Interpolation Example
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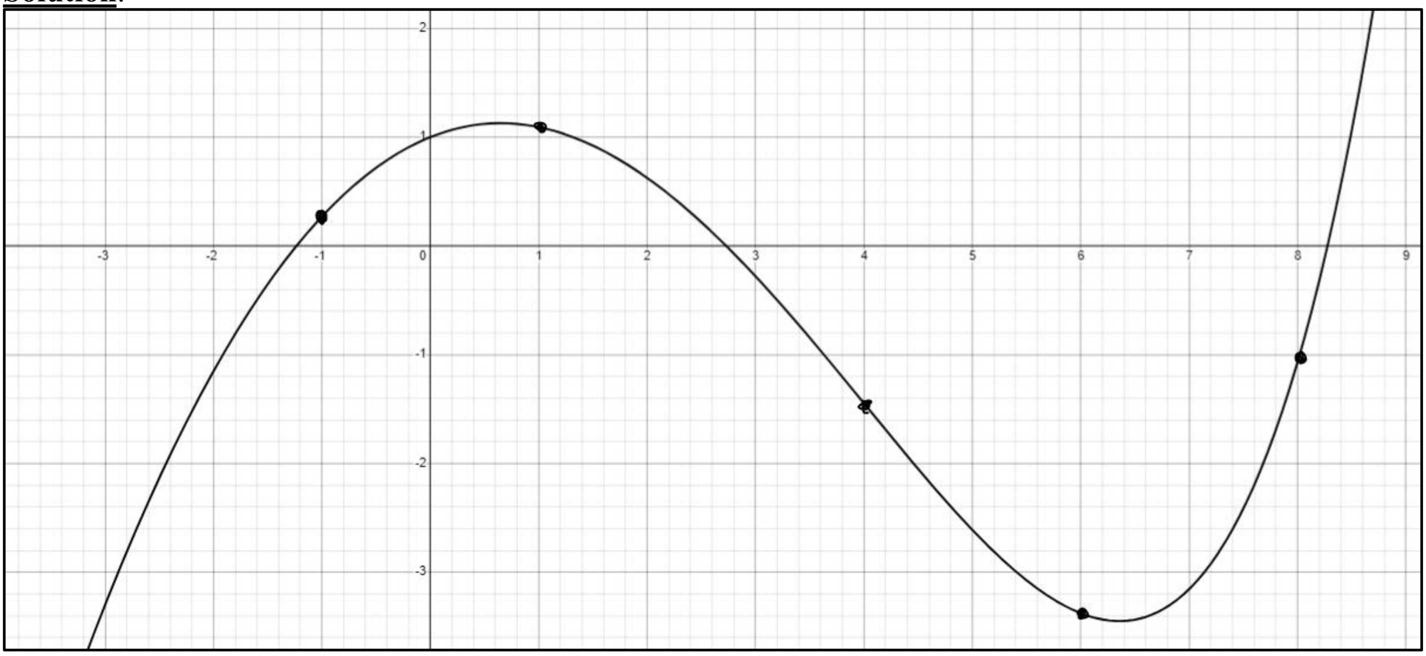
Given a function as
$y = g(x) = e^{0.4x} - 0.4x^2$ (exponential plus quadratic).
Find a polynomial function that interpolates $g(x)$ at $x = -1$, $x = 1$, $x = 4$, $x = 6$, and $x = 8$ by using
A. Newton's Divided-Difference Interpolating Polynomial
B. Lagrange Interpolating Polynomial
C. Linear Splines
D. Cubic Splines (with natural spline condition)

Evaluate the function at $x = 3$

Solution:



From the problem, we have the following data:

$x_0 = -1$	$f(x_0) = 0.27032$
$x_1 = 1$	$f(x_1) = 1.0918247$
$x_2 = 4$	$f(x_2) = -1.4469676$
$x_3 = 6$	$f(x_3) = -3.3768236$
$x_4 = 8$	$f(x_4) = -1.0674698$

A. Newton's Divided-Difference Interpolating Polynomial
Since we have 5 data points, then the polynomial order is 4 ($n = 4$). Accordingly, the polynomial form is expressed as:

$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2) + b_4(x - x_0)(x - x_1)(x - x_2)(x - x_3)$$

Here, our task is to determine the values of b_0 until b_4. Before we calculate them, let's calculate the finite divided differences corresponding to the data points.
First finite divided difference equations:

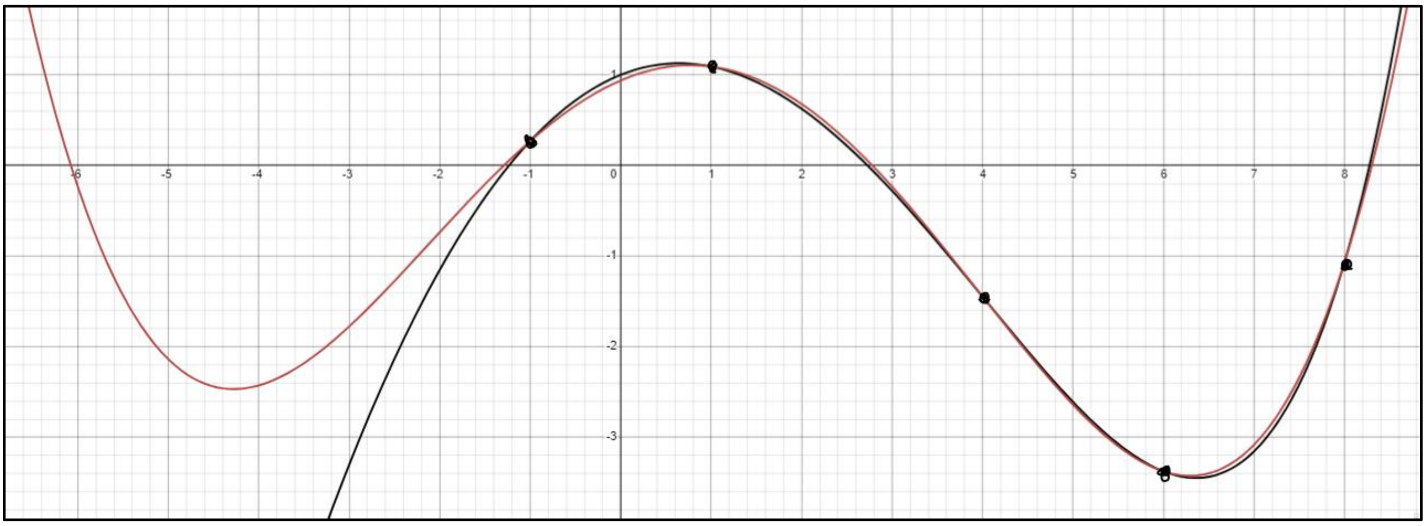
$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1.0918247 - 0.27032}{1 - (-1)} = 0.4107523$$

$$f[x_2, x_1] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{-1.4469676 - 1.0918247}{4 - 1} = -0.8462641$$

$$f[x_3,x_2]=\frac{f(x_3)-f(x_2)}{x_3-x_2}=\frac{-3.3768236-(-1.4469676)}{6-4}=-0.964928$$
$$f[x_4,x_3]=\frac{f(x_4)-f(x_3)}{x_4-x_3}=\frac{-1.0674698-(-3.3768236)}{8-6}=1.1546769$$
Second finite divided difference equations:
$$f[x_2,x_1,x_0]=\frac{f[x_2,x_1]-f[x_1,x_0]}{x_2-x_0}=\frac{-0.8462641-0.4107523}{4-(-1)}=-0.2514033$$
$$f[x_3,x_2,x_1]=\frac{f[x_3,x_2]-f[x_2,x_1]}{x_3-x_1}=\frac{-0.964928-(-0.8462641)}{6-1}=-0.0237328$$
$$f[x_4,x_3,x_2]=\frac{f[x_4,x_3]-f[x_3,x_2]}{x_4-x_2}=\frac{1.1546769-(-0.964928)}{8-4}=0.5299012$$
Third finite divided difference equations:
$$f[x_3,x_2,x_1,x_0]=\frac{f[x_3,x_2,x_1]-f[x_2,x_1,x_0]}{x_3-x_0}=\frac{-0.0237328-(-6.2570164)}{6-(-1)}=0.0325244$$
$$f[x_4,x_3,x_2,x_1]=\frac{f[x_4,x_3,x_2]-f[x_3,x_2,x_1]}{x_4-x_1}=\frac{0.5299012-(-0.0237328)}{8-1}=0.0790906$$
Fourth finite divided difference equations:
$$f[x_4,x_3,x_2,x_1,x_0]=\frac{f[x_4,x_3,x_2,x_1]-f[x_3,x_2,x_1,x_0]}{x_4-x_0}=\frac{0.0790906-0.8904691}{8-(-1)}=0.005174$$

Now, we can compute the values of b_0 until b_4.
$b_0=f(x_0)=0.27032$
$b_1=f[x_1,x_0]=0.4107523$
$b_2=f[x_2,x_1,x_0]=-0.2514033$
$b_3=f[x_3,x_2,x_1,x_0]=0.0325244$
$b_4=f[x_4,x_3,x_2,x_1,x_0]=0.005174$

