Gaussian Elimination Method and Gauss-Jordan Elimination Method to Solve a System of Linear Equation

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Solve the following system of linear equations (finding the values of x_1, x_2, x_3, and x_4) by using (a) The Gaussian Elimination Method followed by back substitution and (b) The Gauss-Jordan Elimination Method.

$$\begin{array}{rrrrl} 2x_1 & -2x_2 & +4x_3 & -4x_4 & =-10 \\ 3x_1 & +3x_2 & -3x_3 & -6x_4 & =-6 \\ -x_1 & -2x_2 & +3x_3 & +2x_4 & =0 \\ 2x_1 & +4x_2 & -4x_3 & -2x_4 & =2 \end{array}$$

<u>Solution</u>:

Either using method (a) or (b), we need to first perform the Gaussian Elimination Method.

Let's construct the augmented matrix from the above equation:

$$\begin{array}{rrrrr} 2 & -2 & 4 & -4 & -10 \\ 3 & 3 & -3 & -6 & -6 \\ -1 & -2 & 3 & 2 & 0 \\ 2 & 4 & -4 & -2 & 2 \end{array}$$

We make **row 1 as a pivot**. Then, the entry at row 1 column 1 must be one. Currently, it is 2. To make it becomes 1, multiply row 1 by 0.5.

$$\begin{array}{rrrrr} 1 & -1 & 2 & -2 & -5 \\ 3 & 3 & -3 & -6 & -6 \\ -1 & -2 & 3 & 2 & 0 \\ 2 & 4 & -4 & -2 & 2 \end{array}$$

Now, we want to make the **1st entry at row 2, 3, and 4 become 0**. To do so:

- row 2 ← row 2 − 3 · row 1
  $(3 \quad 3 \quad -3 \quad -6 \quad -6) \quad - \quad 3\text{x} \quad (1 \quad -1 \quad 2 \quad -2 \quad -5)$
  $=(0 \quad 6 \quad -9 \quad 0 \quad 9)$
- row 3 ← row 3 + row 1
  $(-1 \quad -2 \quad 3 \quad 2 \quad 0) \quad + \quad 1\text{x} \quad (1 \quad -1 \quad 2 \quad -2 \quad -5)$
  $=(0 \quad -3 \quad 5 \quad 0 \quad -5)$
- row 4 ← row 4 − 2 · row 1
  $(2 \quad 4 \quad -4 \quad -2 \quad 2) \quad - \quad 2\text{x} \quad (1 \quad -1 \quad 2 \quad -2 \quad -5)$
  $=(0 \quad 6 \quad -8 \quad 2 \quad 12)$

Then, the augmented matrix becomes as follows:

$$\begin{matrix} 1 & -1 & 2 & -2 & -5 \\ 0 & 6 & -9 & 0 & 9 \\ 0 & -3 & 5 & 0 & -5 \\ 0 & 6 & -8 & 2 & 12 \end{matrix}$$

We make **row 2 as a pivot**. Then, the entry at row 2 column 2 must be one. Currently, it is 6. To make it becomes 1, multiply row 2 by 1/6.

$$\begin{matrix} 1 & -1 & 2 & -2 & -5 \\ 0 & 1 & -3/2 & 0 & 3/2 \\ 0 & -3 & 5 & 0 & -5 \\ 0 & 6 & -8 & 2 & 12 \end{matrix}$$

Now, we want to make the **2nd entry at row 3 and 4 become 0**. To do so:

- row 3 ← row 3 + 3 · row 2

  $(0 \quad -3 \quad 5 \quad 0 \quad -5) \quad + \quad 3\mathrm{x} \quad (0 \quad 1 \quad -3/2 \quad 0 \quad 3/2)$

  $=(0 \quad 0 \quad 0.5 \quad 0 \quad -0.5)$

- row 4 ← row 4 − 6 · row 2

  $(0 \quad -3 \quad 5 \quad 0 \quad -5) \quad + \quad 3\mathrm{x} \quad (0 \quad 1 \quad -3/2 \quad 0 \quad 3/2)$

  $=(0 \quad 0 \quad 1 \quad 2 \quad 3)$

Then, the augmented matrix becomes as follows:

$$\begin{matrix} 1 & -1 & 2 & -2 & -5 \\ 0 & 1 & -3/2 & 0 & 3/2 \\ 0 & 0 & 0.5 & 0 & -0.5 \\ 0 & 0 & 1 & 2 & 3 \end{matrix}$$

We make **row 3 as a pivot**. Then, the entry at row 3 column 3 must be one. Currently, it is 0.5. To make it becomes 1, multiply row 3 by 2.

$$\begin{matrix} 1 & -1 & 2 & -2 & -5 \\ 0 & 1 & -3/2 & 0 & 3/2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 3 \end{matrix}$$

