Logic Circuits Design

Lecture Five

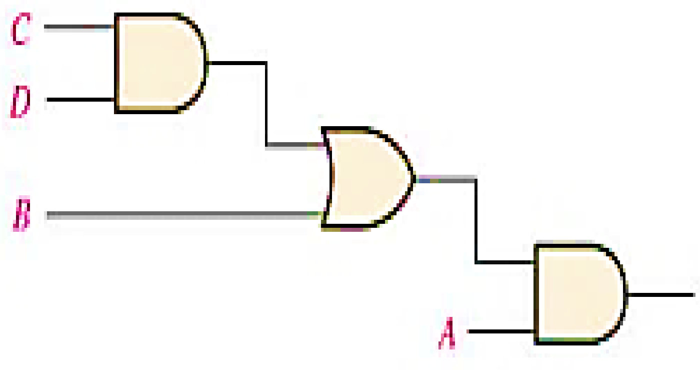
By
Dr. Noor Abdul Khaleq Z.

Content

- Boolean Analysis of Logic Circuits
- Simplification using Boolean Algebra
- Canonical and Standard Forms of Boolean Expressions
- Boolean Expression, and Truth Tables

Boolean Expression for a Logic Circuit

- To derive the Boolean expression for a given combinational logic circuit, begin at the left-most inputs and work toward the final output, writing the expression for each gate.
- **Example:** Determine the Boolean expression for the following logic Circuit



- **Solution:**

A(B + CD)

Constructing a Truth Table for a Logic Circuit

- Once the Boolean expression for a given logic circuit has been determined, a truth table that shows the output for all possible values of the input variables can be developed.

Step1: Evaluating the Expression

For example, to evaluate the expression A(B + CD), find the values of the variables that make the expression equal to 1 using the rules for Boolean addition and multiplication.

- Thus, the expression A(B + CD) equals 1 only if A = 1 and B + CD = 1.
- Now determine when the B + CD term equals 1. The term B + CD = 1 if either B = 1 or CD = 1 or if both B and CD equal 1.
- The term CD = 1 only if C = 1 and D = 1.
- Therefore, the expression A(B + CD) = 1 when A = 1 and B = 1 regardless of the values of C and D or when A = 1 and C = 1 and D = 1 regardless of the value of B. And the expression A(B + CD) = 0 for all other value combinations of the variable

Constructing a Truth Table for a Logic Circuit

Step2: Putting the Results in Truth Table Format

- ❑ First, list the sixteen input variable combinations of 1s and 0s in a binary sequence. Next, place a 1 in the output column for each combination of input variables that was determined in the evaluation. Finally, place a 0 in the output column for all other combinations of input variables. These results are shown in the truth table belo

Inputs				Output
A	B	C	D	$A(B + CD)$
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	1
1	1	0	0	1
1	1	0	1	1
1	1	1	0	1
1	1	1	1	1

Logic Simplification Using Boolean Algebra

- **Example:** Using Boolean algebra techniques, simplify this expression:

$$AB + A(B + C) + B(B + C)$$

- **Solution:** Note: The following is not necessarily the only approach.

