

# Probability—the Science of Uncertainty and Data

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## PROBABILITY

### Probability models and axioms

**Definition (Sample space)** A sample space  $\Omega$  is the set of all possible outcomes. The set's elements must be mutually exclusive, collectively exhaustive and at the right granularity.

**Definition (Event)** An event is a subset of the sample space. Probability is assigned to events.

**Definition (Probability axioms)** A probability law  $\mathbb{P}$  assigns probabilities to events and satisfies the following axioms:

**Nonnegativity**  $\mathbb{P}(A) \geq 0$  for all events  $A$ .

**Normalization**  $\mathbb{P}(\Omega) = 1$ .

**(Countable) additivity** For every sequence of events  $A_1, A_2, \dots$  such that  $A_i \cap A_j = \emptyset$ :  $\mathbb{P}\left(\bigcup_i A_i\right) = \sum_i \mathbb{P}(A_i)$ .

**Corollaries (Consequences of the axioms)**

- $\mathbb{P}(\emptyset) = 0$ .
- For any finite collection of disjoint events  $A_1, \dots, A_n$ ,  
$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mathbb{P}(A_i)$$
.
- $\mathbb{P}(A) + \mathbb{P}(A^c) = 1$ .
- $\mathbb{P}(A) \leq 1$ .
- If  $A \subset B$ , then  $\mathbb{P}(A) \leq \mathbb{P}(B)$ .
- $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ .
- $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$ .

**Example (Discrete uniform law)** Assume  $\Omega$  is finite and consists of  $n$  equally likely elements. Also, assume that  $A \subset \Omega$  with  $k$  elements. Then  $\mathbb{P}(A) = \frac{k}{n}$ .

### Conditioning and Bayes' rule

**Definition (Conditional probability)** Given that event  $B$  has occurred and that  $\mathbb{P}(B) > 0$ , the probability that  $A$  occurs is

$$\mathbb{P}(A|B) \triangleq \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

**Remark (Conditional probabilities properties)** They are the same as ordinary probabilities. Assuming  $\mathbb{P}(B) > 0$ :

- $\mathbb{P}(A|B) \geq 0$ .
- $\mathbb{P}(\Omega|B) = 1$
- $\mathbb{P}(B|B) = 1$ .
- If  $A \cap C = \emptyset$ ,  $\mathbb{P}(A \cup C|B) = \mathbb{P}(A|B) + \mathbb{P}(C|B)$ .

**Proposition (Multiplication rule)**

$$\mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_n) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2|A_1) \cdots \mathbb{P}(A_n|A_1 \cap A_2 \cap \dots \cap A_{n-1}).$$

**Theorem (Total probability theorem)** Given a partition  $\{A_1, A_2, \dots\}$  of the sample space, meaning that  $\bigcup_i A_i = \Omega$  and the events are disjoint, and for every event  $B$ , we have

$$\mathbb{P}(B) = \sum_i \mathbb{P}(A_i) \mathbb{P}(B|A_i).$$

**Theorem (Bayes' rule)** Given a partition  $\{A_1, A_2, \dots\}$  of the sample space, meaning that  $\bigcup_i A_i = \Omega$  and the events are disjoint, and if  $\mathbb{P}(A_i) > 0$  for all  $i$ , then for every event  $B$ , the conditional probabilities  $\mathbb{P}(A_i|B)$  can be obtained from the conditional probabilities  $\mathbb{P}(B|A_i)$  and the initial probabilities  $\mathbb{P}(A_i)$  as follows:

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(A_i) \mathbb{P}(B|A_i)}{\sum_j \mathbb{P}(A_j) \mathbb{P}(B|A_j)}.$$

### Independence

**Definition (Independence of events)** Two events are independent if occurrence of one provides no information about the other. We say that  $A$  and  $B$  are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B).$$

Equivalently, as long as  $\mathbb{P}(A) > 0$  and  $\mathbb{P}(B) > 0$ ,

$$\mathbb{P}(B|A) = \mathbb{P}(B) \quad \mathbb{P}(A|B) = \mathbb{P}(A).$$

**Remarks**

- The definition of independence is symmetric with respect to  $A$  and  $B$ .
- The product definition applies even if  $\mathbb{P}(A) = 0$  or  $\mathbb{P}(B) = 0$ .

