easy = 60
medium = 30
Hard = 10.

National University of Computer and Emerging Sciences, Lahore Campus

0030

Course Name:	Applied Calculus	Course Code:	MT1001
Program:	BEE	Semester:	Fall 2023
Duration:	3 hours	Total Marks:	100
Exam Date:	27th December, 2023	Weight:	50
Section:	ALL	Page(s):	2
Exam Type:	Final Exam		

Student Name:________________ **Roll No.**____________ **Section:**________

Instruction/Notes:
1. All CLOs are covered in this paper
2. If need arises, make valid assumptions and clearly mention it with your answer.

Question 1 [10 marks, CLO1]

a. **[3 marks] Illustrate** the given inequality in terms of intervals and find the solution set on real line.

$$2x + 7 > 3.$$

Very Easy.

b. **[3 marks] Illustrate** the given inequality in terms of intervals and find the solution set on real line.

$$x^2 + x > 1.$$

c. **[4 marks] Identify** whether the given function is even, odd, or neither and also sketch the graph of the function $y = |\sqrt{x} - 3|$, not by plotting points, but by starting with the graph of one of the standard function.

Question 2 [10 marks, CLO2]

a. **[5 marks] Determine** the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f.

Easy.

$$f(x) = \begin{cases} x^2 - 4, & x < 1 \\ 1, & x = 1 \\ -\frac{1}{2}x + 1, & x > 1 \end{cases}$$

b. **[5 marks] Evaluate** the followings. If the symbol [[ ]] denotes the greatest integer function.

(i) $\lim\limits_{x \to -3.1} [[x]]$ (ii) $\lim\limits_{x \to -3.9} [[x]]$ (iii) $\lim\limits_{x \to 4.5} [[x]]$

Question 3 [20 marks, CLO3]

a. **[10 marks] Compute** the derivative of the given function and sketch the curve by using differentiation

$$y = x^3 + 3x^2$$

b. **[10 marks]** Air is being pumped into a spherical balloon so that its volume increases at a rate of $100\ cm^3/s$. How fast is the radius of the balloon increasing when the diameter is $50\ cm$?

Easy

Question 4 [40 marks, CLO4]

a. **[30 marks] Integrate** the given functions by using any appropriate methods.

(i). $\dfrac{1}{2\sin^2 x + 3\cos^2 x}$ (ii). $\dfrac{\sec x}{1 + \csc x}$ (iii). $\displaystyle\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{\frac{3}{2}}} dx.$

easy very hard hard

very easy b. **[10 marks] Sketch** the region enclosed by the given curves and find its area.

$$y = x^2 \quad , \quad y = 2x - x^2.$$

Question 5 [20 marks, CLO5]

very easy (a). **[5 marks] Write** a vector equation, parametric equations and symmetric equation for the line. The line through the point $(\mathbf{0}, \mathbf{14}, -\mathbf{10})$ and parallel to the line $x = -1 + 2t, y = 6 - 3t, z = 3 + 9t$.

easy (b). **[5 marks] Formulate** the distance between the given planes, if they are parallel.

$$2x - 3y + z = 4 \quad , \quad 4x + 2z - 6y = 3.$$

(c). **[10 marks] Write** an equation of the plane. The plane that passes through the point $(3, 1, 4)$ and contains the line of intersection of the planes $x + y + 3z = 1$ and $2x - y + z = -3$.

hard.

2

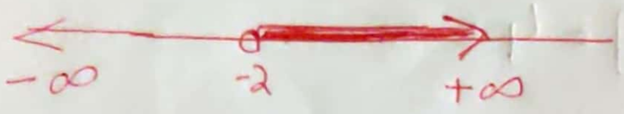
Final Exam Solution.

Q# 1 (a). Sol:

$$2x > -4$$

$$x > -2$$

$$S.S. = (-2, +\infty)$$



Q#1 (b). Sol:

$$x^2 + x - 1 > 0$$

To identify values of x, by equating to zero.

$$x^2 + x - 1 = 0$$

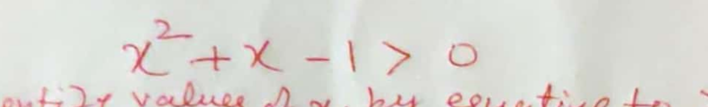
Using quadratic formula.

$$x = \frac{-1 \pm \sqrt{5}}{2}$$

Intervals in which the polynomial makes the inequality true.

