4.6 Variation of Parameters

The method of variation of parameters applies to solve

$$a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = g(x) \quad \text{——} \quad (1)$$

where a_2, a_1, a_0 and g are continuous and $a_2(x) \neq 0$.

⇒ This method is important because it solves the largest classes of equation. Specially included are functions $g(x)$ like $\ln|x|$, $|x|$, e^{x^2}.

Also we know that the general solution of eq (1) is of the form.

$$y = y_c + y_p$$

The Method of Variation of Parameter for 2nd order Equation

Consider the 2nd order linear non-homogeneous D.Eq

$$a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = g(x)$$

We can write this equation in standard form by dividing $a_2(x)$.

$$y'' + P(x)y' + Q(x)y = f(x) \quad — \quad (2)$$

where $P(x)$, $Q(x)$ and $f(x)$ are continuous function on some Interval I.

(*) For the complementary function we consider the associated Homogeneous Equation.

$$y'' + P(x)y' + Q(x)y = 0 \quad — \quad (3)$$

The complementary function is of the form.

$$y_c = c_1 y_1(x) + c_2 y_2(x).$$

Since y_1 and y_2 are solutions of Homogeneous Eq.

(*) For the particular solution y_p, We replace c_1 and c_2 in complementary function with unknown variables $u_1(x)$ and $u_2(x)$. So that the particular integral is

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) \quad — \quad (A)$$

Differentiate the above equation twice.

$$y_p' = u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2$$

$$y_p'' = u_1 y_1'' + u_1' y_1' + u_1' y_1' + u_1'' y_1 + u_2' y_2' + u_2 y_2'' + u_2' y_2' + u_2'' y_2.$$

Now putting the values in eq (2)

$$y_p'' + P(x) y_p' + Q(x) y_p = f(x)$$

$$u_1 y_1'' + u_1' y_1' + u_1' y_1' + u_1'' y_1 + u_2' y_2' + u_2 y_2'' + u_2' y_2' + u_2'' y_2$$

$$+ P(x)\left[u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2\right] + Q(x)\left[u_1 y_1 + u_2 y_2\right] = f(x)$$

$$u_1 \cancel{\left[y_1'' + P y_1' + Q y_1\right]}^{0} + u_2 \cancel{\left[y_2'' + P y_2' + Q y_2\right]}^{0} + u_1' y_1'$$

$$+ u_1'' y_1 + u_2' y_2' + u_2'' y_2 + P\left[u_1' y_1 + y_2 u_2'\right]$$

(∵ from eq 3)

$$+ u_1' y_1' + u_2' y_2' = f(x)$$

$$\Rightarrow \underbrace{u_1' y_1' + u_1'' y_1} + \underbrace{u_2' y_2' + u_2'' y_2} + P\left[u_1' y_1 + u_2' y_2\right]$$

$$+ u_1' y_1' + u_2' y_2' = f(x)$$

$$\frac{d}{dx}(y_1 u_1') + \frac{d}{dx}(u_2' y_2) + P(u_1' y_1 + u_2' y_2)$$

$$+ u_1' y_1' + u_2' y_2' = f(x)$$

$$\Rightarrow \frac{d}{dx}\left[y_1 u_1' + y_2 u_2'\right] + P(u_1' y_1 + u_2' y_2) + u_1' y_1' + u_2' y_2' = f(x) \quad (4)$$

Here we assume $y_1 u_1' + y_2 u_2' = 0$ The above eq reduces to

$$u_1' y_1' + u_2' y_2' = f(x).$$

Hence u_1 and u_2 must be functions that satisfy the equation.

$$y_1 u_1' + y_2 u_2' = 0 \quad \text{——— (5)}$$
$$y_1' u_1' + y_2' u_2' = f(x) \quad \text{——— (6)}$$

By using the Cramer's Rule, we get the solution
(OR)
To eliminate u_2' we multiply eq (5) with y_2' and (6) with $-y_2$ and then adding the equations.

