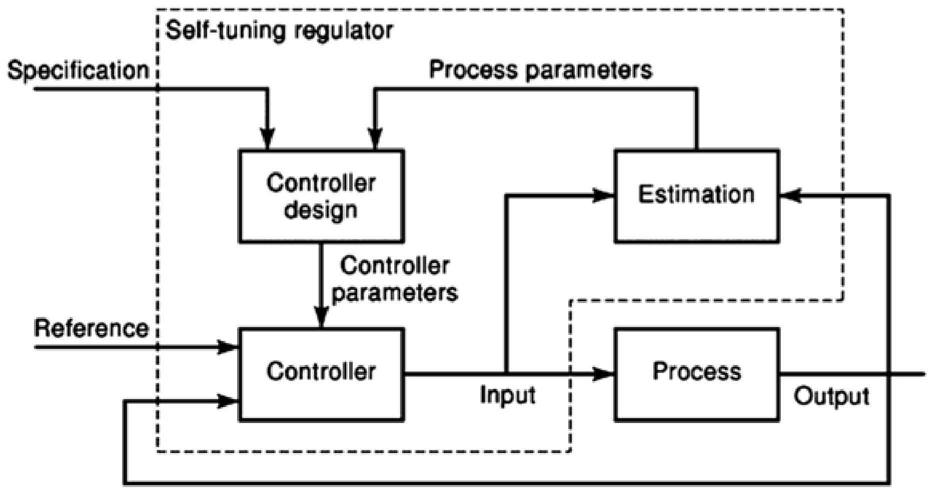
Contents

Indirect Self Tuning Regulator: one and two degrees of freedom controllers

Introduction

This document illustrates how to apply a self-tuning regulator to a system. Beginning by estimating the process model using the recursive least squares method for deterministic models. Selecting the specifications to have the required performance plus specifying the control design approach which is the model following approach, we can continuously update the controller parameters.



Estimation

In this example I am going to assume a model that I know its parameters. Then, adding the functions of estimation and the Diophantine solver in order to update the controller parameters.

Estimation

Transfer function model

$$A(z)y(t) = B(z)u(t) + e(t)$$

Where

$$A(z) = z^{na} + a_1 z^{na-1} + \cdots + a_{na}\,, \quad B(z) = b_0 z^{nb} + b_1 z^{nb-1} + \cdots + b_{nb}$$
$$A(z^{-1}) = 1 + a_1 z^{-1} + \cdots + a_{na} z^{-na}\,, \quad B(z^{-1}) = b_0 + b_1 z^{-1} + \cdots + b_{nb} z^{-nb}$$

$e(t)$ is a white noise

Writing the difference equation, z is a shift operator ($zy(t) = y(t+1),\ \ z^{-1}y(t) = y(t-1)$)

$$(1 + a_1 z^{-1} + \cdots + a_{na} z^{-na})y(t) = z^{-d}\left(b_0 + b_1 z^{-1} + \cdots + b_{nb} z^{-nb}\right)u(t)$$
$$y(t) = -a_1 y(t-1) - a_2 y(t-2) - \cdots - a_{na} y(t-na) + b_1 u(t-d) + \cdots + b_{nb} u(t-d-nb)$$

$$y(t) = \varphi^T(t-1)\theta^0$$

$$\varphi^T(t-1) = [-y(t-1) \ldots - y(t-na) \quad u(t-1) \ldots \quad u(t-nb)], \qquad \theta^0 = \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_{na} \\ b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_{nb} \end{bmatrix}$$

