Numerical Differentiation Example
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Given the following function:
$f(x) = 2e^{0.5x}$

Find the value of $f'(x)$ and $f''(x)$ at $x = 3$ by using

By using:
a) An analytical approach
b) Forward finite-divided-difference with $h = 0.1$, $h = 0.01$, and $h = 0.001$ by using the two approximation with $\mathcal{O}(h)$ and $\mathcal{O}(h^2)$
c) Backward finite-divided-difference with $h = 0.1$, $h = 0.01$, and $h = 0.001$ by using the two approximation with $\mathcal{O}(h)$ and $\mathcal{O}(h^2)$
d) Centered finite-divided-difference with $h = 0.1$, $h = 0.01$, and $h = 0.001$ by using the two approximation with $\mathcal{O}(h)$ and $\mathcal{O}(h^2)$

Write a Python program to implement the above problem and print its output.

Solution:
a) $f'(x) = 2 \cdot 0.5 \cdot e^{0.5x} = e^{0.5x}$
$f'(3) = e^{0.5 \cdot 3} = 4.4816890703380645$
$f''(x) = 0.5e^{0.5x}$
$f''(3) = 0.5e^{0.5 \cdot 3} = 2.2408445351690323$
b) Forward finite-divided-difference
First derivative:
- With $\mathcal{O}(h)$:

$$h = 0.1 \ \rightarrow f'(3) \approx \frac{f(3.1) - f(3)}{0.1} = \mathbf{4.}595622245053548$$

$$h = 0.01 \ \rightarrow f'(3) \approx \frac{f(3.01) - f(3)}{0.01} = \mathbf{4.4}92911990083748$$

$$h = 0.001 \ \rightarrow f'(3) \approx \frac{f(3.001) - f(3)}{0.001} = \mathbf{4.48}28096793665395$$

- With $\mathcal{O}(h^2)$:

$$h = 0.1 \ \rightarrow f'(3) \approx \frac{-f(3.2) + 4 \cdot f(3.1) - 3 \cdot f(3)}{0.2} = \mathbf{4.4}778109495365825$$

$$h = 0.01 \ \rightarrow f'(3) \approx \frac{-f(3.02) + 4 \cdot f(3.01) - 3 \cdot f(3)}{0.02} = \mathbf{4.4816}51582548629$$

$$h = 0.001 \ \rightarrow f'(3) \approx \frac{-f(3.002) + 4 \cdot f(3.001) - 3 \cdot f(3)}{0.002} = \mathbf{4.48168}8696724717$$

Second derivative:
- With $\mathcal{O}(h)$:

$$h = 0.1 \ \rightarrow f''(3) \approx \frac{f(3.2) - 2 \cdot f(3.1) + f(3)}{0.1^2} = \mathbf{2.}3562259103391265$$

$$h = 0.01 \ \rightarrow f''(3) \approx \frac{f(3.02) - 2 \cdot f(3.01) + f(3)}{0.01^2} = \mathbf{2.2}52081507005954$$

$$h = 0.001 \ \rightarrow f''(3) \approx \frac{f(3.002) - 2 \cdot f(3.001) + f(3)}{0.001^2} = \mathbf{2.24}19652836447312$$

- With $\mathcal{O}(h^2)$:

$$h = 0.1 \ \rightarrow f''(3) \approx \frac{-f(3.3) + 4 \cdot f(3.2) - 5 \cdot f(3.1) + 2 \cdot f(3)}{0.1^2} = \mathbf{2.2}354196246062936$$

$$h = 0.01 \rightarrow f''(3) \approx \frac{-f(3.03) + 4 \cdot f(3.02) - 5 \cdot f(3.01) + 2 \cdot f(3)}{0.01^2}$$
$$= \mathbf{2.240}7929015244576$$
$$h = 0.001 \rightarrow f''(3) \approx \frac{-f(3.003) + 4 \cdot f(3.002) - 5 \cdot f(3.001) + 2 \cdot f(3)}{0.001^2}$$
$$= \mathbf{2.240844}022338706$$

c) Backward finite-divided-difference

First derivative:

- With $\mathcal{O}(h)$:

$$h = 0.1 \rightarrow f'(3) \approx \frac{f(3) - f(2.9)}{0.1} = \mathbf{4.}371491103384955$$
$$h = 0.01 \rightarrow f'(3) \approx \frac{f(3.) - f(2.99)}{0.01} = \mathbf{4.4}70503498047762$$
$$h = 0.001 \rightarrow f'(3) \approx \frac{f(3) - f(2.999)}{0.001} = \mathbf{4.48}0568834782872$$

