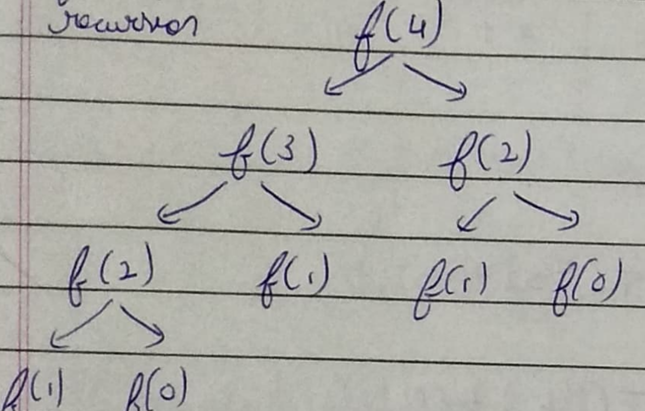
Time and Space Complexity Analysis of Recursive Algorithm

At any particular time, no two function call at the same level of recursion will be at the stack at same time. Only calls that are interlinked will be in the stack at the same time

Space Complexity of Recursion tree = Height of the tree

fibbonacci recursion



Auxillary space = $O(n)$ for fibbonacci using recursion

Types of Recursion

1) Linear
2) Divide and Conquer

⇒ Divide and Conquer Recurrences

Form :- $T(x) = a_1 T(b_1 x + \varepsilon_1(n)) + a_2 T(b_2 x + \varepsilon_2(x)) + \dots + a_K T(b_K x + \varepsilon_K(x)) + \cancel{g(x)} + g(N)$

for $x \geq x_0$ — some constant

$\varepsilon(n)$ and $g(N)$ are some function

e.g. $T(N) = T(N/2) + C$

$a_1 = 1 \quad b_1 = 1/2 \quad \varepsilon_1(n) = 0 \quad g(x) = C$

How to actually solve to get complexity

1) Plug and chug
2) Masters Theorem
3) Akra Bazzi Formula

⇒ Akra Bazzi Formula

$$T(x) = \theta\left(x^p + x^p \int_1^x \frac{g(u)}{u^{p+1}}\, du\right)$$

What is P?

$$a_1 b_1^p + a_2 b_2^p + \cdots\cdots = 1$$

$$\boxed{\sum_{i=1}^{K} a_i b_i^p = 1}$$

eg Binary Search ~~⇒ T(N) = T(N/2) + C~~

eg Merge Sort ⇒ $T(N) = 2T(N/2) + (N-1)$

$a_1 = 2 \quad b_1 = 1/2 \qquad g(x) = x-1$

$2 \times \left(\frac{1}{2}\right)^p = 1$

$\rightarrow \underline{\underline{P = 1}}$

put p in formula

$$T(x) = \theta\left(x^1 + x^1 \int_1^x \frac{u-1}{u^2}\, du\right)$$

$$= \theta\left(x + x \int_1^x \left(\frac{1}{u} - \frac{1}{u^2}\right) du\right)$$

$$= \theta\left(x + x \left[\log u + \frac{1}{u}\right]_1^x\right)$$

$$= \theta\left(x + x\left[\log x + \frac{1}{x} - 1\right]\right)$$

$$= \theta\left(\cancel{x} + x\log x + 1 - \cancel{x}\right)$$

$$= \theta(x\log x + 1)$$

$$T(x) = \underline{\underline{\theta(x\log x)}}$$

eg $T(N) = 2T(N/2) + \frac{8}{9} T\left(\frac{3N}{4}\right) + N^2$

$a_1 = 2 \quad b_1 = 1/2 \quad a_2 = 8/9 \quad b_2 = 3/4$

$$2 \times \left(\frac{1}{2}\right)^p + \frac{8}{9} \times \left(\frac{3}{4}\right)^p = 1$$

$$\cancel{2}^{1} \times \left(\frac{1}{\cancel{4}_2}\right) + \frac{\cancel{8}^{1}}{\cancel{9}} \times \left(\frac{\cancel{9}}{\cancel{16}_2}\right) = 1 = 1$$

$$\underline{\underline{p = 2}}$$

$$T(n) = \theta\left(x^2 + x^2 \int_1^x \frac{u^2}{u^3}\, du\right)$$

$$= \theta\left(x^2 + x^2 \log x\right) \quad \text{[Ignoring lower order terms]}$$

$$= \underline{\underline{\theta\left(x^2 \log x\right)}}$$

e.g If you cant find value of p

$$T(n) = 3T\left(\frac{n}{3}\right) + 4T\left(\frac{n}{4}\right) + n^2$$

$p=1 \qquad 3 \times \left(\frac{1}{3}\right)^1 + 4\left(\frac{1}{4}\right)^1 = 2 > 1 \Rightarrow p > 1$

