# MAT139 Notes

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February 13, 2024

## Contents

# 1 The Theory of Integration

The integral of $f$ is the area under the graph of $f$. The proper definition follows from the development of helper definitions and theorems, which are collected below.

## 1.1 Infimum and Supremum

**Definition 1.1.1** (Bounds)**.** A set $A \in \mathbb{R}$ is *bounded above* if there exists a number $b \in \mathbb{R}$ such that for all $a \in A, a \leq b$. Here, $b$ is called an *upper bound* of $A$. Likewise, $A$ is *bounded below* if there exists some $b \in \mathbb{R}$ such that for all $a \in A, a \geq b$. Here, $b$ is *lower bound* of $A$.

**Definition 1.1.2** (Infimum)**.** Let $A \subseteq \mathbb{R}$. A real number $i$ is the *infimum*, or *greatest lower bound* of $A$ if:

(i) $i$ is a lower bound of $A$;

(ii) if $b$ is any lower bound of $A$, then $i \geq b$.

We write $i = \inf A$ to denote that $i$ is the infimum of $A$.

**Definition 1.1.3** (Supremum)**.** Let $A \subseteq \mathbb{R}$. A real number $c$ is the *supremum,* or *least upper bound* of $A$ if:

(i) $c$ is an upper bound of $A$;

(ii) if $b$ is any upper bound for $A$, then $c \leq b$.

We write $c = \sup A$ to denote that $c$ is the supremum of $A$.

When discussing functions, the supremum and infimum of a function $f$ are the supremum and infimum of its *range*. Following are some useful propositions about suprema:

## 1.2 Sums and Sigma Notation

**Definition 1.2.1** (Sigma Notation)**.** Let $m$ and $n$ be integers, and $a_m, a_{m+1} + \ldots + a_n$ be real numbers. The *sum* of $a_m, a_{m+1}, \ldots, a_n$ is defined as

$$a_m + a_{m+1} + \ldots + a_n = \sum_{i=m}^{n} a_i$$

A summation can be thought of as a simple additive for-loop, which adds the value of the indexed variable $a_i$ to a running total. Just like in a for-loop, the letter $i$ is a bound variable, and can be replaced with any other letter.

The notation is most comfortably read (at least by myself) as follows: "the sum as $i$ goes from $m$ to $n$ of $a_i$."

**Proposition 1.2.1** (Properties of sums)**.** Suppose $c, a_i$, and $b_i$ are real numbers for $i = m, m+1, \ldots, m_n$, where $n$ and $m$ are integers. Then,

(i) $\displaystyle\sum_{i=m}^{n} (c \cdot a_i) = c \sum_{i=m}^{n} a_i$

(ii) $\displaystyle\sum_{i=m}^{n}(a_i+b_i)=\sum_{i=m}^{n}a_i+\sum_{i=m}^{n}b_i$

**Example 1.2.1.** The sum

$$3+9+15+21+27+33+\ldots+297+303$$

can be expressed in the following, equivalent ways:

(i) $\sum_{i=1}^{51}3(2i-1)$

(ii) $\sum_{i=7}^{57}(2i-3)$

(iii) $3\sum_{i=0}^{50}(2i+1)$

(iv) $3\sum_{n=0}^{50}(2n+1)$

## 1.3 Partitions, Heights, and Sums

To construct our rectangles (TBA), we must first talk about *partitions*:

**Definition 1.3.1** (Partitions)**.** A *partition* of an interval $[a,b]$ is a finite set $P$ such that

$$P=\{x_0,x_1,x_2,\ldots,x_n\}$$

where $x_0=a, x_n=b$, and $x_0<x_1<x_2<\ldots<x_n$.

The points in $P$ need not be equidistant; for example, a possible partition of the interval $[1,2]$ is $\{1,1.2,1.314,\phi,1.9,2\}$. Therefore, our rectangles can have varying widths.

Next, we need to define the heights of our rectangles. This requires a two-pronged approach; our rectangles can either be big enough to completely cover the curve, or be small enough to be completely covered by the curve.

**Definition 1.3.2.** For each sub-interval $[x_{i-1},x_i]$ of $P$, where $i\geq 1$, define

$$m_i=\inf\{f(x)\mid x\in[x_{i-1},x_i]\}$$
$$M_i=\sup\{f(x)\mid x\in[x_{i-1},x_i]\}$$

(No visuals; I hate TikZ. Sorry, future me.)