Therefore, the polynomial function that fit data given in the problem is stated as
$$\begin{aligned}f(x)=&\ 0.27032+0.4107523(x-x_0)-0.2514033(x-x_0)(x-x_1)\\&+0.0325244(x-x_0)(x-x_1)(x-x_2)\\&+0.005174(x-x_0)(x-x_1)(x-x_2)(x-x_3)\end{aligned}$$
$$\begin{aligned}f(x)=&\ 0.27032+0.4107523(x+1)-0.2514033(x+1)(x-1)\\&+0.0325244(x+1)(x-1)(x-4)\\&+0.005174(x+1)(x-1)(x-4)(x-6)\end{aligned}$$



The figure above shows the plot of original function $g(x)$ (black curve) and the interpolating polynomial function $f(x)$ (red curve) found by using the Newton's divided-difference method. The conclusion of the figure:

- $g(x)$ and $f(x)$ intersect at $x = -1$, $x = 1$, $x = 4$, $x = 6$, and $x = 8$.
- Within interval $-1 \le x \le 8$, the curve of $g(x)$ and $f(x)$ is near.
- Within interval $x < -1$ or $x > 8$, we cannot guarantee that $g(x)$ and $f(x)$ is near.

The value of the function at $x = 3$

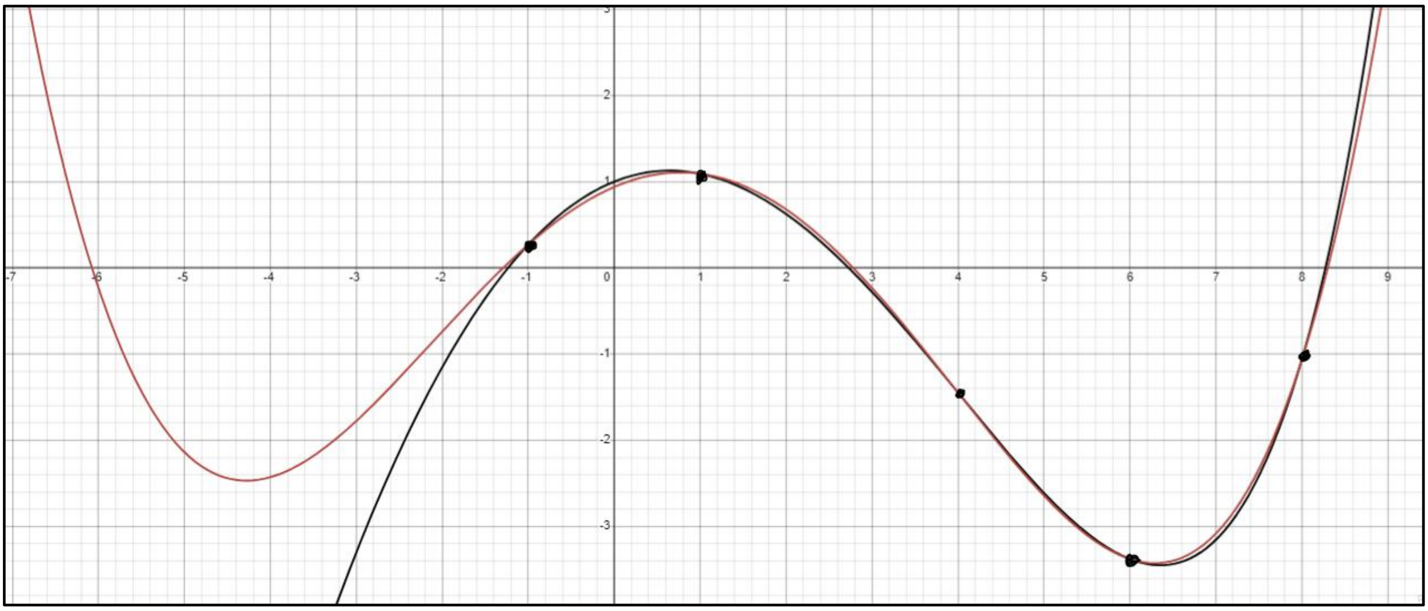
$$\begin{aligned} f(x) = {} & 0.27032 + 0.4107523(3+1) - 0.2514033(3+1)(3-1) \\ & +0.0325244(3+1)(3-1)(3-4) \\ & +0.005174(3+1)(3-1)(3-4)(3-6) \\ = {} & -0.23389 \end{aligned}$$