Algorithm

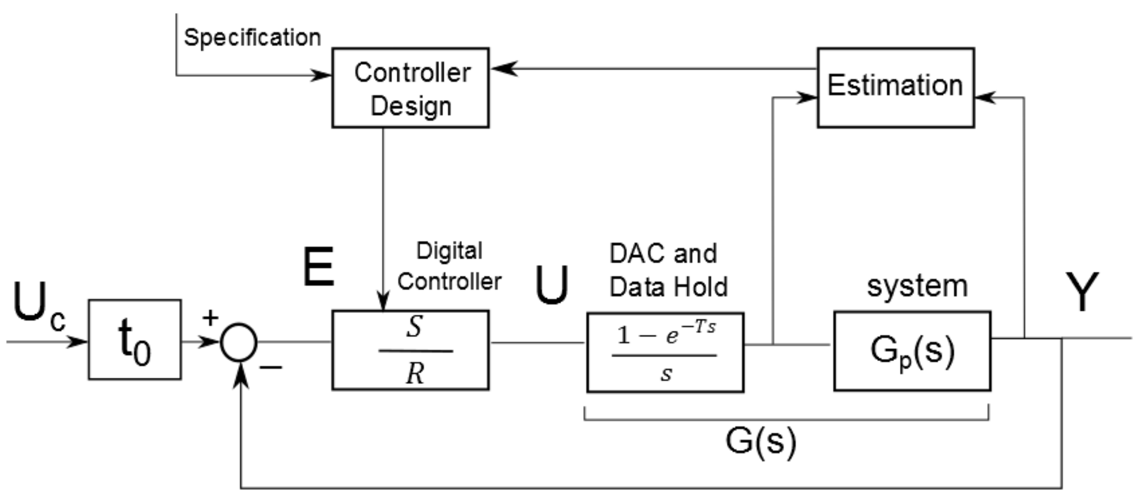
At $t = 0, P(0) = \alpha I$, where I is the identity matrix $n \times n$ and α is large number. $\theta(0) = 0$. For t > 0 do the following

1. $\varphi^T(t-1) = [-y(t-1) \ldots - y(t-na) \quad u(t-1) \ldots \quad u(t-nb)]$ Regressors
2. $P(t) = P(t-1) - \frac{P(t-1)}{\left(1+\varphi^T(t-1)P(t-1)\varphi(t-1)\right)}\varphi^T(t-1)P(t-1)$ Covariance matrix
3. $K(t) = P(t)\varphi(t-1) = \frac{P(t-1)\varphi(t-1)}{\left(1+\varphi^T(t-1)P(t-1)\varphi(t-1)\right)}$ Weighting (scaling) factor
4. $\varepsilon(t) = y(t) - \varphi^T(t-1)\hat{\theta}(t-1)$ Predicted error
5. $\hat{\theta}(t) = \hat{\theta}(t-1) + K(t)\varepsilon(t)$ Estimation
6. $e(t) = y(t) - \varphi^T(t-1)\hat{\theta}(t)$ Residual error (not used here)

Where $\left(1 + \varphi^T(t)P(t-1)\varphi(t)\right)$, is scalar for single output system.

Controller

One Degree of freedom controller



Assume the model

$$G(s) = \frac{Y(s)}{U(s)}$$

The pulse transfer function is given by

$$G(z) = \frac{Y(z)}{U(z)} = \frac{z^{-d}B(z^{-1})}{A(z^{-1})}$$

The controller is given by

$$D(z) = \frac{S(z^{-1})}{R(z^{-1})} = \frac{U_c}{E}$$

Where

$$\begin{aligned}
A(z^{-1}) &= 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_{na} z^{-na} \\
B(z^{-1}) &= b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + a_{nb} z^{-nb} \\
S(z^{-1}) &= s_0 + s_1 z^{-1} + s_2 z^{-2} + \cdots + a_{ns} z^{-ns} \\
R(z^{-1}) &= 1 + r_1 z^{-1} + r_2 z^{-2} + \cdots + a_{nr} z^{-nr}
\end{aligned}$$

A and R polynomials are assumed to be monic

t_0: is 1 divided by the dc gain in order to follow the input at its magnitude

The characteristic equation is given by

$$AR + z^{-d}BS = A_m A_0 = \alpha$$

Where A_m is the required poles of the closed loop transfer function and A_0 are the poles of the observer. The observer poles can be interpreted as, for instance, how fast the system will recover after it has been influenced by a disturbance. That is, it has to be faster by four to five times of the fastest closed loop poles. Their orders are determined by the order of α (i.e. $na_m + na_n = n\alpha$)

where

$$\alpha(z^{-1}) = 1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \cdots + \alpha_{n\alpha} z^{-n\alpha}$$

