1.622 \times 10^{-6} \\ 4.821 \times 10^{-4} \\ 9.468 \times 10^{-2} \end{bmatrix} \quad C = [1 \quad 0 \quad 0]$$

The output matrix C is in the required form, and there is no need for similarity transformation. The second and third state variables must be estimated. The state matrix is partitioned as

$$A = \left[\begin{array}{c|c} a_1 & \mathbf{a}_2^T \\ \hline \mathbf{a}_3 & A_4 \end{array}\right] = \left[\begin{array}{c|cc} 1.0 & 0.1 & 0.0 \\ \hline 0.0 & 0.9995 & 0.0095 \\ 0.0 & -0.0947 & 0.8954 \end{array}\right]$$

The similarity transformation can be selected as an identity matrix; that is,

$$Q_o = Q_o^{-1} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

and therefore we have $A_t = A$, $B_t = B$, and $C_t = C$. Hence, we need to solve the linear equation

$$A_o = \begin{bmatrix} 0.9995 & 0.0095 \\ -0.0947 & 0.8954 \end{bmatrix} - \bar{\mathbf{l}}[0.1 \quad 0]$$

to obtain the observer gain

$$\bar{\mathbf{l}} = [14.949 \quad 550.191]^T$$

The corresponding observer matrices are

$$A_o = \begin{bmatrix} -0.4954 & 0.0095 \\ -55.1138 & 0.8954 \end{bmatrix}$$

$$\mathbf{b}_o = \bar{\mathbf{b}}_2 - \mathbf{l}\bar{\mathbf{b}}_1 = \begin{bmatrix} 4.821 \times 10^{-4} \\ 9.468 \times 10^{-2} \end{bmatrix} - \begin{bmatrix} 14.949 \\ 550.191 \end{bmatrix} \times 1.622 \times 10^{-6} = \begin{bmatrix} 0.04579 \\ 9.37876 \end{bmatrix} \times 10^{-2}$$

$$\mathbf{a}_y = A_o\mathbf{l} + \mathbf{a}_3 - \mathbf{l}a_1$$

$$= \begin{bmatrix} -0.4954 & 0.0095 \\ -55.1138 & 0.8954 \end{bmatrix} \begin{bmatrix} 14.949 \\ 550.191 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 14.949 \\ 550.191 \end{bmatrix} \times 1 = \begin{bmatrix} -0.1713 \\ -8.8145 \end{bmatrix} \times 10^2$$

The state estimate can be computed using

$$\hat{\mathbf{x}}(k) = Q_o \begin{bmatrix} 1 & \mathbf{0}_{1\times 2}^T \\ \mathbf{l} & I_2 \end{bmatrix} \begin{bmatrix} \mathbf{y}(k) \\ \bar{\mathbf{x}}(k) \end{bmatrix} = \left[\begin{array}{cc} 1 & \mathbf{0}_{1\times 2}^T \\ \hline 14.949 & \\ 550.191 & I_2 \end{array}\right] \begin{bmatrix} \mathbf{y}(k) \\ \bar{\mathbf{x}}(k) \end{bmatrix}$$

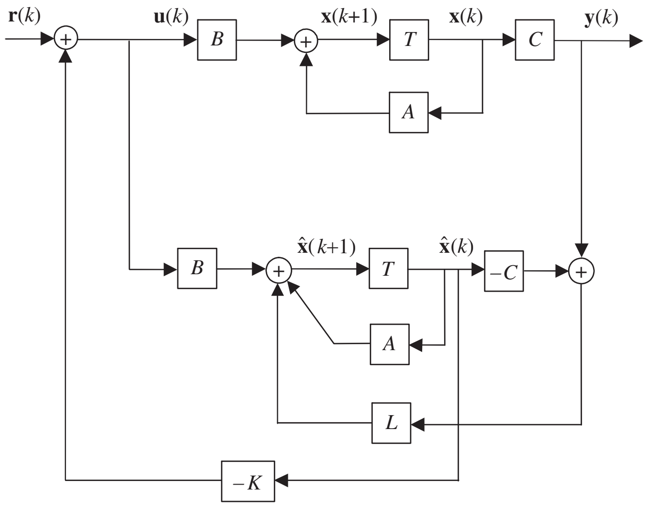

Figure 9.14
Block diagram of a system with observer state feedback.

9.6 Observer state feedback

If the state vector is not available for feedback control, a state estimator can be used to generate the control action as shown in Fig. 9.14. The corresponding control vector is

$$\mathbf{u}(k) = -K\hat{\mathbf{x}}(k) + \mathbf{v}(k) \tag{9.51}$$

Substituting in the state Eq. (9.1) gives

$$\mathbf{x}(k+1) = A\mathbf{x}(k) - BK\hat{\mathbf{x}}(k) + B\mathbf{v}(k) \tag{9.52}$$

Adding and subtracting the term $BK\mathbf{x}(k)$, we rewrite Eq. (9.52) in terms of the estimation error $\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$ as

$$\mathbf{x}(k+1) = (A - BK)\mathbf{x}(k) + BK\tilde{\mathbf{x}}(k) + B\mathbf{v}(k) \tag{9.53}$$

If a full-order (predictor) observer is used, by combining Eq. (9.53) with Eq. (9.35), we obtain

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \tilde{\mathbf{x}}(k+1) \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \tilde{\mathbf{x}}(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \mathbf{v}(k) \tag{9.54}$$

The state matrix of Eq. (9.54) is block triangular and its characteristic polynomial is

$$\begin{aligned} \Delta_{cl}(\lambda) &= \det[\lambda I - (A - BK)]\det[\lambda I - (A - LC)] \\ &= \Delta_c(\lambda)\Delta_o(\lambda) \end{aligned} \tag{9.55}$$

Thus, the eigenvalues of the closed-loop system can be selected separately from those of the observer. This important result is known as the **separation theorem** or the **uncertainty equivalence principle**.

