

Water Filling Algorithm with Torch Pyton

Ahmad Sirojuddin

March 4, 2022

Consider the following optimization problem:

$$\begin{aligned} \min_{\mathbf{p}} \quad & r = -\sum_{n=1}^N a_n \log(b_n + p_n c_n) \\ \text{s.t.} \quad & p_n \geq d_n, \quad \forall n \in \mathcal{N}, \\ & \mathbf{f}^T \mathbf{p} = P, \end{aligned} \quad (1)$$

where $\mathcal{N} = \{1, \dots, N\}$, $\mathbf{p} = \{p_1, \dots, p_N\}$. Here, we want to solve it using the duality concept for convex optimization.

The Lagrangian function (which is the lower bound to problem (1)) is given by

$$L(\mathbf{p}, \boldsymbol{\lambda}, \nu) = -\sum_{n=1}^N a_n \log(b_n + p_n c_n) + \sum_{n=1}^N \lambda_n (d_n - p_n) + \nu (\mathbf{f}^T \mathbf{p} - P) \quad (2)$$

By using KKT, we need to solve the following root of equation problem (see [BV04, Sec. 5.5.3]),

$$d_n - p_n \leq 0, \quad \forall n \in \mathcal{N}, \quad (3)$$

$$\mathbf{f}^T \mathbf{p} - P = 0, \quad (4)$$

$$\lambda_n \geq 0, \quad \forall n \in \mathcal{N}, \quad (5)$$

$$\lambda_n (d_n - p_n) = 0, \quad \forall n \in \mathcal{N}, \quad (6)$$

$$\nabla_{\mathbf{p}} L(\mathbf{p}, \boldsymbol{\lambda}, \nu) = 0. \quad (7)$$

Let's focus on solving (7). Note that optimal p_n^* does not depend on the other $p_{\hat{n}}, \forall \hat{n} \in \mathcal{N}, \hat{n} \neq n$. Hence, the optimal \mathbf{p}^* can be found from individual p_n^* . The first derivative of L w.r.t. p_n equals zero can be expressed as

$$\nabla_{p_n} L(\mathbf{p}, \boldsymbol{\lambda}, \nu) = \frac{a_n c_n}{b_n + p_n c_n} - \lambda_n + \nu f_n = 0 \quad (8)$$

Solving (8) for p_n yields

$$\begin{aligned} \frac{b_n + p_n c_n}{a_n c_n} &= \frac{1}{\lambda_n - \nu f_n} \\ p_n &= \left(\frac{a_n c_n}{\lambda_n - \nu f_n} - b_n \right) \frac{1}{c_n} = \frac{a_n}{\lambda_n - \nu f_n} - \frac{b_n}{c_n} \end{aligned} \quad (9)$$

Substitute (9) to (7) yields the following KKT conditions to solve

$$d_n - p_n \leq 0, \quad \forall n \in \mathcal{N}, \quad (10)$$

$$\mathbf{f}^T \mathbf{p} - P = 0, \quad (11)$$

$$\lambda_n \geq 0, \quad \forall n \in \mathcal{N}, \quad (12)$$

$$\lambda_n (d_n - p_n) = 0, \quad \forall n \in \mathcal{N}, \quad (13)$$

$$p_n = \frac{a_n}{\lambda_n - \nu f_n} - \frac{b_n}{c_n}, \quad \forall n \in \mathcal{N}, \quad (14)$$

Let's focus on (10), (11), (12), and (13). It is obvious that we have two choices: $\lambda_n \geq 0; d_n - p_n = 0$ or $\lambda_n = 0; d_n - p_n \leq 0$. But if we take the first choice, we get $p_n = d_n$, and this expression may

inconsistent with (11) since we not sure whether $\mathbf{f}^T \mathbf{d} - P = 0$ is satisfied since \mathbf{f} , \mathbf{d} , and P are constants. Hence, we take the second choice. Consequently, the KKT conditions are reduced as follow:

$$d_n - p_n \leq 0, \quad \forall n \in \mathcal{N}, \quad (15)$$

$$\mathbf{f}^T \mathbf{p} - P = 0, \quad (16)$$

$$p_n = \frac{a_n}{-\nu f_n} - \frac{b_n}{c_n}, \quad \forall n \in \mathcal{N}, \quad (17)$$