$$AB + A(B + C) + B(B + C) = AB + AB + AC + BB + BC$$

$$= AB + AB + AC + B + BC$$

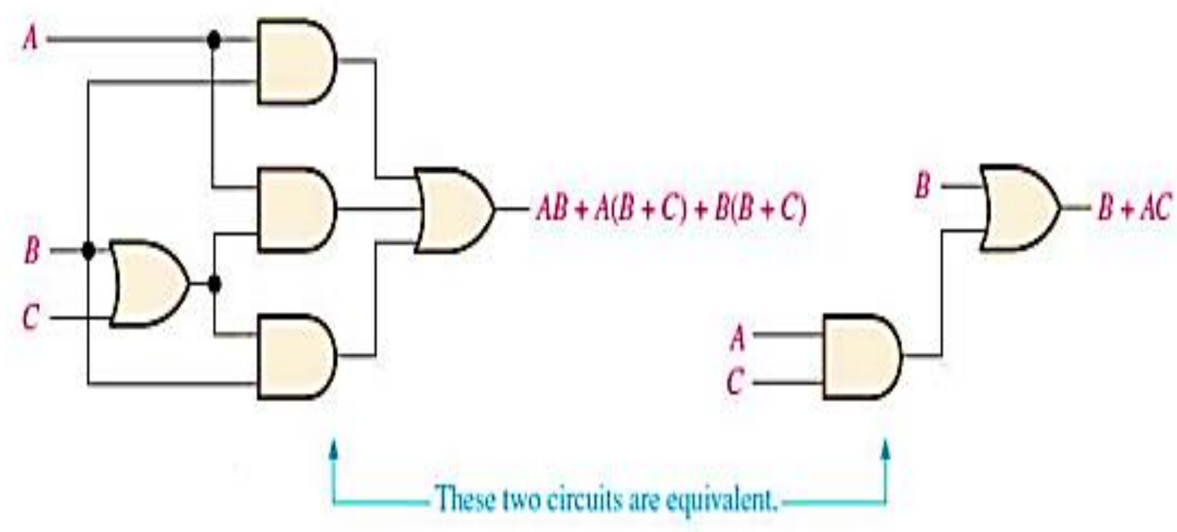
$$= AB + AC + B + BC$$

$$= AB + AC + B$$

$$= B + AC$$

Simplification Reduces Gates for the Same Function

- The figure below shows that five gates are required to implement the expression AB + A(B + C) + B(B + C) in its original form; however, only two gates are needed for the simplified expression (B + AC).



Simplification Reduces Gates for the Same Function

- **Example:** Simplify the following Boolean expression

$$[A\overline{B}\,(C + BD) + \overline{A}\,\overline{B}\,]\,C$$

- **Solution:**

$$
\begin{aligned}
[A\overline{B}(C + BD) + \overline{A}\,\overline{B}]C &= (A\overline{B}C + A\overline{B}BD + \overline{A}\,\overline{B})C \\
&= (A\overline{B}C + A \cdot 0 \cdot D + \overline{A}\,\overline{B})C \\
&= (A\overline{B}C + 0 + \overline{A}\,\overline{B})C \\
&= (A\overline{B}C + \overline{A}\,\overline{B})C \\
&= A\overline{B}CC + \overline{A}\,\overline{B}C \\
&= A\overline{B}C + \overline{A}\,\overline{B}C \\
&= \overline{B}C(A + \overline{A}) \\
&= \overline{B}C \cdot 1 = \overline{B}C
\end{aligned}
$$

Simplification Reduces Gates for the Same Function

- **Example:** Simplify the following Boolean expression:

$$\overline{A}BC + A\overline{B}\,\overline{C} + \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}C + ABC$$

- **Solution:**

$$\overline{A}BC + A\overline{B}\,\overline{C} + \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}C + ABC = BC(\overline{A} + A) + A\overline{B}\,\overline{C} + \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}C$$

$$= BC \cdot 1 + A\overline{B}(\overline{C} + C) + \overline{A}\,\overline{B}\,\overline{C}$$

$$= BC + A\overline{B} \cdot 1 + \overline{A}\,\overline{B}\,\overline{C}$$

$$= BC + A\overline{B} + \overline{A}\,\overline{B}\,\overline{C}$$

$$= BC + \overline{B}(A + \overline{A}\,\overline{C})$$

$$= BC + \overline{B}(A + \overline{C})$$ (using rule 11 $(A + \overline{A}\,\overline{C} = A + \overline{C})$)

$$= BC + A\overline{B} + \overline{B}\,\overline{C}$$

Simplification Reduces Gates for the Same Function

- **Example:** Simplify the following Boolean expression:

$$\overline{AB + AC} + \overline{A}\,\overline{B}C$$

- **Solution:**

$$\overline{AB + AC} + \overline{A}\,\overline{B}C = (\overline{AB})(\overline{AC}) + \overline{A}\,\overline{B}C$$ (By applying DeMorgan’s theorem)

$$= (\overline{A} + \overline{B})(\overline{A} + \overline{C}) + \overline{A}\,\overline{B}C$$ (By applying DeMorgan’s theorem to each term in the parentheses)

$$= \overline{A}\,\overline{A} + \overline{A}\,\overline{C} + \overline{A}\,\overline{B} + \overline{B}\,\overline{C} + \overline{A}\,\overline{B}C$$

