Could there be Symplectic Projective Integration?

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August 17, 2018

Two symplectic-like scenarios occur to me.

- The first is when the macro-scale dynamics are nearly those of a 'slow' wave system (Hamiltonian etc), but the micro-scale has enough dissipation to damp fast waves on a micro-scale time. For example, patch simulations of shallow water waves where the micro-scale in patches feels 'turbulent eddy' viscosity.
- The second is where all modes are (near) wave-like, but we want to effectively 'average' over the fast waves to find the slow waves. Maybe DMD would be good here in that it might better detect and 'average'-out the fast waves.

We may need to assume a user can flag a set of 'position' variables p, and a complementary set of 'momentum' variables q.

The most basic problem and the basic scheme (symplectic Euler) is

$$\begin{aligned} \dot p &= -aq\,, & p_{k+1} &= p_k + h(-aq_k), \\ \dot q &= +bp\,, & q_{k+1} &= q_k + hbp_{k+1}\,, \end{aligned}$$

for time-step h.

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- This differential problem has solutions with frequency $\sqrt{ab}$. Over a time-step h the solutions thus have characteristic multiplier of
$$\lambda_{\text{exact}} = e^{\pm i\sqrt{ab}h} = 1 - \tfrac{1}{2}abh^2 \pm i\sqrt{ab}h(1 - \tfrac{1}{6}abh^2) + O(h^4).$$
- The symplectic Euler scheme is semi-implicit:
$$\begin{bmatrix} 1 & 0 \\ -bh & 1 \end{bmatrix} \begin{bmatrix} p_{k+1} \\ q_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & -ah \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_k \\ q_k \end{bmatrix}.$$
Its characteristic multipliers λ then satisfy
$$\begin{vmatrix} \lambda - 1 & ah \\ -bh\lambda & \lambda - 1 \end{vmatrix} = \lambda^2 - (2 - abh^2)\lambda + 1 = 0\,,$$
with solution
$$\lambda = 1 - \tfrac{1}{2}abh^2 \pm i\sqrt{ab}h\sqrt{1 - \tfrac{1}{4}abh^2} = \lambda_{\text{exact}} + O(h^3).$$
So the scheme is globally $O(h^2)$ for such a system.
Further, when $ab \in \mathbb{R}^+$ and time-step $h < \frac{1}{2}\sqrt{ab}$, then the magnitude of the multiplier is beautifully one:
$$|\lambda|^2 = 1 - abh^2 + \tfrac{1}{4}(ab)^2h^4 + abh^2(1 - \tfrac{1}{4}abh^2) = 1\,.$$

More generally, symplectic Euler methods for $\dot{p} = f(p, q)$ and $\dot{q} = g(p, q)$ are
$$\begin{cases} p_{k+1} = p_k + hf(p_{k+1}, q_k) \\ q_{k+1} = q_k + hg(p_{k+1}, q_k) \end{cases} \quad \text{or} \quad \begin{cases} p_{k+1} = p_k + hf(p_k, q_{k+1}) \\ q_{k+1} = q_k + hg(p_k, q_{k+1}) \end{cases}$$
The first method applied to $\dot{p} = \lambda p$ and $\dot{q} = \mu q$ gives a mix of implicit and explicit: $p_{k+1} = \frac{1}{1-h\lambda}p_k$ and $q_{k+1} = (1 + h\mu)q_k$; and complementary formula for the second method. These are locally $O(h^2)$, so globally $O(h)$.

Challenge *Is there a projective integration version of this basic scheme? that also works for weakly dissipative dynamics? when there is not a clean separation between 'position' and 'momentum'?*