Define $\hat{\nu} = -1/\nu$, then by combining (15) and (17), we have the closed-form solution for optimal $p_n^*, \forall n \in \mathcal{N}$ in terms of dual variable $\hat{\nu}$ as

$$p_n^* = \max \left(\hat{\nu} \frac{a_n}{f_n} - \frac{b_n}{c_n}, d_n \right) \quad (18)$$

Note that although we already find the equation to find optimal $p_n^*, \forall n \in \mathcal{N}$, it is in term of $\hat{\nu}$ that is not one of constant given by the problem. To find $\hat{\nu}$, we substitute (18) to (16) and get the following root of equation problem

$$\sum_{n=1}^N f_n \max \left(\hat{\nu} \frac{a_n}{f_n} - \frac{b_n}{c_n}, d_n \right) - P = 0. \quad (19)$$

It is more convenient if we structure a little bit so we have the following expression to solve

$$g(\hat{\nu}) = \sum_{n=1}^N f_n \max \left(\hat{\nu} \frac{a_n}{f_n} - \frac{b_n}{c_n} - d_n, 0 \right) + \mathbf{f}^T \mathbf{d} - P = 0. \quad (20)$$

In this article, we assume that all entries of \mathbf{a} , \mathbf{b} , \mathbf{c} , \mathbf{d} , and \mathbf{f} are non-negative, and $f_n \neq 0$, $c_n \neq 0$, $\forall n \in \mathcal{N}$. From (20), it obvious that $g(\hat{\nu})$ is a piece-wise linear function composing N fraction with non-negative gradient, and $\nabla_{\hat{\nu}} g(\hat{\nu})$ is non-decrease with $\hat{\nu}$.

In the literature, many papers use the bisection method to find $\hat{\nu}$. This method, although robust, has a slower convergence. By leveraging the fact that $g(\hat{\nu})$ is a monotonically increasing piece-wise linear function with increasing gradient, $g(\hat{\nu})$ can efficiently be solve using the Newton-Raphson Method (see [CC02, Sec. 6.2]). By using this method, we have the updating rule $\hat{\nu}^{(i+1)} = \hat{\nu}^{(i)} - g(\hat{\nu}) / \nabla_{\hat{\nu}} g(\hat{\nu})$, where $\nabla_{\hat{\nu}} g(\hat{\nu})$ is given by

$$\nabla_{\hat{\nu}} g(\hat{\nu}) = \sum_{n=1}^N h_n(\hat{\nu}) \quad (21)$$

$$h_n(\hat{\nu}) = \begin{cases} a_n; & \hat{\nu} \frac{a_n}{f_n} - \frac{b_n}{c_n} - d_n \geq 0 \\ 0; & \text{otherwise.} \end{cases} \quad (22)$$

Finding a proper initial guess for $\hat{\nu}^{(0)}$ is important since it affects the convergence rate. It can be seen from (20) that the solution to $\hat{\nu}$ increases as P increases. In this article, we determine $\hat{\nu}^{(0)}$ by assuming that P is high such that $\hat{\nu}$ is big enough to make sure that $\hat{\nu} a_n / f_n - b_n / c_n - d_n \geq 0$ are always satisfied for all $n \in \mathcal{N}$. In this case, the terms $\max(\hat{\nu} a_n / f_n - b_n / c_n - d_n, 0)$ can be replaced with $\hat{\nu} a_n / f_n - b_n / c_n - d_n, \forall n \in \mathcal{N}$, and thus the solution to (20) is given by

$$\hat{\nu}^{(0)} = \frac{-\mathbf{f}^T \mathbf{d} + P + \sum_{n=1}^N f_n \left(\frac{b_n}{c_n} + d_n \right)}{\sum_{n=1}^N a_n} \quad (23)$$

1 Some examples to get started

1.1 How to create Sections and Subsections

Simply use the section and subsection commands, as in this example document! With Overleaf, all the formatting and numbering is handled automatically according to the template you've chosen. If you're using Rich Text mode, you can also create new section and subsections via the buttons in the editor toolbar.