Apply rule 7 ($\overline{A}\,\overline{A} = \overline{A}$) to the first term, and apply rule 10 [$\overline{A}\,\overline{B} + \overline{A}\,\overline{B}C = \overline{A}\,\overline{B}(1 + C) = \overline{A}\,\overline{B}$] to the third and last terms.

$$= \overline{A} + \overline{A}\,\overline{C} + \overline{A}\,\overline{B} + \overline{B}\,\overline{C} = \overline{A} + \overline{A}\,\overline{B} + \overline{B}\,\overline{C} = \overline{A} + \overline{B}\,\overline{C}$$

Standard Forms of Boolean Expressions

- All Boolean expressions can be converted into either of two standard forms:
  - The sum-of-products form (SOP).
  - The product-of-sums form (POS).

- Standardization makes the evaluation, simplification, and implementation of Boolean expressions much more systematic and easier

The Sum-of-Products (SOP) Form

- When two or more product terms are summed by Boolean addition, the resulting expression is a sum-of-products (SOP).

Some examples are:

- $AB + ABC$
- $ABC + CDE + \bar{B}\bar{C}D$
- $\bar{A}B + \bar{A}B\bar{C} + AC$

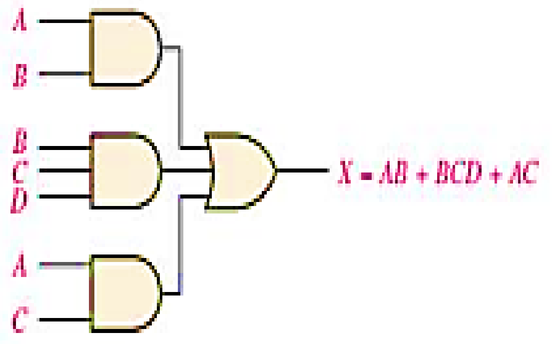
- Also, an SOP expression can contain a single-variable term, as in

$A + \bar{A}\bar{B}C + B\bar{C}D$

- **Note:**
- In an SOP expression, a single overbar cannot extend over more than one variable; however, more than one variable in a term can have an overbar.
- For example, an SOP expression can have the term $\bar{A}\bar{B}\bar{C}$ but not $\overline{ABC}$

AND/OR Implementation of an SOP Expression

- Implementing an SOP expression simply requires ORing the outputs of two or more AND gates.
- A product term is produced by an AND operation, and the sum (addition) of two or more product terms is produced by an OR operation.
- Therefore, an SOP expression can be implemented by AND-OR logic in which the outputs of a number (equal to the number of product terms in the expression) of AND gates connect to the inputs of an OR gate, as shown in the figure below for the expression AB+BCD + AC.



Conversion of a General Expression to SOP Form

- Any logic expression can be changed into SOP form by applying Boolean algebra techniques.
- For example, the expression A(B + CD) can be converted to SOP form by applying the distributive law: A(B +CD) = AB + ACD
- **Example:** Convert each of the following Boolean expressions to SOP form

(a) $AB + B(CD + EF)$ (b) $(A + B)(B + C + D)$ (c) $\overline{\overline{(A + B)} + C}$

- **Solution:**

$$\text{(a)}\quad AB + B(CD + EF) = AB + BCD + BEF$$

$$\text{(b)}\quad (A + B)(B + C + D) = AB + AC + AD + BB + BC + BD$$

$$\text{(c)}\quad \overline{\overline{(A + B)} + C} = \overline{\overline{(A + B)}}\,\overline{C} = (A + B)\overline{C} = A\overline{C} + B\overline{C}$$

The Standard SOP Form

- A standard SOP expression is one in which all the variables in the domain appear in each product term in the expression.
- For example, the expression $\bar{A}B\bar{C} + A\bar{B}D + \bar{A}B\bar{C}D$ is not in the standard SOP $A\bar{B}CD + \bar{A}\bar{B}C\bar{D} + AB\bar{C}\bar{D}$ while is in a standard SOP expression.
- **Note:** $\bar{A}B\bar{C} + A\bar{B}D + \bar{A}B\bar{C}D$ has a domain made up of the variables A, B, C, and D.
- However, the complete set of variables in the domain is not represented in the first two terms of the expression; that is, D or $\bar{D}$ is missing from the first term and C or $\bar{C}$ is missing from the second term.

