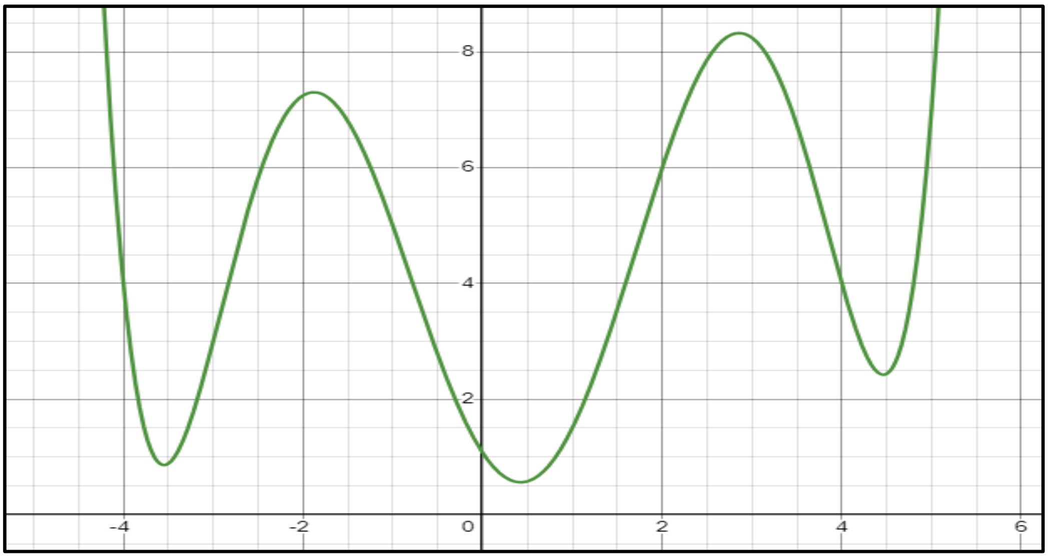
Numerical Integration Example
By: Ahmad Sirojuddin
sirojuddin@its.ac.id

Given the following polynomial function:
$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6,$
with
$c_0 = 1.07142857$
$c_1 = -2.35$
$c_2 = 2.48541667$
$c_3 = 0.63020833$
$c_4 = -0.31770833$
$c_5 = -0.03020833$
$c_6 = 0.0108631$

Find the integration of y from $x = -4$ to $x = 5$ by
1) Using the analytical approach
2) The trapezoidal rule
3) The multiple-application Trapezoidal Rule, divided into 6 segments with equal length
4) Simpson's 1/3 rule
5) The multiple-application Simpson's 1/3 Rule, divided into 6 segments with equal length
6) Simpson's 3/8 Rule
7) The combination of multiple-application Simpson's 1/3 Rule and Simpson's 3/8 Rule, where the integration span is divided into 9 segments with equal length.

Write a Python code to compute the seven problem above

Solution:



1) Using the analytical approach

$$I_1 = \int_a^b c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 \, dx$$

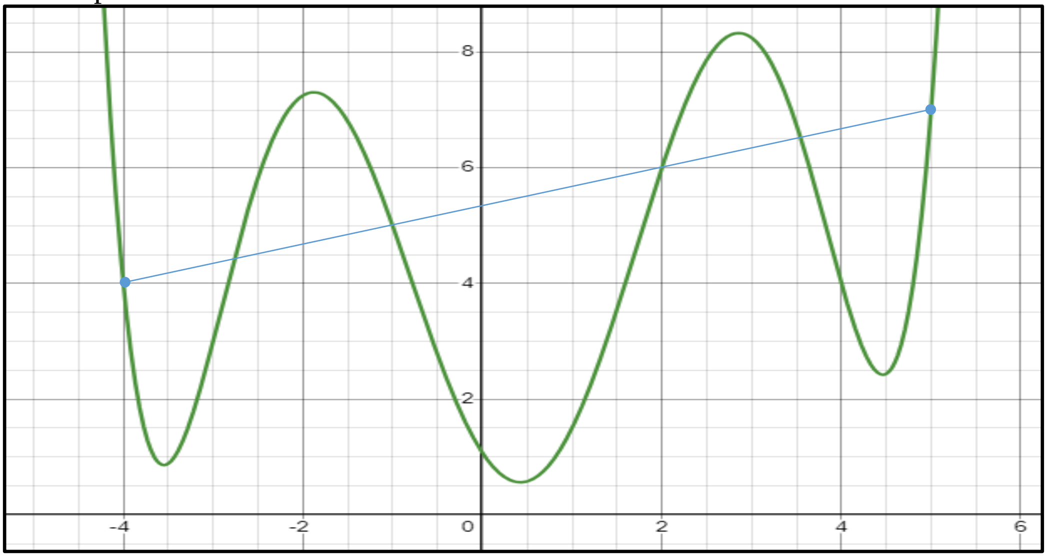
$$I_1 = c_0 x + \frac{c_1}{2} x^2 + \frac{c_2}{3} x^3 + \frac{c_3}{4} x^4 + \frac{c_4}{5} x^5 + \frac{c_5}{6} x^6 + \frac{c_6}{7} x^7 \Big|_{x=a}^{x=b}$$

