Cairo University

Faculty of Engineering
Aerospace Department

Task (2)

NUMERICAL SOLUTION OF ODE (RK4) AIRPLANE SIMULATOR PART I

Prepared by: Team 15

Name	Sec	Bn
Abdallah Ahmed Abdallah	1	31
Mohamed Rajeh Mohamed	2	10
Mohamed Rabie Walaan	2	11
Ahmed Ramadan Kamal	1	2
Mohamed Ahmed Mohamed Khalil	2	5
Mohamed Emad Mahmoud	2	20

Submitted To: Eng. Mahmoud Elewa

Table of contents:

Research questions:

Autopilot

An autopilot is a software or tool that can only manage theaircraft under certain conditions using the vehicle's hydraulic, mechanical and electronic systems. This system, which can follow the flight plan, can stabilize speed and height as well as the location of the front of the aircraft (heading). Pilots mostly lead the aircraft in a controlled manner by autopilot except for departure and landing. Autopilot is mostly used on passenger aircrafts.(It's main objective is managing the aircraft under certain conditions using the vehicle's hydraulic, mechanical andelectronic systems).

The autopilot can take part in most of the control mechanisms except takeoff. In general, it controls the movement of the aircraft around the center of gravity and directs the aircraft according to safety parameters. Route data prepared before the flight is uploaded to this software. From the moment the autopilot is instructed by the pilot, it controls the aircraft within this route. Planes; can have three different types of autopilot software: one-axis, two-axis, and three-axis. The next-generation aircraft can be guided by improved three-axis autopilots. New generation autopilots can also direct the yaw by controlling the rudder along with rotation and reclining movements. In newer systems, the autopilot can perform most of the classic flight maneuvers. The climbing flight and the descent flight are guided by the pilots except in extreme cases. Autopilot performs all operations in accordance with the pilot's commands.

first autopilots invention

The first autopilot, or autopilot, was developed in 1912 by the Sperry Corporation by the inventor Elmer Ambrose Sperry, co-inventor with the German physicist Hermann Anschütz-Kaempfe of the gyro-compass, a type of modern gyroscope which indicates geographic north.

The system itself was quite simple: a gyro altitude indicator and compass were connected to the rudder, lifts and ailerons, operated hydraulically (the hydraulic system moves some actuators that, in turn, activate the three main types of control and control surfaces: tail rudder, tail lifts and wing flaps, which allow the aircraft to change direction). In addition, the system was supplemented by ADF, radiomagnetic locator, or radioaid (requiring NDB radio beacons on the ground) so that the aircraft was maintained at constant speed in straight and level flight (no yaw, no pitch, no twist) without the pilot putting hands on the joystick.The system proved its operation in a real flight in 1914. In 1920 it was first employed on a ship.

inputs & outputs of an Autopilot system

Inputs :
1.Flight Path Command:
. Desired altitude
. Desired heading orcourse (yaw)
.Desired speed or Mach number
. Desired vertical speed (climb rate or descent rate)
. Desired attitude (pitch and roll angles)
. Desired flight mode (e.g., altitude hold, navigation mode, approach mode)
2.Sensors Feedback (for closed-loop control):
. Current aircraft altitude
. Current heading (yaw)
. Current speed
. Current vertical speed
. Current attitude (pitch, roll)
. Airspeed and angle of attack (if needed for specific control tasks)
. Flight plan data (waypoints, approach, etc.)
. Navigation data (GPS or inertial navigation systems)

Outputs (Actuator Outputs):

1.Primary Control Surfaces:

.Elevator: Controls the pitch of the aircraft (up and down movement of the nose).

.Ailerons: Control the roll of the aircraft (banking the wings).

. Rudder: Controls the yaw of the aircraft (left or right movement of the nose).

2.Throttle (Engine Power):

Adjusts the engine thrust, controlling the aircraft's speed and rate of climb or descent.

3.Autopilot Steering Commands:

The autopilot computes the necessary adjustments to the flight controls (elevator, ailerons, rudder, and throttle) based on the desired flight parameters. These commands are sent to the aircraft’s flight control system, which drives the corresponding actuators.

Autopilot buttons and controls :
FD: navigation display

HDG: allows to set the course, essential to navigate precise routes.