Analogously, if a reduced-order observer is employed, the estimation error $\widetilde{\mathbf{x}}$ can be expressed in terms of the errors in estimating $\mathbf{y}$ and $\mathbf{z}$ as

$$\begin{gathered}\widetilde{\mathbf{x}}(k) = Q_o\begin{bmatrix}\widetilde{\mathbf{y}}(k)\\ \widetilde{\mathbf{z}}(k)\end{bmatrix}\\ \widetilde{\mathbf{y}}(k) = \mathbf{y}(k) - \widehat{\mathbf{y}}(k) \quad \widetilde{\mathbf{z}}(k) = \mathbf{z}(k) - \widehat{\mathbf{z}}(k)\end{gathered} \tag{9.56}$$

We partition the matrix Q_o into an $n\times l$ matrix Q_y and an $n\times n-l$ matrix Q_z to allow the separation of the two error terms and rewrite the estimation error as

$$\tilde{\mathbf{x}}(k) = \left[\, Q_y \;\middle|\; Q_z \,\right]\begin{bmatrix}\tilde{\mathbf{y}}(k)\\ \hline \tilde{\mathbf{z}}(k)\end{bmatrix} = Q_y\tilde{\mathbf{y}}(k) + Q_z\tilde{\mathbf{z}}(k)$$

Assuming negligible measurement error $\widetilde{\mathbf{y}}$, the estimation error reduces to

$$\widetilde{\mathbf{x}}(k) = Q_z\widetilde{\mathbf{z}}(k) \tag{9.58}$$

Substituting from Eq. (9.58) into the closed-loop Eq. (9.53) gives

$$\mathbf{x}(k+1) = (A - BK)\mathbf{x}(k) + BKQ_z\widetilde{\mathbf{z}}(k) + B\mathbf{v}(k) \tag{9.59}$$

We evaluate $\widetilde{z}(k+1) = z(k+1) - \widehat{z}(k+1)$ by subtracting Eq. (9.45) from Eq. (9.43) and use Eq. (9.44) to substitute $\overline{A}_2\mathbf{z}(k)$ for $\mathbf{y}_z$

$$\widetilde{\mathbf{z}}(k+1) = \left(\overline{A}_4 - L\overline{A}_2\right)\widetilde{\mathbf{z}}(k) \tag{9.60}$$

Combining Eqs. (9.59) and (9.60), we obtain the equation

$$\begin{bmatrix}\mathbf{x}(k+1)\\ \widetilde{\mathbf{z}}(k+1)\end{bmatrix} = \begin{bmatrix}A - BK & BKQ_z\\ 0_{n-l\times n} & \overline{A}_4 - L\overline{A}_2\end{bmatrix}\begin{bmatrix}\mathbf{x}(k)\\ \widetilde{\mathbf{z}}(k)\end{bmatrix} + \begin{bmatrix}B\\ 0\end{bmatrix}\mathbf{v}(k) \tag{9.61}$$

The state matrix of Eq. (9.61) is block triangular and its characteristic polynomial is

$$\det\left[\lambda I - \left(\overline{A}_4 - L\overline{A}_2\right)\right]\det[\lambda I - (A - BK)] \tag{9.62}$$

Thus, as for the full-order observer, the closed-loop eigenvalues for the reduced-order observer state feedback can be selected separately from those of the reduced-order observer. The separation theorem therefore applies for reduced-order observers as well as for full-order observers.

In addition, combining the plant state equation and the estimator Eq. (9.47) and using the output equation, we have

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \overline{\mathbf{x}}(k+1) \end{bmatrix} = \begin{bmatrix} A & 0_{n\times n-l} \\ A_y C & A_o \end{bmatrix} \begin{bmatrix} x(k) \\ \overline{x}(k) \end{bmatrix} + \begin{bmatrix} B \\ B_0 \end{bmatrix} \mathbf{u}(k) \tag{9.63}$$

We express the estimator state feedback of Eq. (9.51) as

$$\begin{aligned} \mathbf{u}(k) &= -K\hat{\mathbf{x}}(k) + \mathbf{v}(k) \\ &= -KT_o \begin{bmatrix} \mathbf{y}(k) \\ \overline{\mathbf{x}}(k) \end{bmatrix} + \mathbf{v}(k) = -K\left[\, T_{oy} \;\vdots\; T_{ox} \,\right] \begin{bmatrix} C\mathbf{x}(k) \\ \overline{\mathbf{x}}(k) \end{bmatrix} + \mathbf{v}(k) \end{aligned}$$

where T_{oy} and T_{ox} are partitions of T_o of Eq. (9.49) of order $n\times l$ and $n\times n-l$, respectively. Substituting in Eq. (9.63), we have

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \overline{\mathbf{x}}(k+1) \end{bmatrix} = \begin{bmatrix} A - BKT_{oy}C & -BKT_{ox} \\ A_y C - B_o KT_{oy} C & A_o - B_o KT_{ox} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \overline{\mathbf{x}}(k) \end{bmatrix} + \begin{bmatrix} B \\ B_0 \end{bmatrix} \mathbf{v}(k) \tag{9.65}$$

Eq. (9.65) can be used to simulate the complete estimator state feedback system.

9.6.1 Choice of observer eigenvalues

In the selection of the observer poles or the associated characteristic polynomial, expression (9.55) or (9.62) must be considered. The choice of observer poles is not based on the constraints related to the control effort discussed in Section 9.2.3. However, the response of the closed-loop system must be dominated by the poles of the controller that meet the performance specifications. Therefore, as a rule of thumb, the poles of the observer should be selected from 3 to 10 times faster than the poles of the controller. An upper bound on the speed of response of the observer is imposed by the presence of the unavoidable measurement noise. Inappropriately fast observer dynamics will result in tracking the noise rather than the actual state of the system. Hence, a deadbeat observer, although appealing in theory, is avoided in practice.

The choice of observer poles is also governed by the same considerations related to the robustness of the system discussed in Section 9.2.3 for the state feedback control. Thus, the sensitivity of the eigenvalues to perturbations in the system matrices must be considered in the selection of the observer poles.