**Corollary** If  $A$  and  $B$  are independent, then  $A$  and  $B^c$  are independent. Similarly for  $A^c$  and  $B$ , or for  $A^c$  and  $B^c$ .

**Definition (Conditional independence)** We say that  $A$  and  $B$  are independent conditioned on  $C$ , where  $\mathbb{P}(C) > 0$ , if

$$\mathbb{P}(A \cap B|C) = \mathbb{P}(A|C) \mathbb{P}(B|C).$$

**Definition (Independence of a collection of events)** We say that events  $A_1, A_2, \dots, A_n$  are independent if for every collection of distinct indices  $i_1, i_2, \dots, i_k$ , we have

$$\mathbb{P}(A_{i_1} \cap \dots \cap A_{i_k}) = \mathbb{P}(A_{i_1}) \cdot \mathbb{P}(A_{i_2}) \cdots \mathbb{P}(A_{i_k}).$$

### Counting

This section deals with finite sets with uniform probability law. In this case, to calculate  $\mathbb{P}(A)$ , we need to count the number of elements in  $A$  and in  $\Omega$ .

**Remark (Basic counting principle)** For a selection that can be done in  $r$  stages, with  $n_i$  choices at each stage  $i$ , the number of possible selections is  $n_1 \cdot n_2 \cdots n_r$ .

**Definition (Permutations)** The number of permutations (orderings) of  $n$  different elements is

$$n! = 1 \cdot 2 \cdot 3 \cdots n.$$

**Definition (Combinations)** Given a set of  $n$  elements, the number of subsets with exactly  $k$  elements is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

**Definition (Partitions)** We are given an  $n$ -element set and nonnegative integers  $n_1, n_2, \dots, n_r$ , whose sum is equal to  $n$ . The number of partitions of the set into  $r$  disjoint subsets, with the  $i^{\text{th}}$  subset containing exactly  $n_i$  elements, is equal to

$$\binom{n}{n_1, \dots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}.$$

**Remark** This is the same as counting how to assign  $n$  distinct elements to  $r$  people, giving each person  $i$  exactly  $n_i$  elements.

## Discrete random variables

*Probability mass function and expectation*

**Definition (Random variable)** A random variable  $X$  is a function of the sample space  $\Omega$  into the real numbers (or  $\mathbb{R}^n$ ). Its range can be discrete or continuous.

**Definition (Probability mass function (PMF))** The probability law of a discrete random variable  $X$  is called its PMF. It is defined as

$$p_X(x) = \mathbb{P}(X = x) = \mathbb{P}(\{\omega \in \Omega : X(\omega) = x\}).$$

**Properties**

$$p_X(x) \geq 0, \forall x.$$

$$\sum_x p_X(x) = 1.$$

**Example (Bernoulli random variable)** A Bernoulli random variable  $X$  with parameter  $0 \leq p \leq 1$  ( $X \sim \text{Ber}(p)$ ) takes the following values:

$$X = \begin{cases} 1 & \text{w.p. } p, \\ 0 & \text{w.p. } 1 - p. \end{cases}$$

An indicator random variable of an event ( $I_A = 1$  if  $A$  occurs) is an example of a Bernoulli random variable.

**Example (Discrete uniform random variable)** A Discrete uniform random variable  $X$  between  $a$  and  $b$  with  $a \leq b$  ( $X \sim \text{Uni}[a, b]$ ) takes any of the values in  $\{a, a+1, \dots, b\}$  with probability  $\frac{1}{b-a+1}$ .

**Example (Binomial random variable)** A Binomial random variable  $X$  with parameters  $n$  (natural number) and  $0 \leq p \leq 1$  ( $X \sim \text{Bin}(n, p)$ ) takes values in the set  $\{0, 1, \dots, n\}$  with probabilities  $p_X(i) = \binom{n}{i} p^i (1-p)^{n-i}$ .

It represents the number of successes in  $n$  independent trials where each trial has a probability of success  $p$ . Therefore, it can also be seen as the sum of  $n$  independent Bernoulli random variables, each with parameter  $p$ .

**Example (Geometric random variable)** A Geometric random variable  $X$  with parameter  $0 \leq p \leq 1$  ( $X \sim \text{Geo}(p)$ ) takes values in the set  $\{1, 2, \dots\}$  with probabilities  $p_X(i) = (1-p)^{i-1} p$ . It represents the number of independent trials until (and including) the first success, when the probability of success in each trial is  $p$ .

