Ex 7.2

Inverse Transforms and Transforms of Derivatives

. In this section, we study the technique for recovering $f(t)$. $F(s)$ is given and we find some function $f(t)$ by taking Inverse.

Inverse Laplace Transform:-

If $F(s)$ represents the Laplace transform of a function $f(t)$, i.e

$$\mathcal{L}\{f(t)\} = F(s)$$

$$\Rightarrow \quad f(t) = \mathcal{L}^{-1}\{F(s)\}$$

we say $f(t)$ is the inverse Laplace Transform of $F(s)$.

Inverse Transform:-

$$\mathcal{L}\{1\} = \frac{1}{s} \quad \Rightarrow \quad 1 = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2} \quad \Rightarrow \quad t = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$\mathcal{L}\{e^{-3t}\} = \frac{1}{s+3} \quad \Rightarrow \quad t = \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

In General,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \Rightarrow \quad t^n = \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad \Rightarrow \quad e^{at} = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}$$

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2} \Rightarrow \sin kt = \mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}$$

$$\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2} \Rightarrow \cos kt = \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}$$

$$\mathcal{L}\{\sinh k\} = \frac{k}{s^2-k^2} \Rightarrow \sinh kt = \mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\}$$

$$\mathcal{L}\{\cosh kt\} = \frac{s}{s^2-k^2} \Rightarrow \cosh kt = \mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$$

Evaluate

Exp 1: $\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\}$

Sol: As we know that

$$t^n = \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}$$

$$n+1 = 5 \Rightarrow \boxed{n=4}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^{4+1}}\right\}$$

$$= \frac{1}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{s^{4+1}}\right\}$$

$$= \frac{1}{4!}t^4$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{24}t^4$$

Evaluate

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+7}\right\}$$

Sol:- As we know that $\sin kt = \mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}$

By comparing $k^2 = 7 \Rightarrow k = \sqrt{7}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\mathcal{L}^{-1}\left\{\frac{\sqrt{7}}{s^2+k^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\sin\sqrt{7}\,t$$

$\mathcal{L}^{-1}$ is a linear Transformation:-

The inverse Laplace Transform is also a linear transform that is for constant α and β

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\}.$$

Find the given Inverse Laplace Transform.

Q3:- Find the given Inverse Laplace Transform

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{48}{s^5}\right\}$$

Sol:- $\mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{48}{s^5}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{48}{s^5}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{48}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\} \qquad \because t^n = \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}$$

$$= t - \frac{48}{4!}t^4$$

$$= t - 2t^4$$

Q23:- Evaluate

$$\mathcal{L}^{-1}\left\{\frac{4s}{4s^2+1}\right\} \qquad \because \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$$

Sol:- $\mathcal{L}^{-1}\left\{\frac{4s/4}{\frac{4s^2+1}{4}}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1/4}\right\} \Rightarrow \mathcal{L}^{-1}\left\{\frac{s}{(s)^2+(1/2)^2}\right\}$$

$$= \cos\frac{1}{2}t$$

Q24:- $\mathcal{L}^{-1}\left\{\frac{s^2+1}{s(s-1)(s+1)(s-2)}\right\}$

Sol:- Now decompose into partial fraction.

$$\frac{s^2+1}{s(s-1)(s+1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} + \frac{D}{s-2} \quad \text{—— (1)}$$

(all are linear factor)

Multiply eq (1) with $s(s-1)(s+1)(s-2)$ on both sides.

$$\frac{s^2+1}{s(s-1)(s+1)(s-2)}\left(s(s-1)(s+1)(s-2)\right) = \left[\frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} + \frac{D}{s-2}\right] s(s-1)(s+1)(s-2)$$

$$s^2+1 = A(s-1)(s+1)(s-2) + B(s)(s+1)(s-2) + C(s)(s-1)(s-2) + D(s)(s-1)(s+1) \quad \text{—— (2)}$$

Now we find the constants.