We emphasize that the selection of the observer poles does not influence the performance of the overall control system if the initial conditions are estimated perfectly. We prove this fact for the full-order observer. However, the result also holds for the reduced-order observer, but the proof is left as an exercise for the reader. To demonstrate this fact, we consider the state Eq. (9.54) with the output Eq. (9.1) modified for the augmented state vector:

$$\begin{aligned} \begin{bmatrix} \mathbf{x}(k+1) \\ \widetilde{\mathbf{x}}(k+1) \end{bmatrix} &= \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \widetilde{\mathbf{x}}(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \mathbf{v}(k) \\ \mathbf{y}(k) &= \left[\, C \quad 0 \,\right] \begin{bmatrix} \mathbf{x}(k) \\ \widetilde{\mathbf{x}}(k) \end{bmatrix} \end{aligned} \tag{9.66}$$

The zero input–output response of the system ($\mathbf{v}(k) = 0$) can be determined iteratively as

$$\mathbf{y}(0) = C\mathbf{x}(0)$$

$$\mathbf{y}(1) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} \mathbf{x}(0) \\ \widetilde{\mathbf{x}}(0) \end{bmatrix} = C(A - BK)\mathbf{x}(0) + CBK\widetilde{\mathbf{x}}(0)$$

$$\begin{aligned} \mathbf{y}(2) &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} \mathbf{x}(1) \\ \widetilde{\mathbf{x}}(1) \end{bmatrix} = C(A - BK)\mathbf{x}(1) + CBK\widetilde{\mathbf{x}}(1) \\ &= C(A - BK)^2\mathbf{x}(0) - C(A - BK)BK\widetilde{\mathbf{x}}(0) - CBK(A - LC)\widetilde{\mathbf{x}}(0) \\ &\vdots \end{aligned}$$

Clearly, the observer matrix L influences the transient response if and only if $\widetilde{x}(k) \neq 0$. This fact is confirmed by the determination of the z-transfer function from Eq. (9.66), which implicitly assumes zero initial conditions

$$G(z) = \begin{bmatrix} C & 0 \end{bmatrix} \left(zI - \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} = C(zI - A + BK)^{-1}B$$

where the observer gain matrix L does not appear.

Another important transfer function relates the output $\mathbf{y}(k)$ to the control $\mathbf{u}(k)$. It is obtained from the observer state Eq. (9.34) with Eq. (9.51) as the output equation and is given by

$$G_{co}(z) = -K(zI_n - A + BK + LC)^{-1}L$$

Note that any realization of this z-transfer function can be used to implement the controller–observer regardless of the realization of the system used to obtain the transfer function. The denominator of the transfer function is

$$det(zI_n - A + BK + LC)$$

which is clearly different from Eq. (9.62). This should come as no surprise, since the transfer function represents the feedback path in Fig. 9.14 and not the closed-loop system dynamics.

Example 9.12

Consider the armature-controlled DC motor of Example 9.4. Let the true initial condition be $\mathbf{x}(0) = [1,\ 1,\ 1]^T$, and let its estimate be the zero vector $\widehat{\mathbf{x}}(0) = [0, 0, 0]^T$. Design a full-order observer state feedback for a zero-input response with a settling time of 0.2 s.

Solution

As Example 9.4 showed, a choice for the control system eigenvalues that meets the design specification is $\{0.6,\ 0.4 \pm j0.33\}$. This yields the gain vector

$$K = 10^3\begin{bmatrix} 1.6985 & -0.70088 & 0.01008 \end{bmatrix}$$

Example 9.12—cont'd

The observer eigenvalues must be selected so that the associated modes are sufficiently faster than those of the controller. We select the eigenvalues $\{0.1,\ 0.1 \pm j0.1\}$. This yields the observer gain vector

$$L = 10^2 \begin{bmatrix} 0.02595 \\ 0.21663 \\ 5.35718 \end{bmatrix}$$

Using Eq. (9.66), we obtain the space—space equations

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \widetilde{\mathbf{x}}(k+1) \end{bmatrix} = \left[\begin{array}{cccccc} 0.9972 & 0.0989 & 0 & 0.0028 & 0.0011 & 0 \\ -0.8188 & 0.6616 & 0.0046 & 0.8188 & 0.3379 & 0.0049 \\ -160.813 & -66.454 & -0.0589 & 160.813 & 66.359 & 0.9543 \\ & & & -1.5949 & 0.1 & 0 \\ & 0 & & -21.663 & 0.9995 & 0.0095 \\ & & & -535.79 & -0.0947 & 0.8954 \end{array}\right] \begin{bmatrix} \mathbf{x}(k) \\ \widetilde{\mathbf{x}}(k) \end{bmatrix}$$

$$\mathbf{y}(k) = \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \mathbf{x}(k) \\ \tilde{\mathbf{x}}(k) \end{bmatrix}$$

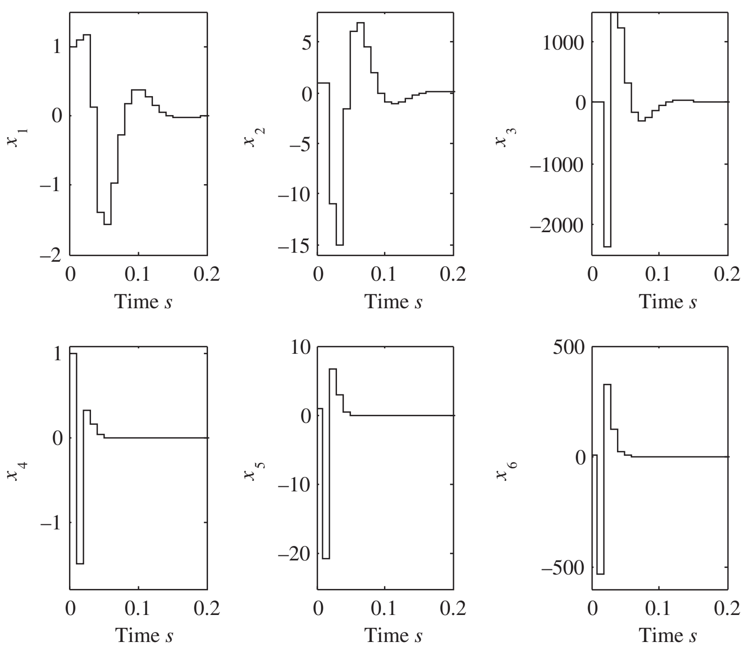

Figure 9.15
Zero-input response for Example 9.12.

Example 9.12—cont'd

The response to the initial condition [1, 1, 1, 1, 1, 1] is plotted in Fig. 9.15. We compare the plant state variables x_i, $i=1, 2, 3$ to the estimation errors x_i, $i=4, 5, 6$. We observe that the estimation errors decay to zero faster than the system states and that the system has an overall settling time less than 0.2 s.

Example 9.13

Solve Example 9.12 using a reduced-order observer.

