Mathematics

Exercises 2024/2025

1. Basic Operations on Matrices

For follwing matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} -1 & 2 & 3 \\ 4 & 0 & 6 \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{pmatrix}$$

1. Calculate: $\mathbf{A} + \mathbf{B}$; $\mathbf{B} - \mathbf{A}$; $\mathbf{A} + \mathbf{C}$; $\mathbf{D} + \mathbf{E}$.
2. Calculate $\frac{1}{2}\mathbf{A}$, $2\mathbf{B}$, $-3\mathbf{C}$, and $4\mathbf{D}$.
3. Calculate the products $\mathbf{A} \cdot \mathbf{B}$; $\mathbf{B} \cdot \mathbf{A}$; $\mathbf{A} \cdot \mathbf{D}$; $\mathbf{D} \cdot \mathbf{E}$.

2. Determinants 2x2 and 3x3

Calculate the determinants for the 2x2 and 3x3 matrices given below.

2x2 Matrices:

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}$$

3x3 Matrices:

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 2 & 4 & -2 \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 4 \\ 5 & 3 & 2 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 4 & -2 \\ 1 & 5 & 3 \end{pmatrix}$$

3. Determinants using Laplace's Expansion

Calculate the determinants of the following matrices:

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 4 & 0 \\ 3 & 2 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 4 & 0 \\ 3 & 2 & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 2 & 3 & 1 & 4 \\ 1 & 0 & 0 & 6 \\ 3 & 2 & 1 & 5 \\ 2 & 1 & 4 & 0 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 2 & 3 & 1 & 4 & 5 \\ 1 & 4 & 0 & 0 & 7 \\ 3 & 0 & 0 & 0 & 0 \\ 2 & 1 & 4 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

4. Determinants from the Gauss Method and Triangular Matrices

Perform row and column operations to reduce the following matrices to an upper triangular form and calculate their determinants by taking the product of the diagonal elements.

$$\mathbf{A} = \begin{pmatrix} 12 & 3 \\ -18 & -4 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

5. Inverse of a Matrix from the formula

1. Find the inverse matrix for matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{pmatrix}$$

and verify if the result is correct.

2. Determine the rank of the matrix:

$$\mathbf{B} = \begin{pmatrix} 4 & -3 & 7 \\ -1 & 6 & 3 \\ 2 & 9 & 1 \end{pmatrix}$$

6. Inverse of a Matrix using the Gauss Method

Find the inverse matrices using the Gauss method:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 2 & 3 & 2 \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

7. Linear Equations old school

Solve the following systems of equations without using matrices:

- $3x - 2y = 5, \quad 2x + 3y = 7,$
- $2x - 3y = 10, \quad 4x + 5y = 20,$
- $2x - y + z = 3, \quad x + 2y - z = 1, \quad 3x - y + 2z = 11.$
- $2x - 3y + 4z + 2t = 2, \quad 3x + 2y - 5z + 3t = 3, \quad 4x - 3y + 2z - 5t = 4, \quad 5x + 4y - 3z + 2t = 5.$

8. Linear equations by Cramer's Rule

1. Solve the system of equations:

$$\begin{cases} 2x_1 - 3x_2 = 7 \\ 3x_1 + 5x_2 = 2 \end{cases}$$

2. Solve the system of equations:

$$\begin{cases} 2x + y - z = 1 \\ x - y + 2z = 4 \\ 3x - 2z = -1 \end{cases}$$

3. Solve the system of equations:

$$\begin{cases} x + y + z - t = 2 \\ x - z + 2t = 6 \\ 2x - 3y + t = 4 \\ 3x + y + 3z - 4t = -2 \end{cases}$$

4. Why can't the following system of equations be solved using Cramer's rule?

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 3 \\ 4x_1 + 5x_2 + 6x_3 = 2 \\ 7x_1 + 8x_2 + 9x_3 = 1 \end{cases}$$

9. Linear equations by Gauss Elimination

$$\begin{cases} x + 2y - 2z = 4 \\ 2x + y + z = 0 \\ 3x + 2y + z = 1 \end{cases} \qquad \begin{cases} x + y + z - t = 2 \\ 2x + y + z = 3 \\ -x + z - t = 0 \\ 3x + 2y - z + 2t = -1 \end{cases} \qquad \begin{cases} x + y - z - t = 0 \\ 2x + 3y - 2z + t = 4 \\ 3x + 5z = 0 \\ -x + y - 3z + 2t = 3 \end{cases}$$

10. Linear equations by Matrix Inversion

1. Solve the system of linear equations using the inverse matrix method:

$$\begin{cases} x + 2y + 3z = 5, \\ 2y + 3z = 4, \\ 3z = 3. \end{cases}$$

2. Solve the system of linear equations using the inverse matrix method:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 41, \\ 4x_1 + 5x_2 + 6x_3 = 93, \\ 7x_1 + 8x_2 + 9x_3 = 145. \end{cases}$$

11. Vectors I

1. By what number should vector $\mathbf{a} = [3, 4]$ be multiplied so that its length is equal to 1?

2. Calculate the length of vector $\mathbf{b} = [1, 1]$ and find the unit vector of this vector.

3. Plot the vector and the unit vector from the previous exercise.

4. Calculate the length of vector $\mathbf{c} = [1, 2, 3]$ and find the unit vector of this vector.

5. Find the Cartesian coordinates of vector $\mathbf{v} = [2, 3, 4]$ in the basis $\{\mathbf{b_1} = [1, 0, 1], \mathbf{b_2} = [0, 1, 0], \mathbf{b_3} = [1, 0, -1]\}$.