**Definition (Expectation/mean of a random variable)** The expectation of a discrete random variable is defined as

$$\mathbb{E}[X] \triangleq \sum_x x p_X(x).$$

assuming  $\sum_x |x| p_X(x) < \infty$ .

**Properties (Properties of expectation)**

- If  $X \geq 0$  then  $\mathbb{E}[X] \geq 0$ .
- If  $a \leq X \leq b$  then  $a \leq \mathbb{E}[X] \leq b$ .
- If  $X = c$  then  $\mathbb{E}[X] = c$ .

**Example** Expected value of know r.v.

- If  $X \sim \text{Ber}(p)$  then  $\mathbb{E}[X] = p$ .
- If  $X = I_A$  then  $\mathbb{E}[X] = \mathbb{P}(A)$ .
- If  $X \sim \text{Uni}[a, b]$  then  $\mathbb{E}[X] = \frac{a+b}{2}$ .
- If  $X \sim \text{Bin}(n, p)$  then  $\mathbb{E}[X] = np$ .
- If  $X \sim \text{Geo}(p)$  then  $\mathbb{E}[X] = \frac{1}{p}$ .

**Theorem (Expected value rule)** Given a random variable  $X$  and a function  $g: \mathbb{R} \rightarrow \mathbb{R}$ , we construct the random variable  $Y = g(X)$ . Then

$$\sum_y y p_Y(y) = \mathbb{E}[Y] = \mathbb{E}[g(X)] = \sum_x g(x) p_X(x).$$

**Remark (PMF of  $Y = g(X)$ )** The PMF of  $Y = g(X)$  is  $p_Y(y) = \sum_{x: g(x)=y} p_X(x)$ .

**Remark** In general  $g(\mathbb{E}[X]) \neq \mathbb{E}[g(X)]$ . They are equal if  $g(x) = ax + b$ .

*Variance, conditioning on an event, multiple r.v.*

**Definition (Variance of a random variable)** Given a random variable  $X$  with  $\mu = \mathbb{E}[X]$ , its variance is a measure of the spread of the random variable and is defined as

$$\text{Var}(X) \triangleq \mathbb{E}[(X - \mu)^2] = \sum_x (x - \mu)^2 p_X(x).$$

**Definition (Standard deviation)**

$$\sigma_X = \sqrt{\text{Var}(X)}.$$

**Properties (Properties of the variance)**

- $\text{Var}(aX) = a^2 \text{Var}(X)$ , for all  $a \in \mathbb{R}$ .
- $\text{Var}(X + b) = \text{Var}(X)$ , for all  $b \in \mathbb{R}$ .
- $\text{Var}(aX + b) = a^2 \text{Var}(X)$ .
- $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$ .

**Example (Variance of known r.v.)**

- If  $X \sim \text{Ber}(p)$ , then  $\text{Var}(X) = p(1 - p)$ .
- If  $X \sim \text{Uni}[a, b]$ , then  $\text{Var}(X) = \frac{(b-a)(b-a+2)}{12}$ .
- If  $X \sim \text{Bin}(n, p)$ , then  $\text{Var}(X) = np(1 - p)$ .
- If  $X \sim \text{Geo}(p)$ , then  $\text{Var}(X) = \frac{1-p}{p^2}$ .

**Proposition (Conditional PMF and expectation, given an event)** Given the event  $A$ , with  $\mathbb{P}(A) > 0$ , we have the following

- $p_{X|A}(x) = \mathbb{P}(X = x|A)$ .
- If  $A$  is a subset of the range of  $X$ , then:
$$p_{X|A}(x) \triangleq p_{X|\{X \in A\}}(x) = \begin{cases} \frac{1}{\mathbb{P}(A)} p_X(x), & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$
- $\sum_x p_{X|A}(x) = 1$ .
- $\mathbb{E}[X|A] = \sum_x x p_{X|A}(x)$ .
- $\mathbb{E}[g(X)|A] = \sum_x g(x) p_{X|A}(x)$ .