For $s=0$, put in (2)

$$0+1 = A(-1)(1)(-2) + 0 + 0 + 0$$

$$2A = 1 \Rightarrow \boxed{A = \frac{1}{2}}$$

For $s=1$, put in (2)

$$1+1 = 0 + B(1)(1+1)(-1) + 0 + 0$$

$$2 = -2B \Rightarrow \boxed{B = -1}$$

For $s=-1$, put in (2)

$$(-1)^2+1 = 0 + 0 + C(-1)(-1-1)(-1-2) + 0$$

$$2 = -6C \Rightarrow \boxed{C = -\frac{1}{3}}$$

For $s=2$, put in (2)

$$2^2+1 = 0 + 0 + 0 + D(2)(2-1)(2+1)$$

$$5 = 6D \Rightarrow \boxed{D = \frac{5}{6}}$$

put all values in eq (1).

$$\frac{s^2+1}{s(s-1)(s+1)(s-2)} = \frac{1/2}{s} + \frac{-1}{s-1} + \frac{-1/3}{s+1} + \frac{5/6}{s-2}.$$

Applying Inverse Laplace.

$$L^{-1}\left\{\frac{s^2+1}{s(s-1)(s+1)(s-2)}\right\} = L^{-1}\left\{\frac{1/2}{s} - \frac{1}{s-1} - \frac{1/3}{s+1} + \frac{5/6}{s-2}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{1}{s}\right\} - 1\,L^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{3}L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$+ \frac{5}{6}L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= \frac{1}{2}(1) - 1e^{t} - \frac{1}{3}e^{-t} + \frac{5}{6}e^{2t}.$$

$$\therefore L^{-1}\left\{\frac{s^2+1}{s(s-1)(s+1)(s-2)}\right\} = \frac{1}{2} - e^{t} - \frac{1}{3}e^{-t} + \frac{5}{6}e^{2t}.$$

Alternate Method (The special technique for finding coefficients).

Cover up Method

$$\frac{s^2+1}{s(s-1)(s+1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} + \frac{D}{s-2}.$$

$$A = \left.\frac{s^2+1}{(s-1)(s+1)(s-2)}\right|_{s=0}$$

$$A = \frac{1}{(-1)(1)(-2)} \Rightarrow \boxed{A = \frac{1}{2}}$$

$$B = \left.\frac{s^2+1}{s(s+1)(s-2)}\right|_{s=1}$$

$$\boxed{B = -1}$$

$$B = \frac{1+1}{1(1+1)(1-2)} \Rightarrow$$

$$C = \left.\frac{s^2+1}{s(s-1)(s-2)}\right|_{s=-1}$$

$$C = \frac{(-1)^2+1}{(-1)(-1-1)(-1-2)} \Rightarrow C = \frac{2}{(-1)(-2)(-3)}$$

$$\boxed{C = -\frac{1}{3}}$$

$$D = \left.\frac{s^2+1}{s(s-1)(s+1)}\right|_{s=2}$$

$$D = \frac{4+1}{(2)(2-1)(2+1)} \Rightarrow \boxed{D = \frac{5}{6}}$$

Transforms of Derivatives:-

Our main goal is to use Laplace transform to solve the Differential equation.

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} \cdot f'(t)\, dt \qquad \text{(Integration by parts)}$$

$$= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty f(t) \frac{d}{dt}(e^{-st})\, dt$$

$$= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty -s e^{-st} f(t)\, dt$$

$$= e^{-\infty} f(t) - e^{0} f(0) + s\int_0^\infty e^{-st} f(t)\, dt$$

$$= 0 - f(0) + s\int_0^\infty e^{-st} f(t)\, dt$$

$$\mathcal{L}\{f'(t)\} = -f(0) + sF(s).$$

$$\Rightarrow \boxed{\mathcal{L}\{f'(t)\} = sF(s) - f(0).} \qquad \lim_{t\to\infty} e^{-st} f(t) = 0.$$

$$\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\, dt$$

$$= e^{-st} f'(t)\Big|_0^\infty - \int_0^\infty f'(t) \frac{d}{dt}(e^{-st})\, dt$$

$$= e^{-\infty} f'(t) - e^{s(0)} f'(0) - \int_0^\infty f'(t)(-s e^{-st})\, dt$$

$$= 0 - f'(0) + s\int_0^\infty e^{-st} \cdot f'(t)\, dt$$

From previous results

$$= -f'(0) + s\left[sF(s) - f(0)\right]$$

$$= -f'(0) + s^2F(s) - sf(0)$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0).$$

