Gauss Seidel Method and Jacobi Method to Solve
a System of Linear Equation

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Solve the following system of linear equations of the form $\mathbf{Ax} - \mathbf{b}$ (finding the values of x_1, x_2, and x_3) by using (a) The Gaussian Seidel Method and (b) The Jacobi Method.

$$\begin{array}{rrrcr} 10x_1 & -x_2 & +x_3 & = & 18 \\ x_1 & +10x_2 & -x_3 & = & 13 \\ x_1 & +x_2 & +10x_3 & = & -7 \end{array}$$

Use the initial point $\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, stopping criterion $\|\mathbf{Ax} - \mathbf{b}\|^2 < \varepsilon$, where $\varepsilon = 10^{-6}$

<u>Solution</u>:

a) Gauss-Seidel Method

Let's rearrange the equations in the problem such that x_1, x_2, and x_3 are in the left hand side.

$$\begin{array}{rcrrr} x_1 & = & 1.8 & +0.1x_2 & -0.1x_3 \\ x_2 & = & 1.3 & -0.1x_1 & +0.1x_3 \\ x_3 & = & -0.7 & -0.1x_1 & -0.1x_2 \end{array}$$

In what follows, the superscript $\mathbf{x}^{(t)}$ means the value of $\mathbf{x}$ at iteration t.

Iteration 1: --

$x_1^{(1)} = 1.8 + 0.1x_2^{(0)} - 0.1x_3^{(0)} = 1.8 + 0.1(0) - 0.1(0) = 1.8$

$x_2^{(1)} = 1.3 - 0.1x_1^{(1)} + 0.1x_3^{(0)} = 1.3 - 0.1(1.8) + 0.1(0) = 1.12$

$x_3^{(1)} =- 0.7 - 0.1x_1^{(1)} - 0.1x_2^{(1)} =- 0.7 - 0.1(1.8) - 0.1(1.12) =- 1$

The value of $\|\mathbf{Ax} - \mathbf{b}\|^2$ is:

$$\left\| \begin{bmatrix} 10 & -1 & 1 \\ 1 & 10 & -1 \\ 1 & 1 & 10 \end{bmatrix} \begin{bmatrix} 1.8 \\ 1.12 \\ -1 \end{bmatrix} - \begin{bmatrix} 18 \\ 13 \\ -7 \end{bmatrix} \right\|^2 = 5.444608$$

Since $\|\mathbf{Ax} - \mathbf{b}\|^2 = 5.444608 > \varepsilon$, we continue the iteration

Iteration 2: --

$x_1^{(2)} = 1.8 + 0.1x_2^{(1)} - 0.1x_3^{(1)} = 1.8 + 0.1(1.12) - 0.1(-1) = 2.012$

$x_2^{(2)} = 1.3 - 0.1x_1^{(2)} + 0.1x_3^{(1)} = 1.3 - 0.1(2.012) + 0.1(-1) = 0.9988$

$x_3^{(2)} =- 0.7 - 0.1x_1^{(2)} - 0.1x_2^{(2)} =- 0.7 - 0.1(2.012) - 0.1(0.9988) =- 1.00108$

The value of $\|\mathbf{Ax} - \mathbf{b}\|^2$ is:

$$\left\| \begin{bmatrix} 10 & -1 & 1 \\ 1 & 10 & -1 \\ 1 & 1 & 10 \end{bmatrix} \begin{bmatrix} 2.012 \\ 0.9988 \\ -1.00108 \end{bmatrix} - \begin{bmatrix} 18 \\ 13 \\ -7 \end{bmatrix} \right\|^2 = 1.2455 \times 10^{-2}$$

Since $\|\mathbf{Ax} - \mathbf{b}\|^2 = 1.2455 \times 10^{-2} > \varepsilon$, we continue the iteration

Iteration 3: --

$x_1^{(3)} = 1.8 + 0.1x_2^{(2)} - 0.1x_3^{(2)} = 1.8 + 0.1(0.9988) - 0.1(-1.00108) = 1.999988$

$x_2^{(3)} = 1.3 - 0.1x_1^{(3)} + 0.1x_3^{(2)} = 1.3 - 0.1(1.999988) + 0.1(-1.00108) = 0.9998932$

$x_3^{(3)} =- 0.7 - 0.1x_1^{(3)} - 0.1x_2^{(3)} =- 0.7 - 0.1(1.999988) - 0.1(0.9998932) =- 0.99998812$