Now, we want to make the **3rd entry at row 4 becomes 0**. To do so:

- row 4 ← row 4 − row 3

  $(0 \quad 0 \quad 1 \quad 2 \quad 3) \quad - \quad 1\mathrm{x} \quad (0 \quad 0 \quad 1 \quad 0 \quad -1)$

  $=(0 \quad 0 \quad 0 \quad 2 \quad 4)$

Then, the augmented matrix becomes as follows:

$$\begin{matrix} 1 & -1 & 2 & -2 & -5 \\ 0 & 1 & -3/2 & 0 & 3/2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 & 4 \end{matrix}$$

We make **row 4 as a pivot**. Then, the entry at row 4 column 4 must be one. Currently, it is 2. To make it becomes 1, multiply row 4 by 1/2.

$$\begin{matrix} 1 & -1 & 2 & -2 & -5 \\ 0 & 1 & -3/2 & 0 & 3/2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{matrix}$$

The above matrix structure whose the elements on the lower triangle are zeros are called **row echelon** form. To this end, we can choose whether we use method (a) The Gaussian Elimination Method followed by back substitution and (b) The Gauss-Jordan Elimination Method.

(a) **The Gaussian Elimination Method followed by back substitution**
Frow the row echelon form of matrix, we can express the system of linear equation as follows:

$$\begin{array}{ccccc}
x_1 & -x_2 & 2x_3 & -2x_4 & =-5 \\
 & x_2 & -\frac{3}{2}x_3 & & =\frac{3}{2} \\
 & & x_3 & & =-1 \\
 & & & x_4 & =2
\end{array}$$

Back substitution means we find the value of x_4, then x_3, then x_2, and then x_1

$x_4 = 2$

$x_3 =- 1$

$x_2 = \frac{3}{2} + \frac{3}{2}x_3 = \frac{3}{2} + \frac{3}{2}(-1) = 0$

$x_1 =- 5 + x_2 - 2x_3 + 2x_4 =- 5 + 0 - 2(-1) + 2(2) = 1$

(b) **The Gauss-Jordan Elimination Method**
We continue the **row echelon form** obtained previously to the **reduced row echelon form**.

We make **row 4 as a pivot**, and the entry of row 4 column 4 is one already. Now, we want to make the **4th entries at row 3, 2, and 1 become zero**. To do so:

- The entry at row 3 column 4 is already zero (keep row 3 as it is)
- The entry at row 2 column 4 is already zero (keep row 2 as it is)
- row 1 ← row 1 + 2 · row 4

$$(1 \quad -1 \quad 2 \quad -2 \quad -5) \quad + \quad 2\text{x} \quad (0 \quad 0 \quad 0 \quad 1 \quad 2)$$
$$=(1 \quad -1 \quad 2 \quad 0 \quad -1)$$

Then, the augmented matrix becomes as follows:

$$\begin{array}{ccccc}
1 & -1 & 2 & 0 & -1 \\
0 & 1 & -3/2 & 0 & 3/2 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & 2
\end{array}$$

We make **row 3 as a pivot**, and the entry of row 3 column 3 is one already. Now, we want to make the **3rd entries at row 2 and 1 become zero**. To do so:

- row 2 ← row 2 + 3/2 · row 3

$$(0 \quad 1 \quad -3/2 \quad 0 \quad 3/2) \quad + \quad (3/2)\text{x} \quad (0 \quad 0 \quad 1 \quad 0 \quad -1)$$
$$=(0 \quad 1 \quad 0 \quad 0 \quad 0)$$

- row 1 ← row 1 − 2 · row 3

$$(1 \quad -1 \quad 2 \quad 0 \quad -1) \quad - \quad 2\text{x} \quad (0 \quad 0 \quad 1 \quad 0 \quad -1)$$
$$=(1 \quad -1 \quad 0 \quad 0 \quad 1)$$

Then, the augmented matrix becomes as follows:

$$\begin{matrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{matrix}$$

We make **row 2 as a pivot**, and the entry of row 2 column 2 is one already. Now, we want to make the **2nd entries at row 1 become zero**. To do so:

- row 1 ← row 1 + row 2

$(1 \quad -1 \quad 0 \quad 0 \quad 1) \quad + \quad 1\text{x} \quad (0 \quad 1 \quad 0 \quad 0 \quad 0)$

$=(1 \quad 0 \quad 0 \quad 0 \quad 1)$

Then, the augmented matrix becomes as follows:

$$\begin{matrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{matrix}$$

Frow the **reduced row echelon form** of matrix above, we can express the system of linear equation as follows:

$$\begin{aligned} x_1 \quad\quad\quad\quad\quad\quad &= 1 \\ x_2 \quad\quad\quad\quad &= \frac{3}{2} \\ x_3 \quad\quad &= -1 \\ x_4 &= 2 \end{aligned}$$