Similarly,

$$\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0).$$

Transform of Derivatives:-

If $f, f', f'', \ldots f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$ then

$$\mathcal{L}\{f^n(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) \cdots - f^{n-1}(0).$$

where $F(s) = \mathcal{L}\{f(t)\}$.

Use Laplace Transform to solve the given Initial value problem.

Q34:- $y' - y = 2\cos 5t \quad ; \quad y(0) = 0$

Sol:- $\frac{dy}{dt} - y = 2\cos 5t$

Taking Laplace on both sides

$$\mathcal{L}\left\{\frac{dy}{dt} - y\right\} = \mathcal{L}\{2\cos 5t\}$$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} - \mathcal{L}\{y\} = 2\mathcal{L}\{\cos 5t\}$$

$\because \cos 5t = \frac{s}{s^2+25}$

$$\overline{sY(s) - y(0) - Y(s)} = 2\left(\frac{s}{s^2+25}\right)$$

$\because y(0) = 0$

$$Y(s)(s-1) - 0 = \frac{2s}{s^2+25}$$

$$\Rightarrow Y(s)\{s-1\} = \frac{2s}{s^2+25}$$

$$Y(s) = \frac{2s}{(s-1)(s^2+25)} \quad \text{——— (1)}$$

$$\frac{2s}{(s-1)(s^2+25)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+25} \quad \text{——— (2)}$$

$(s-1)$ linear, (s^2+25) Quadratic

$$A = \left.\frac{2s}{s^2+25}\right|_{s=1} \Rightarrow A = \frac{2(1)}{(1)^2+25}$$

$$\boxed{A = \frac{1}{13}}$$

b)

Multiply eq (2) with $(s-1)(s^2+25)$

$$2s = A(s^2+25) + (Bs+C)(s-1).$$

$$2s = As^2 + 25A + Bs^2 - Bs + Cs - C$$

By comparing coefficients.

$$s^2: \quad 0 = A + B$$

$$\Rightarrow \frac{1}{13} - B = 0 \Rightarrow \boxed{B = \frac{-1}{13}}$$

$$s: \quad 2 = -B + C$$

$$2 = -(-\frac{1}{13}) + C \Rightarrow \quad C + \frac{1}{13} = 2$$

$$C = 2 - \frac{1}{13}.$$

$$\boxed{C = \frac{25}{13}}$$

put all values in eq (2) and taking Laplace.

$$\frac{2s}{(s-1)(s^2+25)} = \frac{1}{13(s-1)} + \frac{-\frac{1}{13} + \frac{25}{13}}{s^2+25}$$

$$L\left\{\frac{2s}{(s-1)(s^2+25)}\right\} = L\left\{\frac{1}{13(s-1)}\right\} + L\left\{\frac{-\frac{1}{13}s}{s^2+25}\right\} + L\left\{\frac{\frac{25}{13}}{s^2+25}\right\}$$

$$= \frac{1}{13} L\left\{\frac{1}{s-1}\right\} - \frac{1}{13} L\left\{\frac{1\, s}{s^2+25}\right\} + \frac{25}{13} L\left\{\frac{1}{s^2+25}\right\}$$

$$= \frac{1}{13}e^t - \frac{1}{13} . L\left\{\frac{s}{s^2+5^2}\right\} + \frac{5}{13} L\left\{\frac{5}{s^2+5^2}\right\}$$

$$= \frac{1 . e^t}{13} - \frac{1}{13}\cos 5t + \frac{5}{13}\sin 5t.$$

Ans

Ex 7.2

P.Q.:

Q8, Q22, Q30, Q33, Q37

Q41, Q42