AN INTRODUCTION TO EXTERIOR ALGEBRA

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ABSTRACT. After taking a course in Vector Calculus, one becomes acquainted with a total of 4 versions of the Fundamental Theorem of Calculus (FTC): the FTC from single variable calculus, Green's Theorem, Stokes' Theorem, and the Divergence Theorem. Here we show that all of these "versions" can be obtained as special cases of what's known as "The Generalized Stokes' Theorem". We do this by first giving the reader an introduction to the language of differential forms. Here, the unifying force behind the Generalized Stokes' Theorem known as the "exterior derivative" will be emphasized. An accelerated introduction to manifolds integration over manifolds will also be provided, and we finish off with the promised proof of the classical theorems of Vector Calculus.

1. INTRODUCTION

In a first course in vector calculus, the fundamental theorem of calculus (FTC), fundamental theorem for line integrals, Green's theorem, and Stokes' theorem are often introduced as 4 separate theorems. Especially within the applied sciences, this treatment is usually not only sufficient, but favourable since it is easily understood by those with a standard background in differential and integral calculus [5]. For those wanting an introduction into the language of differential geometry, this paper presents a quick introduction to the basic definitions and theorems that one should become familiar with. Section 2 provides an introduction to k-forms over $\mathbb{R}^n$. Definitions 2.1 and 2.2 are there for completeness in the case that the reader does not have knowledge of group theory, and in particular, the *symmetric group*. Readers wanting a more thorough explanation should refer to Dummit and Foote [2] or Loring Tu [6]. We do not go over k-forms over arbitrary vector spaces, but briefly mention how this can be done at the end. For the purposes of understanding why the classical theorems of vector calculus are special cases of the same theorem, examples 3.4, 3.6, and 3.7 are key to the idea of the proof.

2. k-FORMS OVER $\mathbb{R}^n$

We begin with a few basic definitions that are not typically covered in analysis.

Definition 2.1 (Group)**.** A *group* is an ordered pair $(G, \cdot)$ where G is a set and $\cdot$ is a binary operation $\cdot : G \times G \to G$ denoted $(G, \cdot)$ satisfying:

(a) $(f \cdot g) \cdot h = f \cdot (g \cdot h)\ \forall f, g, h \in G$.
(b) $\exists$ an element $1 \in G$, called the 'identity element', with the property that $\forall g \in G$, $G \cdot 1 = 1 \cdot G = G$.
(c) $\forall g \in G$, $\exists$ an element g^{-1}, called the inverse of g, with the property that $g \cdot g^{-1} = g^{-1} \cdot g = 1$.

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If in addition to (a),(b), and (c) $(G, \cdot)$ satisfies $g \cdot h = h \cdot g \ \forall g, h \in G$, then we call the group an *abelian group*.

Definition 2.2 (Symmetric Group [2])**.** Let $\Omega = \{1, 2, \dots, n\}$. Then

$$S_n := \{\sigma : \Omega \to \Omega : \sigma \text{ is a bijection}\}$$

is called the *symmetric group* of order n. (the group operation here is composition). We call the elements σ of a given symmetric group *permutations*.

An easy way to visualize some $\sigma \in S_n$ is by the matrix

$$\begin{bmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{bmatrix}$$

It is easy to see that a permutation σ may be written in terms of transpositions (also known as reversals) (a, b) (i.e. maps that send a to b, b to a, and fix all other elements).

Definition 2.3 (Sign of a Permuation [6])**.** The *sign* of a permutation $\sigma \in S_n$, denoted by $\text{sgn}(\sigma)$, is given by

$$\text{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is the product of an } \textbf{even} \text{ number of transpositions} \\ -1 & \text{if } \sigma \text{ is the product of an } \textbf{odd} \text{ number of transpositions} \end{cases}$$

A very brief exploration into symmetric groups will reveal that an even permutation can never be written as a product of an odd number of transpositions, and that an odd permutation can never be written as a product of an even number of transpositions. In other words, the sgn operation is well defined.

We now begin a series of definitions that are more specific to Exterior Algebra, and will build up very quickly into what we need to immediately start deriving the more interesting results that a first dive into Exterior Algebra can give.

Definition 2.4 (1-form)**.** $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a **1-form** (also known as a linear functional or a 1-covector) if $\forall x, y \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$:

$$\varphi(ax + by) = a\varphi(x) + b\varphi(y)$$

For an arbitrary vector space V, this is simply describing the already familiar linear transformation, which we will henceforth refer to as a linear functional.

Definition 2.5 (Dual Space)**.** Let V be a vector space over a field F. The collection of all linear functionals, denoted V^*, is the *Dual Space* of V.
It is also common to denote the dual space by $V^* = \text{Hom}(V, \mathbb{R})$.

The dual space of $\mathbb{R}^n$ will become of special interest, and so we denote it by the symbol $\Lambda_1(\mathbb{R}^n)$ [1]. Recall the canonical orthonormal basis of $\mathbb{R}^n$ $\mathcal{B} = \{e_1, e_2, \dots, e_n\}$, where each e_i has 1 as its i-th entry, and 0 for all other entries. The following proposition gives a useful way of representing any element in $\Lambda_1(\mathbb{R}^n)$, and in some sense, gives an equivalence between the dot product and linear functionals.

Proposition 2.6. *$\forall a, x \in \mathbb{R}^n$,*

$$\psi(x) := a \cdot x \in \Lambda_1(\mathbb{R}^n)$$

. Furthermore, $\forall \varphi \in \Lambda_1(\mathbb{R}^n)$, $\exists b = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$ such that

$$\varphi(x) = b \cdot x := \sum_{i=1}^{n} b_i x_i \ \forall x = (x_1, x_2, \dots x_n) \in \mathbb{R}^n$$

Proof. The first part of the proposition is clear since the dot product is a linear transformation.

For the second part, suppose $\varphi \in \Lambda_1(\mathbb{R}^n)$ and let $x = (x_1, x_2, \dots x_n) \in \mathbb{R}^n$. Then

$$\begin{aligned}
\varphi(x) &= \varphi(x_1 x_2, \dots x_n) \\
&= \varphi(x_1 e_1 + x_2 e_2 + \dots x_n e_n) \\
&= x_1 \varphi(e_1) + x_2 \varphi(e_2) + \cdots + x_n \varphi(e_n) \\
&= (\varphi(e_1), \varphi(e_2), \dots, \varphi(e_n)) \cdot x
\end{aligned}$$

Hence, if we choose $b = (\varphi(e_1), \varphi(e_2), \dots, \varphi(e_n))$, then the second part of the proposition follows.