Using the following two equations,

$$\begin{aligned}
nr &= nb + d - 1 \\
ns &= na - 1
\end{aligned}$$

we can find nr and ns

the final order of the characteristic equation is

$$n\alpha = na + nb + d - 1$$

Final closed loop transfer function is given by

$$\frac{Y(z)}{Uc(z)} = \frac{z^{-d}B(z^{-1})S(z^{-1})}{A(z^{-1})R(z^{-1}) + B(z^{-1})S(z^{-1})} = \frac{B_m(z^{-1})\,A_0(z^{-1})}{A_m(z^{-1})\,A_0(z^{-1})} = \frac{B_m(z^{-1})A_0(z^{-1})}{\alpha(z^{-1})}$$

The characteristic equation of the system which is $AR + z^{-d}BS = A_m A_0 = \alpha$ is called the Diophantine equation we can find the controller parameters.

Example

$$G(s) = \frac{e^{-T_s s}(s + 1.5)}{(s + 3)(s - 1)}$$

The requirements are to have 2 poles at $z_1 = 0.8187$ and $z_2 = 0.6703$ and the observer poles at $z = 0.2$.

Discretizing with sampling time of Ts = 0.1 seconds including the zero order hold

$$G(z) = \frac{z^{-2}(0.09813 \; - \; 0.0845\, z^{-1})}{1 \; - 1.846\, z\, z^{-1} \; + \; 0.8187\, z^{-2}}$$

$$b_0 = 0.09813, \quad b_1 = -\, 0.0845$$
$$a_1 = -1.846\, z, \quad a_2 = 0.8187$$

in difference equation form

$$y(k) = -a_1\, y(k-1) - a_2\, y(k-2) + b_0\, u(k-2) + b_1 u(k-3)$$

$$na = 2, nb = 1, d = 2$$

For estimation we use

$$\varphi^T(t-1) = [-y(t-1), -y(t-2), u(t-2), u(t-3)]$$

After estimating, we can find θ which gas the coefficients of polynomial A and B.

Now we get ns and nr

$$nr = 1 + 2 - 1 = 2 \rightarrow \; R = 1 + r_1\, z^{-1} + \mathrm{r}_2 \mathrm{z}^{-2}$$
$$ns = 2 - 1 = 1 \rightarrow S = s_0 + s_1\, z^{-1}$$

$$n\alpha = 2 + 1 + 2 - 1 = 4$$

$$A_m = 1 - 1.4891\, z^{-1} + 0.5488\, z^{-2}$$

$$na_0 = n\alpha - na_m = 3 - 2 = 1$$
$$A_0 = (1 - 0.2\, z^{-1})^2$$
$$A_m A_0 = \alpha = 1 - 1.8891\, z^{-1} + 1.1844\, z^{-2} - 0.2791\, z^{-3} + 0.0220\, \mathrm{z}^{-4}$$

The controller is in the form of

$$D(z) = \frac{s_0 + s_1\, z^{-1}}{1 + r_1\, z^{-1} + r_2\, z^{-2}}$$

In difference equation form

$$e = u_c(k) - y(k)$$
$$u(k) = -r_1\, u(k-1) - r_2\, u(k-2) + s_0\, e(k) + s_1\, e(k-1)$$

Diophantine equation can be written as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & b_0 & 0 \\ a_2 & a_1 & b_1 & b_0 \\ 0 & a_2 & 0 & b_1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ s_0 \\ s_1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} - \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{bmatrix}$$

