4.7 Cauchy Euler Equation

Differential Equation with variable Coeffecient

We have been solving Linear Differential Equation with constant coefficients. Now we will solve the D.Eqs with variable coefficients.

Note:- The D.Eqs with variable coefficients cannot be solved so easily.

Cauchy Euler Equation:-

A linear D.Eq of the form.

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

where $a_n, a_{n-1}, a_{n-2}, \ldots a_1, a_0$ are constants, is said to be Cauchy Euler Equation.

Note: The power of x must match the order of derivatives.

Consider the 2nd order non homogeneous D.Eq

$$a_2 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a y = g(x).$$

(or) $$a x^2 y'' + b x y' + c y = g(x)$$

Method of Solution:-

Let $y = x^m$ be a solution of

$$ax^2y'' + bxy' + cy = 0 \quad \text{—— (1)}$$

where m to be determined.

Then first and 2nd order derivatives are

$$y' = mx^{m-1}, \quad y'' = m(m-1)x^{m-2}.$$

Subsitute values in D.Eq (1)

$$ax^2(m(m-1))x^{m-2} + bx(mx^{m-1}) + cx^m = 0$$

$$\Rightarrow \quad am(m-1)x^m + bmx^m + cx^m = 0$$

$$\Rightarrow \quad x^m(am(m-1) + bm + c) = 0.$$

Thus $y = x^m$ is a solution of D.Eq.

$$am(m-1) + bm + c = 0$$

$$am^2 - am + bm + c = 0$$

$$am^2 + (b-a)m + c = 0$$

The solution of D.Eq depends on the roots

There are 3 cases.

Case 1 :-
Distinct roots:-

Let m_1 and m_2 denote distinct roots of the equation. Then $y = x^{m_1}$ and $y = x^{m_2}$ are fundamental set of solutions.

Hence the general solution is

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

Case 2: (Repeated Roots):-

If the roots are repeated. Then we obtain only one $y = x^{m_1}$, the second solution can be found by using reduction of order.

The general solution will be of the form.

$$y = c_1 x^{m_1} + c_2 x^{m_1} \ln x.$$

Case 3:- Complex Conjugate Roots:-

If the roots are complex conjugate

$$m_1 = \alpha + i\beta \quad , \quad m_2 = \alpha - i\beta.$$

Then the general solution is

$$y = x^{\alpha} \left[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x) \right].$$

Ex# 4.7

Solve the given D.Eq.

Q5:- $x^2y'' + xy' + 4y = 0$

Sol. The auxiliary eq

$$m(m-1) + m + 4 = 0$$

$$m^2 - m + m + 4 = 0$$

$$m^2 + 4 = 0$$

$$\sqrt{m^2} = \sqrt{-4} \Rightarrow m = \pm 2i \Rightarrow m = 0 \pm 2i$$

$\alpha = 0$
$\beta = 2$

General Solution is

$$y = x^{\alpha}\left[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)\right]$$

$$y = x^{0}\left[c_1 \cos(2\ln x) + c_2 \sin(2\ln x)\right]$$

$$y = c_1 \cos(2\ln x) + c_2 \sin(2\ln x)$$

Q7:- $x^2y'' - 3xy' - 2y = 0$

Sol:- The auxiliary equation

$$m(m-1) - 3m - 2 = 0$$

$$m^2 - m - 3m - 2 = 0$$

$$m^2 - 4m - 2 = 0$$

Using Quadratic Formula.

$a = 1, \quad b = -4, \quad c = -2$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{4 \pm \sqrt{16 - 4(1)(-2)}}{2(1)}$$

$$m = \frac{4 \pm \sqrt{16+8}}{2}$$

$$m = \frac{4 \pm \sqrt{24}}{2}$$

$$m = \frac{4 \pm 2\sqrt{6}}{2}$$

$$m = \frac{\cancel{2}(2 \pm \sqrt{6})}{\cancel{2}}$$

$$m_1 = 2 + \sqrt{6}, \quad m_2 = 2 - \sqrt{6}$$

General Solution

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

$$y = c_1 x^{2+\sqrt{6}} + c_2 x^{2-\sqrt{6}}.$$

Q11:- $x^2y'' + 5xy' + 4y = 0$

Sol: The auxilicarly equation.

$$m(m-1) + 5m + 4 = 0$$

$$m^2 - m + 5m + 4 = 0$$

$$m^2 + 4m + 4 = 0$$

$$(m+2)^2 = 0 \implies m = -2, -2.$$

General Solution.

$$y = c_1 x^m + c_2 x^m \ln x.$$

$$y = c_1 x^{-2} + c_2 x^{-2} \ln x$$

Ex#4.7:

Q23: Solve the given D.Eq by variation of paramete

$$x^2y'' + xy' - y = \ln x.$$

Sol: Assume $y = x^m$ is a solution of D.Eq.