□

Proposition 2.7. $\Lambda_1(\mathbb{R}^n)$ *is a vector space over* $\mathbb{R}$.

Proof. It is sufficient to show that $\Lambda_1(\mathbb{R}^n)$ is closed under scalar multiplication and vector addition (vector addition here is the same addition used in summing functions).
Suppose $\alpha, \beta \in \mathbb{R}$ and $\varphi, \psi \in \Lambda_1(\mathbb{R}^n)$. From proposition 2.6, $\exists a, b \in \mathbb{R}^n$ such that $\varphi(x) = a \cdot x$ and $\psi(x) = b \cdot x \ \forall x \in \mathbb{R}^n$.
Let $x \in \mathbb{R}^n$. Then for closure under scalar multiplication,

$$\alpha \varphi(x) = \alpha(a \cdot x) = a \cdot (\alpha x) = \varphi(\alpha x)$$

For closure under addition,

$$(\varphi + \psi)(x) = (a + b) \cdot x = a \cdot x + b \cdot x = \varphi(x) + \psi(x)$$

□

Now that we know that the set of 1-forms on $\mathbb{R}^n$ form a vector space, a natural follow-up question is to ask what the basis of such a vector space is. The answer to this question is actually what we have been after this whole time, as it turns out that we can use *differentials* as basis vectors for this space. Since the word 'differential' is often used in a loose, imprecise sense - especially within the applied sciences - we will start the process of setting a precise meaning for the word with the following definition.

Definition 2.8. (Differential 1-form). Suppose $v \in \mathbb{R}^n$. Let v_i denote it's ith component. Then for $1 \leq i \leq n$, let $dx_i \in \Lambda_1(\mathbb{R}^n)$ denote the 1-form defined by

$$dx_i(v) = v_i \ \forall v_i \in \mathbb{R}^n$$

We call dx_i a *differential 1-form*, which we abbreviate as a *differential*.

Our claim that we can use differentials as basis vectors for $\Lambda_1(\mathbb{R}^n)$ follows immediately from propositions 2.6 and 2.7 since $\forall \varphi \in \Lambda_1(\mathbb{R}^n)$ and some $v \in \mathbb{R}^n$,

$$\begin{aligned} \varphi(v) &= \varphi(\sum_{i=1}^n v_i e_i) \\ &= \sum_{i=1}^n \varphi(v_i e_i) \\ &= \sum_{i=1}^n v_i \varphi(e_i) \\ &= \sum_{i=1}^n \varphi(e_i) dx_i(v) \end{aligned}$$

So we can write $\varphi = \sum_{i=1}^n \varphi(e_i) dx_i$. The basis $\{dx_1, dx_2, \dots dx_n\}$ will be referred to as the standard basis of $\Lambda_1(\mathbb{R}^n)$.

We can now make things more interesting by extending the idea of a 1-form to a more general multilinear definition.

Definition 2.9 (k-linear [6])**.** A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is *k-linear* (also simply called multilinear) on $\mathbb{R}^n$ if it satisfies:

$$\begin{aligned} &\varphi(v_1, v_2, \dots, v_{i-1}, au + bw, v_{i+1}, \dots, v_k) = \\ &\quad = a\varphi(v_1, v_2, \dots, v_{i-1}, u, v_{i+1}, \dots, v_k) + \\ &\quad\quad + b\varphi(v_1, v_2, \dots, v_{i-1}, w, v_{i+1}, \dots, v_k) \end{aligned}$$

$\forall a, b \in \mathbb{R}$ and $v, w \in \mathbb{R}^n$.

2-linear and 3-linear functions are often called, respectively, *bilinear* and *trilinear*. A k-linear function is also called a *k-tensor* [6].

Definition 2.10 (Symmetric and Alternating [6])**.** A k-linear function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is *symmetric* if

$$\varphi(v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(k)}) = \varphi(v_1, v_2, \dots v_k)$$

$\forall \sigma \in S_k$. φ is *alternating* if

$$\varphi(v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(k)}) = \mathrm{sgn}(\sigma)\varphi(v_1, v_2, \dots v_k)$$

$\forall \sigma \in S_k$.

Definition 2.11 (k-form [1])**.** A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a *k-form* if it is both k-linear and alternating. A k-form is also called an *alternating k-tensor* or a *k-covector*.

The vector space formed by the set of all k-forms on $\mathbb{R}^n$ will be denoted by $\Lambda_k(\mathbb{R}^n)$. Note that a 0-form is defined to be a constant function. With this definition, $\Lambda_0(\mathbb{R}^n)$ is the vector space $\mathbb{R}$.

Example 2.12. Let $\varphi, \psi \in \Lambda_1(\mathbb{R}^n)$. We denote and define the map $\xi = \varphi \wedge \psi$ by

$$\xi(u, v) = (\varphi \wedge \psi) := \begin{vmatrix} \varphi(u) & \varphi(v) \\ \psi(u) & \psi(v) \end{vmatrix} = \varphi(u)\psi(v) - \varphi(v)\psi(u)$$

From the properties of the determinant map, it immediately follows that ξ is both 2-linear and alternating, and hence a 2-form. The symbol $\wedge$ is called a *wedge product*.

