7.3:- Operational properties I

First Translation theorem:-

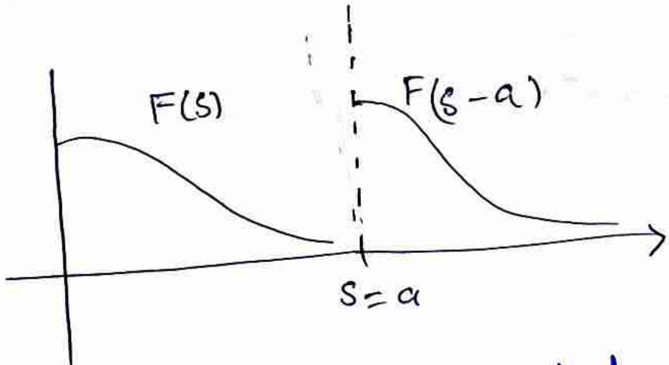
If $\mathcal{L}\{f(t)\} = F(s)$ and a is real number,

then
$$\boxed{\mathcal{L}\{e^{at}\cdot f(t)\} = F(s-a)}$$

Proof:- $\mathcal{L}\{e^{at}\cdot f(t)\} = \int_0^\infty e^{-st}\cdot e^{at} f(t)\, dt$ (∵ Laplace def)

$$= \int_0^\infty e^{-t(s-a)} f(t)\, dt$$

$$\Rightarrow \quad = F(s-a).$$



Shift on s-axis.

Exp1:- Using First Translation theorem find.
$$\mathcal{L}\{e^{5t}t^3\}.$$

Sol:- $\mathcal{L}\{e^{5t}t^3\}$.

Apply the first translation theorem

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a).$$

where

$$F(s) = \mathcal{L}\{f(t)\}.$$

$a = 5$, $f(t) = t^3$.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^3\} = \frac{3!}{s^{3+1}} = \frac{3!}{s^4}.$$

$$F(s-a) = F(s-5) = \frac{3!}{(s-5)^4}$$

$$\mathcal{L}\{e^{5t} t^3\} = \frac{6}{(s-5)^4}$$

Q8: $\mathcal{L}\{e^{-2t}\cos 4t\}$

Sol: $\mathcal{L}\{e^{-2t}\cos 4t\}$

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

$a = -2$

$\because$ Replace s with $s-a$.

$$\because \mathcal{L}\{\cos 4t\} = \frac{s}{s^2+4^2}$$

$$= \mathcal{L}\{\cos 4t\}_{s \to s-(-2)}.$$

$$= \left.\frac{s}{s^2+4^2}\right|_{s \to s+2}$$

$$\mathcal{L}\{e^{-2t}\cos 4t\} = \frac{s+2}{(s+2)^2+16}$$

Inverse form of Laplace using Translation Theorem.

$$\mathcal{L}^{-1}\{F(s-a)\} = \mathcal{L}^{-1}\{F(s)|_{s\to s-a}\}$$

$$= e^{at} f(t), \text{ where}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

Q 17:- $\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2}\right\}$

Sol:- $\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{s+1-1}{(s+1)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2} - \frac{1}{(s+1)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} \qquad \text{Replace } \{\because s \to s-a\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}_{s\to s-(-1)} - \mathcal{L}^{-1}\left\{\frac{1}{(s)^2}\right\}_{s\to s-(-1)}$$

$$= e^{-t}\cdot 1 - e^{-t}t.$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2}\right\} = e^{-t} - e^{-t}t.$$

Q18:- $\mathcal{L}^{-1}\left\{\frac{1}{s^2-6s+10}\right\}$

Sol:- $\mathcal{L}^{-1}\left\{\frac{1}{(s)^2-2(s)(3)+3^2-3^2+10}\right\}$ ∵ By completing square.

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2+1^2}\right\}$$

$\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t.$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\}_{s \to s-3}$$

$$= e^{3t}\sin t.$$

Q19:- $\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2(s+1)^3}\right\}$

Sol:- $\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2(s+1)^3}\right\}$

$$\frac{2s-1}{s^2(s+1)^3} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)^3} \quad \text{—(1)}$$

Multiply (1) with $s^2(s+1)^3$ on both sides.