Solution

In this case, we have $l=1$, and because the measured output corresponds to the first element of the state vector, we do not need similarity transformation, that is, $Q_0 = Q_0^{-1} = I_3$. Thus, we obtain

$$\overline{A}_1 = 1 \quad \overline{A}_2 = 0.1 \quad \overline{A}_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \overline{A}_4 = \begin{bmatrix} 0.9995 & 0.0095 \\ -0.0947 & 0.8954 \end{bmatrix}$$

$$B_1 = 1.622 \times 10^{-6} \quad \overline{B}_2 = \begin{bmatrix} 4.821 \times 10^{-4} \\ 9.468 \times 10^{-2} \end{bmatrix}$$

We select the reduced-order observer eigenvalues as $\{0.1 \pm j0.1\}$ and obtain the observer gain vector

$$L = 10^2 \begin{bmatrix} 0.16949 \\ 6.75538 \end{bmatrix}$$

and the associated matrices

$$A_o = \begin{bmatrix} -0.6954 & 0.0095 \\ -67.6485 & 0.8954 \end{bmatrix} \quad A_y = 10^3 \begin{bmatrix} -0.02232 \\ -1.21724 \end{bmatrix} \quad B_o = 10^{-2} \begin{bmatrix} 0.045461 \\ 9.358428 \end{bmatrix}$$

Partitioning $Q_0 = T_o = I_3$ gives

$$Q_z = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad T_{ox} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad T_{oy} = 10^2 \begin{bmatrix} 0.01 \\ 0.16949 \\ 6.75538 \end{bmatrix}$$

We have that the state–space equation of Eq. (9.61) is

$$\begin{bmatrix} x(k+1) \\ \tilde{z}(k+1) \end{bmatrix} = \begin{bmatrix} 0.99725 & 0.09886 & 0.00002 & 0.00114 & 0.00002 \\ -0.81884 & 0.66161 & 0.00464 & 0.33789 & 0.00486 \\ -160.813 & -66.4540 & -0.05885 & 66.3593 & 0.95435 \\ & 0 & & -0.6954 & 0.0095 \\ & & & -67.6485 & 0.8954 \end{bmatrix} \begin{bmatrix} x(k) \\ \tilde{z}(k) \end{bmatrix} \tag{9.67}$$

whereas the state–space equation of Eq. (9.65) is

Example 9.13—cont'd

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \overline{\mathbf{x}}(k+1) \end{bmatrix} = \begin{bmatrix} 0.96693 & 0.1 & 0 & -0.00114 & -0.00002 \\ -9.82821 & 0.9995 & 0.0095 & -0.33789 & -0.00486 \\ -1930.17 & -0.0947 & 0.8954 & -66.3593 & -0.95425 \\ -31.5855 & 0 & 0 & -1.01403 & 0.00492 \\ -3125.07 & 0 & 0 & -133.240 & 0.04781 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \overline{\mathbf{x}}(k) \end{bmatrix}$$

The response of the state–space system Eq. (9.67) to the initial condition [1, 1, 1, 1, 1] is plotted in Fig. 9.16. As in Example 9.12, we observe that the estimation errors x_i, $i = 4$, and 5 decay to zero faster than the system states x_i, $i = 1$, 2, and 3, and that the system has an overall settling time less than 0.2 s.

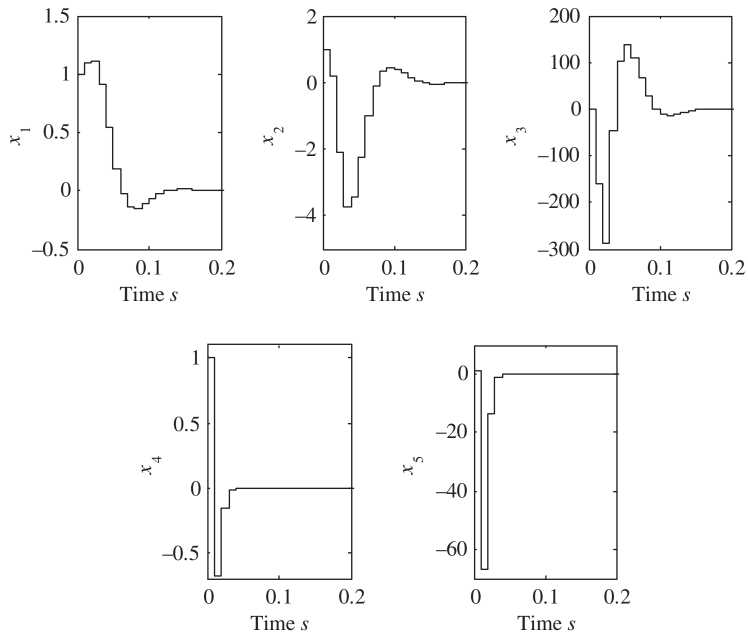

Figure 9.16
Zero-input response for Example 9.13.

Example 9.14

Design a full-order observer state feedback for the armature controller DC motor of Example 9.4 with feedforward action, a settling time of 0.2 s, and overshoot less than 10%.

Example 9.14—cont'd

Solution

As in Example 9.12, we select the control system eigenvalues as $\{0.4, 0.6 \pm j0.33\}$ and the observer eigenvalues as $\{0.1, 0.1 \pm j0.1\}$. This yields the gain vectors

$$K = 10^3\begin{bmatrix} 1.6985 & -0.70088 & 0.01008 \end{bmatrix}, \quad L = 10^2\begin{bmatrix} 0.02595 \\ 0.21663 \\ 5.35718 \end{bmatrix}$$

From the matrix $A_{cl} = A - BK$, we determine the feedforward term using Eq. (9.28) as

$$F = \left[C\left(I_n - (A - BK)\right)^{-1}B\right]^{-1} = 1698.49$$

Substituting $Fr(k)$ for $v(k)$ in Eq. (9.66), we obtain the closed-loop state–space equations

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \tilde{\mathbf{x}}(k+1) \end{bmatrix} = \left[\begin{array}{ccc:ccc} 0.9972 & 0.0989 & 0 & 0.0028 & 0.0011 & 0 \\ -0.8188 & 0.6616 & 0.0046 & 0.8188 & 0.3379 & 0.0049 \\ -160.813 & -66.454 & -0.0589 & 160.813 & 66.359 & 0.9543 \\ \hdashline & & & -1.5949 & 0.1 & 0 \\ & 0 & & -21.663 & 0.9995 & 0.0095 \\ & & & -535.79 & -0.0947 & 0.8954 \end{array}\right] \begin{bmatrix} \mathbf{x}(k) \\ \tilde{\mathbf{x}}(k) \end{bmatrix} + \left[\begin{array}{c} 0.0028 \\ 0.8188 \\ 160.813 \\ \hdashline \\ 0 \\ \\ \end{array}\right] r(k)$$

$$\mathbf{y}(k) = \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \mathbf{x}(k) \\ \tilde{\mathbf{x}}(k) \end{bmatrix}$$

The step response of the system shown in Fig. 9.17 has a settling time of about 0.1 and a percentage overshoot of about 6%. The controller meets the design specification.