$$y_2' y_1 u_1' - y_1' u_1' y_2 = -y_2 f(x)$$

$$u_1' (y_1 y_2' - y_1' y_2) = -y_2 f(x)$$

$$u_1' W = -y_2 f(x)$$

$$u_1' = \frac{-y_2 f(x)}{W}$$

$\left\{ \because W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \right.$

$W = y_1 y_2' - y_2 y_1'$

Similarly, to eliminate u_1' we multiply eq (5) with y_1' and (6) with $-y_1$ and adding them

$$y_1' y_2 u_2' - y_1 y_2' u_2' = -y_1 f(x)$$

$$u_2' (y_1' y_2 - y_1 y_2') = -y_1 f(x)$$

$$-u_2' (\underbrace{y_1 y_2' - y_1' y_2}_{W}) = -y_1 f(x)$$

$$u_2' = -\frac{y_1 f(x)}{W}$$

The functions u_1 and u_2 are found by Integrating the above results.

$$u_1 = -\int \frac{y_2 f(x)}{W} dx \quad ; \quad W \neq 0.$$

$$u_2 = \int \frac{y_1 f(x)}{W} dx.$$

and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.

Higher Order Equation :-

This method can be generalized to the linear nth order Non homogeneous equation.

We put the equation in standard form.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$\frac{d^n y}{dx^n} + P_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + P_1(x)\frac{dy}{dx} + P_0(x)y = f(x)$$

or

$$y^n + P_{n-1}(x)y^{n-1} + \dots + P_1(x)y' + P_0(x)y = f(x). \quad \text{(A)}$$

The complementary function for eq (A) will be

$$y_c = c_1 y_1 + c_2 y_2 + \dots + c_n y_n.$$

and particular solution is

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) + \cdots + u_n(x)y_n(x)$$

where $u_1, u_2, \ldots u_n$ are found by n equations.

$$y_1 u_1' + y_2 u_2' + \cdots + y_n u_n' = 0$$

$$y_1' u_1' + y_2' u_2' + \cdots + y_n' u_n' = 0$$

$$\vdots$$

$$y_1^{(n-1)} u_1' + y_2^{(n-1)} u_2' + \cdots + y_n^{(n-1)} u_n' = f(x)$$

By using Cramer's Rule. we find the results.

$$u_n' = \frac{W_n}{W}$$

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}, \quad u_3' = \frac{W_3}{W}$$

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f(x) & y_3'' \end{vmatrix}, \quad W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f(x) \end{vmatrix}$$

and $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$

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Ex # 4.6 :-

Q11:- Solve each differential Eq by variation of parameters

$$y'' + 3y' + 2y = \frac{1}{1+e^x}$$ (already in standard form).

Sol:- $y'' + 3y' + 2y = \frac{1}{1+e^x}$

The auxiliary eq is

$$m^2 + 3m + 2 = 0$$

$$m^2 + 2m + m + 2 = 0$$

$$m(m+2) + 1(m+2) = 0$$

$$(m+1)(m+2) = 0$$

$$\Rightarrow \boxed{m_1 = -1}, \boxed{m_2 = -2}$$

The complementary solution is

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$y_c = c_1 e^{-x} + c_2 e^{-2x}.$$

with the identifications $y_1 = e^{-x}$ and $y_2 = e^{-2x}$

Now Wronskian is

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$$

$$= -2e^{-3x} + e^{-3x} = -e^{-3x} \neq 0$$

$$u_1 = \int -\frac{y_2 f(x)}{W}\,dx \quad , \quad u_2 = \int \frac{y_1 f(x)}{W}\,dx$$