**Proposition (Total expectation rule)** Given a partition of disjoint events  $A_1, \dots, A_n$  such that  $\sum_i \mathbb{P}(A_i) = 1$ , and  $\mathbb{P}(A_i) > 0$ ,

$$\mathbb{E}[X] = \mathbb{P}(A_1) \mathbb{E}[X|A_1] + \dots + \mathbb{P}(A_n) \mathbb{E}[X|A_n].$$

**Definition (Memorylessness of the geometric random variable)**

When we condition a geometric random variable  $X$  on the event  $X > n$  we have memorylessness, meaning that the “remaining time”  $X - n$ , given that  $X > n$ , is also geometric with the same parameter. Formally,

$$p_{X-n|X>n}(i) = p_X(i).$$

**Definition (Joint PMF)** The joint PMF of random variables

$X_1, X_2, \dots, X_n$  is  $p_{X_1, X_2, \dots, X_n}(x_1, \dots, x_n) = \mathbb{P}(X_1 = x_1, \dots, X_n = x_n)$ .

**Properties (Properties of joint PMF)**

- $\sum_{x_1} \dots \sum_{x_n} p_{X_1, \dots, X_n}(x_1, \dots, x_n) = 1$ .
- $p_{X_1}(x_1) = \sum_{x_2} \dots \sum_{x_n} p_{X_1, \dots, X_n}(x_1, x_2, \dots, x_n)$ .
- $p_{X_2, \dots, X_n}(x_2, \dots, x_n) = \sum_{x_1} p_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n)$ .

**Definition (Functions of multiple r.v.)** If  $Z = g(X_1, \dots, X_n)$ , where  $g: \mathbb{R}^n \rightarrow \mathbb{R}$ , then  $p_Z(z) = \mathbb{P}(g(X_1, \dots, X_n) = z)$ .

**Proposition (Expected value rule for multiple r.v.)** Given  $g: \mathbb{R}^n \rightarrow \mathbb{R}$ ,

$$\mathbb{E}[g(X_1, \dots, X_n)] = \sum_{x_1, \dots, x_n} g(x_1, \dots, x_n) p_{X_1, \dots, X_n}(x_1, \dots, x_n).$$

**Properties (Linearity of expectations)**

- $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$ .
- $\mathbb{E}[X_1 + \dots + X_n] = \mathbb{E}[X_1] + \dots + \mathbb{E}[X_n]$ .

*Conditioning on a random variable, independence*

**Definition (Conditional PMF given another random variable)**

Given discrete random variables  $X, Y$  and  $y$  such that  $p_Y(y) > 0$  we define

$$p_{X|Y}(x|y) \triangleq \frac{p_{X,Y}(x, y)}{p_Y(y)}.$$

**Proposition (Multiplication rule)** Given jointly discrete random variables  $X, Y$ , and whenever the conditional probabilities are defined,

$$p_{X,Y}(x, y) = p_X(x) p_{Y|X}(y|x) = p_Y(y) p_{X|Y}(x|y).$$

**Definition (Conditional expectation)** Given discrete random variables  $X, Y$  and  $y$  such that  $p_Y(y) > 0$  we define

$$\mathbb{E}[X|Y = y] = \sum_x x p_{X|Y}(x|y).$$

Additionally we have

$$\mathbb{E}[g(X)|Y = y] = \sum_x g(x) p_{X|Y}(x|y).$$

**Theorem (Total probability and expectation theorems)**

If  $p_Y(y) > 0$ , then

$$p_X(x) = \sum_y p_Y(y) p_{X|Y}(x|y),$$

$$\mathbb{E}[X] = \sum_y p_Y(y) \mathbb{E}[X|Y = y].$$

**Definition (Independence of a random variable and an event)** A discrete random variable  $X$  and an event  $A$  are independent if  $\mathbb{P}(X = x \text{ and } A) = p_X(x) \mathbb{P}(A)$ , for all  $x$ .

**Definition (Independence of two random variables)** Two discrete random variables  $X$  and  $Y$  are independent if

$$p_{X,Y}(x, y) = p_X(x) p_Y(y) \text{ for all } x, y.$$

**Remark (Independence of a collection of random variables)** A collection  $X_1, X_2, \dots, X_n$  of random variables are independent if

$$p_{X_1, \dots, X_n}(x_1, \dots, x_n) = p_{X_1}(x_1) \dots p_{X_n}(x_n), \quad \forall x_1, \dots, x_n.$$

**Remark (Independence and expectation)** In general,  $\mathbb{E}[g(X, Y)] \neq g(\mathbb{E}[X], \mathbb{E}[Y])$ . An exception is for linear functions:  $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$ .