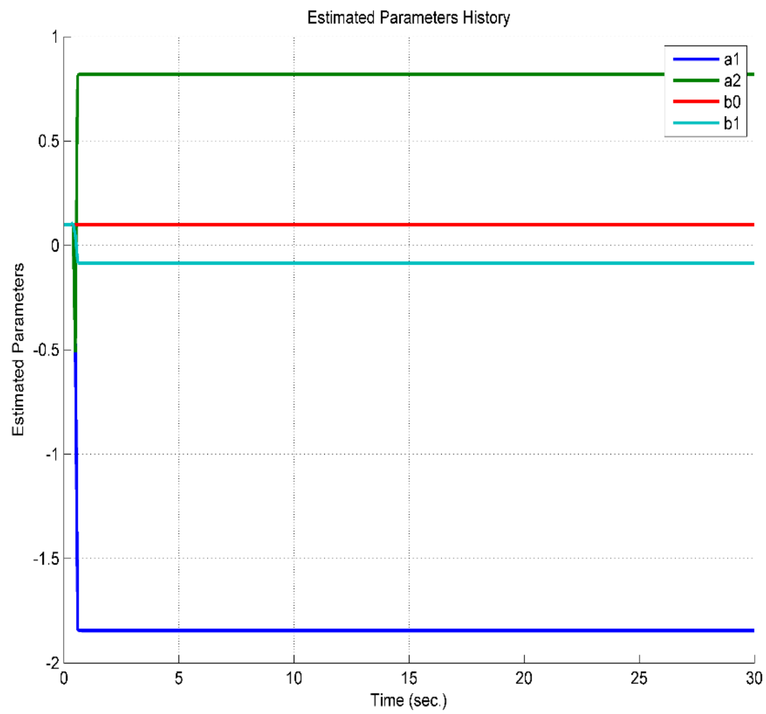
$$SV = M \rightarrow V = S^{-1} M$$

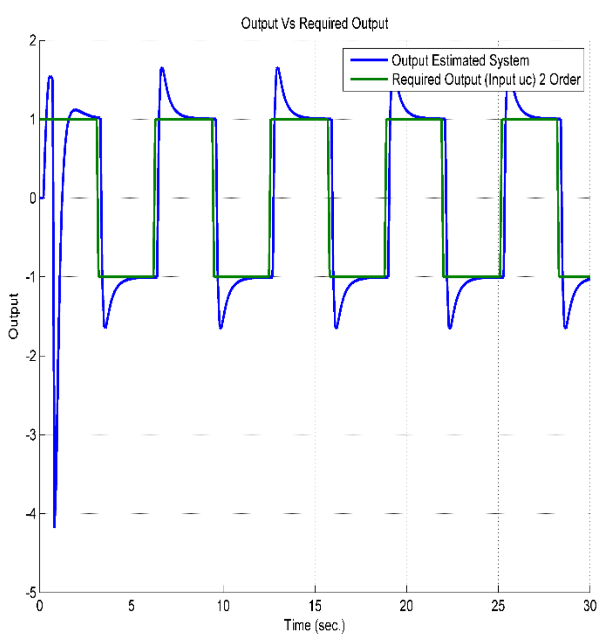
$$M = \begin{bmatrix} r_1 \\ r_2 \\ s_0 \\ s_1 \end{bmatrix}$$

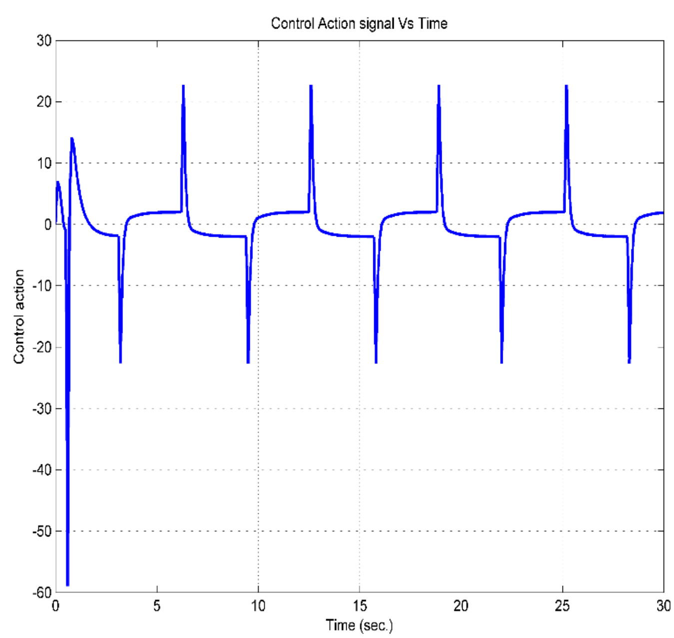
Using the given algorithm and estimated A and B vector M can be estimated to get the coefficient of polynomials S and R.