Theorem 2.13 ([1]). *For $1 \leq i < j \leq n$, the 2-forms $dx_i \wedge dx_j$, called elementary 2-forms, constitute a basis for $\Lambda_2(\mathbb{R}^n)$. Moreover, dim$(\Lambda_2(\mathbb{R}^n)) = \binom{n}{2}$.*

Proof. Let $\varphi \in \Lambda_2(\mathbb{R}^n)$ and $u = \sum_{i=1}^n u_i e_i$, $v = \sum_{j=1}^n v_j e_j \in \mathbb{R}^n$.
Define $a_{ij} := \varphi(u, v)$. Then since φ is alternating, $a_{ij} = \varphi(u, v) = -\varphi(v, u) = -a_{ji}$. Moreover, if $i = j$ then $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$. Hence, the matrix (a_{ij}) is skew-symmetric. i.e.,

$$(a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} 0 & -a_{12} & -a_{13} & \dots & -a_{1n} \\ a_{12} & 0 & -a_{23} & \dots & -a_{2n} \\ a_{13} & a_{23} & 0 & \dots & -a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \dots & 0 \end{bmatrix}$$

Now observe,

$$\begin{aligned} \varphi(u, v) &= \varphi(\sum_{i=1}^n u_i e_i, \sum_{j=1}^n v_j e_j) \\ &= \sum_{i=1}^n \sum_{j=1}^n u_i v_j \varphi(e_i, e_j) \\ &= \sum_{i=1}^n \sum_{j=1}^n a_{ij} u_i v_j \\ &= \sum_{1 \leq i < j \leq n} (a_{ij} u_i v_j - a_{ij} u_j v_i) \\ &= \sum_{1 \leq i < j \leq n} a_{ij} (dx_i(u) dx_j(v) - dx_j(u) dx_i(v)) \\ &= \sum_{1 \leq i < j \leq n} a_{ij} (dx_i \wedge dx_j)(u, v) \end{aligned}$$

Which shows that $\{dx_i \wedge dx_j : 1 \leq i < j \leq n\}$ spans $\Lambda_2(\mathbb{R}^n)$. To see why this set is linearly independent, suppose that $\sum_{1 \leq i < j \leq n} a_{ij} (dx_i \wedge dx_j)(u, v) = 0 \ \forall u, v \in \mathbb{R}^n$. Then choosing $u = e_i$ and $v = e_j$, the above implies that his is exactly identity in $\Lambda_2(\mathbb{R}^n)$, 0(u,v). Thus, $a_{ij} = 0(u, v) = 0 \ \forall 1 \leq i < j \leq n$. Thus, $\{dx_i \wedge dx_j : 1 \leq i < j \leq n\}$ is linearly independent, and hence a basis for $\Lambda_2(\mathbb{R}^n)$.
The fact that $\dim(\Lambda_2(\mathbb{R}^n)) = \binom{n}{2}$ follows immediately since the cardinality of the set $\{dx_i \wedge dx_j : 1 \leq i < j \leq n\}$ is determined by choosing 2 different indices for i and j from the n that we have available. $\square$

The definitions of the wedge product and elementary 2-forms can be easily extended to higher order tensors. As one might expect, we will also see that we can use elementary k- forms as basis vectors for $\Lambda_k(\mathbb{R}^n)$.

Definition 2.14 (Elementary k-forms [1])**.** Recall dx_i, the differential 1-form over $\mathbb{R}^n$. Let $v_1, v_2, \dots, v_k \in \mathbb{R}^n$ We denote and define the *elementary k-forms* by:

$$
\begin{aligned}
(dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k})(v_1, v_2, \dots, v_k) &= \begin{vmatrix} dx_{i_1}(v_1) & dx_{i_1}(v_2) & \dots & dx_{i_1}(v_k) \\ dx_{i_2}(v_1) & dx_{i_2}(v_2) & \dots & dx_{i_2}(v_k) \\ \vdots & \vdots & \ddots & \vdots \\ dx_{i_k}(v_1) & dx_{i_k}(v_2) & \dots & dx_{i_k}(v_k) \end{vmatrix} \\
&= \begin{vmatrix} v_{1_{i_1}} & v_{2_{i_1}} & \dots & v_{k_{i_1}} \\ v_{1_{i_2}} & v_{2_{i_2}} & \dots & v_{k_{i_2}} \\ \vdots & \vdots & \ddots & \vdots \\ v_{1_{i_k}} & v_{2_{i_k}} & \dots & v_{k_{i_k}} \end{vmatrix}
\end{aligned}
$$

Note that in the above definition, each $i_1, i_2, \dots, i_k$ represents an index; that is, we are indexing the index i. From the properties of the determinant, we know that interchanging columns alternates the sign of the determinant. Another way to express this property using what we have discussed before is that

$$dx_{\sigma(i_1)} \wedge dx_{\sigma(i_2)} \wedge \cdots \wedge dx_{\sigma(i_1)} = \operatorname{sgn}(\sigma) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k},\ \forall \sigma \in S_k$$

this means that any rearrangement of the wedge product above yields either same k-form, or a scalar multiple of that k-form by -1. As we've already seen in the case of 1-forms and 2-forms, the set

$$\mathcal{B} := \{dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} | 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

forms a basis for $\Lambda_k(\mathbb{R}^n)$ (as fact whose proof we omit here). We know from before that dx_i is the 1-form that takes in a vector $v \in \mathbb{R}^n$ and returns its i-th component, so there are exactly n distinct dx_i. Hence, $|\mathcal{B}| = \binom{n}{k}$.

With the knowledge that any k-form can be expressed as a linear combination of wedge products of elementary 1-forms, we can now look at how to take the wedge product of two multilinear functions. Let $\varphi \in \Lambda_k(\mathbb{R}^n)$ and $\psi \in \Lambda_\ell(\mathbb{R}^n)$. Then we can write

$$
\begin{aligned}
\varphi &= \sum_{1 \leq i_1 < i_2 < \dots i_k \leq n} a_{i_1 i_2 \dots i_k} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} \text{ (has up to } \binom{n}{k} \text{ terms)} \\
\psi &= \sum_{1 \leq j_1 < j_2 < \dots j_k \leq \ell} b_{j_1 j_2 \dots j_\ell} dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_\ell} \text{ (has up to } \binom{n}{\ell} \text{ terms)}
\end{aligned}
$$

The wedge product of φ and ψ is then given by

$$\varphi \wedge \psi = \sum_{\substack{1 \leq i_1 < i_2 < \dots i_k \leq n \\ 1 \leq j_1 < j_2 < \dots j_k \leq \ell}} a_{i_1 i_2 \dots i_k} b_{j_1 j_2 \dots j_\ell} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_\ell}$$

Which is a $(k+\ell)$-form. Now, let $\varphi, \varphi_1, \varphi_2 \in \Lambda_k(\mathbb{R}^n)$, $\psi \in \Lambda_\ell(\mathbb{R}^n)$, and $\chi \in \Lambda_m(\mathbb{R}^n)$ with $a, b \in \mathbb{R}$. Some immediate results defining the wedge product this way (that follow again from the properties of the determinant are) [1]