B. Lagrange Interpolating Polynomial

By using Lagrange Interpolating Polynomial, the polynomial function that fit with 5 data points (order 4) is expressed as

$$\begin{aligned} f(x) = {} & \frac{(x-x_1)}{(x_0-x_1)}\frac{(x-x_2)}{(x_0-x_2)}\frac{(x-x_3)}{(x_0-x_3)}\frac{(x-x_4)}{(x_0-x_4)}f(x_0) + \frac{(x-x_0)}{(x_1-x_0)}\frac{(x-x_2)}{(x_1-x_2)}\frac{(x-x_3)}{(x_1-x_3)}\frac{(x-x_4)}{(x_1-x_4)}f(x_1) \\ & + \frac{(x-x_0)}{(x_2-x_0)}\frac{(x-x_1)}{(x_2-x_1)}\frac{(x-x_3)}{(x_2-x_3)}\frac{(x-x_4)}{(x_2-x_4)}f(x_2) + \frac{(x-x_0)}{(x_3-x_0)}\frac{(x-x_1)}{(x_3-x_1)}\frac{(x-x_2)}{(x_3-x_2)}\frac{(x-x_4)}{(x_3-x_4)}f(x_3) \\ & + \frac{(x-x_0)}{(x_4-x_0)}\frac{(x-x_1)}{(x_4-x_1)}\frac{(x-x_2)}{(x_4-x_2)}\frac{(x-x_3)}{(x_4-x_3)}f(x_4) \\ = {} & \frac{(x-1)}{(-1-1)}\frac{(x-4)}{(-1-4)}\frac{(x-6)}{(-1-6)}\frac{(x-8)}{(-1-8)}0.27032 + \frac{(x+1)}{(1+1)}\frac{(x-4)}{(1-4)}\frac{(x-6)}{(1-6)}\frac{(x-8)}{(1-8)}1.0918247 \\ & + \frac{(x+1)}{(4+1)}\frac{(x-1)}{(4-1)}\frac{(x-6)}{(4-6)}\frac{(x-8)}{(4-8)}(-1.4469676) + \frac{(x+1)}{(6+1)}\frac{(x-1)}{(6-1)}\frac{(x-4)}{(6-4)}\frac{(x-8)}{(6-8)}(-3.3768236) \\ & + \frac{(x+1)}{(8+1)}\frac{(x-1)}{(8-1)}\frac{(x-4)}{(8-4)}\frac{(x-6)}{(8-6)}(-1.0674698) \end{aligned}$$

$$\begin{aligned} f(x) = {} & 0.0004290793(x-1)(x-4)(x-6)(x-8) - 0.005199165(x+1)(x-4)(x-6)(x-8) \\ & -0.01205806(x+1)(x-1)(x-6)(x-8) + 0.02412016(x+1)(x-1)(x-4)(x-8) \\ & -0.002117996(x+1)(x-1)(x-4)(x-6) \end{aligned}$$



The figure above shows the plot of original function $g(x)$ (black curve) and the interpolating polynomial function $f(x)$ (red curve) found by using the Lagrange method. As seen, $f(x)$ obtained by using the Lagrange method is the same as $f(x)$ obtained by using the Newton's Divided-Difference method (see the previous figure).

The value of the function at $x = 3$

$$
\begin{aligned}
f(x) &= 0.0004290793(3-1)(3-4)(3-6)(3-8) - 0.005199165(3+1)(3-4)(3-6)(3-8) \\
&\quad -0.01205806(3+1)(3-1)(3-6)(3-8) + 0.02412016(3+1)(3-1)(3-4)(3-8) \\
&\quad -0.002117996(3+1)(3-1)(3-4)(3-6) \\
&= -0.2339152
\end{aligned}
$$

C. Linear Splines

Interpolating using Splines means we need to create a function for each interval of data points. Since the problem gives us 5 data points, then there are 4 four intervals with different equations.
The general equation form for interval *i* by using linear splines is

$$f_i(x) = f(x_{i-1}) + \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}(x - x_{i-1})$$

Interval 1 $(x_0 \le x \le x_1)$, $i = 1$:

$x_0 = -1$	$f(x_0) = 0.27032$
$x_1 = 1$	$f(x_1) = 1.0918247$

$$
\begin{aligned}
f(x) &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) = 0.27032 + \frac{1.0918247 - 0.27032}{1-(-1)}\big(x - (-1)\big) \\
&= 0.27032 + 0.4107524(x+1); \qquad -1 \le x \le 1
\end{aligned}
$$

Interval 2 $(x_1 \le x \le x_2)$, $i = 2$:

$x_1 = 1$	$f(x_1) = 1.0918247$
$x_2 = 4$	$f(x_2) = -1.4469676$

$$
\begin{aligned}
f(x) &= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) = 1.0918247 + \frac{-1.4469676 - 1.0918247}{4-1}(x - 1) \\
&= 1.0918247 - 0.8462641(x-1); \qquad 1 \le x \le 4
\end{aligned}
$$