Using Matlab Code we find, we get the closed loop system response for square wave and sine wave, respectively

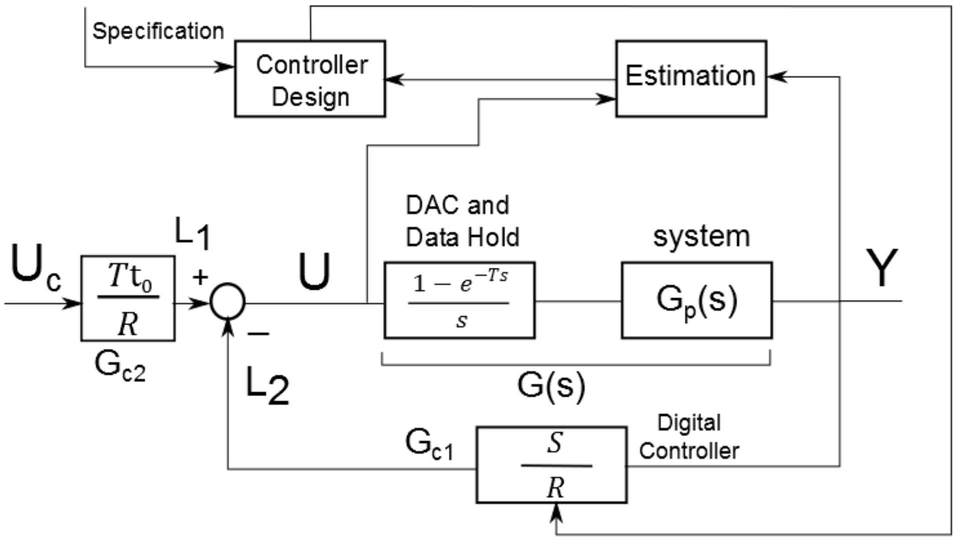
```
Gz =

             0.09813 z - 0.0845
  z^(-1) * ----------------------
           z^2 - 1.846 z + 0.8187

Gz_estm =

      0.09813 z - 0.0845
  -------------------------
  z^3 - 1.846 z^2 + 0.8187 z


Gc =

    11.32 z^2 - 8.251 z
  -----------------------
  z^2 - 0.04306 z - 0.8248


Gcl =

            1.015 z^2 - 1.614 z + 0.6372
  --------------------------------------------
  z^4 - 1.889 z^3 + 1.184 z^2 - 0.2791 z + 0.0
```







Two Degrees of freedom Controller



Assume the model

$$G(s) = \frac{Y(s)}{U(s)}$$

The pulse transfer function is given by

$$G(z) = \frac{Y(z)}{U(z)} = \frac{z^{-d}B(z^{-1})}{A(z^{-1})}$$

The first controller is given by

$$D_1(z) = \frac{S(z^{-1})}{R(z^{-1})} = \frac{L_2}{Y}$$

While the Second controller is given by

$$D_1(z) = \frac{S(z^{-1})}{R(z^{-1})} = \frac{L_1}{U_c}$$

Where

$$A(z^{-1}) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_{na} z^{-na}$$
$$B(z^{-1}) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + a_{nb} z^{-nb}$$
$$S(z^{-1}) = s_0 + s_1 z^{-1} + s_2 z^{-2} + \cdots + a_{ns} z^{-ns}$$
$$R(z^{-1}) = 1 + r_1 z^{-1} + r_2 z^{-2} + \cdots + a_{nr} z^{-nr}$$

A and R polynomials are assumed to be monic

t_0: is 1 divided by the dc gain in order to follow the input at its magnitude

T: is a polynomial that is to be chosen to specify the zeros of the closed loop transfer function

We say that this controller is two degrees of freedom as we can specify the polynomials both S, R form Diophantine equation and T.