(1) (Linearity).

$$(a\varphi_1 + b\varphi_2) \wedge \psi = a\varphi_1 \wedge \psi + b\varphi_2 \wedge \psi$$

(2) (Associativity).

$$(\varphi \wedge \psi) \wedge \chi = \varphi \wedge (\psi \wedge \chi)$$

(3) (Skew-Commutativity).

$$\varphi \wedge \psi = (-1)^{k\ell} \psi \wedge \varphi$$

Remark 2.15. The wedge product is also often defined in terms of an operation called the *tensor product*. Given a k-linear function f and an ℓ-linear function g, their tensor product is denoted and defined by:

$$(f \otimes g)(v_1, \ldots, v_{k+\ell}) := f(v_1, \ldots, v_k) g(v_{k+1}, \ldots, v_{k+\ell})$$

It is not too hard to show that the tensor product is associative. We can also define two operators, called the *symmetrizing* and *alternating* operators, that make, respectively, symmetric and alternating k-linear functions from f. These are denoted and defined by [6]:

$$(Sf)(v_1, \ldots, v_k) := \sum_{\sigma \in S_k} f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \text{ (Symmetrizing)}$$

$$(Af)(v_1, \ldots, v_k) := \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \text{ (Alternating)}$$

Which we will abbreviate as $Sf = \sum_{\sigma \in S_k} \sigma f$ and $Af = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \sigma f$. For $f \in \Lambda_k(\mathbb{R}^n)$ and $g \in \Lambda_\ell(\mathbb{R})$, we can define the wedge product (also known as the exterior product) as

$$f \wedge g = \frac{1}{k!\ell!} A(f \otimes g)$$

This definition is more appropriate if we are looking to explore multilinear functions over any arbitrary vector space, and not just $\mathbb{R}^n$, and is equivalent to the one we have given earlier (and will continue to use). It is only mentioned here for extra emphasis on the fact that all of what we have mentioned here about forms may be extended to any vector space.

3. Differential Forms on $\mathbb{R}^n$

When making the jump from single-variable calculus to multi-variable calculus, one attempts to extend functions that take in one input and return a single output to ones that take in multiple inputs, but still only return a single output. In a first course in vector calculus, we then make another extension by considering functions that take in several inputs and return several outputs. The derivative of these maps from $\mathbb{R}^n \to \mathbb{R}^m$ have, in Math 447, already been considered. In an analogous sense to that of defining a vector-valued function that maps $\mathbb{R}^n \to \mathbb{R}^m$ called vector fields, we now introduce the notion of a k-form-valued function that maps $\mathbb{R}^n \to \Lambda_k(\mathbb{R}^n)$ called a *differential form*.

Definition 3.1 (C^k/C^∞ Function)**.** Let $U \subseteq \mathbb{R}^n$ be an open set. If $f : U \to \mathbb{R}^m$ is has all of its partial derivatives up to order k, and if these partials are all continuous, we say that f is C^k. If $f : U \to \mathbb{R}^m$ has continuous partials of all orders, we call f C^∞.

In this report, we refer to $f : \mathbb{R}^n \to \mathbb{R}$ as *smooth* if it is C^k up to the desired order partials that we want to compute (in some cases, smoothness will require f to be C^∞).

Definition 3.2 (Differential k-form [1])**.** Let $U \subseteq \mathbb{R}^n$ be an open set. With $k \geq 1$ and some $\Phi : U \to \Lambda_k(\mathbb{R}^n)$ given by

$$\Phi(x) = \sum_{1 \leq i_1 < i_2 < \ldots i_k \leq n} a_{i_1 i_2 \ldots i_k}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} \ \forall x \in U$$

with the coefficients $a_{i_1 i_2 \ldots i_k}$, $1 \leq i_1 < i_2 < \ldots i_k \leq n$, smooth functions, we call Φ a *differential k- form.* In the case of $k = 0$, a *differential 0-form* is a smooth, real-valued function on D.

For $k \geq 0$, the set of all differential k-forms one some open set $D \subseteq \mathbb{R}^n$ is denoted by $\mathcal{F}_k(D)$. Since differential k-forms send into $\Lambda_k(\mathbb{R}^n)$, we can define the wedge product point-wise using the same definition as that given in section 2, with one added requirement for the case where we are trying to take the wedge product with a differential 0-form (since we have not defined how to take the wedge product with a 0-form). If $f \in \mathcal{F}_0(D)$, $\Phi \in \mathcal{F}_k(D)$, and $\Psi \in \mathcal{F}_\ell(D)$, then we require that $f \wedge \Phi = f\Phi$ (i.e., we just multiply the coefficients of Φ by f). An immediate consequence of this is that $(f\Phi) \wedge \Psi = f(\Phi \wedge \Psi) = \Phi \wedge (f\Psi)$ since the multiplication of coefficients of the k-forms is commutative and associative.

We are finally ready to give a precise definition of what people mean by the "differential of a function" (which is really the exterior derivative of a differential 0-form) that we mentioned at the beginning is so often used in loose terms. In fact, we can do even better than this and compute the exterior derivative of an arbitrary k-form.

Definition 3.3 (Exterior Derivative [1])**.** Let $D \in \mathbb{R}^n$ be open and $f \in \mathcal{F}_0(D)$. The *exterior derivative* of f, denoted by df is given by

$$df(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

If $\Phi \in \mathcal{F}_k(D)$, then the *exterior derivative* of Φ, denoted $d\Phi$ is given by

$$d\Phi(x) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} (da_{i_1 i_2 \ldots i_k}(x)) \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

where $a_{i_1 i_2 \ldots i_k}(x)$, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, are the coefficients of Φ (which vary with $x \in D$) written in terms of the standard basis of $\Lambda_k(\mathbb{R}^n)$. Notice that the exterior differential operator "d" maps $\mathcal{F}_k(D)$ into $\mathcal{F}_{k+1}(D)$.