**Proposition (Expectation of product of independent r.v.)** If  $X$  and  $Y$  are discrete independent random variables,

$$\mathbb{E}[XY] = \mathbb{E}[X] \mathbb{E}[Y].$$

**Remark** If  $X$  and  $Y$  are independent,  $\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)] \mathbb{E}[h(Y)]$ .

**Proposition (Variance of sum of independent random variables)** If  $X$  and  $Y$  are discrete independent random variables,

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y).$$

**Continuous random variables**

*PDF, Expectation, Variance, CDF*

**Definition (Probability density function (PDF))** A probability density function of a r.v.  $X$  is a non-negative real valued function  $f_X$  that satisfies the following

- $\int_{-\infty}^{\infty} f_X(x) dx = 1$ .
- $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x) dx$  for some random variable  $X$ .

**Definition (Continuous random variable)** A random variable  $X$  is continuous if its probability law can be described by a PDF  $f_X$ .

**Remark** Continuous random variables satisfy:

- For small  $\delta > 0$ ,  $\mathbb{P}(a \leq X \leq a + \delta) \approx f_X(a) \delta$ .
- $\mathbb{P}(X = a) = 0, \forall a \in \mathbb{R}$ .

**Definition (Expectation of a continuous random variable)** The expectation of a continuous random variable is

$$\mathbb{E}[X] \triangleq \int_{-\infty}^{\infty} x f_X(x) dx.$$

assuming  $\int_{-\infty}^{\infty} |x| f_X(x) dx < \infty$ .

**Properties (Properties of expectation)**

- If  $X \geq 0$  then  $\mathbb{E}[X] \geq 0$ .
- If  $a \leq X \leq b$  then  $a \leq \mathbb{E}[X] \leq b$ .
- $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$ .
- $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$ .

**Definition (Variance of a continuous random variable)** Given a continuous random variable  $X$  with  $\mu = \mathbb{E}[X]$ , its variance is

$$\text{Var}(X) = \mathbb{E}[(X - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f_X(x) dx.$$

It has the same properties as the variance of a discrete random variable.

**Example (Uniform continuous random variable)** A Uniform continuous random variable  $X$  between  $a$  and  $b$ , with  $a < b$ , ( $X \sim \text{Uni}(a, b)$ ) has PDF

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a < x < b, \\ 0, & \text{otherwise.} \end{cases}$$

We have  $\mathbb{E}[X] = \frac{a+b}{2}$  and  $\text{Var}(X) = \frac{(b-a)^2}{12}$ .

**Example (Exponential random variable)** An Exponential random variable  $X$  with parameter  $\lambda > 0$  ( $X \sim \text{Exp}(\lambda)$ ) has PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

We have  $E[X] = \frac{1}{\lambda}$  and  $\text{Var}(X) = \frac{1}{\lambda^2}$ .

**Definition (Cumulative Distribution Function (CDF))** The CDF of a random variable  $X$  is  $F_X(x) = \mathbb{P}(X \leq x)$ .

In particular, for a continuous random variable, we have

$$F_X(x) = \int_{-\infty}^x f_X(x) dx, \\ f_X(x) = \frac{dF_X(x)}{dx}.$$

**Properties (Properties of CDF)**

- If  $y \geq x$ , then  $F_X(y) \geq F_X(x)$ .
- $\lim_{x \rightarrow -\infty} F_X(x) = 0$ .
- $\lim_{x \rightarrow \infty} F_X(x) = 1$ .

**Definition (Normal/Gaussian random variable)** A Normal random variable  $X$  with mean  $\mu$  and variance  $\sigma^2 > 0$  ( $X \sim \mathcal{N}(\mu, \sigma^2)$ ) has PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}.$$

We have  $E[X] = \mu$  and  $\text{Var}(X) = \sigma^2$ .

**Remark (Standard Normal)** The standard Normal is  $\mathcal{N}(0, 1)$ .

**Proposition (Linearity of Gaussians)** Given  $X \sim \mathcal{N}(\mu, \sigma^2)$ , and if  $a \neq 0$ , then  $aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$ .

Using this  $Y = \frac{X-\mu}{\sigma}$  is a standard gaussian.