The characteristic equation is given by

$$AR + z^{-d}BS = A_m A_0 = \alpha$$

Where A_m is the required poles of the closed loop transfer function and A_0 are the poles of the observer. The observer poles can be interpreted as, for instance, how fast the system will recover after it has been influenced by a disturbance. That is, it has to be faster by four to five times of the fastest closed loop poles. Their orders are determined by the order of α (i.e. $na_m + na_n = n\alpha$)

where

$$\alpha(z^{-1}) = 1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \cdots + \alpha_{n\alpha} z^{-n\alpha}$$

Using the following two equations,

$$nr = nb + d - 1$$
$$ns = na - 1$$

we can find nr and ns

the final order of the characteristic equation is

$$n\alpha = na + nb + d - 1$$

Final closed loop transfer function is given by

$$\frac{Y(z)}{Uc(z)} = \frac{z^{-d}B(z^{-1})\,T}{A(z^{-1})R(z^{-1}) + B(z^{-1})S(z^{-1})} = \frac{B_m(z^{-1})\,A_0(z^{-1})}{A_m(z^{-1})\,A_0(z^{-1})} = \frac{B_m(z^{-1})A_0(z^{-1})}{\alpha(z^{-1})}$$

The characteristic equation of the system which is $AR + z^{-d}BS = A_m A_0 = \alpha$ is called the Diophantine equation we can find the controller parameters.

There are different methods to choose the polynomial T such as:

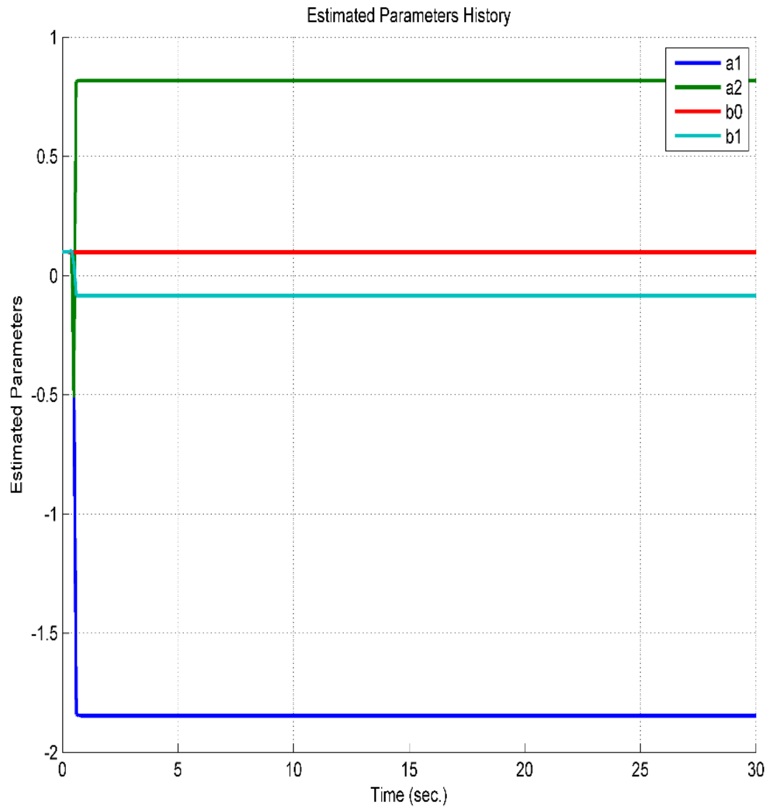
1- $T = A_0$, it’s chosen here to cancel the polynomial A_0 from the closed loop transfer function
2- $BT = z^{-n}, n \geq d$, it’s chosen to cancel the zeros of the closed loop transfer function which is polynomial B and this is valid for stable system zeros (non minimum phase zeros).
3- It’s chosen to cancel the dynamics of the error