NAV: browser

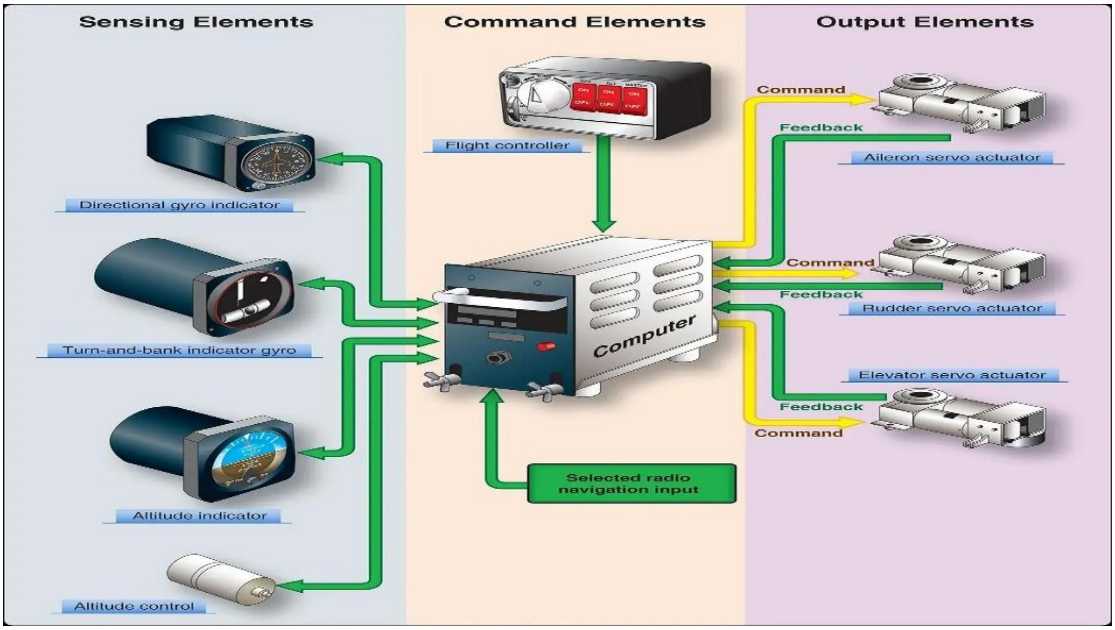
ALT: tells the pilot how to track altitude, crucial for airspace management.

HDG Knob: to control direction

APPR: for approach

LVL: keeps the plane level

SPD MACH: Activates speed control, essential for efficient fuel management and flight duration.



the role of the pilot in an airplane equipped with an autopilot

Before takeoff, the pilot will enter the route into the computer, giving it a start and end position and exactly how to get there. Throughout that route there are a series of points that the computer will note, each having its own speed and altitude.

The autopilot does not steer the airplane on the ground or taxi the plane at the gate. Generally, the pilot will handle takeoff and then initiate the autopilot to take over for most of the flight. In some newer aircraft models, autopilot systems will even land the plane.

the difference between an Autopilot & SAS (stability augmentation system)

Stability Augmentation System (SAS):

is an inertially stabilized platform that maintains an Aero plane or helicopter's altitude and heading in a fixed position. Originally conceived as an SAS for helicopters to relieve pilot fatigue when hovering for an extended amount of time.

Autopilot :

Many of the same functions as an SAS-equipped aircraft will be performed by autopilots. Autopilot in helicopters would function similarly to SAS. However, autopilot has a number of other features, like altitude hold. Hold your airspeed. Tracking the VOR (Localizer approaches) ILS techniques that are linked.

Point-to-point navigation, "R" navigation capabilities, Flight Management Systems. Autopilots have grown in sophistication over time, and will continue to do so as aeronautical and navigation engineers enhance their products.

Stability Augmentation System (SAS) differs from autopilot systems, which take full control of the aircraft and plan entire flights, as SAS focuses solely on maintaining stable flight without directing the aircraft's path. This system is crucial for both manned and unmanned aircraft to ensure safety and pilot control.

the role of the onboard sensors

.

Sensor	Quantities they measure	Typical sampling rates
GPS	Position (Latitude, Longitude, Altitude), Velocity	1 – 10 Hz
Gyroscope	Angular velocity (roll, pitch, yaw)	100 – 1000 Hz
Pitot Tube (Airspeed Sensor)	Airspeed (Dynamic pressure)	10 – 50 Hz
Magnetometer	Heading (Magnetic North direction)	10 – 100 Hz
Accelerometer	Linear acceleration (X, Y, Z) axes	10 – 1000 Hz
LIDAR / Rader	Altitude (above ground) Obstacle direction	1 – 100 Hz
Optical Flow Sensor	Motion relative to the ground	10 – 200 Hz
Sonar / Ultrasonic Sensor	Proximity to the ground (for landing)	10 – 50 Hz

Estimation techniques

If a state is not directly measured by a sensor, we can estimate it using state estimation techniques .

Mathematical Modeling

Use known system dynamics (e.g., differential equations) to predict the unmeasured state based on measured inputs and outputs

Soft Sensors (Virtual Sensors)

Combine multiple sensor readings with models to infer the unmeasured state.

