learning

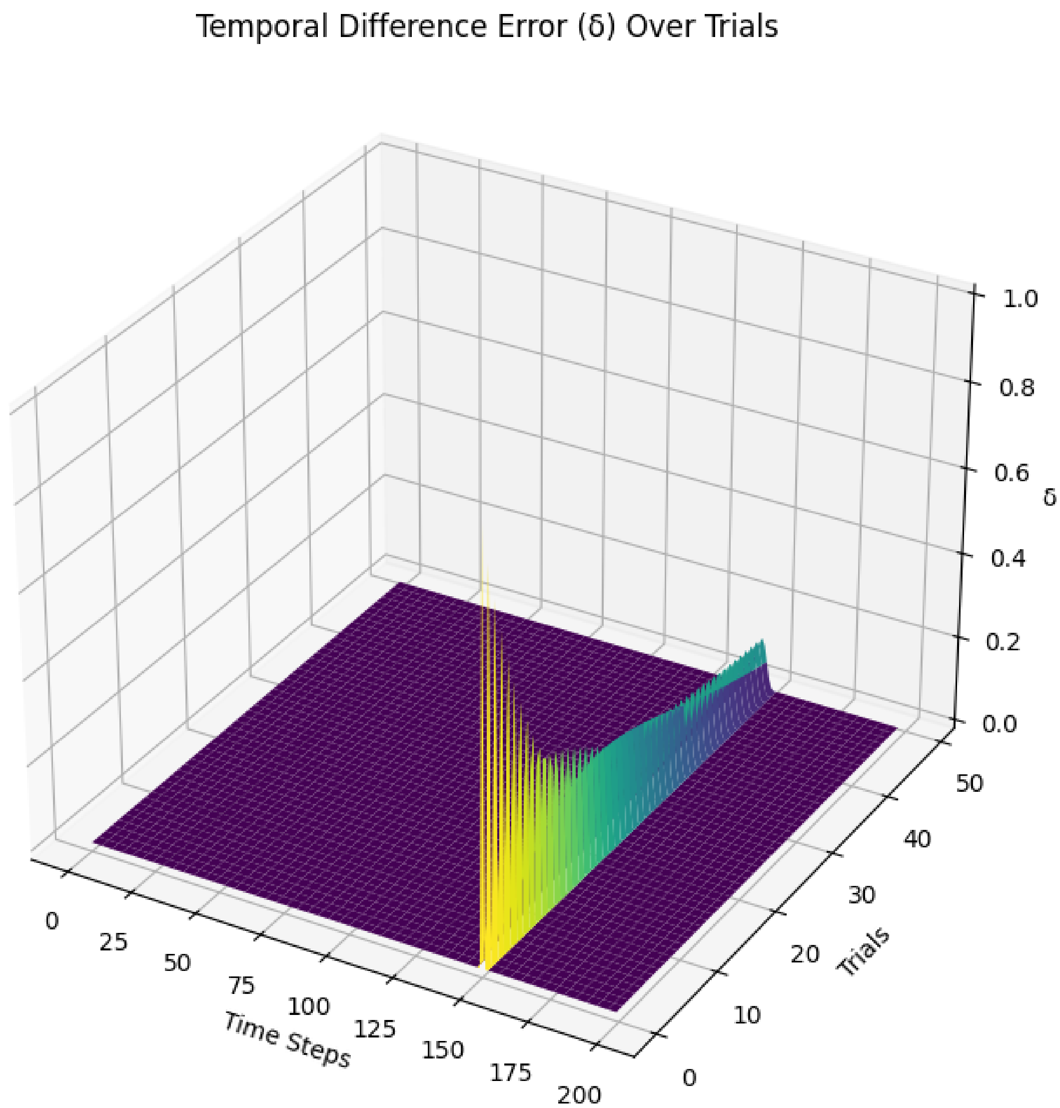
November 22, 2024

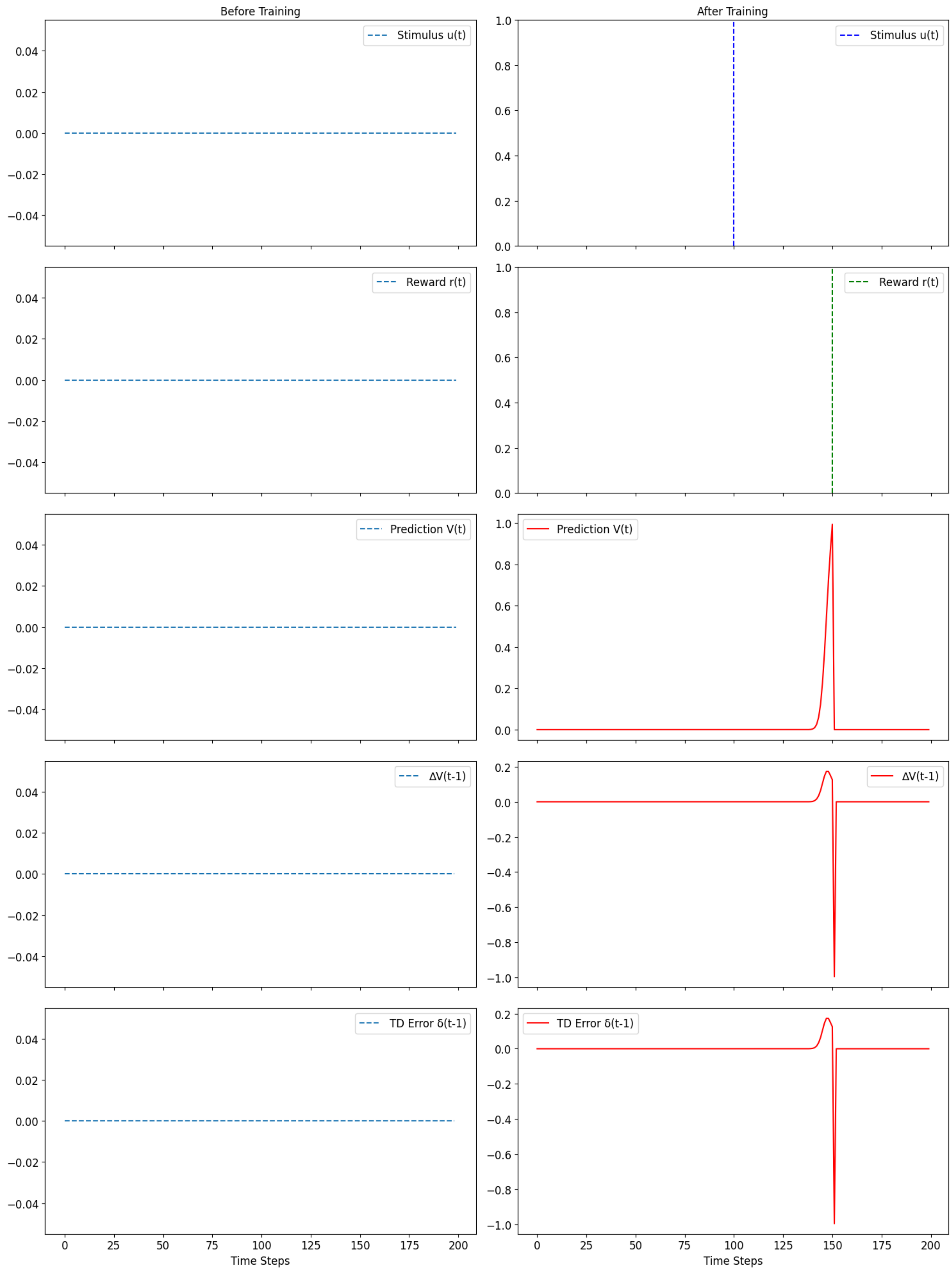
1 Temporal Difference Learning (TDE)