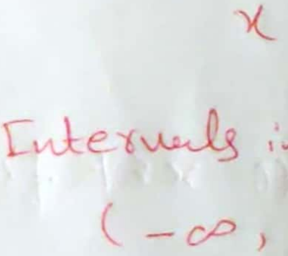
$$\left(-\infty, \frac{-1-\sqrt{5}}{2}\right) \cup \left(\frac{-1+\sqrt{5}}{2}, +\infty\right)$$

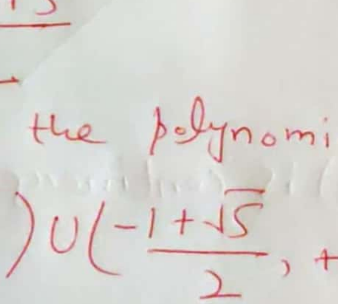
on number line,

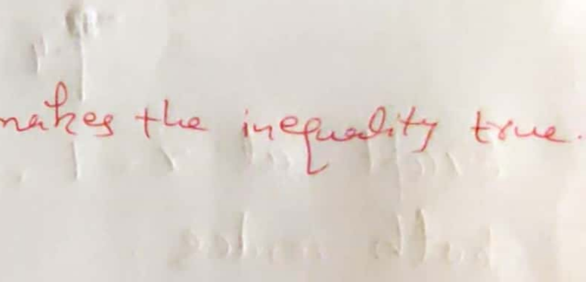


Q#1(c). Sol:

$y = \sqrt{x}$ $\qquad$ $y = \sqrt{x} - 3$ $\qquad$ $y = |\sqrt{x} - 3|$







$f(x) = |\sqrt{x} - 3|$ and $f(-x) = |\sqrt{-x} - 3|$ are not comparable in traditional sense, as $\sqrt{-x}$ is not valid $\forall x \in R$. But, we know $\sqrt{x} = \sqrt{-x}$ is an even ftn. for all x in domain where the expression is defined. Subtracting 3 & taking absolute does not affect whether the ftn. is even or odd. So ftn. f(x) is even

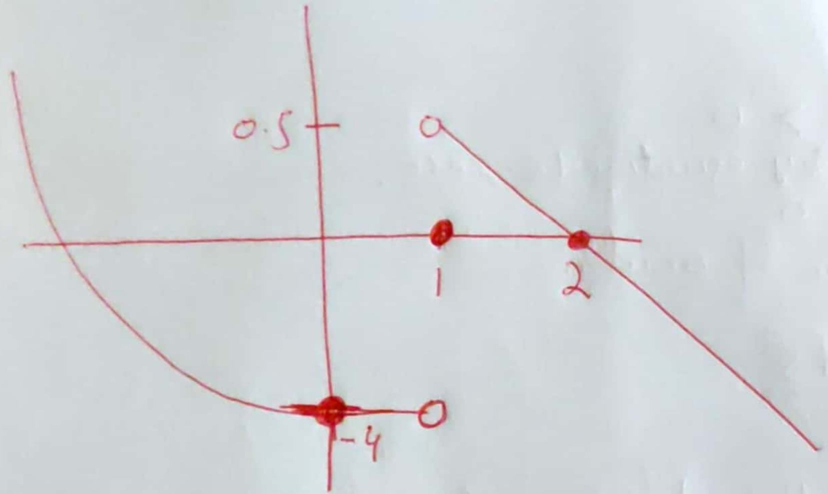
Q#2(b). sol:

(i). $\lim_{x \to -3.1} [[x]] = -4$

(ii). $\lim_{x \to -3.9} [[x]] = -4$

(iii). $\lim_{x \to 4.5} [[x]] = 4$

Q#2(a). sol:



Except at $x = 1$, $f(x)$ is continuous at all $x \in R$ from both sides.

Q#3(b). sol:

Let V be the volume of the baloon & r be its radius.

Given $\frac{dv}{dt} = 100$, $\frac{dr}{dt} = ?$ when $r = 25$.

we know

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

&

$$\frac{dr}{dt} = \frac{1}{25\pi} \approx 0.0127 \text{ cm/s}$$

Q#3.(a).sol:

a). Domain: $D_f = (-\infty, +\infty)$

b). x-intercepts,

$x = -3$ & $x = 0$.

y-intercepts,

$y = 0$.

c). Symmetry:

Neither even nor odd function.

d). Assymptote:

No horizontal & No vertical assymptote.

e). Increase/Decrease:

$f'(x) = 3x^2 + 6x = 0$

$x = 0$ & $x = -2$ are the critical pt.s.

f is increasing on $(-\infty, -2)$ & $(0, \infty)$ & decreasing on $(-2, 0)$.

f). Max./min.:

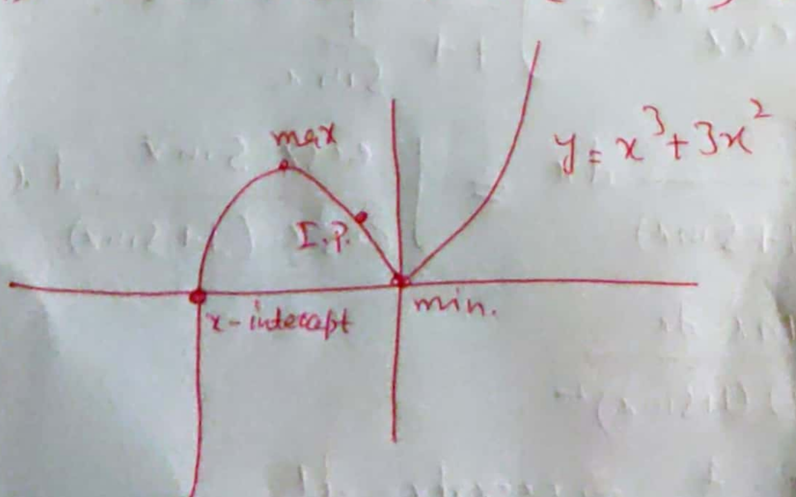
f is local max. at $(-2, 4)$ and min. at $(0, 0)$.

g). Concavity: $f''(x) = 6x + 6$

$(-1, 2)$ is the pt. of inflection.

f is concave down on $(-\infty, -1)$ and upward on $(-1, \infty)$.

h).



Q#4(a). Sol:

$$\int \frac{dx}{2\sin^2x + 3\cos^2x}$$

(we solve it by substitution,)

For simplification, divide Numerator and Denominator by $\cos^2x$

$$= \int \frac{\sec^2x\,dx}{2\tan^2x + 3}$$

Put $\tan x = t$

$\sec^2x\,dx = dt$

$$= \int \frac{dt}{2t^2+3} = \frac{1}{2}\int \frac{dt}{t^2 + \frac{3}{2}} = \frac{1}{2}\int \frac{dt}{(t)^2 + \left(\sqrt{\frac{3}{2}}\right)^2}$$

$$= \frac{1}{2}\,\frac{\sqrt{2}}{\sqrt{3}}\,\tan^{-1}\frac{t}{\sqrt{\frac{3}{2}}}$$

$$= \frac{1}{\sqrt{6}}\tan^{-1}\left(\sqrt{\frac{2}{3}}\tan x\right)$$

Q#4(b). Sol: $\displaystyle\int \frac{\sec x}{1+\csc x}\,dx = \int \frac{1/\cos x}{1 + \frac{1}{\sin x}}\,dx$

$$= \int \frac{\sin x}{\cos x(1+\sin x)}\,dx = \int \frac{\cos x \sin x}{\cos^2x\,(1+\sin x)}\,dx$$

$$= \int \frac{\cos x \sin x\,dx}{(1-\sin x)(1+\sin x)^2}$$

Put $\sin x = t \Rightarrow \cos x\,dx = dt$

$$= \int \frac{t\,dt}{(1-t)(1+t)^2}$$

By using Partial fraction

$$\frac{t}{(1-t)(1+t)^2} = \frac{1}{4(1-t)} + \frac{1}{4(1+t)} - \frac{1}{2(1+t)^2}$$

therefore,

$$\int \frac{t}{(1-t)(1+t)^2} = \frac{1}{4}\int \frac{dt}{1-t} + \frac{1}{4}\int \frac{dt}{1+t} - \frac{1}{2}\int \frac{dt}{(1+t)^2}$$

$$= \frac{1}{4}\ln(1-t) + \frac{1}{4}\ln(1+t) + \frac{1}{2}\,\frac{1}{1+t}$$

$$= \frac{1}{4}\ln(1-\sin x) + \frac{1}{4}\ln(1+\sin x) + \frac{1}{2}\,\frac{1}{(1+\sin x)}$$

$$= \frac{1}{4}\ln\left(\frac{1+\sin x}{1-\sin x}\right) + \frac{1}{2}\,\frac{1}{1+\sin x}$$

Q#4(a).(iii). Sol:

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{((2x)^2+3^2)^{3/2}}\,dx$$

Put $2x = 3\tan\theta \Rightarrow x = \frac{3}{2}\tan\theta$

$\Rightarrow dx = \frac{3}{2}\sec^2\theta\,d\theta$

$x = 0 \Rightarrow \theta = 0$ & $x = \frac{3\sqrt{3}}{2} \Rightarrow \theta = \pi/3$

