**Final Examination: Mathematical Thinking – Section 2**

**(Dec 1, 2023): 14:16-16:15**

**2 hours (180 Marks)**

**Instructions**

1. The Exam is open textbook only if you have bought one☺
2. In all questions, show your complete rough work to illustrate your thought process and then write a reason.
3. Remember, this course rewards the entire process of thinking and not a particular endpoint and result. The thinking journey and discovery of different paths is more important than reaching the destination itself (that is also important in real life though).
4. BEST OF LUCK on your thinking endeavor

Table

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**Question 1 (20 Marks): Please score using the rubric and provide reasons for your evaluation.**

Lemma: For any two propositions P, Q, ¬P ∧ ¬Q is equivalent to ¬[P ∧ Q].

Proof: Suppose that ¬P ∧ ¬Q is true. Then both ¬P and ¬Q are true.  
So P and Q are both false. Thus P ∧Q is false. Hence ¬[P ∧Q] is true. This argument clearly works the other way. So we have implication in both directions, which proves the claim.

**Reasons:**

**Question 2 (15 Marks):** Which of the following formal propositions says that there is no largest prime. (There may be more than one. You have to select all correct propositions.) The variables denote natural numbers.

(a) ¬∃x∃y[Prime(x) ∧ ¬Prime(y) ∧ (x < y)]

(b) ∀x∃y[Prime(x) ∧ Prime(y)∧(x<y)]  
(c) ∀x∀y[Prime(x) ∧ Prime(y)∧(x<y)]

(d) ∀x∃y[Prime(y) ∧ (x < y)]

(e) ∃x∀y[Prime(y) ∧ (x < y)]

(f) ∀x∃y[Prime(x) ∧ (x < y)]

**Reason:**

**Question 3 (20 Marks):**

This theorem is obviously false. Enter the line number of the (incorrect) statement where the proof logically breaks down.

***Theorem All Americans are the same age.***

**Proof:**

Let S(n) be the statement: In any group of n Americans, everyone in that group has the same age.

2. We prove S(n) by induction on n.  
3. Since everyone in a group of one American has the same age, S(1) is true.  
4. Assume S(n) is true for some n.  
5. We prove S(n + 1).  
6. Let G be an arbitrary group of n + 1 Americans.  
7. We show that everyone in G has the same age.  
8. We do this by showing that any two members of G have the same age.  
9. Let a, b ∈ G.  
10. Let Ga be the result of removing a from G.  
11. Since Ga has n members, b (which is in Ga) has the same age as any other person in Ga. 12. Similarly, if Gb is G with b removed, then a has the same age as any other person in Gb. 13. Now let c be any person in G other than a and b.  
14. Thenc∈Ga andc∈Gb.  
15. So, a and b both have the same age as c.  
16. Hence a and b have the same age.  
17. This proves S(n + 1).  
18. Hence, by induction, S(n) is true for all n.  
19. This implies that all Americans have the same age.

**Reasons**

**Question 4: (45 Marks)** Consider a clause *C* ≡ (*L*1 ∨...∨*Lk*), where the *Li*’s are literals.

1. Show that *C* is valid if and only if, for some *i*, *j*∈{1,...,*k*}, *Li* ≡ ¬*Lj*.
2. Use the result from part (a) to design (and justify) an efficient algorithm that checks the validity of an input formula given in CNF (*in CNF, the statements in Boolean logic are conjunctions of clauses with clauses of disjunctions. In other words, a statement is a series of ORs connected by ANDs*) [2 Examples: *(1) (A OR B) AND (C OR D), (2)* (A OR B) AND (NOT C OR B)]
3. Convert the expression ((*A*∨*B*) ⇒ *C*) ⇒ (*A* ⇒ *C*)) directly into CNF using standard logical equivalences (do not simplify along the way). Hence check its validity using your algorithm of part (b).

**Question 5: (45 Marks)**

A Boolean expression is in *Kowalski Normal Form* (KNF) if it is a conjunction of implications, where the premise of each implication has a conjunction of variables and the conclusion of each implication is a disjunction of variables. (Note: *variables*, not *literals*.) For example, the expression ((*B* ∧ *C*) ⇒ (*A* ∨ *D*)) ∧ (C ⇒ E) is in KNF.

(a) Prove that (B ∧ C) ⇒ (A ∨ D) is logically equivalent to (A ∨ ¬B ∨¬C ∨ D).

(b) Prove by induction that ¬(X1 ∨···∨Xn) ≡ (¬X1 ∧···∧¬Xn).

(c) Now use part (b) to prove that every Boolean expression is logically equivalent to an expression in KNF.

**Question 6: (20 Marks)**

Geoff Poshingten is out at a fancy pizza joint, and decides to order a calzone. When the waiter asks what he would like in it, he replies, “I want either pepperoni or sausage. Also, if I have sausage, then I must also include quail. Oh, and if I have pepperoni or quail then I must also have ricotta cheese.”

1. Translate Geoff's order into logical symbols.
2. The waiter knows that Geoff is either a liar or a truth-teller (so either everything he says is false, or everything is true). Which is it?
3. What, if anything, can the waiter conclude about the ingredients in Geoff's desired calzone?

**Question 7 (15 Marks)**: Tommy Flanagan was telling you what he ate yesterday afternoon. He tells you, “I had either popcorn or raisins. Also, if I had cucumber sandwiches, then I had soda. But I didn't drink soda or tea.” Of course you know that Tommy is the worlds worst liar, and everything he says is false. What did Tommy eat?

Justify your answer by writing all of Tommy's statements using sentence variables (P,Q,R,S, T) taking their negations, and using these to deduce what Tommy actually ate.