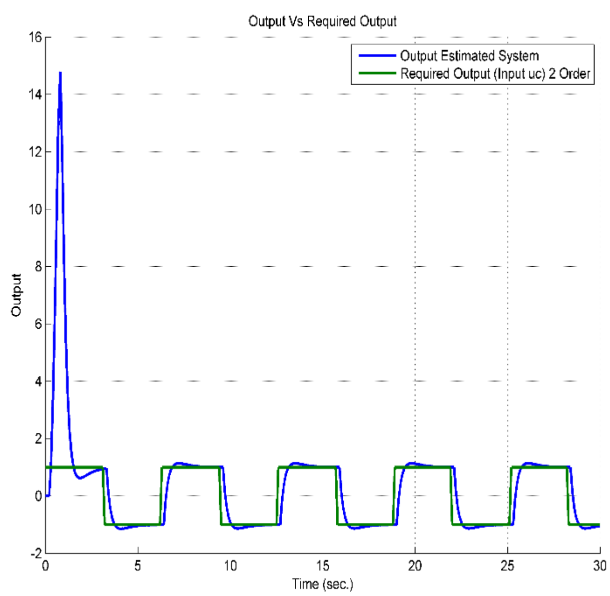
First Method $T = A_0$

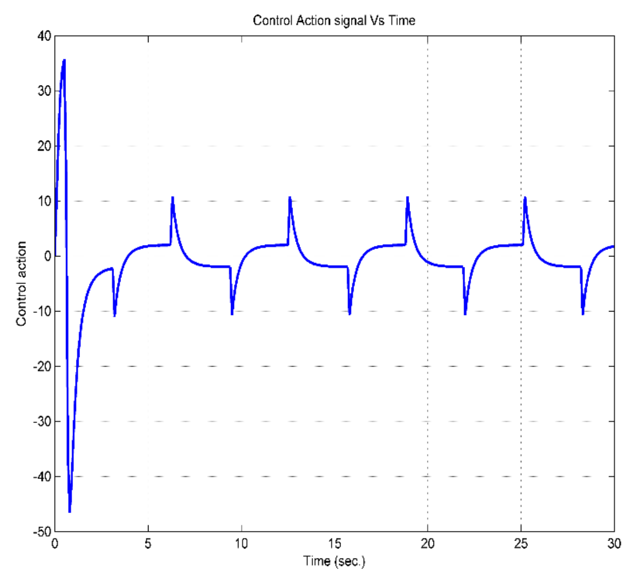
Using Matlab Code we find, we get the closed loop system response for square wave and sine wave, respectively

Square Wave

```
Gz_estm =

      0.09813 z - 0.0845
  --------------------------
  z^3 - 1.846 z^2 + 0.8187 z

Gc1 =

    11.32 z^2 - 8.251 z
  ------------------------
  z^2 - 0.04306 z - 0.8248

Gc2 =

  4.385 z^2 - 1.754 z + 0.1754
  ----------------------------
    z^2 - 0.04306 z - 0.8248

Gcl =

    0.4303 z - 0.3705
  ---------------------
  z^2 - 1.489 z + 0.5488
```







Second Method $BT = z^{-n}$

Using Matlab Code we find, we get the closed loop system response for square wave and sine wave, respectively with $n = d$

```
Gz_estm =

      0.09813 z - 0.0845
  -------------------------
  z^3 - 1.846 z^2 + 0.8187 z


Gc1 =

    11.32 z^2 - 8.251 z
  -----------------------
  z^2 - 0.04306 z - 0.8248

Gc2 =

                  0.03825
  --------------------------------------------
  0.09813 z^3 - 0.08873 z^2 - 0.0773 z + 0.0697

Gcl =

                   0.03825
  -----------------------------------------------
  z^4 - 1.889 z^3 + 1.184 z^2 - 0.2791 z + 0.02195
```