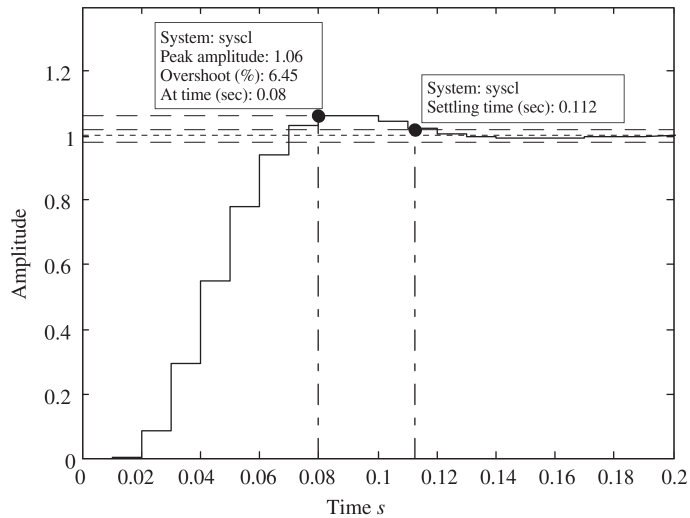

Figure 9.17
Step response for Example 9.14.

9.7 Pole assignment using transfer functions

The pole assignment problem can be solved in the framework of transfer functions. Consider the state–space equations of the two-degree-of-freedom controller shown in Fig. 9.7 with the state vector estimated using a full-order observer. For a SISO plant with observer state feedback, we have

$$\begin{aligned}\widehat{\mathbf{x}}(k+1) &= A\widehat{\mathbf{x}}(k) + Bu(k) + L(y(k) - C\widehat{\mathbf{x}}(k))\\ u(k) &= -K\widehat{\mathbf{x}}(k) + Fr(k)\end{aligned}$$

or equivalently,

$$\begin{aligned}\widehat{\mathbf{x}}(k+1) &= (A - BK - LC)\widehat{\mathbf{x}}(k) + \begin{bmatrix} BF & L \end{bmatrix}\begin{bmatrix} r(k) \\ y(k) \end{bmatrix}\\ u(k) &= -K\widehat{\mathbf{x}}(k) + Fr(k)\end{aligned}$$

The corresponding z-transfer function from $[r, y]$ to $\widehat{\boldsymbol{x}}$ is

$$\widehat{\mathbf{X}}(z) = (zI - A + BK + LC)^{-1}\begin{bmatrix} BF & L \end{bmatrix}\begin{bmatrix} R(z) \\ Y(z) \end{bmatrix}$$

The transfer function from $[r,\ y]$ to u is

$$U(z) = \left[-K(zI - A + BK + LC)^{-1}BF + F\right]R(z) - \left[K(zI - A + BK + LC)^{-1}L\right]Y(z)$$

Thus, the full-order observer state feedback is equivalent to the transfer function model depicted in Fig. 9.18 with

$$U(z) = G_f(z)R(z) - H(z)Y(z)$$

where

$$H(z) = \frac{S(z)}{D(z)}$$

is the feedback gain with

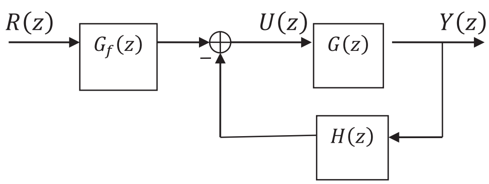

Figure 9.18
Block diagram for pole assignment with transfer functions.

$$H(z) = K(zI - A + BK + LC)^{-1}L$$

and

$$G_f(z) = \frac{N(z)}{D(z)} = -K(zI - A + BK + LC)^{-1}BF + F$$

is the prefilter gain.

The plant transfer function $G(z) = P(z)/Q(z)$ is assumed strictly realizable; that is, the degree of $P(z)$ is less than the degree of $Q(z)$. We also assume that $P(z)$ and $Q(z)$ are **coprime** (i.e., they have no common factors). Further, it is assumed that $Q(z)$ is **monic,** that is, the coefficient of the term with the highest power in z is one.

From the block diagram of Fig. 9.18, simple block diagram manipulations give the closed-loop transfer function

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + G(z)H(z)} G_f(z) = \frac{P(z)N(z)}{Q(z)D(z) + P(z)S(z)}$$

and the polynomial equation

$$(Q(z)D(z) + P(z)S(z))Y(z) = P(z)N(z)R(z) \tag{9.68}$$

Therefore, the closed-loop characteristic polynomial is

$$\Delta_{cl}(z) = Q(z)D(z) + P(z)S(z) \tag{9.69}$$

The pole placement problem thus reduces to finding polynomials $D(z)$ and $S(z)$ that satisfy Eq. (9.69) for given $P(z)$, $Q(z)$, and for a given desired characteristic polynomial $\Delta_{cl}(z)$. Eq. (9.69) is called a **Diophantine equation**, and its solution can be found by first expanding its RHS terms as

$$\begin{aligned}
P(z) &= p_{n-1}z^{n-1} + p_{n-2}z^{n-2} + \ldots + p_1 z + p_0 \\
Q(z) &= z^n + q_{n-1}z^{n-1} + \ldots + q_1 z + q_0 \\
D(z) &= d_m z^m + d_{m-1}z^{m-1} + \ldots + d_1 z + d_0 \\
S(z) &= s_m z^m + s_{m-1}z^{m-1} + \ldots + s_1 z + s_0
\end{aligned}$$

The closed-loop characteristic polynomial $\Delta_{cl}(z)$, which can be obtained from the desired pole locations, is of degree $n + m$ and has the form

$$\Delta_{cl}(z) = z^{n+m} + \delta_{n+m-1}z^{n+m-1} + \ldots + \delta_1 z + \delta_0$$