$\therefore \; 2s-1 = As(s+1)^3 + B(s+1)^3 + Cs^2(s+1)^2 +$
$D s^2(s+1) + E(s^2)$ —— (1)

$2s-1 = As(s^3+3s^2+3s+1) + B(s^3+3s^2+3s+1)$
$+ Cs^2(s^2+1+2s) + Ds^2(s+1) + E(s^2)$ —— (2)

put $s=0$ in eq (2)

$2(0)-1 = 0 + B(0+0+0+1) + 0 + 0 + 0$

$-1 = B \Rightarrow \boxed{B=-1}$

For $s=-1$ put in eq (2)

$2(-1)-1 = A(-1)((-1)^3 + 3(-1)^2 + 3(-1) + 1) + 0 + 0$
$+ 0 + E(-1)^2$

$-3 = 0 + E \Rightarrow \boxed{E=-3}$

Now by comparing coeffecients.

$s^4: \quad 0 = A + C$ —— (3)

$s^3: \quad 0 = 3A + B + 2C + D$

$0 = 3A - 1 + 2C + D$

$\Rightarrow 3A + 2C + D = 1$ —— (4)

s:

$$2 = A + 3B$$

$$2 = A + 3(-1) \Rightarrow \boxed{A = 5}$$

put in (3)

$$A + C = 0$$

$$5 + C = 0 \Rightarrow \boxed{C = -5}$$

put value of c in eq (4)

$$3A + 2C + D = 1$$

$$3(5) + 2(-5) + D = 1$$

$$15 - 10 + D = 1$$

$$5 + D = 1$$

$$\boxed{D = -4}$$

put all values in eq (1).

$$\frac{2s-1}{s^2(s+1)^3} = \frac{5}{s} + \frac{-1}{s^2} + \frac{-5}{s+1} + \frac{-4}{(s+1)^2} + \frac{-3}{(s+1)^3}$$

Taking L^{-1} on both side.

$$L^{-1}\left\{\frac{2s-1}{s^2(s+1)^3}\right\} = 5L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s^2}\right\} - 5L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$-4L^{-1}\left\{\frac{1}{(s+1)^2}\right\} - 3L^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$

$$= 5(1) - t - 5e^{-t} - 4L^{-1}\left\{\frac{1}{(s+1)^2}\right\} - 3L^{-1}\left\{\frac{1}{(s+1)^3}\right\}.$$

Now using first translation theorem.

$$= 5 - t - 5e^{-t} - 4L^{-1}\left\{\frac{1}{s^2}\right\}_{s\to(s+1)} - 3L^{-1}\left\{\frac{1}{s^3}\right\}_{s\to s+1}$$

using $L^{-1}\{F(s-a)\} = e^{at}f(t)$

$$= 5 - t - 5e^{-t} - 4e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\} - 3e^{-t}L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= 5 - t - 5e^{-t} - 4e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{3}{2!}e^{-t}L^{-1}\left\{\frac{2!}{s^3}\right\}$$

$$= 5 - t - 5e^{-t} - 4e^{-t}t - \frac{3}{2!}e^{-t}t^2 \qquad \because L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$$

Q#27: Use Laplace transform to solve the given initial value problem.

$$y'' - 6y' + 13y = 0 \quad ; \quad y(0) = 0, \quad y'(0) = -3$$

Sol: $y'' - 6y' + 13y = 0$

Taking Laplace on both sides.

$$L\{y''\} - 6L\{y'\} + 13L\{y\} = L\{0\}$$

$$s^2 \mathcal{L}\{y\} - s\,y(0) - y'(0) - 6\left[s\,\mathcal{L}\{y\} - y(0)\right] + 13\,\mathcal{L}\{y\} = 0$$