$p=2 \qquad 3 \times \left(\frac{1}{3}\right)^2 + 4\left(\frac{1}{4}\right)^2 = \frac{7}{12} < 1$

$$\Rightarrow p < 2$$

Note :- when $p <$ power of $(g(n))$ then ans $= g(n)$

Here $g(n) = n^2$

$p < 2$ (i.e power of $g(n)$)

hence, ans $= O(g(n))$.

Proof :- $T(n) = \Theta\left(x^p + x^p \int_1^x \frac{u^2}{u^{p+1}} du\right)$

$= \Theta\left(x^p + x^p \int_1^x u^{1-p} du\right)$

$= \Theta(x^p + x^2)$

$\because p < 2$, x^p becomes less dominating term

$= \Theta(x^2)$

⇒ Linear Recurrences

e.g. Fibbonacci numbers $F(N) = F(N-1) + F(N-2)$ — ①

Form - $f(x) = a_1 f(x-1) + a_2 f(x-2) + a_3 f(x-3) + \cdots + a_n f(x-n)$

Homogenous Equation

$f(x) = \sum_{i=1}^{n} a_i f(x-i)$, for a_i, n is fixed

n = order of recurrence

Soln) 1) Put $f(n) = \alpha^n$ for some constant α

$\alpha^n = \alpha^{n-1} + \alpha^{n-2}$

$\alpha^n - \alpha^{n-1} - \alpha^{n-2} = 0$

dividing by α^{n-2}

$\alpha^2 - \alpha - 1 = 0$

2) Roots of quadratic equation

$\alpha = \frac{1 \pm \sqrt{5}}{2}$

$\alpha_1 = \frac{1+\sqrt{5}}{2}$, $\alpha_2 = \frac{1-\sqrt{5}}{2}$

3) $f(n) = c_1\alpha_1^n + c_2\alpha_2^n$ is a soln for fibonacci

$= f(n-1) + f(n-2)$

$f(n) = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$ — ②

4) No of roots = No of ans you have already
Here we have 2 roots α_1 & α_2
Here we should have 2 ans already
$F(0) = 0$ & $F(1) = 1$

$f(0) = 0 = c_1 + c_2 \Rightarrow c_1 = -c_2$ — ③

$f(1) = 1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right)$

from ③

$$1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) - c_1\left(\frac{1-\sqrt{5}}{2}\right)$$

$$c_1 = \frac{1}{\sqrt{5}} \quad c_2 = \frac{-1}{\sqrt{5}}$$

putting this in eq^n ②

$$\boxed{f(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

↳ formula for n^{th} fibonacci numbers

$$f(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \underbrace{\left(\frac{1-\sqrt{5}}{2}\right)^n}_{\text{as } n \to \infty\text{, this will be } 0}\right]$$

Hence this is less dominating term, ignore

Complexity = $O\left(\frac{1+\sqrt{5}}{2}\right)^N$ → golden ratio

$T(N) = O(1.6180)^N$

Code :- int fiboFormula(int n){

```
    return (int)(Math.pow(((1+Math.sqrt(5))/2),n) /
                 Math.sqrt(5));
}
```

If we get equal roots

eg $f(n) = 2f(n-1) + f(n-2)$

1) $f(n) = 2^{\alpha}$

$$\alpha^n = 2\alpha^{n-1} + \alpha^{n-2}$$

$$\alpha^n - 2\alpha^{n-1} - \alpha^{n-2} = 0$$

$$\alpha^2 - 2\alpha - 1 = 0$$

$$\alpha = 1 \quad \text{(double root)}$$

If α is repeated r times then $\alpha^n, n\alpha^n, n^2\alpha^n, \ldots\ldots n^{r-1}\alpha^n$ are all solution to the recurrence.