$$u_1 = -\int \frac{e^{-2x} \cdot \frac{1}{(1+e^x)}}{-e^{-3x}}\,dx$$

$$u_1 = +\int \frac{e^x}{1+e^x}\,dx.$$

$$u_1 = \ln|1+e^x|.$$

Similarly,

$$u_2 = \int \frac{e^{-x} \cdot \frac{1}{1+e^x}}{-e^{-3x}}\,dx.$$

$$u_2 = -\int \frac{e^{2x}}{1+e^x}\,dx$$

$$u_2 = -\int \left[e^x - \frac{e^x}{1+e^x}\right] dx$$

$$u_2 = \int \frac{e^x}{1+e^x}\,dx - \int e^x\,dx$$

$$u_2 = \ln|1+e^x| - e^x.$$

$$y_p = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$$

$$y_p = \ln|1+e^x|\,e^{-x} + \left(\ln|1+e^x| - e^x\right) e^{-2x}.$$

General Solution is

5)

$$y = y_c + y_p$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-x}\ln|1+e^{x}| + e^{-2x}\ln|1+e^{x}| - e^{-x}.$$

Q15: $y'' + 2y' + y = e^{-t}\ln t$ ∵ y is function of t.

Sol: The auxiliary equation is

$$m^2 + 2m + 1 = 0$$

$$m^2 + m + m + 1 = 0$$

$$m(m+1) + 1(m+1) = 0$$

$$(m+1)^2 = 0 \Rightarrow m = -1, -1$$ (roots are repeated)

$$y_c = c_1 e^{mt} + c_2 t e^{mt}$$

$$y_c = c_1 e^{-t} + c_2 t e^{-t}$$

∴ Here $y_1 = e^{-t}$, $y_2 = t e^{-t}$

Now we find y_p.

$$y_p = u_1(t) y_1(t) + u_2(t) y_2(t)$$

$$y_p = u_1(t) e^{-t} + u_2(t) t e^{-t} \quad \text{—— (1)}$$

Now the wronskian

$$W = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix} = e^{-2t} - \cancel{te^{-2t}} + \cancel{te^{-2t}} = e^{-2t} \neq 0$$

$$f(t) = e^{-t}\ln t$$

$$u_1 = \int \frac{-y_2 g(t)}{W} dt \quad , \quad u_2 = \int \frac{y_1 g(t)}{W} dt$$

$$u_1 = -\int \frac{t e^{-t} (e^{-t} \ln t)}{e^{-2t}} dt$$

$$u_1 = -\int t \ln t \, dt$$

$$= -\left[\ln t \frac{t^2}{2} - \int \frac{t^2}{2}\left(\frac{1}{t}\right) dt\right]$$

$$= -\ln t \frac{t^2}{2} + \int \frac{t}{2} dt$$

$$u_1 = -\ln t \frac{t^2}{2} + \frac{t^2}{4}$$

$$u_2 = \int \frac{e^{-t} e^{-t} \ln t \, dt}{e^{-2t}}$$

$$u_2 = \int \ln t \, dt \Rightarrow u_2 = \int 1 . \ln t \, dt$$

$$u_2 = t \ln t - \int t . \frac{1}{t} \, dt$$

$$u_2 = t \ln t - t$$

$$y_p = \left(-\ln t \frac{t^2}{2} + \frac{t^2}{4}\right) e^{-t} + (t \ln t - t) t e^{-t}$$

General Solution is

$$y = y_c + y_p$$

$$y = c_1 e^{-t} + c_2 t e^{-t} + \left(-\frac{1}{2} t^2 \ln t + \frac{t^2}{4}\right) e^{-t} + (t \ln t - t) t e^{-t}$$

Q19:- Solve each D.Eq by variation of parameter subject to Initial condition

$y(0) = 1, \quad y'(0) = 0$

$4y'' - y = xe^{x/2}$

Sol:. $4y'' - y = xe^{x/2}$

We first put equation in standard form by dividing it 4.