*Conditioning on an event, and multiple continuous r.v.*

**Definition (Conditional PDF given an event)** Given a continuous random variable  $X$  and event  $A$  with  $P(A) > 0$ , we define the conditional PDF as the function that satisfies

$$\mathbb{P}(X \in B|A) = \int_B f_{X|A}(x) dx.$$

**Definition (Conditional PDF given  $X \in A$ )** Given a continuous random variable  $X$  and an  $A \subset \mathbb{R}$ , with  $P(A) > 0$ :

$$f_{X|X \in A}(x) = \begin{cases} \frac{1}{\mathbb{P}(A)} f_X(x), & x \in A, \\ 0, & x \notin A. \end{cases}$$

**Definition (Conditional expectation)** Given a continuous random variable  $X$  and an event  $A$ , with  $P(A) > 0$ :

$$\mathbb{E}[X|A] = \int_{-\infty}^{\infty} x f_{X|A}(x) dx.$$

**Definition (Memorylessness of the exponential random variable)**

When we condition an exponential random variable  $X$  on the event  $X > t$  we have memorylessness, meaning that the “remaining time”  $X - t$  given that  $X > t$  is also geometric with the same parameter i.e.,

$$\mathbb{P}(X - t > x | X > t) = \mathbb{P}(X > x).$$

**Theorem (Total probability and expectation theorems)** Given a partition of the space into disjoint events  $A_1, A_2, \dots, A_n$  such that  $\sum_i \mathbb{P}(A_i) = 1$  we have the following:

$$F_X(x) = \mathbb{P}(A_1)F_{X|A_1}(x) + \dots + \mathbb{P}(A_n)F_{X|A_n}(x), \\ f_X(x) = \mathbb{P}(A_1)f_{X|A_1}(x) + \dots + \mathbb{P}(A_n)f_{X|A_n}(x), \\ \mathbb{E}[X] = \mathbb{P}(A_1)\mathbb{E}[X|A_1] + \dots + \mathbb{P}(A_n)\mathbb{E}[X|A_n].$$

**Definition (Jointly continuous random variables)** A pair (collection) of random variables is jointly continuous if there exists a joint PDF  $f_{X,Y}$  that describes them, that is, for every set  $B \subset \mathbb{R}^n$

$$\mathbb{P}((X, Y) \in B) = \iint_B f_{X,Y}(x, y) dx dy.$$

**Properties (Properties of joint PDFs)**

- $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy$ .
- $F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y) = \int_{-\infty}^x \left[ \int_{-\infty}^y f_{X,Y}(u, v) dv \right] du$ .
- $f_{X,Y}(x) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y}$ .

**Example (Uniform joint PDF on a set  $S$ )** Let  $S \subset \mathbb{R}^2$  with area  $s > 0$ , then the random variable  $(X, Y)$  is uniform over  $S$  if it has PDF

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{s}, & (x, y) \in S, \\ 0, & (x, y) \notin S. \end{cases}$$

*Conditioning on a random variable, independence, Bayes' rule*

**Definition (Conditional PDF given another random variable)**

Given jointly continuous random variables  $X, Y$  and a value  $y$  such that  $f_Y(y) > 0$ , we define the conditional PDF as

$$f_{X|Y}(x|y) \triangleq \frac{f_{X,Y}(x, y)}{f_Y(y)}.$$

Additionally we define  $\mathbb{P}(X \in A|Y = y) = \int_A f_{X|Y}(x|y) dx$ .

**Proposition (Multiplication rule)** Given jointly continuous random variables  $X, Y$ , whenever possible we have

$$f_{X,Y}(x, y) = f_X(x)f_{Y|X}(y|x) = f_Y(y)f_{X|Y}(x|y).$$

**Definition (Conditional expectation)** Given jointly continuous random variables  $X, Y$ , and  $y$  such that  $f_Y(y) > 0$ , we define the conditional expected value as

$$\mathbb{E}[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx.$$

Additionally we have

$$\mathbb{E}[g(X)|Y = y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y) dx.$$

**Theorem (Total probability and total expectation theorems)**

$$f_X(x) = \int_{-\infty}^{\infty} f_Y(y) f_{X|Y}(x|y) dy, \\ \mathbb{E}[X] = \int_{-\infty}^{\infty} f_Y(y) \mathbb{E}[X|Y = y] dy.$$

**Definition (Independence)** Jointly continuous random variables  $X, Y$  are independent if  $f_{X,Y}(x, y) = f_X(x)f_Y(y)$  for all  $x, y$ .