By substituting the value of $c_0, \cdots, c_6$ and set $a = -4$, $b = 5$, we get

$$I_1 = (1.07142857x + \frac{-2.35}{2} x^2 + \frac{2.48541667}{3} x^3 + \frac{0.63020833}{4} x^4 + \frac{-0.31770833}{5} x^5 + \frac{-0.03020833}{6} x^6 + \frac{0.0108631}{7} x^7)\Big|_{x=-4}^{x=5}$$

$$I_1 = 22.0159453946429 - \ -16.756018901714288$$
$$= \ 38.77196429635718$$

2) The trapezoidal rule

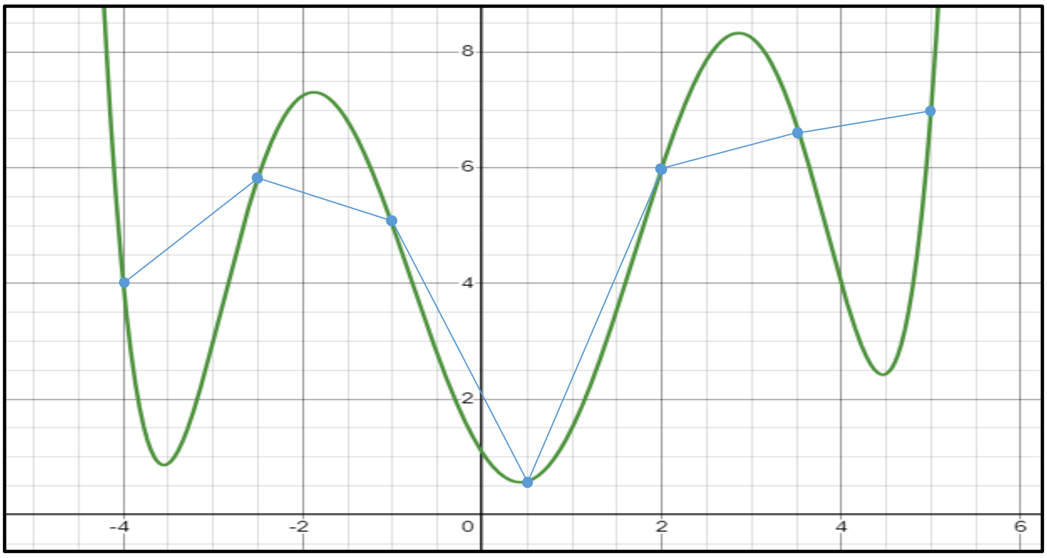


$a = -4$ and $b = 5$. Therefore,

$$I_2 = (b - a)\frac{f(a) + f(b)}{2} = (5 - (-4))\frac{f(-4) + f(5)}{2} = 9\frac{4 + 7}{2} = 49.5$$

3) The multiple-application Trapezoidal Rule, divided into 6 segments with equal length. Then, we have:

$x_0 = -4$	$x_1 = -2.5$	$x_2 = -1$	$x_3 = 0.5$	$x_4 = 2$	$x_5 = 3.5$	$x_6 = 5$
$f(x_0) = 4$	$f(x_1) = 5.825$	$f(x_2) = 5$	$f(x_3) = 0.576$	$f(x_4) = 6$	$f(x_5) = 6.74$	$f(x_6) = 7$



$$I_3 = (b-a)\frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2n}$$
$$= (b-a)\frac{f(x_0) + 2[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)] + f(x_6)}{2n}$$
$$= (5-(-4))\frac{f(-4) + 2[f(-2.5) + f(-1) + f(0.5) + f(2) + f(3.5)] + f(5)}{2\cdot 6}$$
$$= 9\frac{4 + 2[5.825 + 5 + 0.576 + 6 + 6.74] + 7}{12} = 9\frac{59.28}{12} = 44.46$$