[3]:
```
from temporal_difference_learning import TemporalDifferenceLearning
import matplotlib.pyplot as plt
import numpy as np
```

1.1 1. Recreate figure 9.2

[18]:
```
n_time_steps = 200
alpha = 0.1
gamma = 0.9
num_trials = 50
stimulus_time = 100
reward_time = 150

td = TemporalDifferenceLearning(n_time_steps, alpha, gamma)
predictions, deltas = td.run_trials(reward_time, stimulus_time, num_trials)
td.plot_results(predictions, deltas, stimulus_time, reward_time)
```


Temporal Difference Error (δ) Over Trials
Time Steps
0
25
50
75
100
125
150
175
200
Trials
0
10
20
30
40
50
δ
0.0
0.2
0.4
0.6
0.8
1.0


Before Training
After Training
Stimulus u(t)
Reward r(t)
Prediction V(t)
ΔV(t-1)
TD Error δ(t-1)
Time Steps
Time Steps

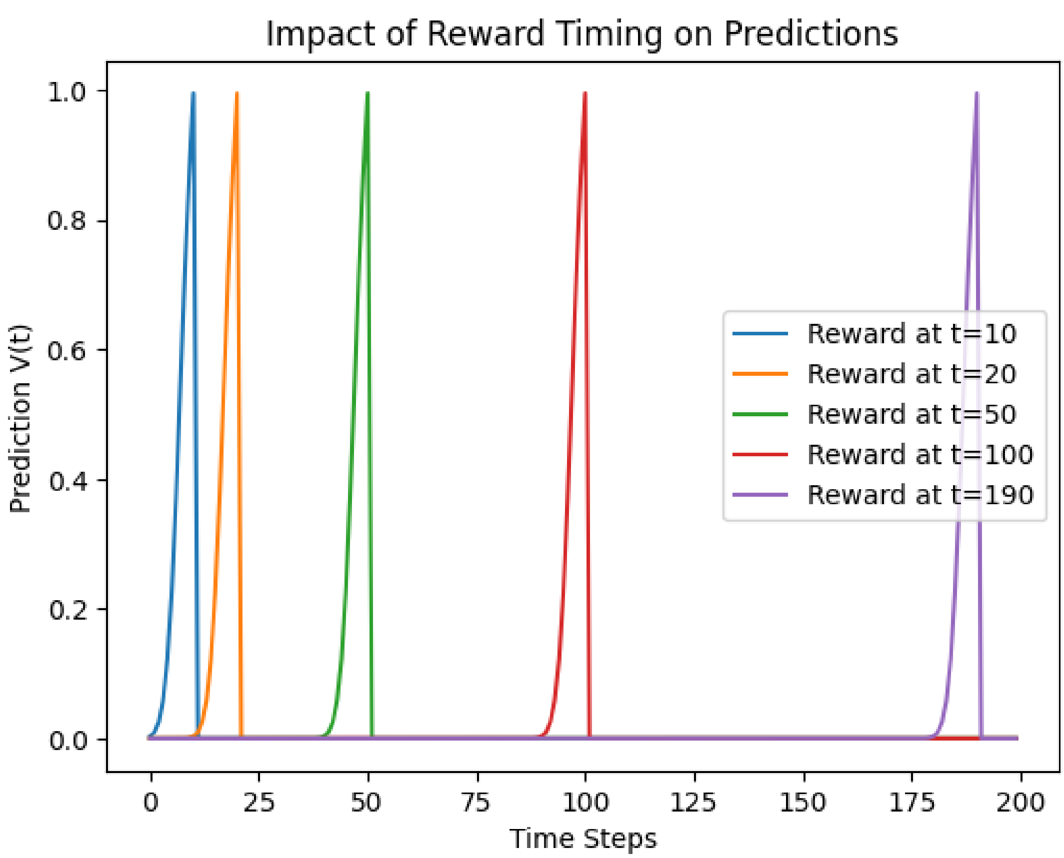
1.1.1 Recreate Figure 9.2

- **3D Surface Plot of Temporal Difference Errors**: Shows how TD errors reduce over trials as the model learns, starting high and converging to near-zero values.

1.2 2. Simulating with different parameters

1.2.1 2.1 Reward timing

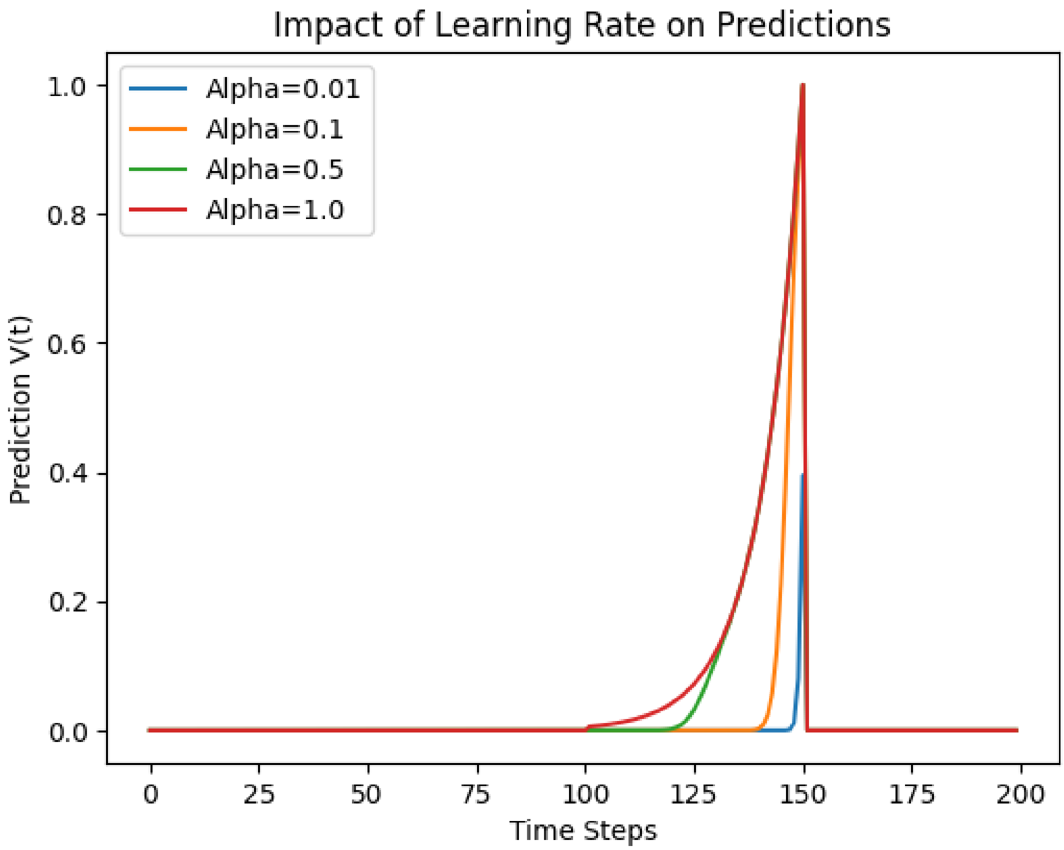
[19]:
```
reward_timings = [10, 20, 50, 100, 190]  # Different reward delays
stimulus_time = 50  # Fixed stimulus time

for reward_time in reward_timings:
    td = TemporalDifferenceLearning(n_time_steps=200, alpha=0.1, gamma=0.9)
    predictions, deltas = td.run_trials(reward_time=reward_time,␣
 ↪stimulus_time=stimulus_time, num_trials=50)
    plt.plot(predictions[-1], label=f'Reward at t={reward_time}')

plt.title("Impact of Reward Timing on Predictions")
plt.xlabel("Time Steps")
plt.ylabel("Prediction V(t)")
plt.legend()
plt.show()
```