So, above integral becomes,

$$= \frac{3}{16}\int_0^{\pi/3} \frac{1-\cos^2\theta}{\cos^2\theta}\sin\theta\,d\theta$$

Again substitute, $u = \cos\theta \Rightarrow du = -\sin\theta\,d\theta$

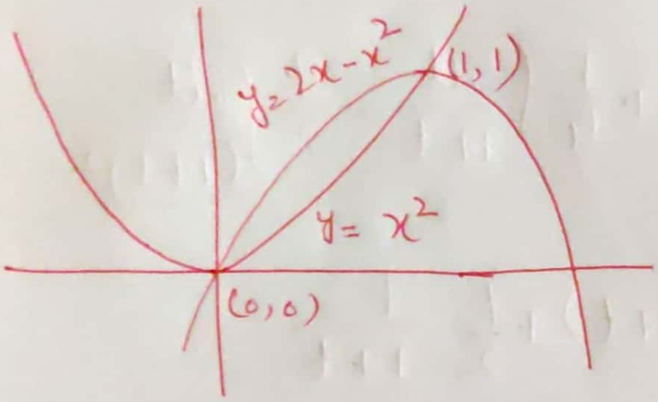
when $\theta = 0 \Rightarrow u = 1$ & $\theta = \pi/3 \Rightarrow u = 1/2$

$$= -3/16\int_1^{1/2} \frac{1-u^2}{u^2}\,du$$

$$= -\frac{3}{16}\int_1^{1/2} \frac{1-u^2}{u^2}\,du$$

$$= \frac{3}{32}$$

Q #4.(b). sol:



$x=0$ & $x=1$ are the pt. of intersection

$$A = \int_0^1 (2x-2x^2)\,dx$$

$$= \frac{1}{3}$$

Q#5. (a). sol:

vector eq. of line

$$\bar{r} = r_0 + t\bar{v}$$

$$= \langle 0,14,-10\rangle + t\langle 2,-3,9\rangle$$

$$\bar{r} = (14j - 10k) + t(2i - 3j + 9k)$$

Parametric eq. of line;

$$x = 2t,\ y = 14-3t,\ z = -10+9t$$

Symmetric eq. of line;

$$t = x/2,\quad t = \frac{14-y}{3},\quad t = \frac{z+10}{9}$$

$$\Rightarrow \frac{x}{2} = \frac{14-y}{3} = \frac{z+10}{9}$$

Q#5. (b). Sol:

$$D = \frac{|ax_0 + by + cz + d|}{\sqrt{a^2+b^2+c^2}}$$

After choosing one pt. from any of the given planes

$D = 5/2\sqrt{14}$ (Note: Answer depend on the choice of pt.)

Q#5. (c). Sol!

First, we will calculate the direction vector of the line,

$$<1,1,3> \times <2,-1,1> = \begin{vmatrix} i & j & k \\ 1 & 1 & 3 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= \hat{i}(1+3) - \hat{j}(1-6) + \hat{k}(-1-2)$$

$$= (4, 5, -3)$$

Secondly, find an pt. on intersecting line,

Set $x = t$, then, $t + y + 3z = 1 \rightarrow (a)$

& $2t - y + z = -3 \rightarrow (b)$

By adding (a) & (b),

$$3t + 4z = -2$$

$$z = \frac{-2-3t}{4}$$

eq. (a) becomes

$$y = 1 - t - 3z$$

$$= 1 - t - 3\left(\frac{-2-3t}{4}\right) = \frac{4-4t+6+9t}{4} = \frac{10+5t}{4}$$

- Parameter, t can take any value, choose t = 0,

$\left(0, \frac{5}{2}, -1/2\right)$

vector b/w two line, is calculated as follows,

$$(3,1,4)-(0,5/2,-1/2)=(3,-3/2,9/2).$$

To obtain normal vector of the surface,

$$\langle 3,-3/2,9/2\rangle \times \langle 4,5,-3\rangle = \begin{vmatrix} i & j & k \\ 3 & -3/2 & 9/2 \\ 4 & 5 & -3 \end{vmatrix}$$

$$= i\left(\frac{9}{2}-\frac{45}{2}\right)-j(-9-18)+\hat{k}(15+6)$$

$$= i(-18)-j(-27)+\hat{k}(21)$$

$$= \langle -18,27,21\rangle = 3\langle -6,9,7\rangle$$

Eq. of the Plane, using $(-6,9,7)$ & $(3,1,4)$

$$-6x+9y+7z=(-18+9+28)=1$$