Figure 1: This frog was uploaded via the file-tree menu.

Item	Quantity
Widgets	42
Gadgets	13

Table 1: An example table.

1.2 How to include Figures

First you have to upload the image file from your computer using the upload link in the file-tree menu. Then use the `includegraphics` command to include it in your document. Use the figure environment and the caption command to add a number and a caption to your figure. See the code for Figure 1 in this section for an example.

Note that your figure will automatically be placed in the most appropriate place for it, given the surrounding text and taking into account other figures or tables that may be close by. You can find out more about adding images to your documents in this help article on [including images on Overleaf](#).

1.3 How to add Tables

Use the table and tabular environments for basic tables — see Table 1, for example. For more information, please see this help article on [tables](#).

1.4 How to add Comments and Track Changes

Comments can be added to your project by highlighting some text and clicking “Add comment” in the top right of the editor pane. To view existing comments, click on the Review menu in the toolbar above. To reply to a comment, click on the Reply button in the lower right corner of the comment. You can close the Review pane by clicking its name on the toolbar when you’re done reviewing for the time being.

Track changes are available on all our [premium plans](#), and can be toggled on or off using the option at the top of the Review pane. Track changes allow you to keep track of every change made to the document, along with the person making the change.

1.5 How to add Lists

You can make lists with automatic numbering ...

1. Like this,
2. and like this.

...or bullet points ...

- Like this,
- and like this.

1.6 How to write Mathematics

L^AT_EX is great at typesetting mathematics. Let X_1, X_2, \dots, X_n be a sequence of independent and identically distributed random variables with $E[X_i] = \mu$ and $\text{Var}[X_i] = \sigma^2 < \infty$, and let

$$S_n = \frac{X_1 + X_2 + \dots + X_n}{n} = \frac{1}{n} \sum_i^n X_i$$

denote their mean. Then as n approaches infinity, the random variables $\sqrt{n}(S_n - \mu)$ converge in distribution to a normal $\mathcal{N}(0, \sigma^2)$.

1.7 How to change the margins and paper size

Usually the template you're using will have the page margins and paper size set correctly for that use-case. For example, if you're using a journal article template provided by the journal publisher, that template will be formatted according to their requirements. In these cases, it's best not to alter the margins directly.

If however you're using a more general template, such as this one, and would like to alter the margins, a common way to do so is via the geometry package. You can find the geometry package loaded in the preamble at the top of this example file, and if you'd like to learn more about how to adjust the settings, please visit this help article on [page size and margins](#).

1.8 How to change the document language and spell check settings

Overleaf supports many different languages, including multiple different languages within one document.

To configure the document language, simply edit the option provided to the babel package in the preamble at the top of this example project. To learn more about the different options, please visit this help article on [international language support](#).

To change the spell check language, simply open the Overleaf menu at the top left of the editor window, scroll down to the spell check setting, and adjust accordingly.

1.9 How to add Citations and a References List

You can simply upload a `.bib` file containing your BibTeX entries, created with a tool such as JabRef. You can then cite entries from it, like this: [\[Gre93\]](#). Just remember to specify a bibliography style, as well as the filename of the `.bib`. You can find a [video tutorial here](#) to learn more about BibTeX.

If you have an [upgraded account](#), you can also import your Mendeley or Zotero library directly as a `.bib` file, via the upload menu in the file-tree.

1.10 Good luck!

We hope you find Overleaf useful, and do take a look at our [help library](#) for more tutorials and user guides! Please also let us know if you have any feedback using the Contact Us link at the bottom of the Overleaf menu — or use the contact form at <https://www.overleaf.com/contact>.

References

- [BV04] Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, March 2004.
- [CC02] Steven C. Chapra and Raymond P. Canale. *Numerical methods for engineers* ∴ McGraw-Hill,, Boston ∴, 4th ed. edition, c2002.
- [Gre93] George D. Greenwade. The Comprehensive Tex Archive Network (CTAN). *TUGBoat*, 14(3):342–351, 1993.