Now that we can talk about the widths and heights of both small and big rectangles, we can define their sums like so:

**Definition 1.3.3** (Upper and Lower Sums)**.** Let $f:[a,b]\to\mathbb{R}$, and consider a partition $P=\{x_0,x_1,x_2,\ldots,x_n\}$ of $[a,b]$. The *lower sum* of $f$ with respect to $P$ is given by

$$L_f(P)=\sum_{i=1}^{n}m_i(x_i-x_{i-1})$$

Similarly, the *upper sum* is defined as

$$U_f(P)=\sum_{i=1}^{n}M_i(x_i-x_{i-1})$$

**Remark.** By virtue of their definitions, it is easy to see that $M_i \geq m_i$ for all $i \geq 1$. Which means that for any partition $P$ of $[a, b]$, $U_f(P) \geq L_f(P)$.

A partition with 10 elements may allow us to calculate upper and lower sums which are decent but imprecise estimates of a function's integral. If we were to keep our 10-element partition, and add 90 more points to it such that the original partition is a *subset* of the new one, we could potentially get a much better estimate of the integral. This is the motivation behind a *refinement*.

**Definition 1.3.4** (Refinements)**.** Let $P$ be a partition of [a,b]. A partition $Q$ of [a,b] is called a *refinement* of $P$ if $P \subseteq Q$.

The following properties follow intuitively from the definitions stated earlier, and are "proved" in the videos. The first assures that refinements do indeed make upper sums smaller (or unchanged) and lower sums larger (or unchanged).

**Lemma 1.3.1.** If $Q$ is a refinement of $P$, then $L(f, P) \leq L(f, Q)$ and $U(f, P) \geq U(f, Q)$.

The next lemma states that all upper sums are greater than or equal to all lower sums.

**Lemma 1.3.2.** Let $f : [a, b] \to \mathbb{R}$. Then, for any partitions $P_1, P_2$ of $[a, b]$, $L(f, P_1) \leq U(f, P_2)$.

## 1.4 The Integral

The idea so far has been to take finer and finer partitions of $[a, b]$, and to calculate the upper and lower sums of $f$ with respect to these partitions. If the upper and lower sums converge to the same value, we say that $f$ is *integrable* on $[a, b]$, and that value is the integral of $f$ on $[a, b]$.

**Definition 1.4.1.** Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and let $\mathcal{P}$ be the set of all partitions of $[a, b]$. The *upper integral* of $f$ is

$$\overline{I_a^b}(f) = \sup\{L(f, P) \mid P \in \mathcal{P}\}$$

The *lower integral* of $f$ is

$$= \underline{I_a^b}(f) \inf\{U(f, P) \mid P \in \mathcal{P}\}$$

**Remark.** By Lemma 1.2.2, it is clear that $\overline{I_a^b}(f) \geq \underline{I_a^b}(f)$.

This brings us to the following paraphrase of the above paragraph:

**Definition 1.4.2** (Integrability)**.** Let $f : [a, b] \to \mathbb{R}$ be a bounded function. If $\overline{I_a^b}(f) = \underline{I_a^b}(f)$, then we say that $f$ is *integrable* on $[a, b]$, and we denote the integral as $\int_a^b f(x)\, dx$.

## 1.5 Integrability

We use the following theorem without proof:

**Theorem 1.5.1** (Continuity $\implies$ Integrability)**.** If $f : [a, b] \to \mathbb{R}$ is continuous, then $f$ is integrable on $[a, b]$.

The fact that non-continuous functions can be integrable is demonstrated by the following example:

**Example 1.5.1.** Let f be a function defined as

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

Prove that $f$ is integrable on $[-1, 1]$.

*Proof.* Let $f$ be the function defined above. Suppose $P$ is an arbitrary partition of [-1,1]. Clearly, $m_i$ = 0 for all $i \geq 1$. Thus,

$$L_f(P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = 0.$$

Since $L_f(P) = 0$ for all partitions $P$ of [-1,1], we have that $\underline{I_{-1}^{1}}(f) = \sup\{0\} = 0$.

Now, consider the upper sums. Since $M_i = 1$ for all $i \geq 1$, we have that

$$0 < U_f(P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n} 1 \cdot (x_i - x_{i-1}) \leq 2$$

for all partitions $P$ of [-1,1]. Thus, $\overline{I_{-1}^{1}}(f) = \inf\{(0, 2]\} = 0$. We have shown that $\overline{I_{-1}^{1}}(f) = \underline{I_{-1}^{1}}(f) = 0$, so $f$ is integrable on [-1,1], as needed. □

Below is an example of a function which is *not integrable*:

**Example 1.5.2.** Let f be a function defined as

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

(What a surprise!)

*Proof.* I'm too lazy to reproduce it here, but the proof is in the videos. (We've used a sneaky proof tactic here, called *proof by cubersome reference*). □

We can solidify our criteria for integrability. For the supremum of one set to be equal to the iniumum of another set, it is intuitive to see that their elements have to be arbitrarily close to each other; hence, the below criteria.

**Definition 1.5.1** (The "$\varepsilon$-characterization" of integrability)**.** Let $f$ be a bounded function on $[a, b]$. Then $f$ is integrable **if and only if** $\forall \varepsilon > 0$ there exists a partition $P_\varepsilon$ of $[a, b]$ such that $U_f(P) - L_f(P) < \varepsilon$.