- With $\mathcal{O}(h^2)$:

$$h = 0.1 \rightarrow f'(3) \approx \frac{3 \cdot f(3) - 4 \cdot f(2.9) + f(2.8)}{0.2} = \mathbf{4.4}78091171836018$$
$$h = 0.01 \rightarrow f'(3) \approx \frac{3 \cdot f(3) - 4 \cdot f(2.99) + f(2.98)}{0.02} = \mathbf{4.4816}51862655589$$
$$h = 0.001 \rightarrow f'(3) \approx \frac{3 \cdot f(3) - 4 \cdot f(2.999) + f(2.998)}{0.002} = \mathbf{4.48168}8697002717$$

Second derivative:

- With $\mathcal{O}(h)$:

$$h = 0.1 \rightarrow f''(3) \approx \frac{f(3) - 2 \cdot f(2.9) + f(2.8)}{0.1^2} = \mathbf{2.}1320013690210966$$
$$h = 0.01 \rightarrow f''(3) \approx \frac{f(3) - 2 \cdot f(2.99) + f(2.98)}{0.01^2} = \mathbf{2.2}29672921547632$$
$$h = 0.001 \rightarrow f''(3) \approx \frac{f(3) - 2 \cdot f(2.99) + f(2.98)}{0.001^2} = \mathbf{2.2}397244379135373$$

- With $\mathcal{O}(h^2)$:

$$h = 0.1 \rightarrow f''(3) \approx \frac{2 \cdot f(3) - 5 \cdot f(2.9) + 4 \cdot f(2.8) - f(2.7)}{0.1^2} = \mathbf{2.2}359803027539367$$
$$h = 0.01 \rightarrow f''(3) \approx \frac{2 \cdot f(3) - 5 \cdot f(2.99) + 4 \cdot f(2.98) - f(2.97)}{0.01^2} = \mathbf{2.240}793461574242$$
$$h = 0.001 \rightarrow f''(3) \approx \frac{2 \cdot f(3) - 5 \cdot f(2.999) + 4 \cdot f(2.998) - f(2.997)}{0.001^2}$$
$$= \mathbf{2.240844}0170096355$$

d) Centered finite-divided-difference

First derivative:

- With $\mathcal{O}(h)$:

$$h = 0.1 \rightarrow f'(3) \approx \frac{f(3.1) - f(2.9)}{0.2} = \mathbf{4.48}3556674219251$$
$$h = 0.01 \rightarrow f'(3) \approx \frac{f(3.01) - f(2.99)}{0.02} = \mathbf{4.481}707744065755$$
$$h = 0.001 \rightarrow f'(3) \approx \frac{f(3.001) - f(2.999)}{0.002} = \mathbf{4.481689}257074706$$

- With $\mathcal{O}(h^2)$:

$$h = 0.1 \rightarrow f'(3) \approx \frac{-f(3.2) + 8 \cdot f(3.1) - 8 \cdot f(2.9) + f(2.8)}{1.2} = \mathbf{4.48168}8136374941$$

$$h = 0.01 \rightarrow f'(3) \approx \frac{-f(3.02) + 8 \cdot f(3.01) - 8 \cdot f(2.99) + f(2.98)}{0.12} = \mathbf{4.481689070}24451$$

$$h = 0.001 \rightarrow f'(3) \approx \frac{-f(3.002) + 8 \cdot f(3.001) - 8 \cdot f(2.999) + f(2.998)}{0.012}$$
$$= \mathbf{4.48168907033}7709$$

Second derivative:

- With $\mathcal{O}(h)$:

$$h = 0.1 \rightarrow f''(3) \approx \frac{f(3.1) - 2 \cdot f(3) + f(2.9)}{0.1^2} = \mathbf{2.24}1311416685931$$

$$h = 0.01 \rightarrow f''(3) \approx \frac{f(3.01) - 2 \cdot f(3) + f(2.99)}{0.01^2} = \mathbf{2.24084}92035985716$$

$$h = 0.001 \rightarrow f''(3) \approx \frac{f(3.001) - 2 \cdot f(3) + f(2.999)}{0.001^2} = \mathbf{2.2408445}836674673$$

- With $\mathcal{O}(h^2)$:

$$h = 0.1 \rightarrow f''(3) \approx \frac{-f(3.2) + 16 \cdot f(3.1) - 30 \cdot f(3) + 16 \cdot f(2.9) - f(2.8)}{12 \cdot 0.1^2}$$
$$= \mathbf{2.240844}3795201456$$

$$h = 0.01 \rightarrow f''(3) \approx \frac{-f(3.02) + 16 \cdot f(3.01) - 30 \cdot f(3) + 16 \cdot f(2.99) - f(2.98)}{12 \cdot 0.01^2}$$
$$= \mathbf{2.2408445351}581223$$

$$h = 0.001 \rightarrow f''(3) \approx \frac{-f(3.002) + 16 \cdot f(3.001) - 30 \cdot f(3) + 16 \cdot f(2.999) - f(2.998)}{12 \cdot 0.001^2}$$
$$= \mathbf{2.240844453}79262787$$