12. Vectors II

1. Perform the addition of vector $[2, 1]$ to vector $[-1, 1]$. Plot both vectors and their sum on a graph.

2. Calculate the area of the triangle spanned by vectors $[2, 1]$ and $[-1, 1]$.

3. Calculate the volume of the parallelepiped spanned by vectors $[2, 1]$, $[-1, 1]$, and $[1, 2]$.

4. Check if vectors $[2, 1]$ and $[-1, 1]$ are perpendicular.

5. Calculate the angle in degrees between vectors $[4, 2, 1]$ and $[1, 3, 2]$.

6. For three-dimensional vectors: $\mathbf{a} = [a_x, a_y, a_z]$, $\mathbf{b} = [b_x, b_y, b_z]$, $\mathbf{c} = [c_x, c_y, c_z]$, prove that the following identity is satisfied:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

13. Vectors III

1. Divide the line segment connecting points $A(-1, 2)$ and $B(3, -2)$ in the ratio $1 : 3$. Illustrate the result on a graph.

2. Project vector $\mathbf{a} = (3, 4)$ onto the OX and OY axes. Illustrate the result on a graph.

3. Project vector $\mathbf{a} = (2, 3)$ onto vector $\mathbf{b} = (1, 1)$. Ilustrate the result on a graph.

4. Project vector $\mathbf{b} = (1, 1)$ onto vector $\mathbf{a} = (2, 3)$. Ilustrate the result on a graph.

14. Equations of lines on a plane

- The line passes through points $A(1, 2)$ and $B(3, 4)$. Find the equation of the line.
- The line passes through point $A(1, 2)$ and is parallel to the line $y = 2x + 3$. Find the equation of the line.
- The line passes through point $A(1, 2)$ and is perpendicular to the line $y = 2x + 3$. Find the equation of the line.
- We have two lines $y = 2x + 3$ and $y = 3x + 2$. Find the intersection point of these lines and calculate the angle between them.
- Write the equation of the line passing through point $A(1, 2)$ and parallel to the vector $\mathbf{v} = [2, 3]$.
- We have the line $y = 2x + 3$. Find an example of a line perpendicular and parallel to it.
- We have the line $y = 2x + 3$ and point $A(1, 2)$. Find the distance from point A to the line.

- The line intersects the coordinate axes at points $A(2,0)$ and $B(0,3)$. Find the equation of the line.
- Calculate the angle between the line $y = x + 3$ and the Ox axis.
- Provide a vector perpendicular to the line $x + y + 1 = 0$.

15. Equations of second-order curves (conic sections)

- Find the equation of a circle with center at point $A(1,2)$ and radius $r = 3$.
- Find the equation of a parabola intersecting the Ox axis at points $x = 2$, $x = 4$, and passing through point $y(3) = 1$.
- Find the center of the ellipse with the equation $x^2 + 4y^2 - 4x - 16y + 16 = 0$.
- Find the slope $(m > 0)$ of the line $y = mx - 5$ that is tangent to the circle with the equation $x^2 + y^2 = 1$.
- Find the intersection points of the hyperbola $x^2 - y^2 = 1$ with the ellipse's line $x^2 + 4y^2 = 6$.
- For the given hyperbola $x^2 - y^2 = 1$, find the distance between its branches.

16. Equations of planes in space

- The plane passes through points $A(1,2,3)$, $B(3,4,5)$, and $C(2,1,4)$. Find the equation of the plane.
- The plane passes through point $A(1,2,3)$ and is parallel to the plane $2x + 3y + 4z = 5$. Find the equation of the plane.
- The plane passes through point $A(1,2,3)$ and is perpendicular to the normal vector $\mathbf{n} = [2,3,4]$. Find the equation of the plane.
- We have two planes $2x + 3y + 4z = 5$ and $3x + 4y + 2z = 6$. Find the line of intersection of these planes.
- Write the equation of the plane passing through point $A(1,2,3)$ and parallel to vectors $\vec{v_1} = [1,0,1]$ and $\vec{v_2} = [0,1,-1]$.
- We have the plane $2x + 3y + 4z = 5$. Find an example of a plane parallel and perpendicular to it.
- We have the plane $2x + 3y + 4z = 5$ and point $A(1,2,3)$. Find the distance from point A to this plane.
- The plane intersects the coordinate axes at points $A(2,0,0)$, $B(0,3,0)$, and $C(0,0,4)$. Find the equation of the plane.
- Calculate the angle between the plane $x + y + z = 1$ and the plane $x = 0$ (i.e., the yz plane).
- Find the vector perpendicular to the plane $x + y + z = 1$.

17. Equations of second-order surfaces

- Write the equation of a sphere with center at point $P = (1, 2, 3)$ and radius $r = 3$.
- Do the spheres with equations $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 2$ have any common points?
- What curve in space is formed by the intersection of the sphere $x^2 + y^2 + z^2 = 1$ with the sphere $(x - 1)^2 + y^2 + z^2 = 1$? Find the equation of this curve.
- Write the equation of the tangent plane to the paraboloid $z = (x - 1)^2 + y^2 + 1$ at point $P(1, 0, 1)$.

Calculus - under construction