- Early reward: Prediction peaks closer to the reward time, with minimal influence on earlier time steps.
- Delayed reward: The prediction spreads further backward, associating the stimulus with the delayed reward.

1.2.2 2.2 Learning rate

[20]:
```
learning_rates = [0.01, 0.1, 0.5, 1.0]  # Test different learning rates
reward_time = 150
stimulus_time = 100

for alpha in learning_rates:
    td = TemporalDifferenceLearning(n_time_steps=200, alpha=alpha, gamma=0.9)
    predictions, deltas = td.run_trials(reward_time=reward_time,␣
 ↪stimulus_time=stimulus_time, num_trials=50)
    plt.plot(predictions[-1], label=f'Alpha={alpha}')

plt.title("Impact of Learning Rate on Predictions")
plt.xlabel("Time Steps")
plt.ylabel("Prediction V(t)")
plt.legend()
plt.show()
```

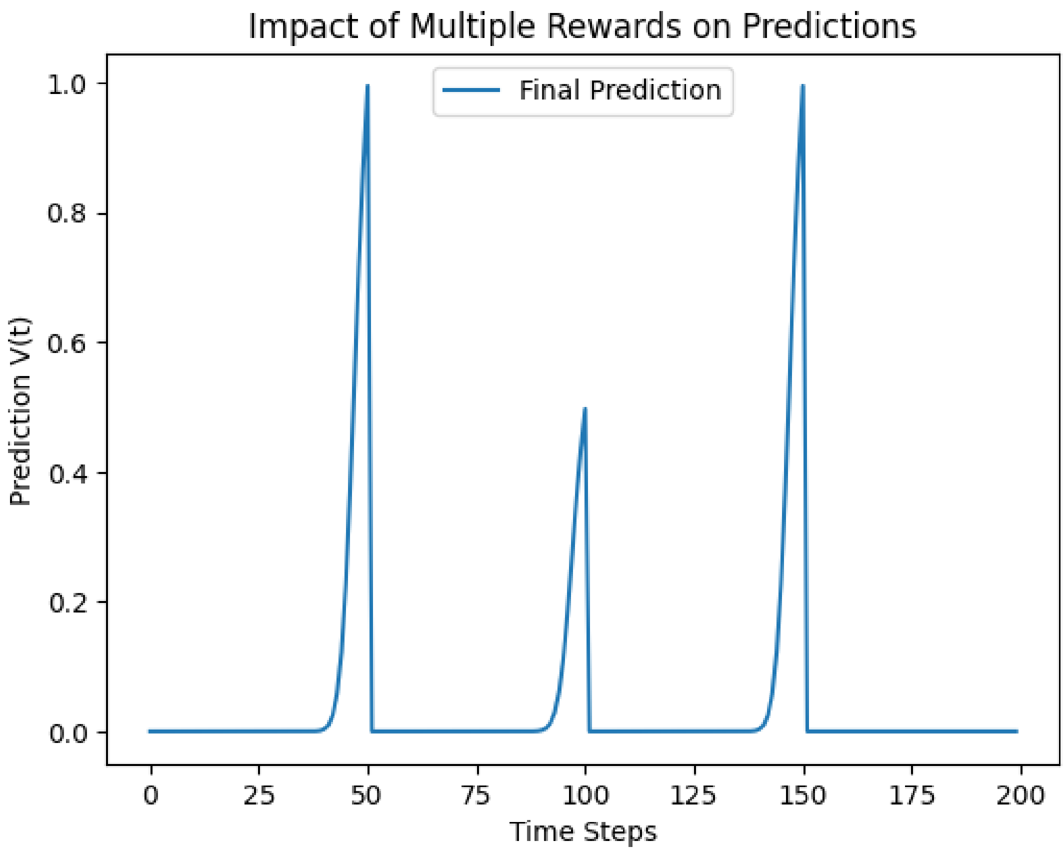


- Small : Slower learning; predictions take more trials to converge.
- Large : Faster learning but risk of instability.

1.2.3 2.3 Multiple rewards

[25]:
```
reward_vector = np.zeros(200)
reward_vector[50] = 1.0  # First reward
reward_vector[100] = 0.5  # Second reward
reward_vector[150] = 1.0 # Third reward
stimulus_time = 100

td = TemporalDifferenceLearning(n_time_steps=200, alpha=0.1, gamma=0.9)
predictions_over_trials = []

for trial in range(50):  # Run 50 trials
    td.update(reward_vector)
    predictions_over_trials.append(np.copy(td.V))

# Plot the predictions after the last trial
plt.plot(predictions_over_trials[-1], label="Final Prediction")
```