**Proposition (Expectation of product of independent r.v.)** If  $X$  and  $Y$  are independent continuous random variables,

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y].$$

**Remark** If  $X$  and  $Y$  are independent,  $\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$ .

**Proposition (Variance of sum of independent random variables)** If  $X$  and  $Y$  are independent continuous random variables,

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y).$$

**Proposition (Bayes' rule summary)**

- For  $X, Y$  discrete:  $p_{X|Y}(x|y) = \frac{p_X(x)p_{Y|X}(y|x)}{p_Y(y)}$ .
- For  $X, Y$  continuous:  $f_{X|Y}(x|y) = \frac{f_X(x)f_{Y|X}(y|x)}{f_Y(y)}$ .
- For  $X$  discrete,  $Y$  continuous:  $p_{X|Y}(x|y) = \frac{p_X(x)f_{Y|X}(y|x)}{f_Y(y)}$ .
- For  $X$  continuous,  $Y$  discrete:  $f_{X|Y}(x|y) = \frac{f_X(x)p_{Y|X}(y|x)}{p_Y(y)}$ .

**Derived distributions**

**Proposition (Discrete case)** Given a discrete random variable  $X$  and a function  $g$ , the r.v.  $Y = g(X)$  has PMF

$$p_Y(y) = \sum_{x:g(x)=y} p_X(x).$$

**Remark (Linear function of discrete random variable)** If  $g(x) = ax + b$ , then  $p_Y(y) = p_X\left(\frac{y-b}{a}\right)$ .

**Proposition (Linear function of continuous r.v.)** Given a continuous random variable  $X$  and  $Y = aX + b$ , with  $a \neq 0$ , we have

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right).$$

**Corollary (Linear function of normal r.v.)** If  $X \sim \mathcal{N}(\mu, \sigma^2)$  and  $Y = aX + b$ , with  $a \neq 0$ , then  $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$ .

**Example (General function of a continuous r.v.)** If  $X$  is a continuous random variable and  $g$  is any function, to obtain the pdf of  $Y = g(X)$  we follow the two-step procedure:

1. Find the CDF of  $Y$ :  $F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(g(X) \leq y)$ .
2. Differentiate the CDF of  $Y$  to obtain the PDF:  $f_Y(y) = \frac{dF_Y(y)}{dy}$ .

**Proposition (General formula for monotonic  $g$ )** Let  $X$  be a continuous random variable and  $g$  a function that is monotonic wherever  $f_X(x) > 0$ . The PDF of  $Y = g(X)$  is given by

$$f_Y(y) = f_X(h(y)) \left| \frac{dh}{dy}(y) \right|,$$

where  $h = g^{-1}$  in the interval where  $g$  is monotonic.

Sums of independent r.v., covariance and correlation

Proposition (Discrete case) Let  $X, Y$  be discrete independent random variables and  $Z = X + Y$ , then the PMF of  $Z$  is

$$p_Z(z) = \sum_x p_X(x)p_Y(z-x).$$

Proposition (Continuous case) Let  $X, Y$  be continuous independent random variables and  $Z = X + Y$ , then the PDF of  $Z$  is

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx.$$

Proposition (Sum of independent normal r.v.) Let  $X \sim \mathcal{N}(\mu_x, \sigma_x^2)$  and  $Y \sim \mathcal{N}(\mu_y, \sigma_y^2)$  independent. Then  $Z = X + Y \sim \mathcal{N}(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$ .

Definition (Covariance) We define the covariance of random variables  $X, Y$  as

$$\text{Cov}(X, Y) \triangleq \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])].$$

Properties (Properties of covariance)

- If  $X, Y$  are independent, then  $\text{Cov}(X, Y) = 0$ .
- $\text{Cov}(X, X) = \text{Var}(X)$ .
- $\text{Cov}(aX + b, Y) = a \text{Cov}(X, Y)$ .
- $\text{Cov}(X, Y + Z) = \text{Cov}(X, Y) + \text{Cov}(X, Z)$ .
- $\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$ .

Proposition (Variance of a sum of r.v.)

$$\text{Var}(X_1 + \dots + X_n) = \sum_i \text{Var}(X_i) + \sum_{i \neq j} \text{Cov}(X_i, X_j).$$

Definition (Correlation coefficient) We define the correlation coefficient of random variables  $X, Y$ , with  $\sigma_X, \sigma_Y > 0$ , as

$$\rho(X, Y) \triangleq \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}.$$

Properties (Properties of the correlation coefficient)

- $-1 \leq \rho \leq 1$ .
- If  $X, Y$  are independent, then  $\rho = 0$ .
- $|\rho| = 1$  if and only if  $X - \mathbb{E}[X] = c(Y - \mathbb{E}[Y])$ .
- $\rho(aX + b, Y) = \text{sign}(a)\rho(X, Y)$ .