The value of $\|\mathbf{Ax} - \mathbf{b}\|^2$ is:

$$\left\| \begin{bmatrix} 10 & -1 & 1 \\ 1 & 10 & -1 \\ 1 & 1 & 10 \end{bmatrix} \begin{bmatrix} 1.999988 \\ 0.9998932 \\ -0.99998812 \end{bmatrix} - \begin{bmatrix} 18 \\ 13 \\ -7 \end{bmatrix} \right\|^2 = 1.9817 \times 10^{-6}$$

Since $\|\mathbf{Ax} - \mathbf{b}\|^2 = 1.9817 \times 10^{-6} > \varepsilon$, we continue the iteration

Iteration 4: ---

$x_1^{(4)} = 1.8 + 0.1x_2^{(3)} - 0.1x_3^{(3)} = 1.8 + 0.1(0.9998932) - 0.1(-0.99998812) = 1.999988132$
$x_2^{(4)} = 1.3 - 0.1x_1^{(4)} + 0.1x_3^{(3)} = 1.3 - 0.1(1.999988132) + 0.1(-0.99998812) = 1.0000023748$
$x_3^{(4)} =- 0.7 - 0.1x_1^{(4)} - 0.1x_2^{(4)} =- 0.7 - 0.1(1.999988132) - 0.1(1.0000023748) =- 0.9999991$

The value of $||\mathbf{Ax} - \mathbf{b}||^2$ is:

$$\left\|\begin{bmatrix} 10 & -1 & 1 \\ 1 & 10 & -1 \\ 1 & 1 & 10 \end{bmatrix}\begin{bmatrix} 1.999988132 \\ 1.0000023748 \\ -0.9999991 \end{bmatrix} - \begin{bmatrix} 18 \\ 13 \\ -7 \end{bmatrix}\right\|^2 = 1.4650 \times 10^{-8}$$

Since $||\mathbf{Ax} - \mathbf{b}||^2 = 1.4650 \times 10^{-8} < \varepsilon$, we stop the iteration.

Hence, the solution is $\mathbf{x} = \begin{bmatrix} 1.999988132 \\ 1.0000023748 \\ -0.9999991 \end{bmatrix}$

b) Jacobi Method

Let's rearrange the equations in the problem such that x_1, x_2, and x_3 are in the left hand side.

$$\begin{array}{ccrrrr} x_1 & = & 1.8 & +0x_1 & +0.1x_2 & -0.1x_3 \\ x_2 & = & 1.3 & -0.1x_1 & +0x_2 & +0.1x_3 \\ x_3 & = & -0.7 & -0.1x_1 & -0.1x_2 & +0x_3 \end{array}$$

Suppose the values of x_1, x_2, and x_3 are to be update at t-th iteration. Thus, the values of x_1, x_2, and x_3 is from the previous iteration $(t-1)$. Therefore, the above equation can be re-arranged as

$$\begin{bmatrix} x_1^{(t)} \\ x_2^{(t)} \\ x_3^{(t)} \end{bmatrix} = \begin{bmatrix} 1.8 & 0 & 0.1 & -0.1 \\ 1.3 & -0.1 & 0 & 0.1 \\ -0.7 & -0.1 & -0.1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ x_1^{(t-1)} \\ x_2^{(t-1)} \\ x_3^{(t-1)} \end{bmatrix}$$

Iteration 1: ---

$$\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} = \begin{bmatrix} 1.8 & 0 & 0.1 & -0.1 \\ 1.3 & -0.1 & 0 & 0.1 \\ -0.7 & -0.1 & -0.1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{bmatrix} = \begin{bmatrix} 1.8 & 0 & 0.1 & -0.1 \\ 1.3 & -0.1 & 0 & 0.1 \\ -0.7 & -0.1 & -0.1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.8 \\ 1.3 \\ -0.7 \end{bmatrix}$$

The value of $||\mathbf{Ax} - \mathbf{b}||^2$ is:

$$\left\|\begin{bmatrix} 10 & -1 & 1 \\ 1 & 10 & -1 \\ 1 & 1 & 10 \end{bmatrix}\begin{bmatrix} 1.8 \\ 1.3 \\ -0.7 \end{bmatrix} - \begin{bmatrix} 18 \\ 13 \\ -7 \end{bmatrix}\right\|^2 = 19.86$$