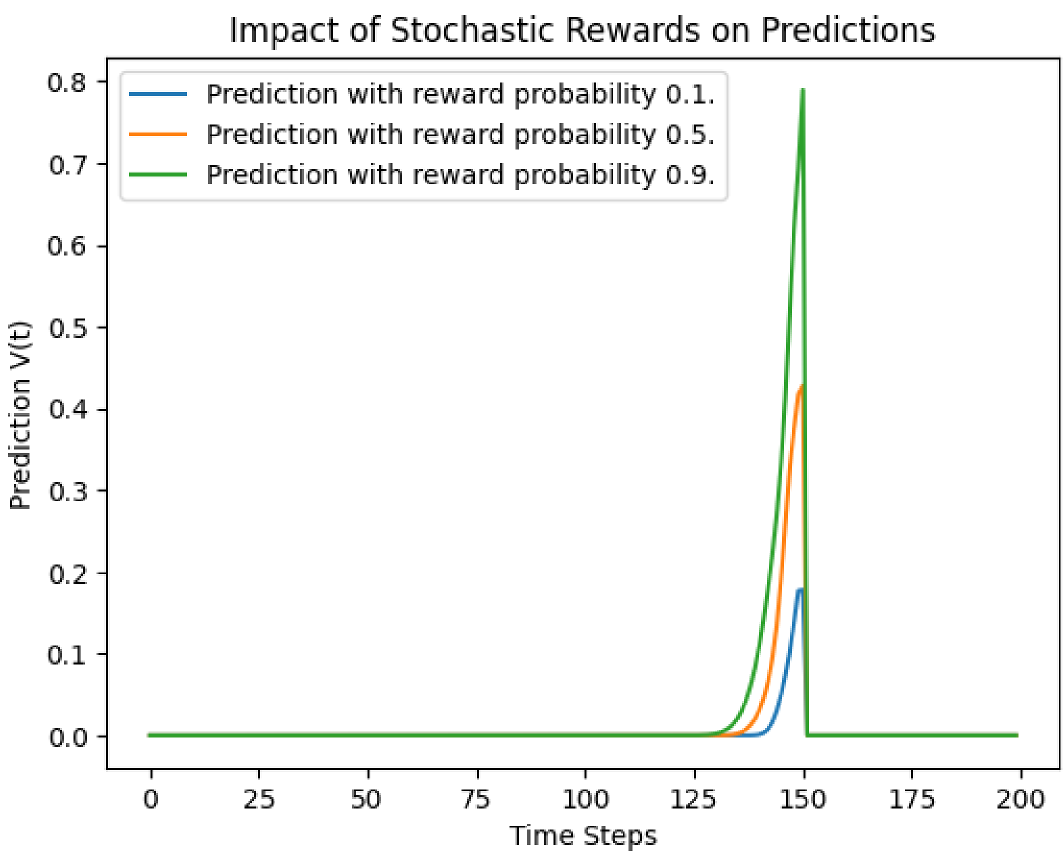
```
plt.title("Impact of Multiple Rewards on Predictions")
plt.xlabel("Time Steps")
plt.ylabel("Prediction V(t)")
plt.legend()
plt.show()
```



1.2.4 2.4 Stochastic rewards

[64]:
```
stimulus_time = 100
reward_time = 150
reward_probabilities = [0.1, 0.5, 0.9]  # 80% chance of delivering reward

td = TemporalDifferenceLearning(n_time_steps=200, alpha=0.1, gamma=0.9)

for reward_probability in reward_probabilities:
    predictions_over_trials = []
    for trial in range(50):  # Run 50 trials
        reward_vector = np.zeros(200)
        if np.random.rand() <= reward_probability:  # Random reward delivery
            reward_vector[reward_time] = 1.0
```

```
        td.update(reward_vector)
        predictions_over_trials.append(np.copy(td.V))
    plt.plot(predictions_over_trials[-1], label=f"Prediction with reward␣
↪probability {reward_probability}.")

# Plot the predictions after the last trial

plt.title("Impact of Stochastic Rewards on Predictions")
plt.xlabel("Time Steps")
plt.ylabel("Prediction V(t)")
plt.legend()
plt.show()
```



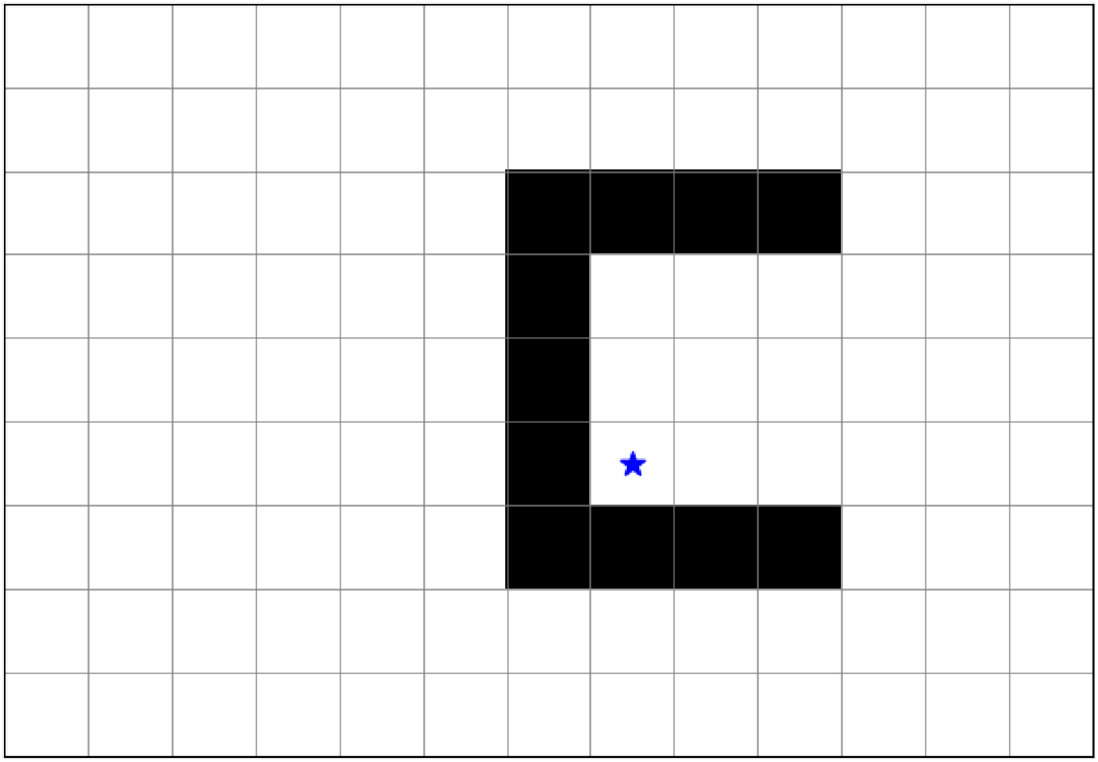
1. **High Reward Probability (p = 0.9):**
    - Predictions converge smoothly to the expected values.
    - Learning is stable as rewards are delivered consistently, reinforcing the value association.
2. **Medium Reward Probability (p = 0.5):**
    - Predictions converge more slowly compared to ($p = 0.9$) due to irregular reward delivery.
    - The value association is noisier as rewards are only delivered half the time.
3. **Low Reward Probability (p = 0.1):**

- Predictions converge very slowly and are noisier.
- The model struggles to associate time steps with future rewards due to the sporadic delivery of rewards.