Thus, Eq. (9.69) can be rewritten as

$$
\begin{aligned}
z^{n+m} + \delta_{n+m-1} z^{n+m-1} + \cdots + \delta_1 z + \delta_0 = {} & \left(z^n - q_{n-1} z^{n-1} + \cdots + q_1 z + q_0\right) \\
& \left(d_m z^m + d_{m-1} z^{m-1} + \cdots + d_1 z + d_0\right) + \left(p_{n-1} z^{n-1} + p_{n-2} z^{n-2} + \cdots + p_1 z + p_0\right) \\
& \left(s_m z^m + s_{m-1} z^{m-1} + \cdots + s_1 z + s_0\right)
\end{aligned}
\tag{9.70}
$$

Eq. (9.70) is linear in the $2m$ unknowns d_i and s_i, $i = 0, 1, 2, \ldots, m-1$, and its LHS is a known polynomial with $n + m-1$ coefficients. The solution of the Diophantine equation is unique if $n + m-1 = 2m$—that is, if $m = n-1$. Eq. (9.70) can be written in the matrix form

$$
\begin{bmatrix}
1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
q_{n-1} & 1 & 0 & \ldots & 0 & p_{n-1} & 0 & \ldots & 0 \\
q_{n-2} & q_{n-1} & 1 & \ldots & 0 & p_{n-2} & p_{n-1} & \ldots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
q_0 & q_1 & q_2 & \ldots & q_{n-1} & p_0 & p_1 & \ldots & p_{n-1} \\
0 & q_0 & q_1 & \ldots & q_{n-2} & 0 & p_0 & \ldots & p_{n-2} \\
0 & 0 & q_0 & \ldots & q_{n-3} & 0 & 0 & \ldots & p_{n-3} \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \ldots & q_0 & 0 & 0 & \ldots & p_0
\end{bmatrix}
\begin{bmatrix}
d_m \\ d_{m-1} \\ d_{m-2} \\ \vdots \\ d_0 \\ s_m \\ s_{m-1} \\ \vdots \\ s_0
\end{bmatrix}
=
\begin{bmatrix}
1 \\ \delta_{sn-2} \\ \delta_{2n-3} \\ \vdots \\ \delta_{n-1} \\ \delta_{n-2} \\ \delta_{n-3} \\ \vdots \\ \delta_0
\end{bmatrix}
\tag{9.71}
$$

It can be shown that the matrix on the LHS is nonsingular if and only if the polynomials $P(z)$ and $Q(z)$ are coprime, which we assume. As discussed in Section 9.2.3, the matrix must have a small condition number for the system to be robust with respect to errors in the known parameters. The condition number becomes larger as the matrix becomes almost singular.

The structure of the matrix shows that it will be almost singular if the coefficients of the numerator polynomial $P(z)$ and denominator polynomial $Q(z)$ are almost identical. We therefore require that the roots of the polynomials $P(z)$ and $Q(z)$ be sufficiently different to avoid an ill-conditioned matrix. From the discussion of pole-zero matching of Section 6.3.2, it can be deduced that the poles of the discretized plant approach the zeros as the sampling interval is reduced (see also Section 12.2.2). Thus, when the controller is designed by pole assignment, the sampling interval must not be excessively short to avoid an ill-conditioned matrix in Eq. (9.71).

We now discuss the choice of the desired characteristic polynomial. From the equivalence of the transfer function design to the state−space design described in Section 9.6, the separation principle implies that $\Delta_{cl}^d(z)$ can be written as the product

$$
\Delta_{cl}^d(z) = \Delta_c^d(z)\Delta_o^d(z)
$$

where $\Delta_c^d(z)$ is the controller characteristic polynomial and $\Delta_o^d(z)$ is the observer characteristic polynomial. We select the polynomial $N(z)$ as

$$N(z) = k_{ff}\Delta_o^d(z) \tag{9.72}$$

so that the observer polynomial $\Delta_o^d(z)$ cancels in the transfer function from the reference input to the system output. The scalar constant k_{ff} is selected so that the steady-state output is equal to the constant reference input

$$\frac{Y(1)}{R(1)} = \frac{P(1)N(1)}{\Delta_c^d(1)\Delta_o^d(1)} = \frac{P(1)k_{ff}\Delta_o^d(1)}{\Delta_c^d(1)\Delta_o^d(1)} = 1$$

The condition for zero steady-state error is

$$k_{ff} = \frac{\Delta_c^d(1)}{P(1)} \tag{9.73}$$

Example 9.15

Solve Example 9.14 using the transfer function approach.

Solution

The plant transfer function is

$$G(z) = \frac{P(z)}{Q(z)} = 10^{-6}\frac{1.622z^2 + 45.14z + 48.23}{z^3 - 2.895z^2 + 2.791z - 0.8959}$$

Thus, we have the polynomials

$$P(z) = 1.622 \times 10^{-6}z^2 + 45.14 \times 10^{-6}z + 48.23 \times 10^{-6}$$

with coefficients

$$p_2 = 1.622 \times 10^{-6}, \quad p_1 = 45.14 \times 10^{-6}, \quad p_0 = 48.23 \times 10^{-6}$$

$$Q(z) = z^3 - 2.895z^2 + 2.791z - 0.8959$$

with coefficients

$$q_2 = -2.895, \quad q_1 = 2.791, \quad q_0 = -0.8959$$

The plant is third order—that is, $n = 3$—and the solvability condition of the Diophantine equation is $m = n-1 = 2$. The order of the desired closed-loop characteristic polynomial is $m + n = 5$. We can therefore select the controller poles as $\{0.6, 0.4 \pm j0.33\}$ and the observer poles as $\{0.1, 0.2\}$ with the corresponding polynomials

$$\Delta_c^d(z) = z^3 - 1.6z^2 + 0.9489z - 0.18756$$

Example 9.15—cont'd

$$\Delta_o^d(z) = z^2 - 0.3z + 0.02$$

$$\Delta_{cl}^d(z) = \Delta_c^d(z)\Delta_o^d(z) = z^5 - 1.9z^4 + 1.4489z^3 - 0.50423z^2 + 0.075246z - 0.0037512$$

In other words, $\delta_4 = -1.9$, $\delta_3 = 1.4489$, $\delta_2 = -0.50423$, $\delta_1 = 0.075246$, and $\delta_0 = -0.0037512$.

