

Change of Basis and Model-view Transformation

- CS174A, discussion 1A, Winter 2020.
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- <https://github.com/luckiday/cs174a-1a-2020w>

Outline

- Change of basis
- Model-view Transformation

Change of basis

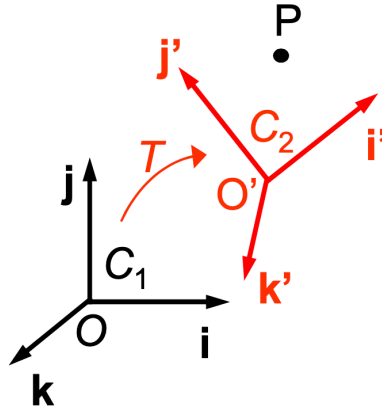
Recall: (Coordinate Systems) In homogeneous coordinate systems, a vector is denoted as:

$$\mathbf{v} = v_1 \mathbf{a} + v_2 \mathbf{b} + v_3 \mathbf{c} \rightarrow \mathbf{v} = [\mathbf{a} \quad \mathbf{b} \quad \mathbf{c} \quad O] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}$$

A point is

$$P = p_1 \mathbf{a} + p_2 \mathbf{b} + p_3 \mathbf{c} + O \rightarrow P = [\mathbf{a} \quad \mathbf{b} \quad \mathbf{c} \quad O] \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix}$$

Transformations as a change of basis



Question: if we apply a transformation, M_1 , to coordinate system C_1 , and get C_2 , what is the new position of P in C_2 ?

We use $[x, y, z, 1]^T$ to represent P 's position in C_1 , and $[x', y', z', 1]^T$ to represent P 's position in C_2 .

Solution:

In coordinate system C_1 :

$$P = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} + O = [\mathbf{i} \quad \mathbf{j} \quad \mathbf{k} \quad O] \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

In coordinate system C_2 :

$$P = x'\mathbf{i}' + y'\mathbf{j}' + z'\mathbf{k}' + O' = [\mathbf{i}' \quad \mathbf{j}' \quad \mathbf{k}' \quad O'] \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

Since C_2 is transformed from C_1 with M_1 ,

$$M_1 [\mathbf{i} \quad \mathbf{j} \quad \mathbf{k} \quad O] = [\mathbf{i}' \quad \mathbf{j}' \quad \mathbf{k}' \quad O'] \quad (*1)$$

Hence

$$[\mathbf{i} \quad \mathbf{j} \quad \mathbf{k} \quad O] \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = M_1 [\mathbf{i} \quad \mathbf{j} \quad \mathbf{k} \quad O] \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

Since $[\mathbf{i} \quad \mathbf{j} \quad \mathbf{k} \quad O]$ is the identity matrix, we have

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = M_1 \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} \rightarrow P_{C_1} = M_1 P_{C_2}$$

i.e., $P_{C_2} = M_1^{-1} P_{C_1}$

Question: what is M_1 with respect to the basis vectors?

Referring to Eq. (*1), and since $\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{O} \end{bmatrix}$ is the identity matrix:

$$M_1 = \begin{bmatrix} \mathbf{i}' & \mathbf{j}' & \mathbf{k}' & \mathbf{O}' \end{bmatrix} = \begin{bmatrix} i'_x & j'_x & k'_x & O'_x \\ i'_y & j'_y & k'_y & O'_y \\ i'_z & j'_z & k'_z & O'_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Successive transformations

If we transform C_2 to C_3 with M_2 . We consider the $\begin{bmatrix} \mathbf{i}'' & \mathbf{j}'' & \mathbf{k}'' & \mathbf{O}'' \end{bmatrix}$ as the identity matrix in the section transformation, then,

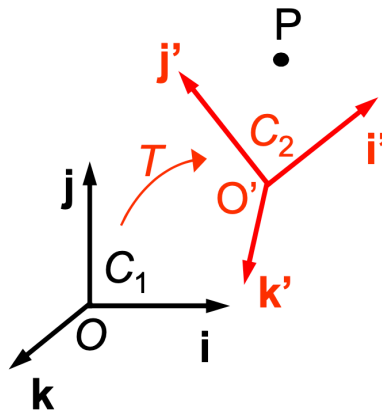
$$P_{C_2} = M_2 P_{C_3}$$

Therefore,

$$P_{C_1} = M_1 P_{C_2} = M_1 M_2 P_{C_3}$$

Notice: $\begin{bmatrix} \mathbf{i}'' & \mathbf{j}'' & \mathbf{k}'' & \mathbf{O}'' \end{bmatrix}$ of C_3 is the matrix representation in the C_2 .