Observers

Luenberger Observer (for linear systems)

Nonlinear Observers (for nonlinear systems)

Sliding Mode Observers (for robust estimation in uncertain environments)

Machine Learning-Based Estimation

Use neural networks or regression models trained on historical data to predict the state.

Kalman Filters and Variants

Kalman Filter (KF) (for linear systems with Gaussian noise)

Extended Kalman Filter (EKF) (for nonlinear systems)

Unscented Kalman Filter (UKF) (for better nonlinear performance)

Particle Filter (for highly nonlinear and non-Gaussian systems).

fly-by-wire flight control system

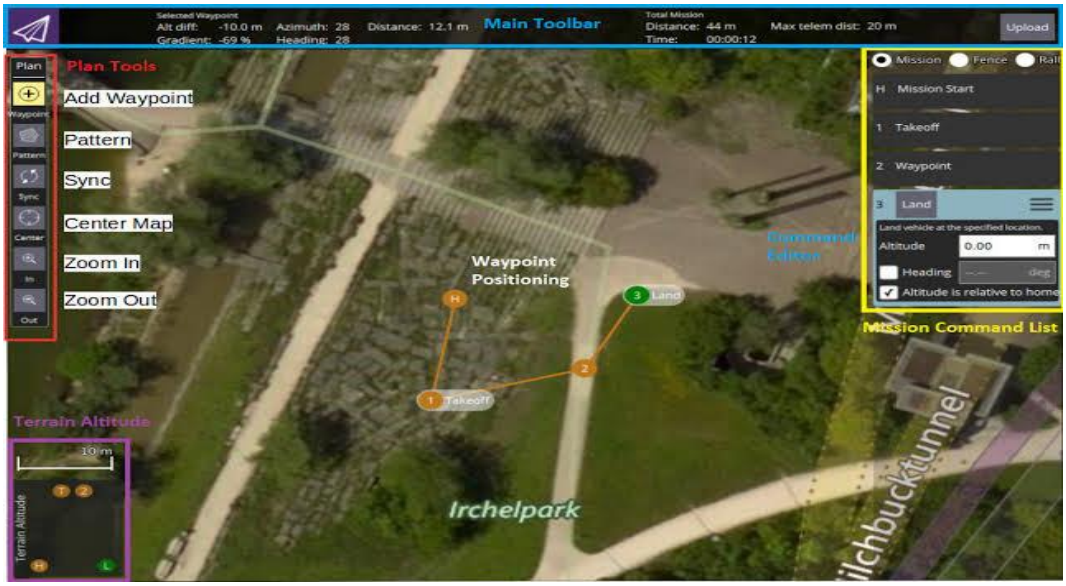
Fly-by-wire the flight control systems which use computers to process the flight control inputs made by the pilot or autopilot, and which send corresponding electrical signals to the flight control surface actuators.

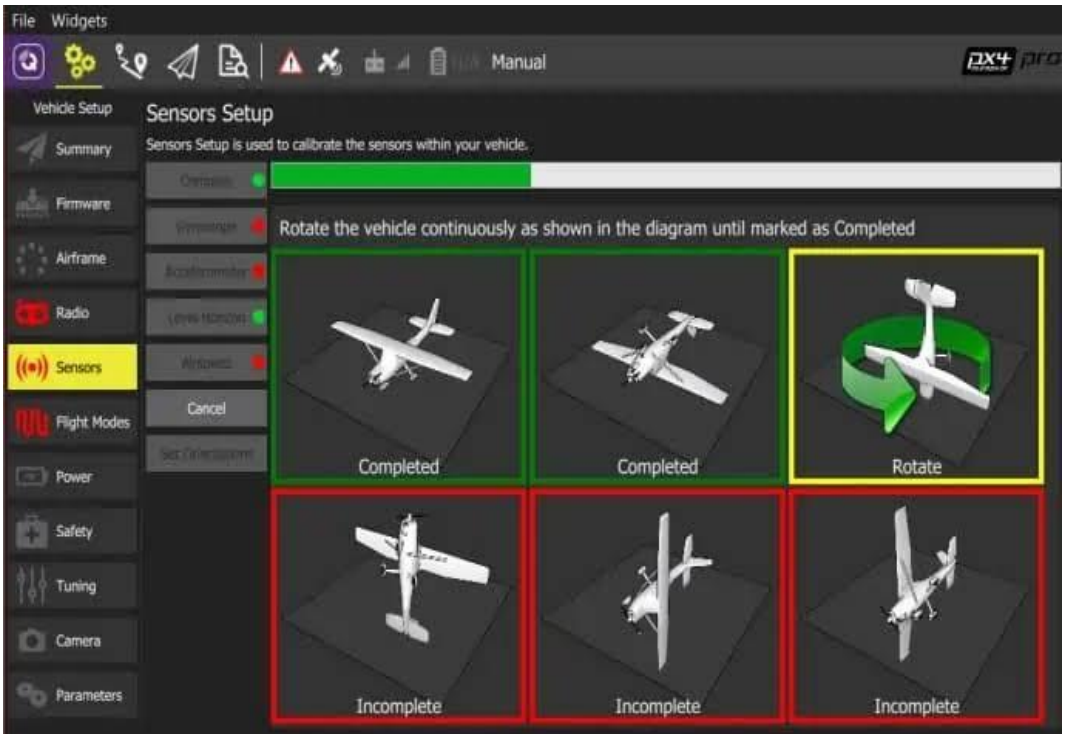
Fly-by-Wire (FBW) is the generally accepted term for those flight control systems which use computers to process the flight control inputs made by the pilot or autopilot, and send corresponding electrical signals to the flight control surface actuators. This arrangement replaces mechanical linkage and means that the pilot inputs do not directly move the control surfaces. Instead, inputs are read by a computer that in turn determines how to move the control surfaces to best achieve what the pilot wants in accordance with which of the available Flight Control Laws is active.

