A Big Data problem arising in Design Theory Cyclic $(v;\, k_1, k_2, k_3;\, \lambda)$ Difference Families with Three Base Blocks

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1 Introduction

Consider the ring $Z_v = \{0, 1, \ldots, v-1\}$ of integers modulo a positive integer v. Let $k_1, \ldots, k_i$ be positive integers and λ an integer such that

$$\lambda(v-1) = \sum_{i=1}^{t} k_i(k_i - 1), \tag{1}$$

and let $X_1, \ldots, X_t$ be subsets of Z_v such that $|X_i| = k_i, \quad i \in \{1, \ldots, t\}$. We say that $X_1, \ldots, X_t$ are **supplementary difference sets** (SDS) with parameters $(v; k_1, \ldots, k_t; \lambda)$, if the multiset of the union of the differences of $X_1, \ldots, X_t$ is equal to λ copies of $\{1, 2, \ldots, v-1\}$.

2 SDS Examples

1. Let $v = 7$, $k_1 = k_2 = k_3 = 2$, $\lambda = 1$, t = 3, producing: (7;2,2,2;1)

Construct $X_i \subseteq Z_i$ st:

1. $|X_i| = k_i$
2. $\bigcup_{i=1}^{t} \Delta(X_i) = \lambda \cdot \{1, 2, \ldots, v-1\}$

Given the requirements above, we get: $X_1 = \{0{,}1\}$, $X_2 = \{0{,}2\}$, $X_3 = \{0{,}3\}$ For each set X_i, we generate the difference all pairs of elements such that $d = (x - y) \bmod v$, $(x, y) \in X_i, x \neq y$. The collection of the union of the multi sets produces: $\{1, 6\} \bigcup \{2, 5\} \bigcup \{3, 4\} = 1$ copy of $\{1, 2, \ldots, 6\}$

- SDSs with $t = 1$ are called cyclic difference sets, (Baumert 1971)
- SDSs with $t = 2$ are called difference families with two base blocs, (Djokovic 2011)
- SDSs with $t = 3$ are called difference families with three base blocks. v prime p $\equiv$ 3(mod 4), (Djokovic, Kotsireas, 2016)

2. Let $v = 19$, $k_1 = 9, k_2 = 7, k_3 = 6$, $\lambda = 8$, t = 3, producing: (19;9,7,6;8)
By following the same requirements as above, we get:
$X_1 = \{0, 1, 2, 3, 5, 7, 12, 13, 16\}$, $X_2 = \{0{,}1{,}2{,}4{,}5{,}10{,}13\}$, $X_3 = \{0{,}1{,}4{,}6{,}8{,}13\}$
There exist $\binom{9}{2}$ combinations of pairs in X_1 alone, computing each difference by hand is rigorous, and instead completed by a program.

The program works slightly differently:

1. Check for the size of all subsets $\sum_{i=1}^{t} |U_i|$

2. Track the frequency of each difference of subsets produced $\{1, \ldots, v-1\}$

3. If we produce λ frequencies of each element, then we have the sufficient elements to create λ sets of unique elements

```python
def is_sds(v, lam, X_sets):
    for X in X_sets:
        for x in X:
            for y in X:
                if x != y:
                    d = (x - y) % v
                    if d != 0:
                        diff_count[d] += 1
    for d in range(1, v):
        if diff_count[d] != lam:
            return False
    return True
```

Click here to see the full code with comments

OPEN PROBLEMS:

1. (167; 83, 76, 73; 107)
2. (167; 79, 76, 74, 104)

PAF and PSD invariants

- **PAF** == Periodic Autocorrelation Function
- **PSD** == Power Spectral Density (DFT)
- For SDS (v; k_1; k_2; k_3; λ) with three base blocks, denote a new parameter as:

$$n = k_1 + k_2 + k_3 - \lambda.$$

- Denote the $\{\pm 1\}$-sequences (of lengths v) associated to X_1, X_2, X_3 by A, B, C
-

$$PSD(A,i) + PSD(B,i) + PSD(C,i) = 4n, \quad i = 1, \ldots, v-1$$

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$$PAF(A,i)+PAF(B,i)+PAF(C,i)=3v-4n, \quad i=1, i=1,\ldots,v-1$$

3 Computational Results

- D. Z Djokovic, I. S. Korsireas

  A class of cyclic $(v;\, k_1, k_2, k_3;\, \lambda)$ difference families with $\mathbf{v} \equiv 3 (\text{mod } 4)$ a prime. Special Matrices 4 (2016), pp. 317 - 325

- Skew hadamard matrices of orders $4 \cdot 239 = 956$ and $4 \cdot 331 = 1324$

- SDS $(v;\, k_1, k_2, k_3;\, \lambda)$ for all parameters set with prime $\mathbf{v} \leq 131$, $\mathbf{v} \equiv 3$ (mod 4).

How about SDS with 3 blocks for prime = 3 mod 3? Say for v = 23.

1. Look for SDS(23;11,10,7;11), n = 11+10+7-11 = 17,

2. Three $\{+1, -1\}$ sequences having PAF = $\text{t}_v - 4n$ = 3 $\cdot$ 23 - 4 $\cdot$ 14 = 1

and PSD $= 4\text{n} = 4 \cdot 17 = 68$

$$(23 - 2 \cdot 11)^2 + (23 - 2 \cdot 10)^2 + (23 - 2 \cdot 7)^2 = a^2 + b^2 + c^2 = 91$$

and it turns out that the equation $a^2 + b^2 + c^2 = 91$ has only one solution (up to sign): $\{1, 3, 9\}$