Using the matrix Eq. (9.71) gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2.8949 & 1 & 0 & 1.622\times10^{-6} & 0 & 0 \\ 2.790752 & -2.8949 & 1 & 45.14\times10^{-6} & 1.622\times10^{-6} & 0 \\ -0.895852 & 2.790752 & -2.8949 & 48.23\times10^{-6} & 45.14\times10^{-6} & 1.622\times10^{-6} \\ 0 & -0.895852 & 2.790752 & 0 & 48.23\times10^{-6} & 45.14\times10^{-6} \\ 0 & 0 & -0.895852 & 0 & 0 & 48.23\times10^{-6} \end{bmatrix}$$

$$\begin{bmatrix} d_2 \\ d_1 \\ d_0 \\ s_2 \\ s_1 \\ s_0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1.9 \\ 1.4489 \\ -0.50423 \\ 0.075246 \\ -0.00375 \end{bmatrix}$$

The MATLAB command **linsolve** gives the solution

$$d_2 = 1,\ \ d_1 = 0.9645,\ \ d_0 = 0.6526,\ \ s_2 = 1.8735\bullet10^4,\ \ s_1 = -2.9556\bullet10^4,\ \ s_0 = 1.2044\bullet10^4$$

and the polynomials

$$D(z) = z^2 + 0.9645z + 0.6526$$

$$S(z) = 1.8735\bullet10^4 z^2 - 2.9556\bullet10^4 z + 1.2044\bullet10^4$$

Then, from Eq. (9.72), we have $k_{ff} = 1698.489$ and the numerator polynomial

$$N(z) = 1698.489\left(z^2 - 0.3z + 0.02\right)$$

The step response of the control system of Fig. 9.19 has a settling time of 0.1 s and a percentage overshoot of less than 7%. The response meets all the design specifications.

Example 9.15—cont'd

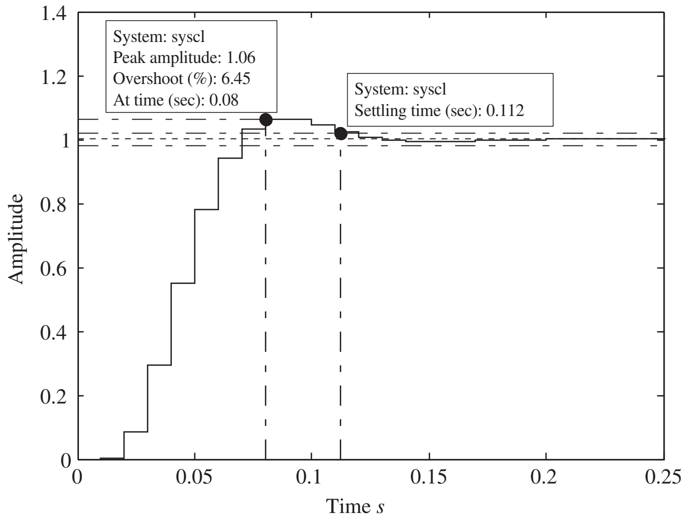

Figure 9.19
Step response for Example 9.15.

Further reading

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Problems

9.1 Show that the closed-loop quadruple for (A, B, C, D) with the state feedback $\mathbf{u}(k) = -K\mathbf{x}(k)+\mathbf{v}(k)$ is $(A-BK, B, C-DK, D)$.

9.2 Show that for the pair (A, B) with state feedback gain matrix K to have the closed-loop state matrix $A_{cl} = A-BK$, a necessary condition is that for any vector $\mathbf{w}T$ satisfying $\mathbf{w}^T B = \mathbf{0}^T$ and $\mathbf{w}^T A = \lambda \mathbf{w}^T$, A_{cl} must satisfy $\mathbf{w}^T A_{cl} = \lambda \mathbf{w}^T$. Explain the significance of this necessary condition. (Note that the condition is also sufficient.)

9.3 Show that for the pair (A, B) with $m\times n$ state feedback gain matrix K to have the closed-loop state matrix $A_{cl} = A-BK$, a sufficient condition is

$$\operatorname{rank}\{B\} = \operatorname{rank}\{[A - A_{cl}|B]\} = m$$

Is the matrix K unique for given matrices A and A_{cl}? Explain.

9.4 Using the results of Problem 9.3, determine if the closed-loop matrix can be obtained using state feedback for the pair

$$A = \begin{bmatrix} 1.0 & 0.1 & 0 \\ 0 & 1 & 0.01 \\ 0 & -0.1 & 0.9 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

a. $A_{cl} = \begin{bmatrix} 1.0 & 0.1 & 0 \\ -12.2 & -1.2 & 0 \\ 0.01 & 0.01 & 0 \end{bmatrix}$

b. $A_{cl} = \begin{bmatrix} 0 & 0.1 & 0 \\ -12.2 & -1.2 & 0 \\ 0.01 & 0.01 & 0 \end{bmatrix}$

9.5 Show that for the pair (A, C) with observer gain matrix L to have the observer matrix $A_o = A-LC$, a necessary condition is that for any vector $\mathbf{v}$ satisfying $C\mathbf{v} = \mathbf{0}$ and $A\mathbf{v} = \lambda\mathbf{v}$, A_o must satisfy $A_o\mathbf{v} = \lambda\mathbf{v}$. Explain the significance of this necessary condition. (Note that the condition is also sufficient.)

9.6 Show that for the pair (A, C) with $n\times l$ observer gain matrix L to have the observer matrix $A_o = A-LC$, a sufficient condition is

$$rank\{C\} = rank\left\{\begin{bmatrix} C \\ --- \\ A - A_o \end{bmatrix}\right\} = l$$

Is the matrix L unique for given matrices A and A_o? Explain.

9.7 Design a state feedback control law to assign the eigenvalues to the set $\{0, 0.1, 0.2\}$ for the systems with

a. $A = \begin{bmatrix} 0.1 & 0.5 & 0 \\ 2 & 0 & 0.2 \\ 0.2 & 1 & 0.4 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0.01 \\ 0 \\ 0.005 \end{bmatrix}$

b. $A = \begin{bmatrix} -0.2 & -0.2 & 0.4 \\ 0.5 & 0 & 1 \\ 0 & -0.4 & -0.4 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0.01 \\ 0 \\ 0 \end{bmatrix}$

9.8 Show that the eigenvalues $\{\lambda_i, i = 1, \ldots, n\}$ of the closed-loop system for the pair (A, B) with state feedback are scaled by α if we use the state feedback gains from the pole placement with the pair $(A/\alpha, B/\alpha)$. Verify this result using MATLAB for the pair

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.005 & -0.11 & -0.7 \end{bmatrix} B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

and the eigenvalues $\{0.1, 0.2, 0.3\}$ with $\alpha = 0.5$.