$$y'' - \frac{1}{4}y = \frac{xe^{x/2}}{4}$$

The Associated Homogeneous eq

$$y'' - \frac{1}{4}y = 0$$

Auxilary Eq : $m^2 - \frac{1}{4} = 0$

$$(m)^2 - \left(\frac{1}{2}\right)^2 = 0$$

$$\left(m - \frac{1}{2}\right)\left(m + \frac{1}{2}\right) = 0$$

$$\Rightarrow m = -\frac{1}{2}, \quad m = -\frac{1}{2}$$

$$y_c = c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{1}{2}x}.$$

$y_1 = e^{\frac{1}{2}x}$
$y_2 = e^{-\frac{1}{2}x}.$

$$\text{Wronskian } W = \begin{vmatrix} e^{\frac{1}{2}x} & e^{-\frac{1}{2}x} \\ \frac{1}{2}e^{\frac{1}{2}x} & -\frac{1}{2}e^{-\frac{1}{2}x} \end{vmatrix}$$

$$= -\frac{1}{2}e^{0x} - \frac{1}{2}e^{0x} = -1 \neq 0$$

$$f(x) = \frac{xe^{x/2}}{4}.$$

$$u_1 = -\int \frac{y_2\, g(x)}{W}\, dx$$

$$= -\int \frac{e^{-\frac{1}{2}x}\, \frac{x e^{x/2}}{4}}{-1}\, dx$$

$$= \int \frac{x}{4}\, dx \Rightarrow \frac{x^2}{8}.$$

$$u_2 = \int \frac{y_1\, g(x)}{W}\, dx$$

$$= \int \frac{e^{\frac{1}{2}x} \left(\frac{x e^{x/2}}{4}\right)}{-1}\, dx$$

$$= -\int \frac{x e^{x}}{4}\, dx.$$

$$= -\left[\frac{e^x}{4} \cdot x - \int \frac{e^x}{4} \cdot \frac{d}{dx}(x)\right) dx\Big]$$

$$= -\frac{e^x \cdot x}{4} + \int \frac{e^x}{4} (1)\, dx$$

$$u_2 = -\frac{x e^x}{4} + \frac{e^x}{4}$$

$$y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$$

$$y_p = \frac{x^2}{8} e^{\frac{1}{2}x} + \left(-\frac{x e^x}{4} + \frac{e^x}{4}\right)\left(e^{-\frac{1}{2}x}\right)$$

General Solution : $y = y_c + y_p$

$$y = c_1 e^{x/2} + c_2 e^{-x/2} + \frac{x^2}{8} e^{x/2} - \frac{x e^{x/2}}{4} - \frac{e^{x/2}}{4}$$

$$y = c_3 e^{x/2} + c_2 e^{-x/2} + \frac{1}{8}x^2 e^{x/2} - \frac{1}{4} x e^{x/2} \quad \text{—— (1)}$$

$$y' = \frac{1}{2}c_3 e^{x/2} - \frac{1}{2} c_2 e^{-x/2} + \frac{2}{8} x e^{x/2} + \frac{1}{16} x^2 e^{x/2} - \frac{1}{4} e^{x/2} - \frac{1}{8} x e^{x/2}$$

$$y' = \frac{1}{2} c_3 e^{x/2} - \frac{1}{2} c_2 e^{-x/2} + \frac{1}{16} x^2 e^{x/2} + \frac{1}{8} x e^{x/2} - \frac{1}{4} e^{x/2} \quad \text{—— (2)}$$

using Initial condition $y(0) = 1$. in eq (1)

$$1 = c_3 + c_2 + 0 + 0 \Rightarrow c_2 + c_3 = 1. \quad \text{—— (3)}$$

using $y'(0) = 0$, put in eq (2)

$$0 = \frac{1}{2} c_3 - \frac{1}{2} c_2 - \frac{1}{4} \Rightarrow \frac{1}{2} c_3 - \frac{1}{2} c_2 - \frac{1}{4} = 0 \quad \text{—— (4)}$$

Solving eq (3) and (4)

$$c_2 = \frac{1}{4}, \quad c_3 = \frac{3}{4}.$$

Hence put values in eq (1)

$$y = \frac{3}{4} e^{x/2} + \frac{1}{4} e^{-x/2} + \frac{1}{8} x^2 e^{x/2} - \frac{1}{4} x e^{x/2}.$$

P. Question Ex # 4.6

Q4, Q8, Q18, Q21