1.3 3. Successor learning

[4]:
```
# define maze
maze = np.zeros((9, 13))

# place walls
maze[2, 6:10] = 1
maze[-3, 6:10] = 1
maze[2:-3, 6] = 1

# define start
start = (5, 7)
```

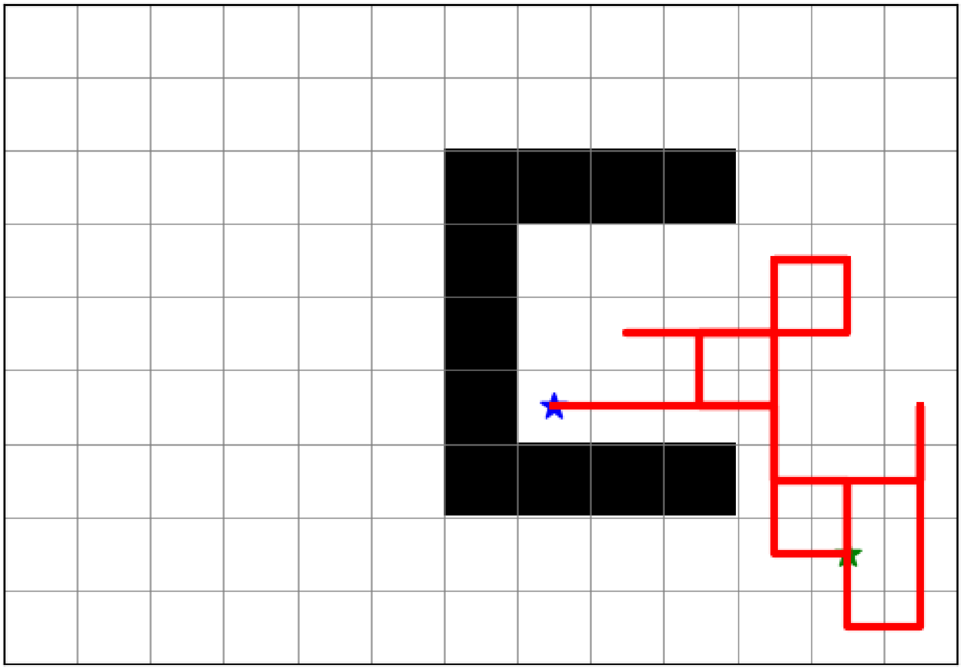
[5]:
```
def plot_maze(maze):
    plt.imshow(maze, cmap='binary')

    # draw thin grid
    for i in range(maze.shape[0]):
        plt.plot([-0.5, maze.shape[1]-0.5], [i-0.5, i-0.5], c='gray', lw=0.5)
    for i in range(maze.shape[1]):
        plt.plot([i-0.5, i-0.5], [-0.5, maze.shape[0]-0.5], c='gray', lw=0.5)

    plt.xticks([])
    plt.yticks([])

plot_maze(maze)
plt.scatter(start[1], start[0], marker='*', color='blue', s=100)
plt.tight_layout()
# plt.savefig('maze.png')
plt.show()
```



1.3.1 4.1 Random Walk

[8]:
```
######################################
############## Part 1 ##############
######################################

def random_walk(maze, start, n_steps):
    # perform a single random walk in the given maze, starting from start,␣
 ↪performing n_steps random moves
    # moves into the wall and out of the maze boundary are not possible

    # initialize list to store positions
    positions = [start]

    # Current Position
    current_position = np.array(start)

    # Define possible moves
    moves = [(-1, 0), (1, 0), (0, -1), (0, 1)]  # up, down, left, right
    # perform random steps...
    for _ in range(n_steps):
        while True:
```

```
            # Pick a random move
            move = moves[np.random.choice(len(moves))]

            # Compute the new position
            next_position = current_position + np.array(move)

            # Check if the move is valid
            if (
                    0 <= next_position[0] < maze.shape[0] and  # Vertical bounds
                    0 <= next_position[1] < maze.shape[1] and  # Horizontal␣
 ↪bounds
                    maze[next_position[0], next_position[1]] == 0  # not a wall
            ):
                current_position = next_position  # Update position
                break

        # Append the valid position
        positions.append(tuple(current_position))

    # return a list of positions
    return positions

def plot_path(maze, path):
    # plot a maze and a path in it
    plot_maze(maze)
    path = np.array(path)
    plt.plot(path[:, 1], path[:, 0], c='red', lw=3)
    plt.scatter(path[0, 1], path[0, 0], marker='*', color='blue', s=100)
    plt.scatter(path[-1, 1], path[-1, 0], marker='*', color='green', s=100)
    plt.show()

# plot a random path
path = random_walk(maze, start, 40)
plot_path(maze, path)
```