Example 3.4. For some open set $D \subseteq \mathbb{R}^n$, let $f \in \mathcal{F}_0(D)$. Then for $v \in \mathbb{R}^n$,

$$df(x)(v) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i(v) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} v_i = \nabla f \cdot v = (Df)v$$

So the exterior derivative of a differential 0-form evaluated at some point is simply the total derivative of f that we know and love from Math 447 multiplied with v. Since the exterior derivative yields another differential form, we are able to take the exterior derivative again, which we denote by $d^2 f = d(df)$. Observe,

$$\begin{aligned} d^2 f(x) &= d\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i\right) \\ &= \sum_{i=1}^{n} d\left(\frac{\partial f}{\partial x_i}\right) \wedge dx_i \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j\right) \wedge dx_i \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i\right) \end{aligned}$$

and since $\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ and $dx_j \wedge dx_i = -dx_i \wedge dx_j$, it follows that $d^2 f(x) = 0$. One may wonder whether this is a phenomenon that occurs only for 0-forms, but closer inspection of what's happening when one takes the exterior derivative of an exterior derivative reveals that this something that happens for all k-forms. This fact, along with a few other properties of the exterior derivative, are given in the following theorem.

Theorem 3.5 (Properties of the Exterior Derivative [1])**.** *Let $\Phi, \Psi \in \mathcal{F}_k(D)$ and $\Omega \in \mathcal{F}_\ell(D)$ on some open set $D \in \mathbb{R}^n$, and let $a, b \in \mathbb{R}$. Then,*

(a) *(Linearity from $\mathcal{F}_k(D) \to \mathcal{F}_{k+1}(D)$).*

$$d(a\Phi + b\Psi) = a\ d\Phi + b\ d\Psi$$

(b) *(Product Rule).*

$$d(\Phi \wedge \Omega) = (d\Phi) \wedge \Omega + (-1)^k \Psi \wedge (d\Omega)$$

(c) *(Exterior Derivative of the Exterior Derivative is the zero Differential Form).*

$$d^2\Phi = d(d\Phi) = \{0\}$$

In the interest of saving space for more interesting results, we omit the proof of the above theorem, but note that parts (a), and (b) are relatively straightforward, while the proof of (c) is essentially contained in example 3.4.

Example 3.4 showed that the exterior derivative of a differential 0-form is substantially equivalent to taking the gradient. In similar regards, as the next two examples will illustrate, the exterior derivative of a differential 1-form is like taking the curl of a vector field, whereas the exterior derivative of a differential 2-form is similar to taking the divergence of a vector field.

Example 3.6 ([1])**.** Let $\Phi = Pdx_1 + Qdx_2 + Rdx_3 \in \mathcal{F}_1(\mathbb{R}^3)$, where P, Q, and R are functions that map $\mathbb{R}^3 \to \mathbb{R}$. Then,

$$\begin{aligned}
d\Phi &= (dP) \wedge dx_1 + (dQ) \wedge dx_2 + (dR) \wedge dx_3 \\
&= \cancel{\frac{\partial P}{\partial x_1} dx_1 \wedge dx_1} + \frac{\partial P}{\partial x_2} dx_2 \wedge dx_1 + \frac{\partial P}{\partial x_3} dx_3 \wedge dx_1 + \\
&\quad + \frac{\partial Q}{\partial x_1} dx_1 \wedge dx_2 + \cancel{\frac{\partial Q}{\partial x_2} dx_2 \wedge dx_2} + \frac{\partial Q}{\partial x_3} dx_3 \wedge dx_2 + \\
&\quad + \frac{\partial R}{\partial x_1} dx_1 \wedge dx_3 + \frac{\partial R}{\partial x_2} dx_2 \wedge dx_3 + \cancel{\frac{\partial R}{\partial x_3} dx_3 \wedge dx_3} \\
&= \left(\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2} \right) dx_1 \wedge dx_2 + \left(\frac{\partial P}{\partial x_3} - \frac{\partial R}{\partial x_1} \right) dx_3 \wedge dx_1 + \\
&\quad + \left(\frac{\partial R}{\partial x_2} - \frac{\partial Q}{\partial x_3} \right) dx_2 \wedge dx_3
\end{aligned}$$

So if we interpret Φ as the vector field $\vec{F} = < P, Q, R >$, and the $d\Phi$ as a vector field whose components are the coefficients of $dx_2 \wedge dx_3$, $dx_3 \wedge dx_1$, and $dx_1 \wedge dx_2$, then it by comparison we see that $d\Phi$ is describing $\nabla \times \vec{F}$.

Example 3.7 ([1])**.** Let $\Psi = Pdx_2 \wedge dx_3 + Rdx_3 \wedge dx_1 + Rdx_1 \wedge dx_2 \in \mathcal{F}_1(\mathbb{R}^3)$. Then,

$$\begin{aligned}
d\Psi &= (dP) \wedge dx_2 \wedge dx_3 + (dR) \wedge dx_3 \wedge dx_1 + (dR) \wedge dx_1 \wedge dx_2 \\
&= \frac{\partial P}{\partial x_1} dx_1 \wedge dx_2 \wedge dx_3 + \cancel{\frac{\partial P}{\partial x_2} dx_2 \wedge dx_2} \wedge dx_3 + \cancel{\frac{\partial P}{\partial x_3} dx_3 \wedge dx_2} \wedge dx_3 + \\
&\quad + \cancel{\frac{\partial Q}{\partial x_1} dx_1 \wedge dx_3} \wedge dx_1 + \frac{\partial Q}{\partial x_2} dx_2 \wedge dx_3 \wedge dx_1 + \cancel{\frac{\partial Q}{\partial x_3} dx_3 \wedge dx_3} \wedge dx_1 + \\
&\quad \cancel{\frac{\partial R}{\partial x_1} dx_1 \wedge dx_1} \wedge dx_2 + \cancel{\frac{\partial R}{\partial x_2} dx_2 \wedge dx_1} \wedge dx_2 + \frac{\partial R}{\partial x_3} dx_3 \wedge dx_1 \wedge dx_2 \\
&= \left(\frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2} + \frac{\partial R}{\partial x_3} \right) dx_1 \wedge dx_2 \wedge dx_3
\end{aligned}$$

Thus if we compare the differential 2-form Ψ with the vector field $\vec{F} = < P, Q, R >$, then the exterior derivative of Ψ maps Ψ into the differential 3-form whose coefficient is $\nabla \cdot \vec{F}$.