Apply the initial condition

$$y(0) = 0 \quad , \quad y'(0) = -3$$

$$(s^2 - 6s + 13)\,\mathcal{L}\{y\} - s\,y(0) - y'(0) - 6\,y(0) = 0$$

$$(s^2 - 6s + 13)\,\mathcal{L}\{y\} - 0 + 3 - 0 = 0$$

$$(s^2 - 6s + 13)\,\mathcal{L}\{y\} = -3.$$

$$\mathcal{L}\{y\} = \frac{-3}{s^2 - 6s + 13}$$

$$y = \mathcal{L}^{-1}\left\{\frac{-3}{s^2 - 6s + 13}\right\}$$

$$y = -3\,\mathcal{L}^{-1}\left\{\frac{1}{\underbrace{(s)^2 - 2(s)(3) + 3^2}\,\underbrace{-\,3^2 + 13}}\right\}$$

$$y = -3\,\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2 + 4}\right\}$$ by completing square

$$y = -3\,\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2 + (2)^2}\right\}$$

use translation theorem.

$$y = -3\,\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 2^2}\right\}_{s \to s-3}$$

$$y = -3e^{+3t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 2^2}\right\}$$

$$y = -3e^{+3t}\frac{1}{2}L^{-1}\left\{\frac{2}{s^2+2^2}\right\} \qquad \because L^{-1}\left\{\frac{K}{s^2+K^2}\right\} = \sin Kt$$

$$y = \frac{-3e^{3t}}{2}\sin 2t$$

Unit Step Function :-

In engineering applications, we frequently encounter functions whose values change abruptly at specified values of time t.
⇒ One example is when a voltage is switched on or off in an electrical circuit at a specified value of time t.

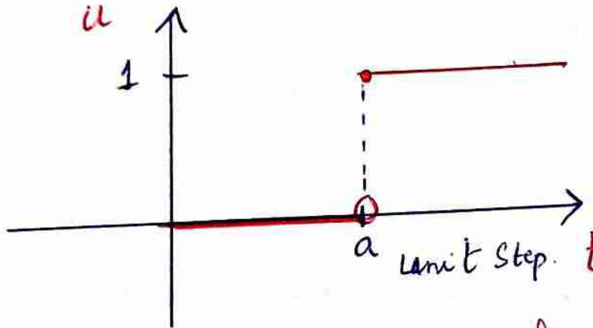
Simply,
The switching process can be described mathematically by the unit step function called function otherwise known as Heaviside function after (Oliver Heaviside).

Def:-

Unit Step Function:-

The unit step function $u(t-a)$ is defined to be

$$u(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a. \end{cases}$$



Unit Step.

Now we study this special function in the finding the Laplace transform.

When u have step function we use 2nd translation theorem.

Second Translation Theorem:-

If $F(s) = \mathcal{L}\{f(t)\}$ and $a > 0$, then.

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-a(s)}F(s).$$

(or) We can also find Laplace Inverse.

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a)$$

Laplace Transform of $u(t-a)$.

$$L\{u(t-a)\} = \int_0^\infty e^{-st} \cdot u(t-a)\, dt$$

$$u(t-a) = \begin{cases} 0 ; & 0 \le t \le a \\ 1 ; & t \ge a \end{cases}$$

$$= \int_0^a e^{-st}(0)\, dt + \int_a^\infty e^{-st}(1)\, dt$$

$$= 0 + \int_a^\infty e^{-st}\, dt$$

$$= \left| \frac{e^{-st}}{-s} \right|_a^\infty$$

$$= -\frac{1}{s}\left[e^{-st} \Big|_a^\infty \right]$$

$$= -\frac{1}{s}\left[e^{-\infty} - e^{-as} \right] \qquad \because e^{-\infty} = 0$$

$$= -\frac{1}{s}\left[0 - e^{-as} \right]$$

$$L\{u(t-a)\} = \frac{e^{-as}}{s}$$

Laplace Transform of $u(t-a)$.

$$L\{u(t-a)\} = \int_0^\infty e^{-st} \cdot u(t-a)\, dt$$

$$u(t-a) = \begin{cases} 0 ; & 0 \le t < a \\ 1 ; & t \ge a \end{cases}$$

$$= \int_0^a e^{-st}(0)\, dt + \int_a^\infty e^{-st}(1)\, dt$$

$$= 0 + \int_a^\infty e^{-st}\, dt$$

$$= \left| \frac{e^{-st}}{-s} \right|_a^\infty$$

$$= -\frac{1}{s}\left[e^{-st} \Big|_a^\infty \right]$$

$$= -\frac{1}{s}\left[e^{-\infty} - e^{-as} \right] \qquad \because e^{-\infty} = 0$$

$$= -\frac{1}{s}\left[0 - e^{-as} \right]$$

$$L\{u(t-a)\} = \frac{e^{-as}}{s}$$

Here, we study how we use second translation theorem to find Laplace and Laplace Inverse.