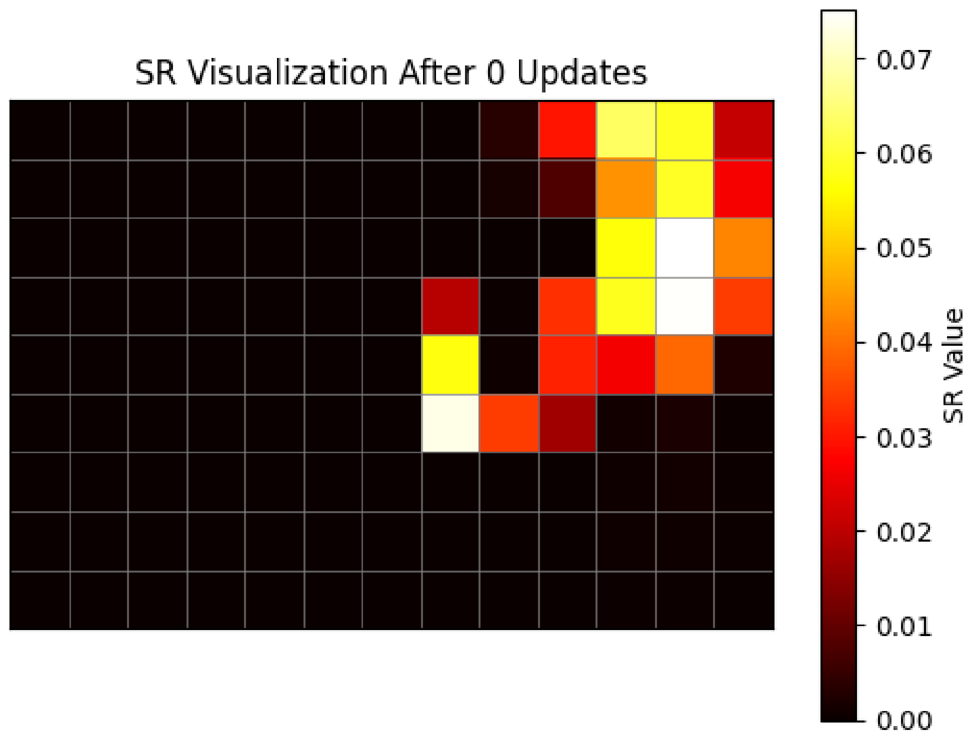


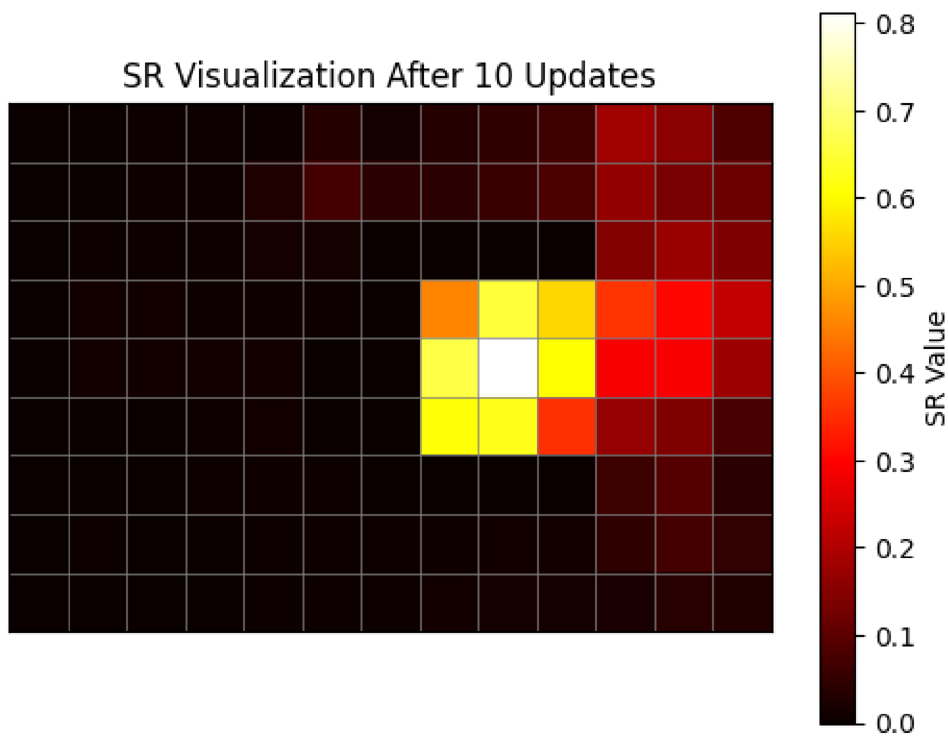
Random Walk Path in the Maze: Demonstrates a valid trajectory through the maze, avoiding walls and boundaries.

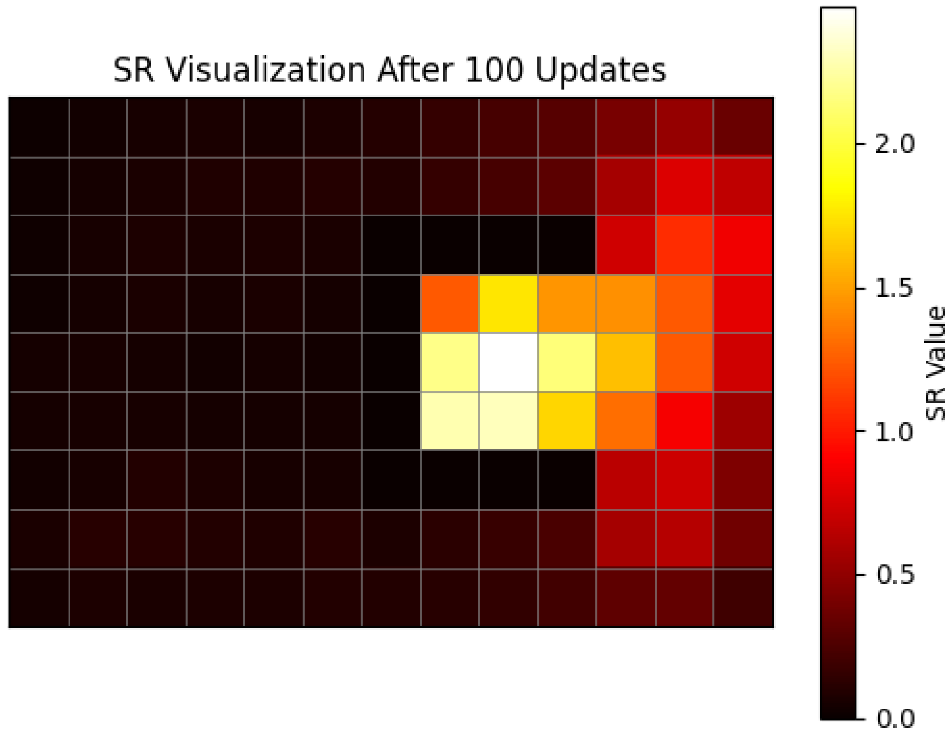
1.3.2 4.2 Learn from Trajectory

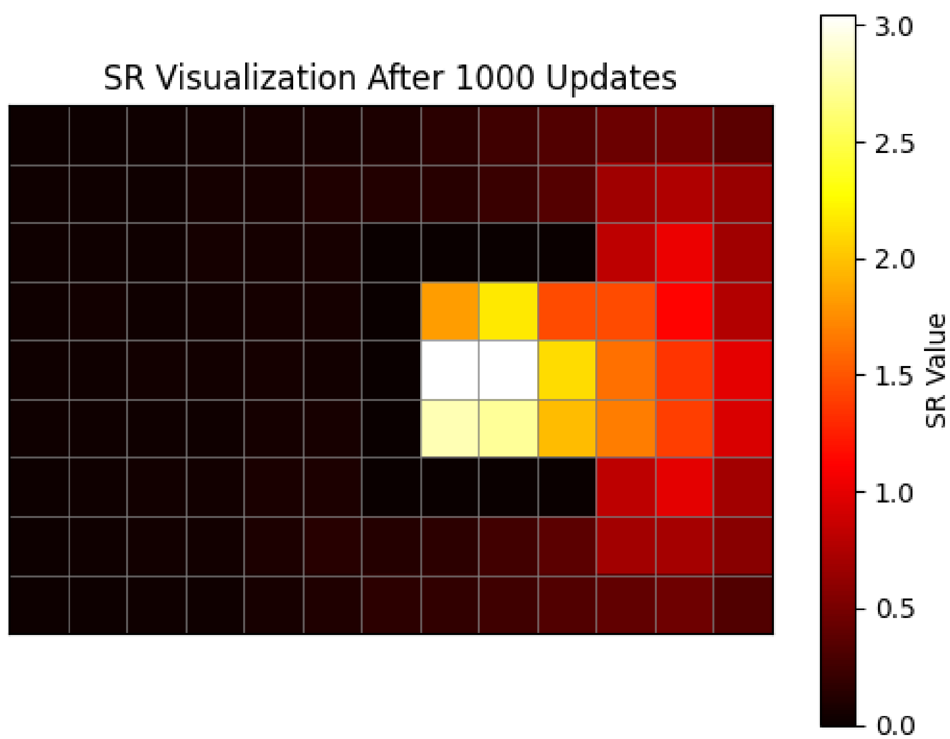
[7]:
```
####################################
############## Part 2 ##############
####################################


def learn_from_traj(succ_repr, trajectory, maze_shape, gamma=0.98, alpha=0.02):
    # Write a function to update a given successor representation (for the
 ↪state at which the trajectory starts) using an example trajectory
    # using discount factor gamma and learning rate alpha
    # Grid shape

    # Convert trajectory grid positions to linear indices
    linear_trajectory = [pos[0] * maze_shape[1] + pos[1] for pos in trajectory]

    # Get the starting state index
    start_state = linear_trajectory[0]

    # Initialize discounted trajectory vector
    discounted_trajectory = np.zeros_like(succ_repr[start_state])
```