Converting Product Terms to Standard SOP

- Each product term in an SOP expression that does not contain all the variables in the domain can be expanded to standard form to include all variables in the domain and their complements.
- As stated in the following steps, a nonstandard SOP expression is converted into standard form using Boolean algebra rule 6 ($A + \overline{A} = 1$).
- **Step 1:** Multiply each nonstandard product term by a term made up of the sum of a missing variable and its complement (you can multiply anything by 1 without changing its value).
- **Step 2:** Repeat Step 1 until all resulting product terms contain all variables in the domain in either complemented or uncomplemented form.
- In converting a product term to standard form, the number of product terms is doubled for each missing variable

Converting Product Terms to Standard SOP

- **Example:** Convert the following Boolean expression into standard SOP form: $A\bar{B}C + \bar{A}\bar{B} + AB\bar{C}D$
- **Solution:**

$$A\bar{B}C + \bar{A}\bar{B} + AB\bar{C}D = A\bar{B}C(D + \bar{D}) + \bar{A}\bar{B}(C + \bar{C}) + AB\bar{C}D$$

$$= A\bar{B}CD + A\bar{B}C\bar{D} + \bar{A}\bar{B}C + \bar{A}\bar{B}\bar{C} + AB\bar{C}D$$

$$= A\bar{B}CD + A\bar{B}C\bar{D} + \bar{A}\bar{B}C(D + \bar{D}) + \bar{A}\bar{B}\bar{C}(D + \bar{D}) + AB\bar{C}D$$

$$= A\bar{B}CD + A\bar{B}C\bar{D} + \bar{A}\bar{B}CD + \bar{A}\bar{B}C\bar{D} + \bar{A}\bar{B}\bar{C}D + \bar{A}\bar{B}\bar{C}\bar{D} + AB\bar{C}D$$

The Product-of-Sums (POS) Form

- When two or more sum terms are multiplied, the resulting expression is a product-of-sums (POS). Some examples are

- $(\overline{A} + B)(A + \overline{B} + C)$
- $(\overline{A} + \overline{B} + \overline{C})(C + \overline{D} + E)(\overline{B} + C + D)$
- $(A + B)(A + \overline{B} + C)(\overline{A} + C)$

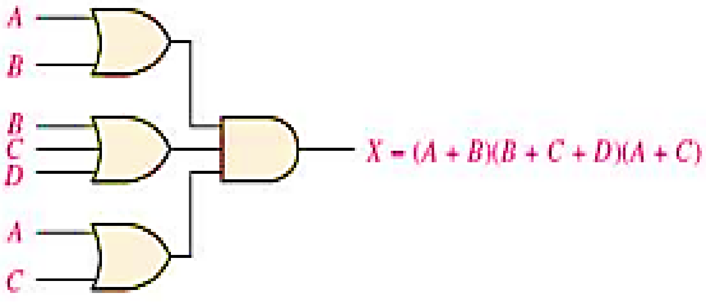
- A POS expression can contain a single-variable term, as in

$$\overline{A}\,(A + \overline{B} + C)\,(\overline{B} + \overline{C} + D)$$

- In a POS expression, a single overbar cannot extend over more than one variable; however, more than one variable in a term can have an overbar.
- For example, a POS expression can have the term $\overline{A} + \overline{B} + \overline{C}$ but not $\overline{A + B + C}$

Implementation of a POS Expression

- Implementing a POS expression simply requires ANDing the outputs of two or more OR gates.
- A sum term is produced by an OR operation, and the product of two or more sum terms is produced by an AND operation.
- For example, the implementation of the POS expression (A + B)(B + C + D)(A + C) is shown in the figure below:



The Standard POS Form

- A standard POS expression is one in which all the variables in the domain appear in each sum term in the expression. For example, $(\overline{A} + \overline{B} + \overline{C} + \overline{D})(A + B + \overline{C} + D)(A + B + C + \overline{D})$ is a standard POS expression.
- Converting a Sum Term to Standard POS
- A nonstandard POS expression is converted into standard form by using the following steps:
- **Step1:** Add to each nonstandard product term a term made up of the product of the missing variable and its complement $(A \cdot \overline{A} = 0)$.
- **Step2:** Apply rule 12: $A + BC = (A + B)(A + C)$
- **Step3:** Repeat Step 1 until all resulting sum terms contain all variables in the domain in either complemented or uncomplemented form.