Interval 3 $(x_2 \le x \le x_3)$, $i = 3$:

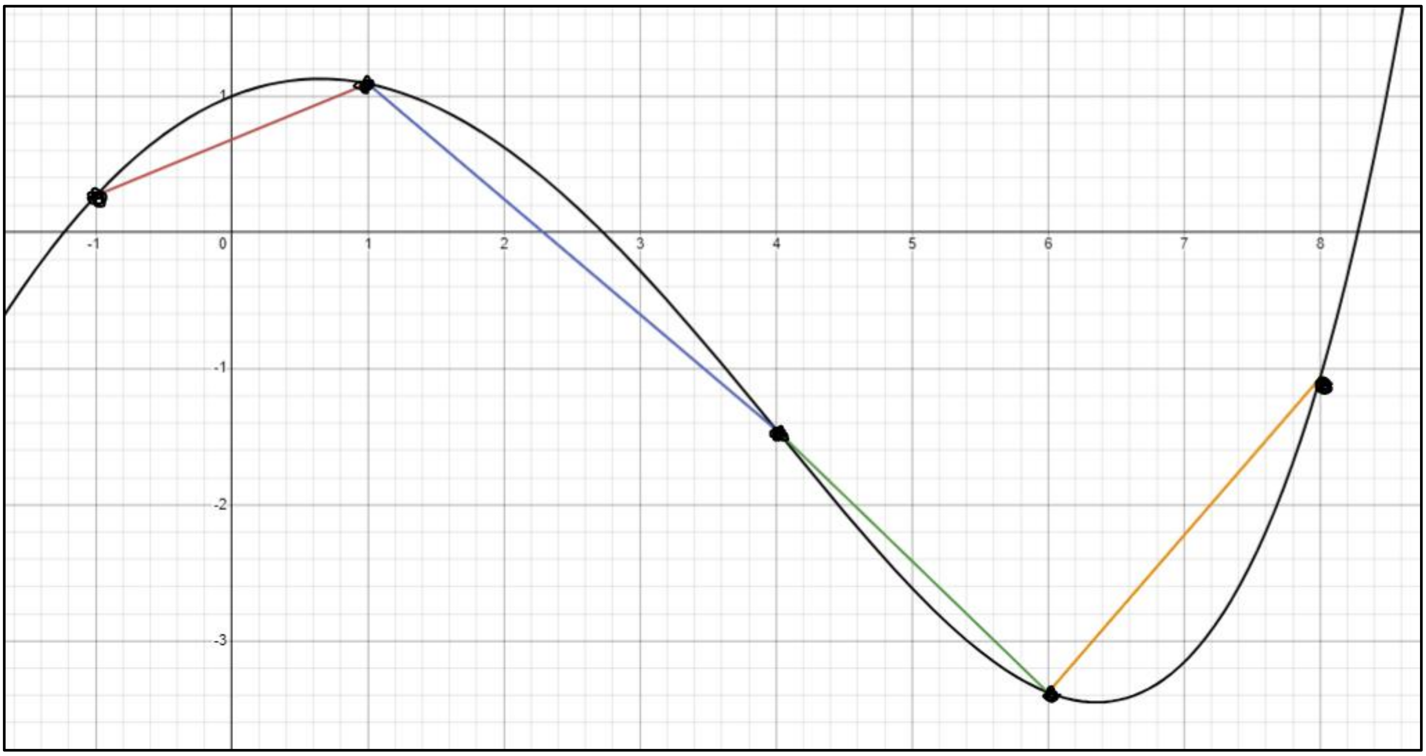
$x_2 = 4$	$f(x_2) = -1.4469676$
$x_3 = 6$	$f(x_3) = -3.3768236$

$$
\begin{aligned}
f(x) &= f(x_2) + \frac{f(x_3) - f(x_2)}{x_3 - x_2}(x - x_2) = -1.4469676 + \frac{-3.3768236 - (-1.4469676)}{6-4}(x - 4) \\
&= -1.4469676 - 0.964928(x-4); \qquad 4 \le x \le 6
\end{aligned}
$$

Interval 4 $(x_3 \le x \le x_4)$, $i = 4$:

$x_3 = 6$	$f(x_3) = -3.3768236$
$x_4 = 8$	$f(x_4) = -1.0674698$

$$
\begin{aligned}
f(x) &= f(x_3) + \frac{f(x_4) - f(x_3)}{x_4 - x_3}(x - x_3) = -3.3768236 + \frac{-1.0674698 - (-3.3768236)}{8-6}(x - 6) \\
&= -3.3768236 + 1.1546769(x-6); \qquad 6 \le x \le 8
\end{aligned}
$$