Q#37: Find either $F(s)$ or $f(t)$, as indicated.

$\mathcal{L}\{(t-1)\,u(t-1)\}$.

Sol: The formula is

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-a(s)} F(s)$$

Here $a = 1$.

$f(t-a) = t-a$

$\because f(t-1) = t-1$

$f(t) = t$

$\because \mathcal{L}\{f(t)\} = F(s)$

$\mathcal{L}\{t\} = \frac{1}{s^2}$

$$\mathcal{L}\{(t-1)\,u(t-1)\} = e^{-s} F(s)$$

$$= e^{-s} \mathcal{L}\{t\}$$

$$= e^{-s}\left(\frac{1}{s^2}\right)$$

$$\mathcal{L}\{(t-1)u(t-1)\} = \frac{e^{-s}}{s^2}.$$

Q40:- $\mathcal{L}\{(3t+1)u(t-1)\}$

Sol.. Since $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$

we write function in terms of $f(t-a)$ we use some algebraic properties to make a proper form

$$F(s) = \mathcal{L}\{f(t)\}, \quad a = 1$$

$$f(t) = 3t+1$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{3t+1\}$$

$$= 3\mathcal{L}\{t\} + \mathcal{L}\{1\}$$

$$= \frac{3}{s^2} + \frac{1}{s}$$

$$\mathcal{L}\{(3t+1)u(t-1)\} = e^{-s}\left(\frac{3}{s^2} + \frac{1}{s}\right)$$

[redacted]

Sol.- Since $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$

$$= \mathcal{L}\{(3t+1)u(t-1)\}$$

$$= \mathcal{L}\{(3(t-1+1)+1)u(t-1)\}$$

$$= \mathcal{L}\{(3(t-1)+4)\cdot u(t-1)\}$$

$$= \mathcal{L}\{3(t-1)u(t-1) + 4u(t-1)\}$$

$$= 3\mathcal{L}\{(t-1)u(t-1)\} + 4\mathcal{L}\{u(t-1)\}$$

$$= 3e^{-s}L\{t\} + 4\frac{e^{-s}}{s}$$

$a=1$

$f(t-1) = t-1$

$f(t) = t$

$$= \frac{3e^{-s}}{s^2} + \frac{4e^{-s}}{s}$$

Q47. $L^{-1}\left\{\frac{e^{-s}}{s(s+1)}\right\}$

Sol: We use 2nd Translation theorem as an Inverse form.

$$L^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a).$$

Here $a = 1$, $F(s) = \frac{1}{s(s+1)}$

$$L^{-1}\left\{\frac{e^{-s}}{s(s+1)}\right\} = u(t-1)f(t-1) \quad \text{— (A)}$$

$$\Rightarrow f(t) = L^{-1}\{F(s)\}$$

$$= L^{-1}\left\{\frac{1}{s(s+1)}\right\} \quad \text{— (1)}$$

$$= \cancel{L}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \quad \text{— (2)}$$

$$1 = A(s+1) + B(s) \quad \text{— (3)}$$

put $s=0$

$$\boxed{A = 1}$$

put $s = -1$

$$1 = B(-1) \Rightarrow \boxed{B = -1}$$

put values in (2)

$$\frac{1}{s(s+1)} = \frac{1}{s} + \frac{-1}{s+1}$$

put in (1)

$$= L^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}$$

$$= L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$f(t) = 1 - e^{-t} \Rightarrow f(t-1) = 1 - e^{-(t-1)}.$$

put all values in (A)

$$L^{-1}\left\{\frac{e^{-s}}{s(s+1)}\right\} = u(t-1)\left(1 - e^{-(t-1)}\right).$$

P.Q Ex 7.3

Q6, Q16, Q17, Q30, Q39, Q48.