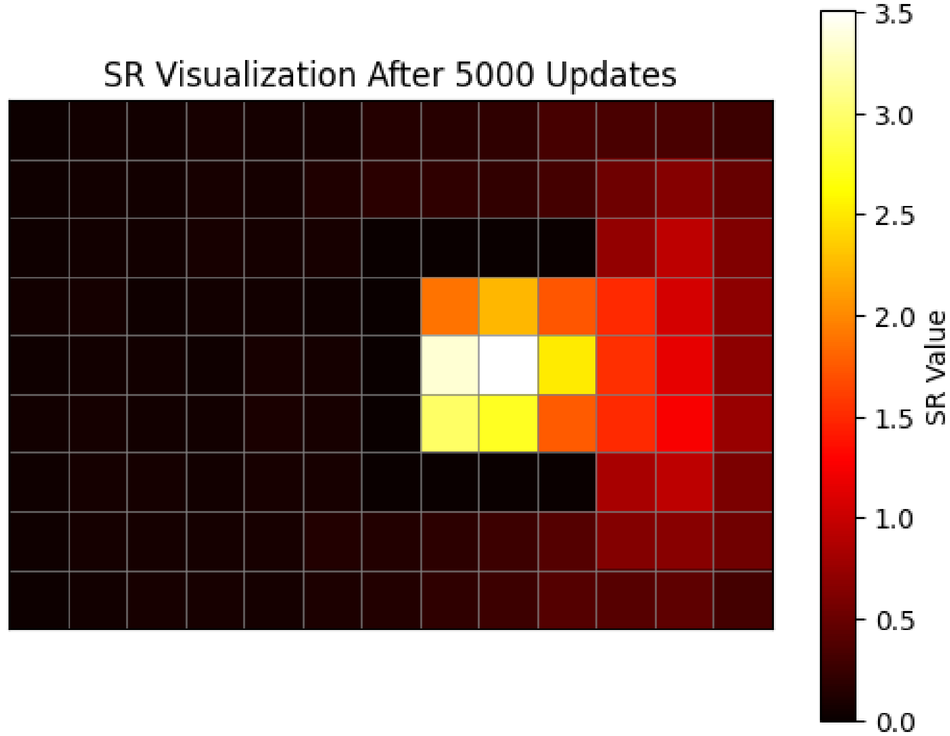
```python
    # Populate the discounted trajectory
    for t, future_state in enumerate(linear_trajectory):
        discounted_trajectory[future_state] += gamma ** t

    # Update the successor representation for the starting state
    succ_repr[start_state] = (1 - alpha) * succ_repr[start_state] + alpha *␣
↪discounted_trajectory

    # return the updated successor representation
    return succ_repr

# initialize successor representation
n_states = maze.shape[0] * maze.shape[1]  # Total number of states (9 * 13 =␣
↪117)
succ_repr = np.zeros((n_states, n_states)) # Initialization as 117x117

# sample a whole bunch of trajectories (reduce this number if this code takes␣
↪too long, but it shouldn't take longer than a minute with reasonable code)
for i in range(5001):
    # sample a path (we use 340 steps here to sample states until the␣
↪discounting becomes very small)
    path = random_walk(maze, start, 340)
    # update the successor representation
    succ_repr = learn_from_traj(succ_repr, path, maze.shape, alpha=0.02)  #␣
↪choose a small learning rate

    # occasionally plot it
    if i in [0, 10, 100, 1000, 5000]:
        start_state_index = start[0] * maze.shape[1] + start[1]
        reshaped_sr = succ_repr[start_state_index].reshape(maze.shape)

        plot_maze(maze)
        plt.imshow(reshaped_sr, cmap='hot')
        plt.title(f"SR Visualization After {i} Updates")
        plt.colorbar(label="SR Value")
        plt.show()
```


SR Visualization After 0 Updates
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
SR Value


SR Visualization After 10 Updates
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
SR Value


SR Visualization After 100 Updates
2.0
1.5
1.0
0.5
0.0
SR Value


SR Visualization After 1000 Updates
3.0
2.5
2.0
1.5
1.0
0.5
0.0
SR Value



SR Visualization After Updates: Successor Representation converges to reflect state visitation probabilities as the model learns from sampled paths.

1.3.3 4.3 Transition Matrix Computation

[15]:
```
####################################
############## Part 3 ##############
####################################



def compute_transition_matrix(maze):
    # for a given maze, compute the transition matrix from any state to any␣
 ↪other state under a random walk policy
    # (you will need to think of a good way to map any 2D grid coordinates onto␣
 ↪a single number for this)

    # create a matrix over all state pairs
    grid_shape = maze.shape
    n_states = grid_shape[0] * grid_shape[1] # Total number of states ( 9x13␣
 ↪grid)
```

```python
    transition_matrix = np.zeros((n_states, n_states)) # Initialize transition␣
↪matrix

    # Define possible moves:
    moves = [(-1, 0), (1, 0), (0, -1), (0, 1)] # up, Down, Left, Right

    # iterate over all states, filling in the transition probabilities to all␣
↪other states on the next step (only one step into the future)
    for row in range(grid_shape[0]):
        for col in range(grid_shape[1]):
            # Linear index for the current state
            current_state = row * grid_shape[1] + col

            # Skip walls
            if maze[row, col] == 1:
                continue  # Leave the row as zeros (no transitions possible)

            # Compute valid transitions
            valid_transitions = []
            for move in moves:
                next_row, next_col = row + move[0], col + move[1]
                if (
                    0 <= next_row < grid_shape[0] and  # Within row bounds
                    0 <= next_col < grid_shape[1] and  # Within column bounds
                    maze[next_row, next_col] == 0  # Not a wall
                ):
                    next_state = next_row * grid_shape[1] + next_col
                    valid_transitions.append(next_state)

                    # Assign equal probabilities to all valid transitions
                    for next_state in valid_transitions:
                        transition_matrix[current_state, next_state] = 1 /␣
↪len(valid_transitions)



    # normalize transitions if necessary
    row_sums = transition_matrix.sum(axis=1, keepdims=True)
    row_sums[row_sums == 0] = 1  # Avoid division by zero for wall states
    transition_matrix = transition_matrix / row_sums

    # remove NaNs if necessary
    transition_matrix = np.nan_to_num(transition_matrix)

    return transition_matrix

transitions = compute_transition_matrix(maze)
```