$y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$.

Firstly we find complementary solution corresponding to Homogeneous eq.

$$x^2y'' + xy' - y = 0$$

$$x^2 m(m-1)x^{m-2} + xm\,x^{m-1} - x^m = 0$$

$$x^m m(m-1) + mx^m - x^m = 0$$

$$x^m(m(m-1) + m - 1) = 0$$

$$\Rightarrow m(m-1) + m - 1 = 0$$

$$m^2 - m + m - 1 = 0$$

$$m^2 - 1 = 0$$

$$(m-1)(m+1) = 0$$

$$m_1 = 1, \quad m_2 = -1$$

$$y_c = c_1 y_1 + c_2 y_2$$

$$y_c = c_1 x^{m_1} + c_2 x^{m_2}$$

$$y_c = c_1 x + c_2 x^{-1}$$

where $y_1 = x$ and $y_2 = x^{-1}$

and Now by using variation of parameter. we find a particular solution.

$$y_p = u_1 y_1 + u_2 y_2.$$

Now we put given D.Eq into standard form by dividing x^2 on both sides.

$$y'' + \frac{y'}{x} - \frac{y}{x^2} = \frac{\ln x}{x^2}$$

Here

$$f(x) = \frac{\ln x}{x^2}$$

We find Wronskian.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -x^{-1} - x^{-1}$$

$$= -2x^{-1} \neq 0$$

We find u_1 and u_2

$$u_1 = -\int \frac{y_2 \cdot f(x)}{W} dx, \quad u_2 = \int \frac{y_1 f(x)}{W} dx.$$

$$u_1 = -\int \frac{\cancel{x^{-1}} \cdot \ln x / x^2}{-2\cancel{x^{-1}}} dx \quad \Rightarrow u_1 = \frac{1}{2} \int \frac{\ln x}{x^2} dx.$$

Now by Integrating by parts.

$$u_1 = \frac{1}{2}\int \frac{\ln x}{x^2}\,dx$$

$$= \frac{1}{2}\left[-\frac{1}{x}\ln x + \int \frac{1}{x}\frac{d\ln x}{dx}\,dx\right]$$

$$= \frac{1}{2}\left[-\frac{1}{x}\ln x + \int \frac{1}{x}\cdot\frac{1}{x}\,dx\right]$$

$$= \frac{1}{2}\left[-\frac{1}{x}\ln x + \int \frac{1}{x^2}\,dx\right]$$

$$= \frac{1}{2}\left[-\frac{1}{x}\ln x + \frac{x^{-1}}{-1}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{x}\ln x - \frac{1}{x}\right]$$

$$u_1 = -\frac{1}{2}x^{-1}\ln x - \frac{1}{2}x^{-1}$$

$$u_2 = \int \frac{y_1 f(x)}{W}\,dx.$$

$$u_2 = \int \frac{x\,\ln x/x^2}{-2x^{-1}}\,dx.$$

$$= -\int \frac{\ln x}{2x\,x^{-1}}\,dx$$

$$u_2 = -\int \frac{\ln x}{2}\,dx.$$

$$u_2 = -\frac{1}{2}\int 1 \cdot \ln x \, dx$$

$$= -\frac{1}{2}\left[x \cdot \ln x - \int x \frac{d}{dx} \ln x \right] dx$$

$$= -\frac{1}{2}\left[x \ln x - \int x \cdot \frac{1}{x} \, dx\right]$$

$$= -\frac{1}{2}\left[x \ln x - \int 1 \, dx\right]$$

$$u_2 = -\frac{1}{2} x \ln x + \frac{1}{2} x$$

$$y_p = \left(-\frac{1}{2} x^{-1} \ln x - \frac{1}{2} x^{-1}\right) x + \left(-\frac{1}{2} x \ln x + \frac{1}{2} x\right) x^{-1}$$

General solution is:-

$$y = y_c + y_p$$

$$y = c_1 x + c_2 x^{-1} + \left(-\frac{1}{2} x^{-1} \ln x - \frac{1}{2} x^{-1}\right) x +$$
$$\left(-\frac{1}{2} x \ln x + \frac{1}{2} x\right) x^{-1}$$

$$y = c_1 x + c_2 x^{-1} - \frac{1}{2} \ln x - \cancel{\frac{1}{2}} - \frac{1}{2} \ln x + \cancel{\frac{1}{2}}$$

$$y = c_1 x + c_2 x^{-1} - \ln x$$

Ans.

Practice Question

Ex # 4.7

Q 14, Q18, Q24, Q29. Q33.