9.9 Using eigenvalues that are two to three times as fast as those of the plant, design a state estimator for the system

a. $A = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0 & 0 & 0.3 \\ 0.3 & 0 & 0.3 \end{bmatrix} \quad C = [1 \quad 1 \quad 0]$

b. $A = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0 & 0 & 0.3 \\ 0.3 & 0 & 0.3 \end{bmatrix} \quad C = [1 \quad 0 \quad 0]$

9.10 Consider the system

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.005 & -0.11 & -0.7 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = [0.5 \quad 1 \quad 0] \quad d = 0$$

a. Design a controller that assigns the eigenvalues $\{-0.8, -0.3 \pm j0.3\}$. Why is the controller guaranteed to exist?

b. Why can we design an observer for the system with the eigenvalues $\{-0.5, -0.1 \pm j0.1\}$? Explain why the value (-0.5) must be assigned. (*Hint:* $(s+0.1)^2(s+0.5) = s^3+0.7\,s^2+0.11\,s + 0.005$.)

c. Obtain a similar system with a second-order observable subsystem, for which an observer can be easily designed, as in Section 8.3.3. Design an observer for the transformed system with two eigenvalues shifted as in (b), and check your design using the MATLAB command **place** or **acker.** Use the result to obtain the observer for the original system. (*Hint:* Obtain an observer gain $\mathbf{l}_r$ for the similar third-order system from your design by setting the first element equal to zero. Then obtain the observer gain for the original system using $\mathbf{l} = T_r\mathbf{l}_r$, where T_r is the similarity transformation matrix.)

d. Design an observer-based feedback controller for the system with the controller and observer eigenvalues selected as in (a) and (b), respectively.

9.11 Design a reduced-order estimator state feedback controller for the discretized system

$$A = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.5 & 0.2 \\ 0.2 & 0 & 0.4 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0.01 \\ 0 \\ 0.005 \end{bmatrix} \quad \mathbf{c}^T = [\,1, \quad 1, \quad 0\,]$$

to obtain the eigenvalues $\{0.1, 0.4 \pm j0.4\}$.

9.12 Consider the following model of an armature-controlled DC motor, which is slightly different from that described in Example 7.7:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -11 & -11.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u$$

$$y = [\,1 \quad 0 \quad 0\,] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

For digital control with $T = 0.02$, apply the state feedback controllers determined in Example 9.4 in order to verify their robustness.

9.13 Consider the following model of a DC motor speed control system, which is slightly different from that described in Example 6.9:

$$G(s) = \frac{1}{(1.2s + 1)(s + 10)}$$

For a sampling period $T = 0.02$, obtain a state−space representation corresponding to the discrete-time system with DAC and ADC, then use it to verify the robustness of the state controller described in Example 9.6.

9.14 Verify the robustness of the state controller determined in Example 9.7 by applying it to the model shown in Problem 9.13.

9.15 Prove that the pair $\left(\widetilde{A}, \widetilde{B}\right)$ of Eq. (9.31) for the system with integral control is controllable if and only if the pair (A, B) is controllable.

9.16 Consider the DC motor position control system described in Example 3.6, where the (type 1) analog plant has the transfer function

$$G(s) = \frac{1}{s(s+1)(s+10)}$$

For the digital control system with $T = 0.02$, design a state feedback controller to obtain a step response with zero steady-state error, zero overshoot, and a settling time of less than 0.5 s.

9.17 Design a digital state feedback controller for the analog system

$$G(s) = \frac{-s+1}{(5s+1)(10s+1)}$$

with $T = 0.1$ to place the closed-loop poles at $\{0.4, 0.6\}$. Show that the zero of the closed-loop system is the same as the zero of the open-loop system.

9.18 Design a state feedback law for the pair

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.005 & -0.11 & -0.7 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

to assign the eigenvalues $\{0,\ 0.25,\ 0.3,\ 0.4\}$.

9.19 Write the closed-loop system state−space equations of a full-order observer state feedback system with integral action.

9.20 Consider the continuous-time model of the overhead crane proposed in Problem 7.11 with $m_c = 1000$ kg, $m_l = 1500$ kg, and $l = 8$ m. Design a discrete full-order observer state feedback control to provide motion of the load without sway.

9.21 We present a model for the female population of a species with a maximum age of 3 years based on an example from[1] The population is divided into three groups according to age; namely the age groups $(0, 1)$, $[1, 2)$, $[2, 3]$. These three populations are state variables of the model $\{x_{1k},\ x_{2k},\ x_{3k}\}$, respectively. The second and third populations can produce offspring but the second population is more fertile.

[1] Mazen Shahin, *Explorations of Mathematical Models in Biology with MATLAB,* J. Wiley, Hoboken, NJ, 2014.

$$\mathbf{x}_{k+1} = \begin{bmatrix} 0 & 6 & 10/3 \\ 0.6 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix} \mathbf{x}_k$$

Because of the difficulty of estimating the age of members of the species, the only measurement available is the total population and the measurement equation is

$$y_k = [1 \quad 1 \quad 1]\mathbf{x}_k$$

(a) Check the observability of the system
(b) Design an observer for the system with observer eigenvalues $\{0, -0.1 \pm j0.1\}$

9.22 Consider the continuous-time model of the overhead crane proposed in Problem 7.11 with $m_c = 1000$ kg, $m_l = 1500$ kg, and $l = 8$ m. Design a control system based on pole assignment using transfer functions in order to provide motion of the load without sway.

Computer exercises

9.23 Write a MATLAB script to evaluate the feedback gains using Ackermann's **formula for any pair** (A, B) and any desired poles $\{\lambda_1, \ldots, \lambda_n\}$.

9.24 Write a MATLAB program that, for a multi-input system with known state and input matrices, determines (i) a set of feasible eigenvectors and (ii) a state feedback matrix that assign a given set of eigenvalues. Test your program using Example 9.5.

9.25 Write a MATLAB function that, given the system state−space matrices A, B, and C, the desired closed-loop poles, and the observer poles, determines the closed-loop system state−space matrices of a full-observer state feedback system with integral action.

9.26 Write a MATLAB function that uses the transfer function approach to determine the closed-loop system transfer function for a given plant transfer function $G(z)$, desired closed-loop system poles, and observer poles.