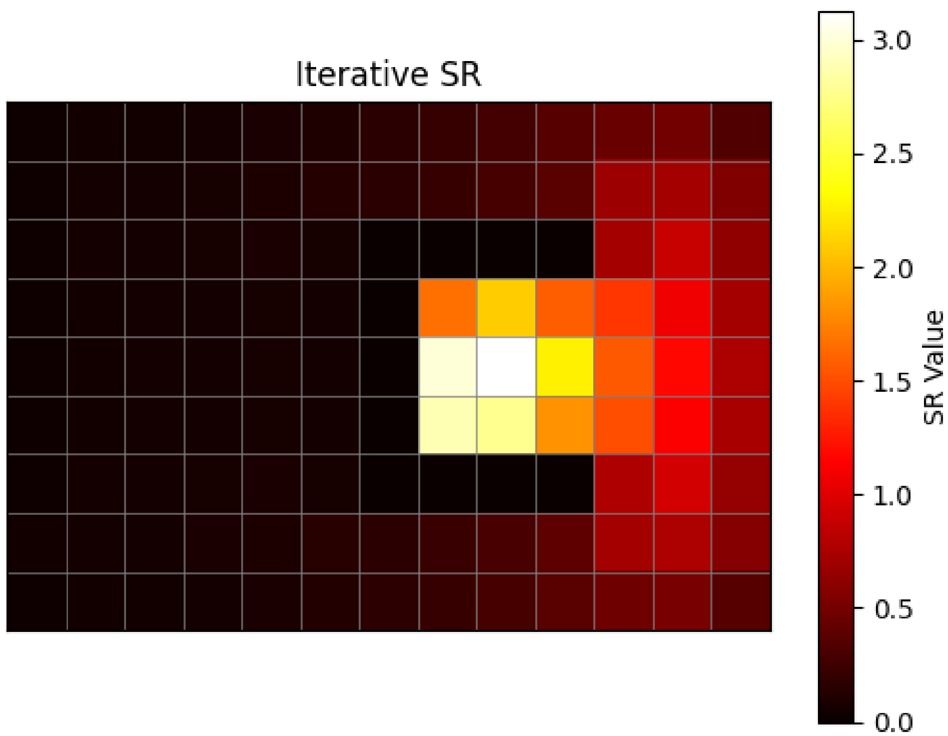
```
transitions
```

```
[15]: array([[0.        , 0.5       , 0.        , …, 0.        , 0.        ,
         0.        ],
        [0.33333333, 0.        , 0.33333333, …, 0.        , 0.        ,
         0.        ],
        [0.        , 0.33333333, 0.        , …, 0.        , 0.        ,
         0.        ],
        …,
        [0.        , 0.        , 0.        , …, 0.        , 0.33333333,
         0.        ],
        [0.        , 0.        , 0.        , …, 0.33333333, 0.        ,
         0.33333333],
        [0.        , 0.        , 0.        , …, 0.        , 0.5       ,
         0.        ]])
```

1.4 4.2 Succesor Representation Computation

```
[11]: ####################################
      ############## Part 4 ##############
      ####################################

      def compute_sr(transitions, i, j, maze_shape, gamma=0.98):
          # given a transition matrix and a specific state (i, j), compute the
       ↪successor representation of that state with discount factor gamma
          n_states = transitions.shape[0]

          # Convert (i, j) coordinates to linear index
          start_state = i * maze_shape[1] + j

          # initialize things (better to represent the current discounted occupancy
       ↪as a vector here)
          current_discounted_occupancy = np.zeros(n_states)
          current_discounted_occupancy[start_state] = 1  # Start state occupancy is 1

          # Initialize total SR vector
          total = np.zeros(n_states)

          # iterate for a number of steps
          for step in range(340):  # Steps to ensure discounting becomes negligible
              # Update total SR
              total += current_discounted_occupancy

              # Propagate to the next step
              current_discounted_occupancy = gamma * (current_discounted_occupancy @
       ↪transitions)
```

```
    # return the successor representation, maybe reshape your vector into the
 ↪maze shape now
    return total.reshape(maze_shape)

transitions = compute_transition_matrix(maze)

# compute state representation for start state
i, j = start
sr = compute_sr(transitions, i, j, maze.shape, 0.98)

# plot state representation
plot_maze(maze)
plt.imshow(sr, cmap='hot')
plt.title("Iterative SR")
plt.colorbar(label="SR Value")
plt.show()
```



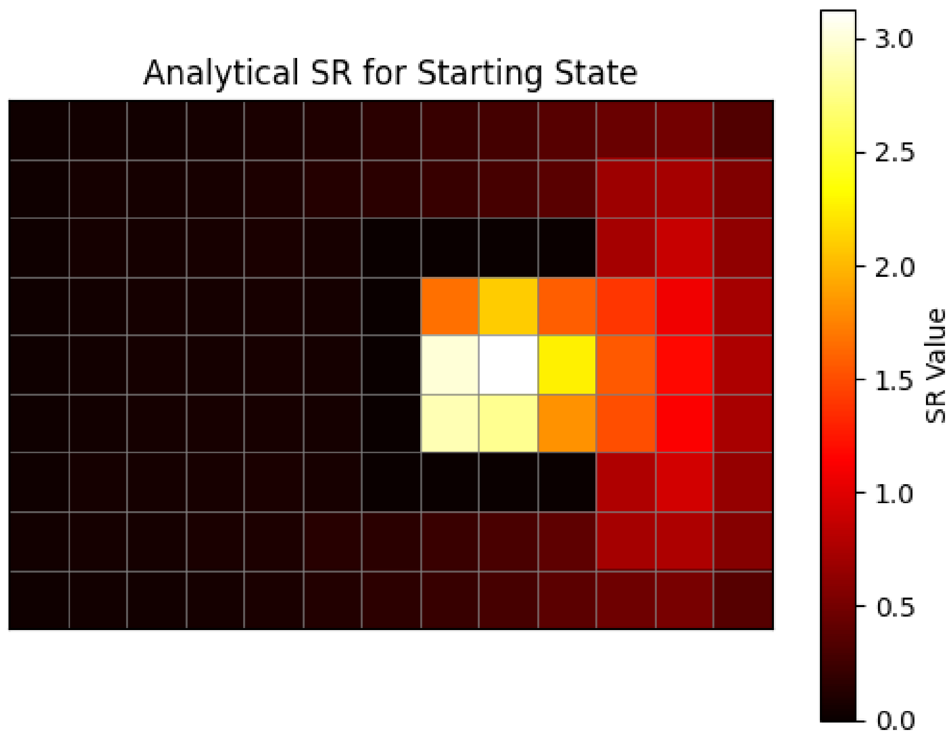
- **Iterative SR Visualization**: Shows how SR values decrease with increasing distance from the starting state, following the discount factor's influence.

[ ]:

```

```

1.4.1 4.5 Analytical Solution

[16]:
```python
############################################
############## Part 5 (Bonus) ##############
############################################

# You're on your own now

def compute_sr_analytical(transitions, gamma=0.98):
    """
        Compute the successor representation (SR) analytically for all states.

        Parameters:
        - transitions: Transition matrix (117 x 117).
        - gamma: Discount factor.

        Returns:
        - sr_matrix: Successor representation matrix (117 x 117).
   """
    # Identity matrix
    identity = np.eye(transitions.shape[0])

    # Compute the SR matrix using (I - gamma * T)^-1
    sr_matrix = np.linalg.inv(identity - gamma * transitions)

    return sr_matrix
# Compute the analytical SR for the entire grid
analytical_sr = compute_sr_analytical(transitions, gamma=0.98)

# Extract and reshape the SR for the starting state
start_state_index = start[0] * maze.shape[1] + start[1]
start_sr_analytical = analytical_sr[start_state_index].reshape(maze.shape)

# Visualize the analytical SR for the starting state
plot_maze(maze)
plt.imshow(start_sr_analytical, cmap='hot')
plt.title("Analytical SR for Starting State")
plt.colorbar(label='SR Value')
plt.show()
```



Analytical SR Visualization: Confirms that the analytical approach provides the same SR structure as the iterative method, with faster computation.

[ ]:

```

```