Since $||\mathbf{Ax} - \mathbf{b}||^2 = 19.86 > \varepsilon$, we continue the iteration

Iteration 2: ---

$$\begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{bmatrix} = \begin{bmatrix} 1.8 & 0 & 0.1 & -0.1 \\ 1.3 & -0.1 & 0 & 0.1 \\ -0.7 & -0.1 & -0.1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} = \begin{bmatrix} 1.8 & 0 & 0.1 & -0.1 \\ 1.3 & -0.1 & 0 & 0.1 \\ -0.7 & -0.1 & -0.1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1.8 \\ 1.3 \\ -0.7 \end{bmatrix} = \begin{bmatrix} 2 \\ 1.05 \\ -1.01 \end{bmatrix}$$

The value of $||\mathbf{Ax} - \mathbf{b}||^2$ is:

$$\left\|\begin{bmatrix} 10 & -1 & 1 \\ 1 & 10 & -1 \\ 1 & 1 & 10 \end{bmatrix}\begin{bmatrix} 2 \\ 1.05 \\ -1.01 \end{bmatrix} - \begin{bmatrix} 18 \\ 13 \\ -7 \end{bmatrix}\right\|^2 = 0.2662$$

Since $||\mathbf{Ax} - \mathbf{b}||^2 = 0.2662 > \varepsilon$, we continue the iteration

Iteration 3: --

$$\begin{bmatrix} x_1^{(3)} \\ x_2^{(3)} \\ x_3^{(3)} \end{bmatrix} = \begin{bmatrix} 1.8 & 0 & 0.1 & -0.1 \\ 1.3 & -0.1 & 0 & 0.1 \\ -0.7 & -0.1 & -0.1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{bmatrix} = \begin{bmatrix} 1.8 & 0 & 0.1 & -0.1 \\ 1.3 & -0.1 & 0 & 0.1 \\ -0.7 & -0.1 & -0.1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1.05 \\ -1.01 \end{bmatrix} = \begin{bmatrix} 2.006 \\ 0.999 \\ -1.005 \end{bmatrix}$$

The value of $\|\mathbf{Ax} - \mathbf{b}\|^2$ is:

$$\left\| \begin{bmatrix} 10 & -1 & 1 \\ 1 & 10 & -1 \\ 1 & 1 & 10 \end{bmatrix} \begin{bmatrix} 2.006 \\ 0.999 \\ -1.005 \end{bmatrix} - \begin{bmatrix} 18 \\ 13 \\ -7 \end{bmatrix} \right\|^2 = 5.16 \times 10^{-3}$$

Since $\|\mathbf{Ax} - \mathbf{b}\|^2 = 0.2662 > \varepsilon$, we continue the iteration

Let's skip iterations 4 and 5, and proceed to iteration 6

Iteration 6: --

$$\begin{bmatrix} x_1^{(6)} \\ x_2^{(6)} \\ x_3^{(6)} \end{bmatrix} = \begin{bmatrix} 1.8 & 0 & 0.1 & -0.1 \\ 1.3 & -0.1 & 0 & 0.1 \\ -0.7 & -0.1 & -0.1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x_1^{(5)} \\ x_2^{(5)} \\ x_3^{(5)} \end{bmatrix} = \begin{bmatrix} 1.8 & 0 & 0.1 & -0.1 \\ 1.3 & -0.1 & 0 & 0.1 \\ -0.7 & -0.1 & -0.1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1.99994 \\ 0.99991 \\ -0.99993 \end{bmatrix} = \begin{bmatrix} 1.999984 \\ 1.000013 \\ -0.999985 \end{bmatrix}$$

The value of $\|\mathbf{Ax} - \mathbf{b}\|^2$ is:

$$\left\| \begin{bmatrix} 10 & -1 & 1 \\ 1 & 10 & -1 \\ 1 & 1 & 10 \end{bmatrix} \begin{bmatrix} 1.999984 \\ 1.000013 \\ -0.999985 \end{bmatrix} - \begin{bmatrix} 18 \\ 13 \\ -7 \end{bmatrix} \right\|^2 = 5.6373 \times 10^{-8}$$

Since $\|\mathbf{Ax} - \mathbf{b}\|^2 = 5.6373 \times 10^{-8} < \varepsilon$, we stop the iteration.

Hence, the solution is $\mathbf{x} = \begin{bmatrix} 1.999984 \\ 1.000013 \\ -0.999985 \end{bmatrix}$