The value of the function at $x = 3$ can be obtained by using interval 2 $(1 \le x \le 4)$

$f(x) = 1.0918247 - 0.8462641(3 - 1) =- 0.6007035$

D. Cubic Splines

By using cubic splines, the equation at each interval i is given by:

$$f_i(x) = \frac{f''(x_{i-1})}{6(x_i - x_{i-1})}(x_i - x)^3 + \frac{f''(x_i)}{6(x_i - x_{i-1})}(x - x_{i-1})^3 + \left[\frac{f(x_{i-1})}{x_i - x_{i-1}} - \frac{f''(x_{i-1})(x_i - x_{i-1})}{6}\right](x_i - x) + \left[\frac{f(x_i)}{x_i - x_{i-1}} - \frac{f''(x_i)(x_i - x_{i-1})}{6}\right](x - x_{i-1})$$

Note that the values of x_{i-1}, x_i, $f(x_{i-1})$, $f(x_i)$ are given by the problem. This equation contains only two unknowns, $f''(x_{i-1})$ and $f''(x_i)$, which can be found by evaluating the following equation:

$$(x_i - x_{i-1})f''(x_{i-1}) + 2(x_{i+1} - x_{i-1})f''(x_i) + (x_{i+1} - x_i)f''(x_{i+1}) = \frac{6}{x_{i+1} - x_i}[f(x_{i+1}) - f(x_i)] + \frac{6}{x_i - x_{i-1}}[f(x_{i-1}) - f(x_i)]$$

Let's re-write the data given by the problem:

$x_0 =- 1$	$f(x_0) = 0.27032$
$x_1 = 1$	$f(x_1) = 1.0918247$
$x_2 = 4$	$f(x_2) =- 1.4469676$
$x_3 = 6$	$f(x_3) =- 3.3768236$
$x_4 = 8$	$f(x_4) =- 1.0674698$

- From the case $i = 1$:

  $$(x_1 - x_0)f''(x_0) + 2(x_2 - x_0)f''(x_1) + (x_2 - x_1)f''(x_2) = \frac{6}{x_2 - x_1}[f(x_2) - f(x_1)] + \frac{6}{x_1 - x_0}[f(x_0) - f(x_1)]$$

  Then,

  $$(1 -- 1)f''(x_0) + 2(4 -- 1)f''(x_1) + (4 - 1)f''(x_2) = \frac{6}{4 - 1}[-1.4469676 - 1.0918247] + \frac{6}{1 -- 1}[0.27032 - 1.0918247]$$

  Then,

$2f''(x_0) + 10f''(x_1) + 3f''(x_2) = -7.5420987$ eq. (1)

- From the case $i = 2$:
  $(x_2 - x_1)f''(x_1) + 2(x_3 - x_1)f''(x_2) + (x_3 - x_2)f''(x_3)$
  $= \frac{6}{x_3 - x_2}[f(x_3) - f(x_2)] + \frac{6}{x_2 - x_1}[f(x_1) - f(x_2)]$
  Then,
  $(4 - 1)f''(x_1) + 2(6 - 1)f''(x_2) + (6 - 4)f''(x_3)$
  $= \frac{6}{6 - 4}[-3.3768236 -- 1.4469676] + \frac{6}{4 - 1}[1.0918247 -- 1.4469676]$
  Then,
  $3f''(x_1) + 10f''(x_2) + 2f''(x_3) = -0.7119834$ eq. (2)

- From the case $i = 3$:
  $(x_3 - x_2)f''(x_2) + 2(x_4 - x_2)f''(x_3) + (x_4 - x_3)f''(x_4)$
  $= \frac{6}{x_4 - x_3}[f(x_4) - f(x_3)] + \frac{6}{x_3 - x_2}[f(x_2) - f(x_3)]$
  Then,
  $(6 - 4)f''(x_2) + 2(8 - 4)f''(x_3) + (8 - 6)f''(x_4)$
  $= \frac{6}{8 - 6}[-1.0674698 -- 3.3768236] + \frac{6}{6 - 4}[-1.4469676 -- 3.3768236]$
  Then,
  $2f''(x_2) + 8f''(x_3) + 2f''(x_4) = 12.7176294$ eq. (3)
- Because of the natural spline condition, the second derivatives of the first and last points are zero, *i.e.*,
  $f''(x_0) = 0$ eq. (4)
  $f''(x_4) = 0$ eq. (5)