4) Simpson's 1/3 rule

x_0	x_1	x_2
-4	0.5	5

$$I_4 = (b-a)\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}$$
$$= (5-(-4))\frac{f(-4) + 4f(0.5) + f(5)}{6}$$
$$= 9\frac{4 + 4\cdot 0.576 + 7}{6} = 19.956$$

5) The multiple-application Simpson's 1/3 Rule, divided into 6 segments with equal length
Then, we have:

$x_0 = -4$	$x_1 = -2.5$	$x_2 = -1$	$x_3 = 0.5$	$x_4 = 2$	$x_5 = 3.5$	$x_6 = 5$
$f(x_0) = 4$	$f(x_1) = 5.825$	$f(x_2) = 5$	$f(x_3) = 0.576$	$f(x_4) = 6$	$f(x_5) = 6.74$	$f(x_6) = 7$

$$I_5 = (b-a)\frac{f(x_0) + 4\sum_{i=1,3,5,\dots}^{n-1} f(x_i) + 2\sum_{j=2,4,6,\dots}^{n-2} f(x_j) + f(x_n)}{3n}$$
$$= (b-a)\frac{f(x_0) + 4[f(x_1) + f(x_3) + f(x_5)] + 2[f(x_2) + f(x_4)] + f(x_6)}{3n}$$
$$= (5-(-4))\frac{f(-4) + 4[f(-2.5) + f(0.5) + f(3.5)] + 2[f(-1) + f(2)] + f(5)}{3\cdot 6}$$
$$= 9\frac{4 + 4[5.825 + 0.576 + 6.74] + 2[5 + 6] + 7}{18} = 42.782$$

6) Simpson's 3/8 Rule

$x_0 = -4$	$x_1 = -1$	$x_2 = 2$	$x_3 = 5$
$f(x_0) = 4$	$f(x_1) = 5.825$	$f(x_2) = 5$	$f(x_3) = 0.576$

$$
\begin{aligned}
I_6 &= (b-a)\frac{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)}{8} \\
&= \big(5-(-4)\big)\frac{f(-4)+3f(-1)+3f(2)+f(5)}{8} \\
&= 9\,\frac{4+3\cdot 5+3\cdot 6+7}{8} = 49.5
\end{aligned}
$$

7) The combination of multiple-application Simpson's 1/3 Rule and Simpson's 3/8 Rule, where the integration span is divided into 9 segments with equal length.

$x_0=-4$	$x_1=-3$	$x_2=-2$	$x_3=-1$	$x_4=0$	$x_5=1$	$x_6=2$	$x_7=3$	$x_8=4$	$x_9=5$
$f(x_0)=4$	$f(x_1)=3$	$f(x_2)=7.25$	$f(x_3)=5$	$f(x_4)=1.071$	$f(x_5)=1.5$	$f(x_6)=6$	$f(x_7)=8.25$	$f(x_8)=4$	$f(x_9)=7$

We use the Simpson's 1/3 rule from x_0 to x_6, and then Simpson's 3/8 rule from x_6 to x_9.

$$
\begin{aligned}
I_{1/3} &= (b-a)\frac{f(x_0)+4\sum_{i=1,3,5,\dots}^{n-1} f(x_i)+2\sum_{j=2,4,6,\dots}^{n-2} f(x_j)+f(x_n)}{3n} \\
&= \big(2-(-4)\big)\frac{\rho}{3n} = 6\,\frac{\rho}{3\cdot 6} = \frac{\rho}{3}
\end{aligned}
$$

Where:

$$
\begin{aligned}
\rho &= f(x_0)+4[f(x_1)+f(x_3)+f(x_5)]+2[f(x_2)+f(x_4)]+f(x_6) \\
&= f(-4)+4[f(-3)+f(-1)+f(1)]+2[f(-2)+f(0)]+f(2) \\
&= 4+4[3+5+1.5]+2[7.25+1.071]+6 = 64.642
\end{aligned}
$$

Then,

$$
I_{1/3} = \frac{\rho}{3} = \frac{64.642}{3} = 21.547
$$

$$
\begin{aligned}
I_{3/8} &= (b-a)\frac{f(x_6)+3f(x_7)+3f(x_8)+f(x_9)}{8} \\
&= (5-2)\frac{f(2)+3f(3)+3f(4)+f(5)}{8} \\
&= 3\,\frac{6+3\cdot 8.25+3\cdot 4+7}{8} = 18.65625
\end{aligned}
$$