Converting a Sum Term to Standard POS

- **Example:** Convert the following Boolean expression into standard POS form:

$$(A + \overline{B} + C)(\overline{B} + C + \overline{D})(A + \overline{B} + \overline{C} + D)$$

- **Solution**

The domain of this POS expression is A, B, C, D. Take one term at a time. The first term, $A + \overline{B} + C$, is missing variable D or $\overline{D}$, so add $D\overline{D}$ and apply rule 12 as follows:

$$A + \overline{B} + C = A + \overline{B} + C + D\overline{D} = (A + \overline{B} + C + D)(A + \overline{B} + C + \overline{D})$$

The second term, $\overline{B} + C + \overline{D}$, is missing variable A or $\overline{A}$, so add $A\overline{A}$ and apply rule 12 as follows:

$$\overline{B} + C + \overline{D} = \overline{B} + C + \overline{D} + A\overline{A} = (A + \overline{B} + C + \overline{D})(\overline{A} + \overline{B} + C + \overline{D})$$

The third term, $A + \overline{B} + \overline{C} + D$, is already in standard form. The standard POS form of the original expression is as follows:

$$(A + \overline{B} + C)(\overline{B} + C + \overline{D})(A + \overline{B} + \overline{C} + D) =$$
$$(A + \overline{B} + C + D)(A + \overline{B} + C + \overline{D})(A + \overline{B} + C + \overline{D})(\overline{A} + \overline{B} + C + \overline{D})(A + \overline{B} + \overline{C} + D)$$

Converting Standard SOP to Standard POS

- **Step 1:** Evaluate each product term in the SOP expression. That is, determine the binary numbers that represent the product terms.

- **Step 2:** Determine all of the binary numbers not included in the evaluation in Step1.

- **Step 3:** Write the equivalent sum term for each binary number from Step 2 and express in POS form.

- **Note:** Using a similar procedure, you can go from POS to SOP

Converting Standard SOP to Standard POS

- Example: Convert the following SOP expression to an equivalent POS expression:

$$\overline{A}\,\overline{B}\,\overline{C} + \overline{A}B\overline{C} + \overline{A}BC + A\overline{B}C + ABC$$

- Solution: The evaluation is as follows:

$$000 + 010 + 011 + 101 + 111$$

- Since there are three variables in the domain of this expression, there are a total of eight (2^3) possible combinations. The SOP expression contains five of these combinations, so the POS must contain the other three which are 001, 100, and 110.
- Remember, these are the binary values that make the sum term 0. The equivalent POS expression is

$$(A + B + \overline{C})\,(\overline{A} + B + C)\,(\overline{A} + \overline{B} + C)$$

Boolean Expressions and Truth Tables

- A truth table is simply a list of the possible combinations of input variable values and the corresponding output values (1 or 0).
- The truth table is a common way of presenting the logical operation of a circuit.
- Standard SOP or POS expressions can be determined from a truth table.

Converting SOP Expressions to Truth Table Format

- **Step1:** construct a truth table by listing all possible combinations of binary values of the variables in the expression.
- **Step2:** convert the SOP expression to standard form if it is not already.
- **Step3:** place a 1 in the output column (X) for each binary value that makes the standard SOP expression a 1 and place a 0 for all the remaining binary values.
- **Note:** an SOP expression is equal to 1 only if at least one of the product terms is equal to 1.