Equation $(1) - (5)$ form a system of linear equation, where the variables are $f''(x_0)$ until $f''(x_4)$. Let's put them together in one place.
$f''(x_0) = 0$
$2f''(x_0) + 10f''(x_1) + 3f''(x_2) = -7.5420987$
$3f''(x_1) + 10f''(x_2) + 2f''(x_3) = -0.7119834$
$2f''(x_2) + 8f''(x_3) + 2f''(x_4) = 12.7176294$
$f''(x_4) = 0$

We can use any methods to find the solution, such as Gaussian elimination and substitution, Gauss-Jordan Elimination, etc. The details are omitted here. The solution to above system of linear equations is:
$f''(x_0) = 0$, $f''(x_1) = -0.69739262$, $f''(x_2) = -0.18939083$, $f''(x_3) = 1.63705138$, $f''(x_4) = 0$

Substituting this result to $f_i(x)$ yields the interpolation functions as follows:

Interval 1 $(x_0 \le x \le x_1)$, $i = 1$:

$$f_1(x) = \frac{f''(x_0)}{6(x_1 - x_0)}(x_1 - x)^3 + \frac{f''(x_1)}{6(x_1 - x_0)}(x - x_0)^3 + \left[\frac{f(x_0)}{x_1 - x_0} - \frac{f''(x_0)(x_1 - x_0)}{6}\right](x_1 - x) + \left[\frac{f(x_1)}{x_1 - x_0} - \frac{f''(x_1)(x_1 - x_0)}{6}\right](x - x_0)$$

$$= \frac{0}{6(1 -- 1)}(1 - x)^3 + \frac{-0.69739262}{6(1 -- 1)}(x -- 1)^3 + \left[\frac{0.27032}{1 -- 1} - \frac{0(1 -- 1)}{6}\right](1 - x) + \left[\frac{1.0918247}{1 -- 1} - \frac{-0.69739262(1 -- 1)}{6}\right](x -- 1)$$

$$= -0.05811605(x + 1)^3 + 0.13516(1 - x) + 0.7783766(x + 1)$$

$$= -0.05811605(x+1)^3 + 0.6432165x + 0.9135366$$

Interval 2 ($x_1 \le x \le x_2$), $i = 2$:

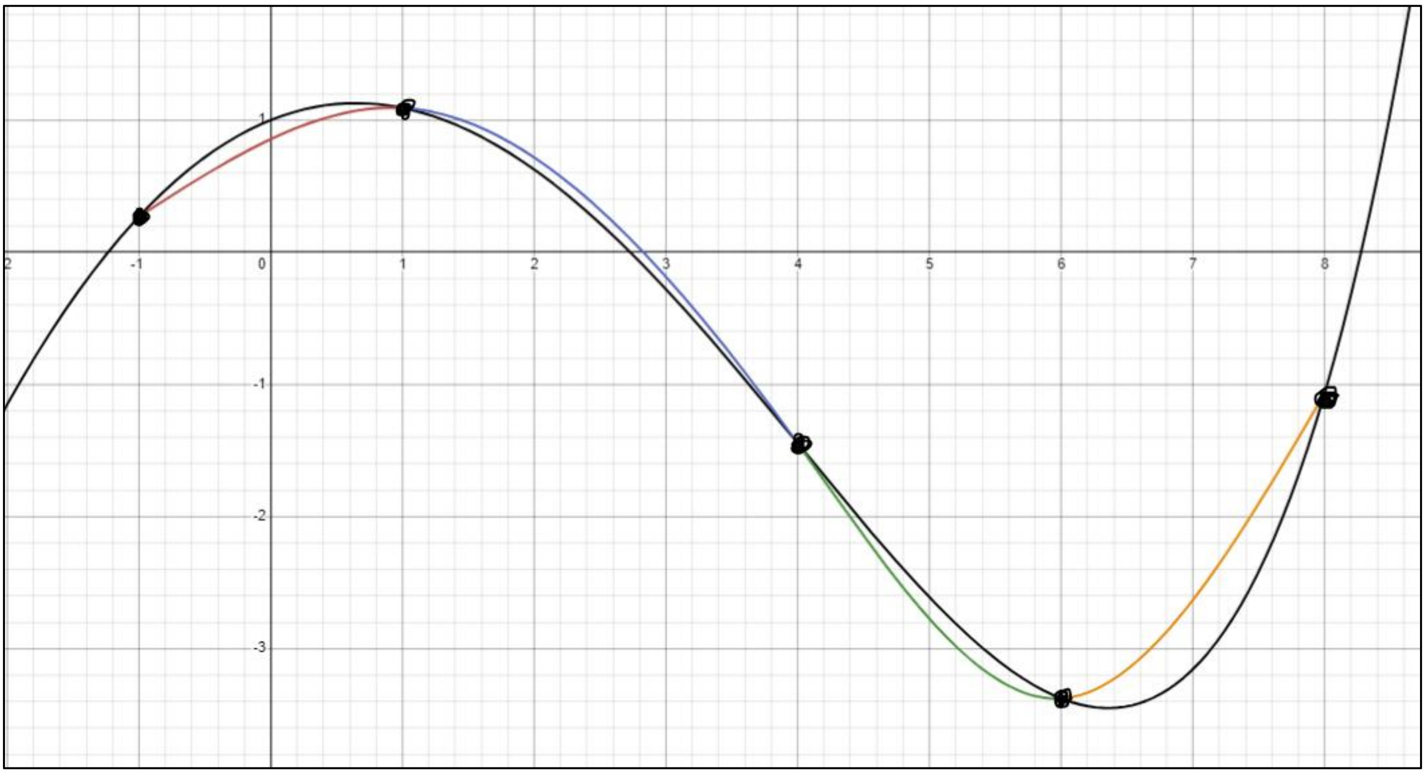
$$f_2(x) = \frac{f''(x_1)}{6(x_2-x_1)}(x_2-x)^3 + \frac{f''(x_2)}{6(x_2-x_1)}(x-x_1)^3 + \left[\frac{f(x_1)}{x_2-x_1} - \frac{f''(x_1)(x_2-x_1)}{6}\right](x_2-x)$$
$$+ \left[\frac{f(x_2)}{x_2-x_1} - \frac{f''(x_2)(x_2-x_1)}{6}\right](x-x_1)$$
$$= \frac{-0.69739262}{6(4-1)}(4-x)^3 + \frac{-0.18939083}{6(4-1)}(x-1)^3$$
$$+ \left[\frac{1.0918247}{4-1} - \frac{-0.69739262(4-1)}{6}\right](4-x)$$
$$+ \left[\frac{-1.4469676}{4-1} - \frac{-0.18939083(4-1)}{6}\right](x-1)$$
$$= -0.03874403(4-x)^3 - 0.01052171(x-1)^3 + 0.7126379(4-x) - 0.3876271(x-1)$$