## 1.6 The Fundamental Theorem of Calculus

We've built the theory that the Integral hinges on; however, so far we haven't developed a general method to determine the value of the integral. Perhaps intuitively, it is impossible to explicitly calculate the upper and lower sums of most functions. Fortunately, the following theorem describes a method for evaluating integrals.

**Theorem 1.6.1** (The Fundamental Theorem of Calculus)**.** Here goes:

(i) Suppose $f : [a, b] \to \mathbb{R}$ is integrable, and $F : [a, b] \to R$ satisfies $F'(x) = f(x)$ for all $x \in [a, b]$. Then,

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a).$$

(ii) Let $g : [a, b] \to \mathbb{R}$ be integrable and define $G : [a, b] \to \mathbb{R}$ by

$$G(x) = \int_a^x g(t)\mathrm{d}t.$$

Then $G$ is continuous. Moreover, if $g$ is continuous at $x_0 \in [a, b]$, then $G$ is differentiable at $x_0$ and $G'(x_0) = g(x_0)$.

**Remark.** Fun fact: one of the people involved with the founding of this theorem was Isaac Newton's PhD advisor!

The following is the proof for part (i) of the *FTC*, which comes down to a clever application of telescoping sums.

***Proof of (i)***. Suppose $f : [a, b] \to \mathbb{R}$ is integrable, and $F : [a, b] \to \mathbb{R}$ satisfies $F'(x) = f(x)$ for all $x \in [a, b]$. Fix $P = \{x_0, \ldots, x_n\}$ to be an arbitrary partition of $[a, b]$. Narrowing down our focus to a single sub-interval $[x_{i-1}, x_i]$, since $F$ is differentiable, by the Mean Value Theorem, there exists a $c_i \in [x_{i-1}, x_i]$ such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}.$$

Rearrange as follows:

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}).$$

But since $F'(c_i) = f(c_i)$, we have that

$$F(x_i) - F(x_{i-1}) = f(c_i)(x_i - x_{i-1}).$$

.

Now, consider the upper sum $U_P(f)$ and lower sum $L_P(f)$. It holds that

$$\begin{aligned}
&\sum_{i=1}^{n} m_i(x_i - x_{i-1}) \leq \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \leq \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\
&\implies L_P(f) \leq \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \leq U_P(f) \\
&\implies L_P(f) \leq \sum_{i=1}^{n} F(x_i) - F(x_{i-1}) \leq U_P(f).
\end{aligned}$$

We can cheekily simplify this becausae the middle sum telescopes. So, we end up with:

$$L_P(f) \leq \sum_{i=1}^{n} F(b) - F(a) \leq U_P(f)$$

Since $P$ is arbitrary, we have that

$$\underline{I_a^b}(f) \leq F(b) - F(a) \leq \overline{I_a^b}(f)$$

Because $f$ is integrable, we have that $\underline{I_a^b}(f) = \overline{I_a^b}(f)$, so $F(b) - F(a) = \int_a^b f(x)\mathrm{d}x$, as needed. $\square$

***Proof of (ii).*** $\square$

# 2 Integration Techniques

The following are direct corollaries to the *Fundamental Theorem of Calculus*:

## 2.1 Integration by Substitution

**Theorem 2.1.1** (u-substitution)**.** Suppose $g$ is a function whose derivative $g'$ is continuous on an $[a, b]$, and suppose that $f$ is a function that is continuous on $g[a, b]$. Then,

$$\int_a^b f(g(x))g'(x)\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\mathrm{d}u.$$

# 3 Sequences

## 3.1 Intro

**Definition 3.1.1** (Sequence)**.** A *sequence* is a function $a : \mathbb{N} \to \mathbb{R}$. Usually (i.e. always) we write $a_n$ instead of $a(n)$ to denote the $n$th term of the sequence. To differentiate it from a function called $a$, we write $(a_n)_{n=0}^{\infty}$.

Sequences, like functions, can be bounded.

**Definition 3.1.2** (Boundedness)**.** A sequence $(a_n)$ is *bounded above* if there exists a number $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \in \mathbb{N}$. Likewise, $(a_n)$ is *bounded below* if there exists a number $m \in \mathbb{R}$ such that $a_n \geq m$ for all $n \in \mathbb{N}$. If both a lower and upper bound exist, the sequence is *bounded.*

## 3.2 Convergent Sequences

**Definition 3.2.1.** A sequence $(a_n)$ *converges* to $L \in \mathbb{R}$ if for all $\varepsilon > 0$ there exists some $N \in \mathbb{R}$ such that for all $n \in \mathbb{N}, n \geq N \implies |a_n - L| < \varepsilon$.