Coordinate system transformation with rotation and transformation



Since the \mathbf{i}' , \mathbf{j}' , and \mathbf{k}' are orthogonal, the rotation from C_1 to C_2 is

$$R = \begin{bmatrix} i'_x & j'_x & k'_x & 0 \\ i'_y & j'_y & k'_y & 0 \\ i'_z & j'_z & k'_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The translation from O to O' is

$$T = \begin{bmatrix} 1 & 0 & 0 & O'_x \\ 0 & 1 & 0 & O'_y \\ 0 & 0 & 1 & O'_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

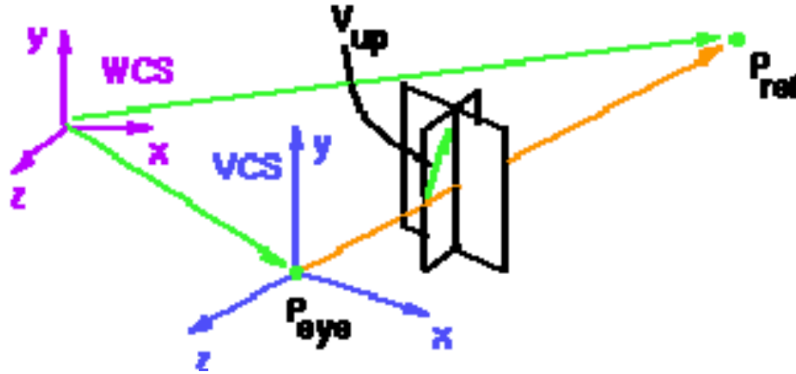
Thus,

$$M_1 = TR = \begin{bmatrix} i'_x & j'_x & k'_x & O'_x \\ i'_y & j'_y & k'_y & O'_y \\ i'_z & j'_z & k'_z & O'_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

P 's position in C_2 , $[x', y', z', 1]^T$, can be calculated with the inverse of the transformations

$$P_{C_2} = M_1^{-1} P_{C_1} = R^{-1} T^{-1} P_{C_1} = \begin{bmatrix} i'_x & j'_x & k'_x & 0 \\ i'_y & j'_y & k'_y & 0 \\ i'_z & j'_z & k'_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & -O'_x \\ 0 & 1 & 0 & -O'_y \\ 0 & 0 & 1 & -O'_z \\ 0 & 0 & 0 & 1 \end{bmatrix} P_{C_1}$$

Model-view transformation



Given eye point P_{eye} , reference point P_{ref} , and up vector v_{up} , build M_{cam} for model-view transformation.

$$\begin{aligned} k' &= \frac{P_{eye} - P_{ref}}{|P_{eye} - P_{ref}|} \\ i' &= \frac{v_{up} \times k'}{|v_{up} \times k'|} \\ j' &= k' \times i' \end{aligned}$$

The new origin $O' = P_{eye}$. Thus,

$$M_{cam} = \begin{bmatrix} i'_x & j'_x & k'_x & O'_x \\ i'_y & j'_y & k'_y & O'_y \\ i'_z & j'_z & k'_z & O'_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transform from WCS (world coordinate system) to VCS (view coordinate system)

$$P_{vcs} = M_{cam}^{-1} P_{wcs} = \begin{bmatrix} i'_x & i'_y & i'_z & -O'_x \\ j'_x & j'_y & j'_z & -O'_y \\ k'_x & k'_y & k'_z & -O'_z \\ 0 & 0 & 0 & 1 \end{bmatrix} P_{wcs}$$

Quiz

Using the following WCS information about the camera location and orientation, derive the 4x4 homogeneous matrix, M_{cam} , needed to transform points from WCS to VCS.

- Camera location is located at: $(-5, 0, 5)$
- P_{ref} is the origin: $(0, 0, 0)$
- Up vector is the y-axis: $(0, 1, 0)$