We end this section with a couple definitions that may be of interest to the reader wishing to apply the theory of Exterior Algebra to objects like conservative vector fields or exact differential equations.

Definition 3.8 (Closed [1])**.** A differential k-form Φ is *closed* if $d\Phi = 0$

Definition 3.9 (Exact [1])**.** A differential k-form Φ is *exact* if $\exists$ a differential $(k-1)$-form Ψ such that $\Phi = d\Psi$

4. Smooth Manifolds

Just as in the previous section we introduced differential k-forms as the analogous object to a vector field, here we introduce k-manifolds as the analogous object to a smooth surface/curve. Moreover, just as we showed that the classic differential operators of the gradient, curl, and divergence can be encompassed by the single exterior derivative operator, we will show that the line, surface, and flux integrals and all the different versions of the fundamental theorem of calculus that go with them can be encompassed by a single notion of integration over an oriented manifold, and a generalized ”Stoke's Theorem”.

Before we introduce the definition of a smooth manifold, we make a quick note on notation: for some $x = (x_1, x_2, \dots, x_{n-k}) \in \mathbb{R}^{n-k}$ and $y = (y_1, y_2, \dots, y_k) \in \mathbb{R}^k$, we will denote the vector $z := (x_1, x_2, \dots, x_{n-k}, y_1, y_2, \dots, y_k) \in \mathbb{R}^{n-k} \times \mathbb{R}^k = \mathbb{R}^n$ by $z = (x, y)$.

Definition 4.1 (Smooth k-Manifold [3])**.** $M \subseteq \mathbb{R}^n$ is a *smooth k-manifold* if $\forall z = (x, y) \in M$ (with $x \in \mathbb{R}^{n-k}$ and $y \in \mathbb{R}^k$), $\exists$ a C^1 function f and an open set $U \in \mathbb{R}^k$ with $f : U \to \mathbb{R}^{n-k}$, $x = f(y)$, and $(x, y) \in M \cap (f(U) \times U)$.

We will abbreviate a smooth k-manifold by simply saying k-manifold (as this will be the only kind of manifold we will study here).

Definition 4.2 (Graph of a Function [3])**.** The *graph of a function* $f : \mathbb{R}^n \to \mathbb{R}^m$ is the set

$$\Gamma(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m} \mid y = f(x)\}$$

Remark 4.3. A smooth manifold can be interpreted as, locally in some open neighbourhood of each $z \in \mathcal{M}$, the graph of some C^1 function $f : \mathbb{R}^k \to \mathbb{R}^{n-k}$. Another interpretation is that $\mathcal{M}$ is locally homeomorphic to $\mathbb{R}^k$ (with additional requirements for smoothness).

Example 4.4 ([3])**.** Consider $\mathcal{M} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. If we let $F(x, y) = x^2 + y^2 - 1 = 0$, we have that $Df = \begin{bmatrix} 2x & 2y \end{bmatrix}$. Then if $x =\neq 0$, we know from the implicit function theorem that we can find some open set $U \in \mathbb{R}$ (with $x \in U$) and a C^1 function $g : U \to \mathbb{R}$ such that $x = f(y)$. If $y \neq 0$, then the implicit function theorem tells us that $\exists$ an open set V (with $y \in V$) and a C^1 function $h : V \to \mathbb{R}$ such that $y = g(x)$. Hence, the circle given by $\mathcal{M}$ is a 1-manifold.

In vector calculus, we are always integrating over objects that have zero “area” with respect the space that we are viewing them from; smooth curves have zero area in $\mathbb{R}^2$, whereas curves and surfaces in $\mathbb{R}^3$ have zero volume. This is not a property that we explicitly demand from curves or surfaces, but something that naturally arises when we parametrize them. Just as one might expect, a k-manifold in $\mathbb{R}^n$ will have a “volume of zero” if we “view” it in a space with more than k dimensions (i.e. if we view it in $\mathbb{R}^m$ with $k < m \leq n$). The following definition reiterates this.

Definition 4.5 (zero k-volume [1])**.** Let $1 \leq k \leq n$. $\forall m \in \mathbb{Z}^+$, let Q_m denote the partition of $\mathbb{R}^m$ into n-dimensional cubes with edge length $1/2^m$. That is,

$$Q_m \quad = \quad \{ \, \{x \in \mathbb{R}^n \mid \|x - p\|_\infty \leq \frac{1}{2^{m+1}}\} \bigm| p = (\frac{N_1 + (1/2)}{2^m}, \frac{N_2 + (1/2)}{2^m}, \dots, \frac{N_n + (1/2)}{2^m}), \text{ and } N_1, N_2, \dots, N_n \in \mathbb{Z}\}$$

(the reader can, if they wish, ignore the above equation and simply use use “partition of $\mathbb{R}^n$ into n-dimensional cubes”).

A bounded set $S \subset \mathbb{R}^n$ has *k-volume 0* if

$$\lim_{m \to \infty} \sum_{\substack{Q \in Q_m \\ Q \cap S \neq 0}} \frac{1}{2^{km}} = 0$$

In the case that S is unbounded, let $S_r = \{x \in S \mid \|x\|_\infty \leq r\}$. Then S has *k-volume 0* if S_r has k-volume zero $\forall r > 0$.

Before we give the next definition, we define yet another analogous object called the "tangent space" (analogous to a tangent line/tangent plane).

Definition 4.6 (Tangent Space [3])**.** Let $\mathcal{M}$ be a k-manifold. Then if $z \in \mathcal{M}$, write $z = (x, y)$, and $\exists f C^1$ with $x = g(y)$ (locally). The *Tangent Space* for $\mathcal{M}$ at $p = (u, v)$ (with $u \in \mathbb{R}^{n-k}$ and $v \in \mathbb{R}^k$) is denoted and defined by

$$T_p(\mathcal{M}) = \{z = (x, y) \in \mathbb{R}^n \mid x - a = Df(b)(y - b),\ \exists f \text{ defined locally at } b\}$$

Definition 4.7 (Smooth Parametrization of a Manifold [1])**.** Let $\mathcal{M} \subset \mathbb{R}^n$ be a k-manifold in $\mathbb{R}^n$. Let $U \subseteq \mathbb{R}^k$ with ∂U having zero k-volume. Let S be the subset of U with zero k-volume such that $U - S = \{x \in S \mid x \notin U\}$ is open in $\mathbb{R}^k$. Let $f : U \to \mathbb{R}^n$ be a map with:

(1) $f(S)$ has zero k-volume,
(2) $\mathcal{M} \subset f(U)$,
(3) $f(U - S) \subset \mathcal{M}$,
(4) f is an injection and differentiable on $U - S$,
(5) $Df(u)$ is injective from $\mathbb{R}^k \to T_{f(u)}(\mathcal{M})$

Then we say that f *is a smooth parametrization of* $\mathcal{M}$ *over* U, and that it is a *strict parametrization over* $U - S$.