Converting SOP Expressions to Truth Table Format

- **Example:** Develop a truth table for the standard SOP expression

$$\overline{A}\,\overline{B}C + A\overline{B}\,\overline{C} + ABC$$

- **Solution:**

Inputs			Output	
A	*B*	*C*	*X*	Product Term
0	0	0	0	
0	0	1	1	$\overline{A}\,\overline{B}C$
0	1	0	0	
0	1	1	0	
1	0	0	1	$A\overline{B}\,\overline{C}$
1	0	1	0	
1	1	0	0	
1	1	1	1	ABC

Boolean Expressions and Truth Tables

Converting POS Expressions to Truth Table Format

- **Step1:** list all the possible combinations of binary values of the variables just as was done for the SOP expression.

- **Step2:** convert the POS expression to standard form if it is not already.

- **Step3:** place a 0 in the output column (X) for each binary value that makes the expression a 0 and place a 1 for all the remaining binary values.

- **Note:** a POS expression is equal to 0 only if at least one of the sum terms is equal to 0.

Converting POS Expressions to Truth Table Format

Example: Determine the truth table for the following standard POS expression

$$(A + B + C)\,(A + \overline{B} + C)\,(A + \overline{B} + \overline{C})\,(\overline{A} + B + \overline{C})\,(\overline{A} + \overline{B} + C)$$

Solution:

Inputs			Output	
A	B	C	X	Sum Term
0	0	0	0	$(A + B + C)$
0	0	1	1	
0	1	0	0	$(A + \overline{B} + C)$
0	1	1	0	$(A + \overline{B} + \overline{C})$
1	0	0	1	
1	0	1	0	$(\overline{A} + B + \overline{C})$
1	1	0	0	$(\overline{A} + \overline{B} + C)$
1	1	1	1	

Determining Standard Expressions from a Truth Table

Determining the standard SOP expression represented by a truth table

- **Step1:** list the binary values of the input variables for which the output is 1.
- **Step2:** Convert each binary value to the corresponding product term by replacing each 1 with the corresponding variable and each 0 with the corresponding variable complement.
- For example, the binary value 1010 is converted to a product term as follows:

$$1010 = A\overline{B}C\overline{D}$$

- If you substitute, you can see that the product term is 1:

$$A\overline{B}C\overline{D} = 1.\overline{0}.1.\overline{0}$$

$$=1 . 1 . 1 .1$$

$$=1$$

Determining Standard Expressions from a Truth Table

Determining the standard POS expression represented by a truth table

- **Step1:** list the binary values for which the output is 0.
- **Step2:** Convert each binary value to the corresponding sum term by replacing each 1 with the corresponding variable complement and each 0 with the corresponding variable.
- For example, the binary value 1001 is converted to a sum term as follows:

$$1001 = \overline{A} + B + C + \overline{D}$$

- If you substitute, you can see that the sum term is 0:

$$\begin{aligned} \overline{A} + B + C + \overline{D} &= \overline{1} + 0 + 0 + \overline{1} \\ &= 0 + 0 + 0 + 0 \\ &= 0 \end{aligned}$$

Determining Standard Expressions from a Truth Table

- **Example:** From the truth table below, determine the standard SOP expression and the equivalent standard POS expression.

- **Solution:**

Inputs			Output
A	B	C	X
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

There are four 1s in the output column and the corresponding binary values are 011, 100, 110, and 111. Convert these binary values to product terms as follows:

$$011 \longrightarrow \overline{A}BC$$
$$100 \longrightarrow A\overline{B}\overline{C}$$
$$110 \longrightarrow AB\overline{C}$$
$$111 \longrightarrow ABC$$

The resulting standard SOP expression for the output X is

$$X = \overline{A}BC + A\overline{B}\overline{C} + AB\overline{C} + ABC$$

For the POS expression, the output is 0 for binary values 000, 001, 010, and 101. Convert these binary values to sum terms as follows:

$$000 \longrightarrow A + B + C$$
$$001 \longrightarrow A + B + \overline{C}$$
$$010 \longrightarrow A + \overline{B} + C$$
$$101 \longrightarrow \overline{A} + B + \overline{C}$$

The resulting standard POS expression for the output X is

$$X = (A + B + C)(A + B + \overline{C})(A + \overline{B} + C)(\overline{A} + B + \overline{C})$$