The Python code:

```
import numpy as np

def numeric_diff(func, x: float, h: float, accuracy: str, method: str):  1 usage
    assert method == 'forward' or method == 'backward' or method == 'centered',\
        f'Input method harus salah satu dari forward, backward, atau centered. Input anda = {method}'
    assert accuracy == 'h' or accuracy == 'h2',\
        f'Input accuracy harus salah satu dari h atau h2. Input Anda = {accuracy}'

    if method == 'forward':
        if accuracy == 'h':
            first_diff = (func(x+h) - func(x))/h
            second_diff = (func(x+2*h) - 2*func(x+h) + func(x))/(h**2)
        else:
            first_diff = (-func(x+2*h) + 4*func(x+h) - 3*func(x))/(2*h)
            second_diff = (-func(x+3*h) + 4*func(x+2*h) - 5*func(x+h) + 2*func(x))/(h**2)
    elif method == 'backward':
        if accuracy == 'h':
            first_diff = (func(x) - func(x-h))/h
            second_diff = (func(x) - 2*func(x-h) + func(x-2*h))/(h**2)
        else:
            first_diff = (3*func(x) - 4*func(x-h) + func(x-2*h))/(2*h)
            second_diff = (2*func(x) - 5*func(x-h) + 4*func(x-2*h) - func(x-3*h))/(h**2)
    else:
        if accuracy == 'h':
            first_diff = (func(x+h) - func(x-h))/(2*h)
            second_diff = (func(x+h) - 2*func(x) + func(x+h))/(h**2)
        else:
            first_diff = (-func(x+2*h) + 8*func(x+h) - 8*func(x-h) + func(x-2*h))/(12*h)
            second_diff = (-func(x+2*h) + 16*func(x+h) - 30*func(x) + 16*func(x-h) - func(x-2*h))/(12*h**2)

    return first_diff, second_diff

def func_test(x):  1 usage
    return 2*np.exp(0.5*x)


methods = ['forward', 'backward', 'centered']
accuracies = ['h', 'h2']
hs = [0.1, 0.01, 0.001]
x = 3

for method in methods:
    for accuracy in accuracies:
        for h in hs:
            print(f'method = {method}, accuracy = {accuracy}, h = {h}')
            first_diff, second_diff = numeric_diff(func=func_test, x=x, h=h, accuracy=accuracy, method=method)
            print(f'first_diff = {first_diff}')
            print(f'second_diff = {second_diff}')
            print(' ')
```

Hasil output running code:

```
Python Console
method = forward, accuracy = h, h = 0.1
first_diff = 4.595622245053548
second_diff = 2.3562259103391265

method = forward, accuracy = h, h = 0.01
first_diff = 4.492911990083748
second_diff = 2.252081507005954

method = forward, accuracy = h, h = 0.001
first_diff = 4.4828096793665395
second_diff = 2.2419652836447312

method = forward, accuracy = h2, h = 0.1
first_diff = 4.4778109495365825
second_diff = 2.2354196246062936

method = forward, accuracy = h2, h = 0.01
first_diff = 4.481651582548629
second_diff = 2.2407929015244576

method = forward, accuracy = h2, h = 0.001
first_diff = 4.481688696724717
second_diff = 2.240844022338706
```

```
method = backward, accuracy = h, h = 0.1
first_diff = 4.371491103384955
second_diff = 2.1320013690210966

method = backward, accuracy = h, h = 0.01
first_diff = 4.470503498047762
second_diff = 2.229672921547632

method = backward, accuracy = h, h = 0.001
first_diff = 4.480568834782872
second_diff = 2.2397244379135373

method = backward, accuracy = h2, h = 0.1
first_diff = 4.478091171836018
second_diff = 2.2359803027539367

method = backward, accuracy = h2, h = 0.01
first_diff = 4.481651862655589
second_diff = 2.240793461574242

method = backward, accuracy = h2, h = 0.001
first_diff = 4.481688697002717
second_diff = 2.2408440170096355
```

```
method = centered, accuracy = h, h = 0.1
first_diff = 4.483556674219251
second_diff = 91.91244490107094

method = centered, accuracy = h, h = 0.01
first_diff = 4.481707744065755
second_diff = 898.5823980167495

method = centered, accuracy = h, h = 0.001
first_diff = 4.481689257074706
second_diff = 8965.61935873308

method = centered, accuracy = h2, h = 0.1
first_diff = 4.48168813637494
second_diff = 2.2408443795201456

method = centered, accuracy = h2, h = 0.01
first_diff = 4.48168907024451
second_diff = 2.2408445351581223

method = centered, accuracy = h2, h = 0.001
first_diff = 4.481689070337709
second_diff = 2.2408445379262787
```