Knowing how to parameterize a k-manifolds allows us to integrate any arbitrary function $F : \mathbb{R}^n \to \mathbb{R}$ over a manifold, but this is not of interest to us. Our goal of proving the classical Green's, Stoke's, and Divergence theorems of vector calculus will be accomplished by integrating k-forms over manifolds. With this in mind, we introduce the next two definitions before showing how to carry out such an integration.

Definition 4.8 (Orientation [1])**.** Let ω be a non-zero k-form over $\mathbb{R}^k$. Then ω defines an *orientation* for $\mathbb{R}^k$. Since $\dim(\Lambda_k(\mathbb{R}^k) = \binom{k}{k} = 1$, any non-zero k-form in $\Lambda_k(\mathbb{R}^k)$ will be either a positive or negative scalar multiple of ω. If a k-form is a positive scalar multiple of ω provide a positive orientation for $\mathbb{R}^k$, whereas negative scalar multiples of ω provide the *opposite*, or negative orientation for $\mathbb{R}^k$.

To be more explicit, any $\varphi \in \Lambda_k(\mathbb{R}^k)$ can be written in the form

$$\varphi = a\ dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k$$

for some $a \in \mathbb{R}$. A $\mathbb{R}$ only has two orientaitons, and ϕ will give the same orientation as

$$\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k$$

if $a > 0$, and will give the opposite orientation as ω if $a < 0$. The tangent space of a k-manifold at at point $x \in \mathcal{M}$, $T_x(\mathcal{M})$ is oriented by assigning a non-zero k-form to it. A k-manifold is oriented as follows:

Definition 4.9 (Oriented Manifold [1])**.** Let $\mathcal{M}$ be a k-manifold. $\forall x \in \mathcal{M}$, suppose that $\exists$ a non-zero k-form w_x that varies smoothly with x and orients $T_x(\mathcal{M})$. Then $\mathcal{M}$ is *orientable* and we call $\omega(x) = w_x$ the differential k-form field that *orients* $\mathcal{M}$.

Note that in the above definition, "$\omega(x)$ varies smoothly with x" means that the real-valued function coefficient of the differential k-form $\omega(x)$ is smooth. Next we combine definitions 4.7 a and 4.9 to define a special kind of parametrization.

Definition 4.10 (Orientation Preserving Parametrization [1])**.** Let $\mathcal{M} \subset \mathbb{R}^n$ be a k-manifold in $\mathbb{R}^n$ oriented by the differential k-form $\omega(x)$. Suppose $\mathcal{M}$ is smoothly parametrized by p over $U \subset \mathbb{R}^k$ and that it is strict on $U - S$ with $S \subset U$ having zero k-volume, just as in the definition of smooth parametrization given previously. Then p is *orientation preserving* if $\forall u \in U - S$

$$\omega(p(u))\left(\frac{\partial p(u)}{\partial u_1}, \frac{\partial p(u)}{\partial u_2}, \dots, \frac{\partial p(u)}{\partial u_k}\right) > 0$$

In the case that

$$\omega(p(u))\left(\frac{\partial p(u)}{\partial u_1}, \frac{\partial p(u)}{\partial u_2}, \dots, \frac{\partial p(u)}{\partial u_k}\right) < 0$$

p is *orientation reversing*
(Note: in the above, $p(u)$ simply shows the functionality of p with some $u \in U$, it not evaluating p at u)

Observe that in the first of the above inequalities, we are evaluating the differential k-form at a point $p(u)$ to obtain a k-form, and then evaluating the resulting k-form at the derivative of p at u. In other words, to evaluate the left-hand-side expression of the first inequality, we need to write

$$\omega(p(u)) = a\ dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k$$

for some $a \in \mathbb{R}$, after which

$$\omega(p(u))\left(\frac{\partial p(u)}{\partial u_1}, \frac{\partial p(u)}{\partial u_2}, \dots, \frac{\partial p(u)}{\partial u_k}\right) = a^k \det \begin{bmatrix} dx_1\left(\frac{\partial p(u)}{\partial u_1}\right) & \dots & dx_1\left(\frac{\partial p(u)}{\partial u_k}\right) \\ \vdots & \ddots & \vdots \\ dx_k\left(\frac{\partial p(u)}{\partial u_1}\right) & \dots & dx_k\left(\frac{\partial p(u)}{\partial u_k}\right) \end{bmatrix}$$

Now comes the final definition of the paper, and the definition that everything we've mentioned has been building up towards.

Definition 4.11 (Integration of a Differential k-form over a k-manifold [1])**.** Let p, mapping $U \subset \mathbb{R}^k \to \mathbb{R}^n$, be an orientation preserving, smooth parametrization of the k-manifold $\mathcal{M} \subset \mathbb{R}^n$ oriented by the differential k-form ω. If Φ is a smooth differential k-form defined on an open set $D \subset \mathbb{R}^n$ for which $\mathcal{M} \subseteq D$, then the integral of Φ over $\mathcal{M}$ is denoted and defined by

$$\int_{\mathcal{M}} \Phi := \int_U \Phi\left(\frac{\partial p(u)}{\partial u_1}, \frac{\partial p(u)}{\partial u_2}, \dots, \frac{\partial p(u)}{\partial u_k}\right) du_1 du_2 \dots du_k$$

where the integral on the right-hand-side above is taken in the same sense that integrals of multivariate functions have been defined in a a multi-variate calculus course.

We end off with a theorem whose proof will not be given, but will be used extensively in the next section.

