STAT 542: Homework 6

Due: April. 19 midnight on Canvas

Please make sure that your solutions are readable and the file size is reasonable. Typing the answers is highly encouraged.

Problem 1.

The following example, due to Feller[1], shows that different probability distributions can have exactly the same finite integer moments: Let $Z = e^{-F}$, where $F \sim \mathcal{N}(0,1)$ is standard normal.

- [1pts] Compute all the integer moments of Z.
- [2pts] Consider the parameter family with density given by $p_\theta(F) = p(F)[1+\theta \sin(2\pi F)]$, where $|\theta| < 1$ and $p()$ denotes the density of the standard normal distribution. Compute all the integer moments of Z_θ.
- [2pts bonus] Is it possible to estimate θ using the empirical moments of samples from $Z_\theta = e^{-F_\theta}$ where $F_\theta \sim p_\theta$? Why? *Hint: Do the integer moments of Z_θ depend on θ? Is Carleman's condition (p10 of the slides) satisfied?*

Problem 2.

[2pts] Consider a naive language generation model that produces a random text composed only of the three words 'machines', 'have', and 'conscious'. Using the bag of words representation, we may assume that the words are generated independently and identically distributed (i.i.d.) according to some unknown distribution $[p_1, p_2, p_3]$ over those three words.

Without any further information, a statistician assumes that the prior distribution of $[p_1, p_2, p_3]$ is uniform on the probability simplex (i.e., the triangle formed by the vertices $[1,0,0]$, $[0,1,0]$, and $[0,0,1]$). Then the statistician obtains a text in which the number of the three words 'machines', 'have', and 'conscious' are 25, 25, 50. What is the posterior distribution of $[p_1, p_2, p_3]$?

Hint: Use the fact that the Dirichlet distribution is the Bayesian conjugate of the multinomial distribution.

[1] Feller, 1968, *An Introduction to Probability Theory and Its Applications.*

Problem 3.

The following example shows that the nonnegative rank of bounded-rank matrices may be unbounded: Let $M \in \mathbb{R}^{n \times n}$ where $M_{ij} = (i-j)^2$.

- [1pts] Show that $\mathrm{rank}(M) \leq 3$ *Hint: it suffices to show that* $M_{ij} = \sum_{k=1}^{3} f_k(i) g_k(j)$ *for some functions* f_k *and* g_k, $k = 1, 2, 3$.
- [1pts] Show that $\mathrm{rank}(M) \geq 3$ for $n \geq 3$. *Hint: find a* 3×3 *submatrix of* M *which has rank 3.*
- [2pts bonus] Show that the nonnegative rank $\mathrm{rank}^+(M) \geq \log_2 n$ *Hint: It suffices to consider just the locations of zeros. Suppose that* $M = AB^\top$ *for some entrywise nonnegative matrices* $A, B \in \mathbb{R}^{n \times r}$. *Can two rows of* A *have the same zero pattern?*

Remark: This problem illustrates that for general nonnegative M*, the nonnegative rank can be much larger than the rank. Curiously, if* M *is restricted to have 0 or 1 entries, the problem becomes much harder, and it is not known whether* $\log \mathrm{rank}^+(M) < (\log \mathrm{rank}(M))^{O(1)}$*! This is a reformulation of the famous log-rank conjecture; see Moitra's book.*