Conditional expectation and variance, sum of random number of r.v.

Definition (Conditional expectation as a random variable) Given random variables  $X, Y$  the conditional expectation  $\mathbb{E}[X|Y]$  is the random variable that takes the value  $\mathbb{E}[X|Y = y]$  whenever  $Y = y$ .

Theorem (Law of iterated expectations)

$$\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X].$$

Definition (Conditional variance as a random variable) Given random variables  $X, Y$  the conditional variance  $\text{Var}(X|Y)$  is the random variable that takes the value  $\text{Var}(X|Y = y)$  whenever  $Y = y$ .

Theorem (Law of total variance)

$$\text{Var}(X) = \mathbb{E}[\text{Var}(X|Y)] + \text{Var}(\mathbb{E}[X|Y]).$$

Proposition (Sum of a random number of independent r.v.)

Let  $N$  be a nonnegative integer random variable. Let  $X, X_1, X_2, \dots, X_N$  be i.i.d. random variables. Let  $Y = \sum_i X_i$ . Then

$$\mathbb{E}[Y] = \mathbb{E}[N]\mathbb{E}[X],$$

$$\text{Var}(Y) = \mathbb{E}[N] \text{Var}(X) + (\mathbb{E}[X])^2 \text{Var}(N).$$

CONVERGENCE OF RANDOM VARIABLES

Inequalities, convergence, and the Weak Law of Large Numbers

Theorem (Markov inequality) Given a random variable  $X \geq 0$  and, for every  $a > 0$  we have

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}.$$

Theorem (Chebyshev inequality) Given a random variable  $X$  with  $\mathbb{E}[X] = \mu$  and  $\text{Var}(X) = \sigma^2$ , for every  $\epsilon > 0$  we have

$$\mathbb{P}(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}.$$

Theorem (Weak Law of Large Number (WLLN)) Given a sequence of i.i.d. random variables  $\{X_1, X_2, \dots\}$  with  $\mathbb{E}[X_i] = \mu$  and  $\text{Var}(X_i) = \sigma^2$ , we define

$$M_n = \frac{1}{n} \sum_{i=1}^n X_i,$$

for every  $\epsilon > 0$  we have

$$\lim_{n \rightarrow \infty} \mathbb{P}(|M_n - \mu| \geq \epsilon) = 0.$$

Definition (Convergence in probability) A sequence of random variables  $\{Y_i\}$  converges in probability to the random variable  $Y$  if

$$\lim_{n \rightarrow \infty} \mathbb{P}(|Y_i - Y| \geq \epsilon) = 0,$$

for every  $\epsilon > 0$ .

Properties (Properties of convergence in probability) If  $X_n \rightarrow a$  and  $Y_n \rightarrow b$  in probability, then

- $X_n + Y_n \rightarrow a + b$ .
- If  $g$  is a continuous function, then  $g(X_n) \rightarrow g(a)$ .
- $\mathbb{E}[X_n]$  does not always converge to  $a$ .

The Central Limit Theorem

Theorem (Central Limit Theorem (CLT)) Given a sequence of independent random variables  $\{X_1, X_2, \dots\}$  with  $\mathbb{E}[X_i] = \mu$  and  $\text{Var}(X_i) = \sigma^2$ , we define

$$Z_n = \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^n (X_i - \mu).$$

Then, for every  $z$ , we have

$$\lim_{n \rightarrow \infty} \mathbb{P}(Z_n \leq z) = \mathbb{P}(Z \leq z),$$

where  $Z \sim \mathcal{N}(0, 1)$ .

Corollary (Normal approximation of a binomial) Let  $X \sim \text{Bin}(n, p)$  with  $n$  large. Then  $S_n$  can be approximated by  $Z \sim \mathcal{N}(np, np(1-p))$ .

Remark (De Moivre-Laplace 1/2 approximation) Let  $X \sim \text{Bin}$ , then  $\mathbb{P}(X = i) = \mathbb{P}\left(i - \frac{1}{2} \leq X \leq i + \frac{1}{2}\right)$  and we can use the CLT to approximate the PMF of  $X$ .