Hence I can take 2 roots as : $1, n\alpha^n$

$$f(n) = c_1(1)^n + c_2(n\alpha)^n$$

$$f(n) = c_1 + c_2 n$$

Let's say $f(0) = 0$ & $f(1) = 1$

$f(0) = 0 = c_1$ $\quad$ $f(1) = 1 = c_1 + c_2$

$$c_2 = 1$$

Ans = $f(n) = n \Rightarrow$ Time complexity $\underline{\underline{O(n)}}$

⇒ Non-Homogeneous Linear recurrence

$$f(n) = a_1 f(n-1) + a_2 f(n-2) + a_3 f(n-3) + \cdots\cdots + \cdots + a_\alpha f(n-\alpha) + g(n)$$

↳ Where this extra function is there, it is non-homogeneous linear recurrence

How to solve

1) Replace $g(n)$ by 0 & solve usually

$$f(n) = 4f(n-1) + \underbrace{3^n}_{=0}, \quad f(1) = 1$$

$$f(n) = 4f(n-1)$$

$$\alpha^n = 4\alpha^{n-1}$$

$$\alpha^n - 4\alpha^{n-1} = 0 \Rightarrow \underline{\underline{\alpha = 4}}$$

Homogeneous soln $\Rightarrow$ $f(n) = c_1 \alpha^n$

$$\boxed{f(n) = c_1 4^n}$$

2) Take $g(n)$ on one side and find particular solution

$f(n) - 4f(n-1) = 3^n$

Guess something that is similar to $g(n)$.

If $g(n) = n^2$, then guess a polynomial of degree 2

guess $\Rightarrow$ $f(n) = c3^n$ $\leftarrow$ put c here

$c3^n - 4c3^{n-1} = 3^n \Rightarrow c = -3$

particular soln $\Rightarrow$ $f(n) = -3^{n+1}$

3) Add both soln together to get general soln

$f(n) = c_1 4^n + (-3^{n+1})$

$f(1) = 1 \Rightarrow c_1 4 - 3^2 = 1$

$c_1 = 5/2$

$$\boxed{f(n) = \frac{5}{2} 4^n - 3^{n+1}}$$

How do we guess a particular solution?

1) If $g(n)$ is exponential, guess of same type

E.g. $g(n) = 2^n + 3^n$

guess :- $f(n) = a2^n + b3^n$

2) If $g(n)$ is polynomial, guess of same degree

E.g. $g(n) = n^2 - 1$ $\Rightarrow$ guess of same degree 2.

guess :- $f(n) = ax^2 + bx + c$

3) If combination of both

$g(n) = 2^n + n$

guess :- $f(n) = a2^n + (bx + c)$

Let say you guessed, $f(n) = a2^n$ and it fails, then try $(an+b)2^n$. If this also fails increase the degree of constants $(a^2n + bn + c)2^n$.

eg $\quad f(n) = 2f(n-1) + 2^n \quad , \quad f(0) = 1$

1) Put $2^n = 0$

$$f(n) = 2f(n-1)$$
$$f(n) = \alpha^n$$
$$\alpha^n = 2\alpha^{n-1}$$
$$\alpha^n - 2\alpha^{n-1} = 0$$
$$\alpha = 2$$

2) ~~type~~ Guess particular soln

$$g(n) = 2^n$$

guess $f(n) = a2^n$

$$a2^n = 2a2^{n-1} + 2^n$$
$$a = a+1 \quad \times \quad \text{wrong}$$

hence, guess another one

$$f(n) = (an+b)2^n$$
$$(an+b)2^n = 2(a(n-1)+b)2^{n-1} + 2^n$$
$$an+b = an - a + b + 1$$
$$\boxed{a = 1} \quad \text{discarded } b$$
$$f(n) = n2^n \quad \text{particular soln}$$

3) General ans

$$f(n) = c_1 2^n + n2^n$$
$$f(0) = 1 \Rightarrow 1 = c_1 + 0$$
$$c_1 = 1$$
$$\boxed{f(n) = 2^n + n2^n}$$

Complexity $= O(n2^n)$