Theorem 4.12 (The Generalized Stokes' Thoerem [1])**.** *If $\mathcal{M}$ is an orientable manifold $\mathcal{M}$ in $\mathbb{R}^n$, and Φ is a smooth differential $(k-1)$-form defined on some open set containing $\mathcal{M}$, then*

$$\int_{\mathcal{M}} d\Phi = \int_{\partial\mathcal{M}} \Phi$$

5. The Three Theorems of Vector Calculus

In this section, we show how the classical theorems of vector calculus are simply special cases of Stokes' theorem. Examples 3.4, 3.6, and 3.6 will be important for doing this. As a warm up, let us prove the fundamental theorem of calculus of single variable calculus using Stokes' Theorem.

Example 5.1 ([4])**.** Suppose $f : [a, b] \to \mathbb{R}$ is a C^1 function on some neighbourhood of $[a, b]$ with. Then we can restate the fundamental theorem of calculus as

$$\int_a^b f'(t)dt = f(b) - f(a)$$

We can rewrite this as

$$\int_{[a,b]} df = \int_{\partial[a,b]} f$$

which is clearly true by the generalized Stokes' theorem. Note that we've implicitly used the fact that $df = \frac{df}{dt}dt$ and that $\mathcal{M} = [a, b]$ is a 1-manifold with boundary $\partial M = \{a, b\}$.

The next three examples show the versions of the fundamental theorem of calculus used in Vector Calculus being deduced from the generalized Stokes' theorem.

Example 5.2 (Fundamental Theorem for Line Integrals [6])**.** Let $\mathcal{C}$ be a curve in $\mathbb{R}^3$ parametrized by $\vec{r}(t) =< x(t), y(t), z(t) >,\ a \leq t \leq b$. Let $\vec{F}(x, y, z) = \nabla f(x, y, z)$ be a vector field on $\mathbb{R}^3$. Then the fundamental theorem of line integrals states that

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = f\left(\vec{r}(b)\right) - f\left(\vec{r}(a)\right)$$

Then in Stokes' theorem, take $\mathcal{M}$ to be the 1-manifold $\mathcal{C}$ with parametrization $\vec{r}(t)$ on $[a, b]$, and $\Phi = f$. Then similar to what we saw in example 3.4,

$$\int_{\mathcal{C}} d\Phi = \int_{\mathcal{C}} df = \int_{\mathcal{C}} \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \int_{\mathcal{C}} \nabla f \cdot d\vec{r}$$

and

$$\int_{\partial\mathcal{C}} \Phi = f|_{\vec{r}(a)}^{\vec{r}(b)} = f\left(\vec{r}(b)\right) - f\left(\vec{r}(a)\right)$$

So the fundamental theorem for line integrals immediately follows.

Example 5.3 (Green's Theorem [6])**.** The statement of Green's theorem is given as: if D is a plane region with boundary ∂D, and P and Q are C^∞ functions on D, then

$$\int_{\partial D} Pdx + Qdy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

where dA is interpreted in the usual way in multivariate calculus (i.e. as $dxdy$). In the view of Stokes' theorem, let $\mathcal{M}$ be the 2-manifold given by D with boundary ∂D and $\Phi = Pdx + Qdy$ is a 1-form. Then,

$$\int_{\partial D} \Phi = \int_{\partial D} Pdx + Qdy$$

and

$$\begin{aligned}\int_D d\Phi = \int_D &= \frac{\partial P}{\cancel{\partial x}}\cancel{dx} \wedge dx + \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy + \frac{\partial Q}{\cancel{\partial y}}\cancel{dy \wedge} dy \\ &= \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy\end{aligned}$$

Which shows Green's theorem.

Example 5.4 (Stokes' Theorem)**.** The statement of Stokes' theorem (the non-general one) is given by: let Σ be an orientable surface in $\mathbb{R}^3$ whose boundary $\partial\Sigma$ is a simple closed curve $\mathcal{C}$, and let $\vec{F} =< P, Q, R >$ be a smooth vector field defined on some subset of $\mathbb{R}^3$ that contains Σ. Let $\mathcal{C}$ be parametrized by $\vec{r}(t)$, $a \le t < b$. Then,

$$\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_{\Sigma} \nabla \times \vec{F} \cdot d\vec{S}$$

In the view of the Generalized Stokes' theorem, we take $\mathcal{M}$ to be the 2-manifold given by Σ and $\partial\mathcal{M} = \partial\Sigma = \mathcal{C}$. We also let $\Phi = Pdx + Qdy + Rdz$. Then,

$$\int_{\partial\mathcal{M}} \Phi = \int_{\mathcal{C}} Pdx + Qdy + Rdz = \oint_{\mathcal{C}} \vec{F} \cdot d\vec{r}$$

the key part of deducing this theorem was already shown in example 3.6, where we saw that $d\Phi$ gives $\nabla \times \vec{F}$. Hence,

$$\int_{\mathcal{M}} d\Phi = \int_{\Sigma} \nabla \times \vec{F} \cdot d\vec{S}$$

Which shows Stokes' theorem.

Example 5.5 (Divergence Theorem)**.** The proof of this will follow just as intuitively as the last 3. We start off with the statement of the Divergence Theorem: Let Σ be a closed surface in $\mathbb{R}^3$ which bounds a solid S (i.e. $\Sigma = \partial S$ where S is a 3-manifold). Let $\vec{F} =< P, Q, R >$ be a vector field defined on some subset of $\mathbb{R}^3$ containing σ. Then,

$$\int_{\Sigma} \vec{F} \cdot d\vec{S} = \int_S \nabla \cdot \vec{F} \; dV$$

where dV is interpreted in the usual way in multivariate calculus (i.e. as $dxdydz$). Then in the view of the generalized Stokes' theorem, we let $\mathcal{M} = S$ so that $\partial\mathcal{M} = \Sigma$. Let $\Phi = Pdx + Qdy + Rdz$. Then by example 3.7, $d\Phi$ yields $\nabla \cdot \vec{F} \; dx \wedge dy \wedge dz$ so that

$$\int_{\partial\mathcal{M}} \Phi = \int_{\Sigma} Pdx + Qdy + Rdz = \int_{\Sigma} \vec{F} \cdot d\vec{S}$$

and

$$\int_{\mathcal{M}} d\Phi = \int_{\mathcal{M}} \nabla \cdot \vec{F} \; dV$$

which shows the Divergence theorem.

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