Therefore,

$$
I_7 = I_{1/3} + I_{3/8} = 21.547 + 18.65625 = 40.20325
$$

The Python code for computing above problems:

```
import numpy as np
import scipy

# ----------- The function we want to integrate ----------- #
def my_poly(x):  8 usages
    c0 = 1.07142857
    c1 = -2.35
    c2 = 2.48541667
    c3 = 0.63020833
    c4 = -0.31770833
    c5 = -0.03020833
    c6 = 0.0108631
    return c0 + c1 * x + c2 * x**2 + c3 * x**3 + c4 * x**4 + c5 * x**5 + c6 * x**6

```

```python
# ----------- (1) The truth value ----------- #
# It is impossible to get an exact value of the integration using numerical integration.
# However, we can obtain a very accurate approximation by using a high number of segment (e.g., 1000)
x_no1 = np.linspace(start=-4, stop=5, num=1000)
y_no1 = my_poly(x_no1)
I1 = scipy.integrate.simpson(y=y_no1, x=x_no1)
print(f'(1) Analytical value (approximation): {I1}')

# ----------- (2) Trapezoidal Rule ----------- #
x_no2 = np.array([-4, 5])
y_no2 = my_poly(x_no2)
I2 = scipy.integrate.trapezoid(y=y_no2, x=x_no2)
print(f'(2) Trapezoidal Rule: {I2}')

# ----------- (3) Multiple Trapezoidal Rule ----------- #
x_no3 = np.array([-4, -2.5, -1, 0.5, 2, 3.5, 5])
y_no3 = my_poly(x_no3)
I3 = scipy.integrate.trapezoid(y_no3, x_no3)
print(f'(3) Multiple Trapezoidal Rule: {I3}')

# ----------- (4) Simpson's 1/3 Rule ----------- #
x_no4 = np.array([-4, 0.5, 5])
y_no4 = my_poly(x_no4)
I4 = scipy.integrate.simpson(y=y_no4, x=x_no4)
print(f'(4) Simpsons 1/3 Rule: {I4}')

# ----------- (5) Multiple Simpson's 1/3 Rule ----------- #
x_no5 = np.array([-4, -2.5, -1, 0.5, 2, 3.5, 5])
y_no5 = my_poly(x_no5)
I5 = scipy.integrate.simpson(y=y_no5, x=x_no5)
print(f'(5) Multiple Simpsons 1/3 Rule: {I5}')

# ----------- (6) Multiple Simpson's 3/8 Rule ----------- #
# There is no default library for computing Simpson's 3/8. Then, we calculate it manually
x_no6 = np.array([-4, -1, 2, 5])
y_no6 = my_poly(x_no6)
I6 = ((x_no6[-1] - x_no6[0])/8)*(y_no6[0] + 3*y_no6[1] + 3*y_no6[2] + y_no6[3])  # calculate manually
print(f'(6) Simpsons 3/8 Rule: {I6}')

# ----------- (7) Combination of Simpson's 1/3 and 3/8 Rule ----------- #
x_no7a = np.array([-4, -3, -2, -1, 0, 1, 2])
y_no7a = my_poly(x_no7a)
I_13 = scipy.integrate.simpson(y=y_no7a, x=x_no7a)

# There is no default library for computing Simpson's 3/8. Then, we calculate it manually
x_no7b = np.array([2, 3, 4, 5])
y_no7b = my_poly(x_no7b)
I_38 = ((x_no7b[-1] - x_no7b[0])/8)*(y_no7b[0] + 3*y_no7b[1] + 3*y_no7b[2] + y_no7b[3])  # calculate manually

I7 = I_13 + I_38
print(f'(7) Combination of Simpsons 1/3 and 3/8: {I7}')
```

The result of executing above code is shown below:

```
Python Console
(1) Analytical value (approximation): 38.77196432494824
(2) Trapezoidal Rule: 49.50046701000018
(3) Multiple Trapezoidal Rule: 44.461609408125035
(4) Simpsons 1/3 Rule: 19.95572207250006
(5) Multiple Simpsons 1/3 Rule: 42.78209352750004
(6) Simpsons 3/8 Rule: 49.500118305000036
(7) Combination of Simpsons 1/3 and 3/8: 40.20394346500002

>>>
```