Interval 3 ($x_2 \le x \le x_3$), $i = 3$:

$$f_3(x) = \frac{f''(x_2)}{6(x_3-x_2)}(x_3-x)^3 + \frac{f''(x_3)}{6(x_3-x_2)}(x-x_2)^3 + \left[\frac{f(x_2)}{x_3-x_2} - \frac{f''(x_2)(x_3-x_2)}{6}\right](x_3-x)$$
$$+ \left[\frac{f(x_3)}{x_3-x_2} - \frac{f''(x_3)(x_3-x_2)}{6}\right](x-x_2)$$
$$= \frac{-0.18939083}{6(6-4)}(6-x)^3 + \frac{1.63705138}{6(6-4)}(x-4)^3 + \left[\frac{-1.4469676}{6-4} - \frac{-0.18939083(6-4)}{6}\right](6-x)$$
$$+ \left[\frac{-3.3768236}{6-4} - \frac{1.63705138(6-4)}{6}\right](x-4)$$
$$= -0.01578257(6-x)^3 + 0.1364209(x-4)^3 - 0.6603535(6-x) - 2.2340956(x-4)$$

Interval 4 ($x_3 \le x \le x_4$), $i = 4$:

$$f_4(x) = \frac{f''(x_3)}{6(x_4-x_3)}(x_4-x)^3 + \frac{f''(x_4)}{6(x_4-x_3)}(x-x_3)^3 + \left[\frac{f(x_3)}{x_4-x_3} - \frac{f''(x_3)(x_4-x_3)}{6}\right](x_4-x)$$
$$+ \left[\frac{f(x_4)}{x_4-x_3} - \frac{f''(x_4)(x_4-x_3)}{6}\right](x-x_3)$$
$$= \frac{1.63705138}{6(8-6)}(8-x)^3 + \frac{0}{6(8-6)}(x-6)^3 + \left[\frac{-3.3768236}{8-6} - \frac{1.63705138(8-6)}{6}\right](8-x)$$
$$+ \left[\frac{-1.0674698}{8-6} - \frac{0(8-6)}{6}\right](x-6)$$
$$= 0.1364209(8-x)^3 - 2.2340956(8-x) - 0.5337349(x-6)$$



The value of the function at $x = 3$ can be obtained by using interval 2 $(1 \leq x \leq 4)$

$$f_2(3) = -0.03874403(4-3)^3 - 0.01052171(3-1)^3 + 0.7126379(4-3) - 0.3876271(3-1)$$
$$= -0.185534$$