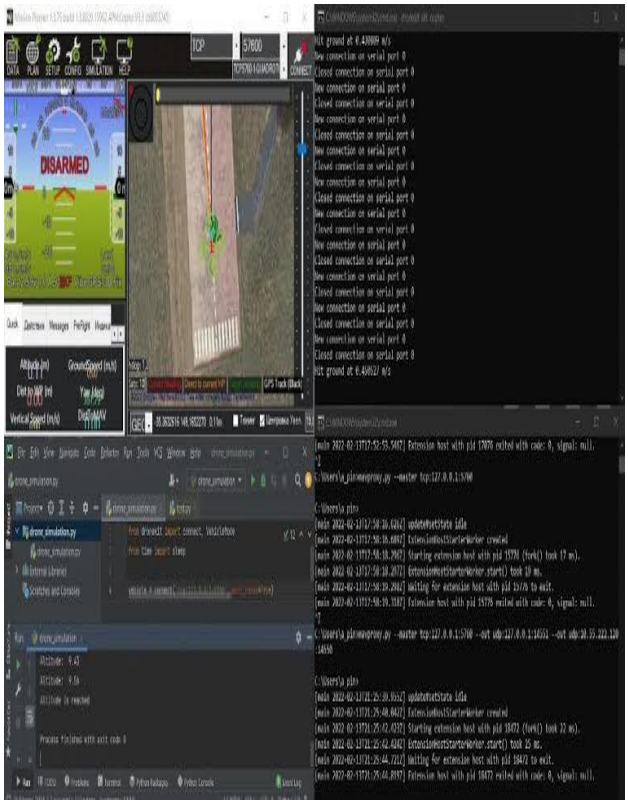
The advantages of reduced weight, improved reliability, damage tolerance, and more effective control of a necessarily highly manoeuverable aircraft, were first recognised in military aircraft design. The first aircraft to have FBW for all its flight controls in place of direct mechanical or hydraulically-assisted operation, was the F-16 in 1973. In the context of military fast jet need for agility, and therefore relatively more unstable aircraft, FBW provides the ability to ensure that unintended increases in angle of attack or sideslip are detected and rapidly, and automatically, resolved by marginally deflecting the control surfaces in the opposite way while the problem is still small. FBW also enables highly reliable flight envelope protection systems which, provided the FBW system functions at its normal level, significantly enhances safety.

open source autopilot softwares used on UAVs

A number of autopilot software packages are open-source, including PX4, Dronekit and ArduPilot which are both used widely in the drone industry. Flight controller hardware may be optimized for use with one or more particular autopilot software stacks.







autopilot hardwares used on UAVs

VECTOR-600 :

The VECTOR-600 is UAV Navigation's most advanced autopilot. VECTOR-600 is a robust and dependable unit, with built-in physical and logical redundancy, allowing it to survive all individual sensor failures while maintaining accurate estimates of attitude and position. This dependability has helped the VECTOR-600 rapidly to become the autopilot of choice for UAV professionals who require the most advanced and reliable autopilot for fixed wing, rotary wing and VTOL UAVs.

Pixhawk Series:

Silicon Errata

Pixhawk Standard/Supported Autopilots:

Holybro Pixhawk 4 (FMUv5)

Holybro Pixhawk 4 Mini (FMUv5)

Drotek Pixhawk 3 Pro (FMUv4pro)

mRo Pixracer (FMUv4)

Hex Cube Black (FMUv3)

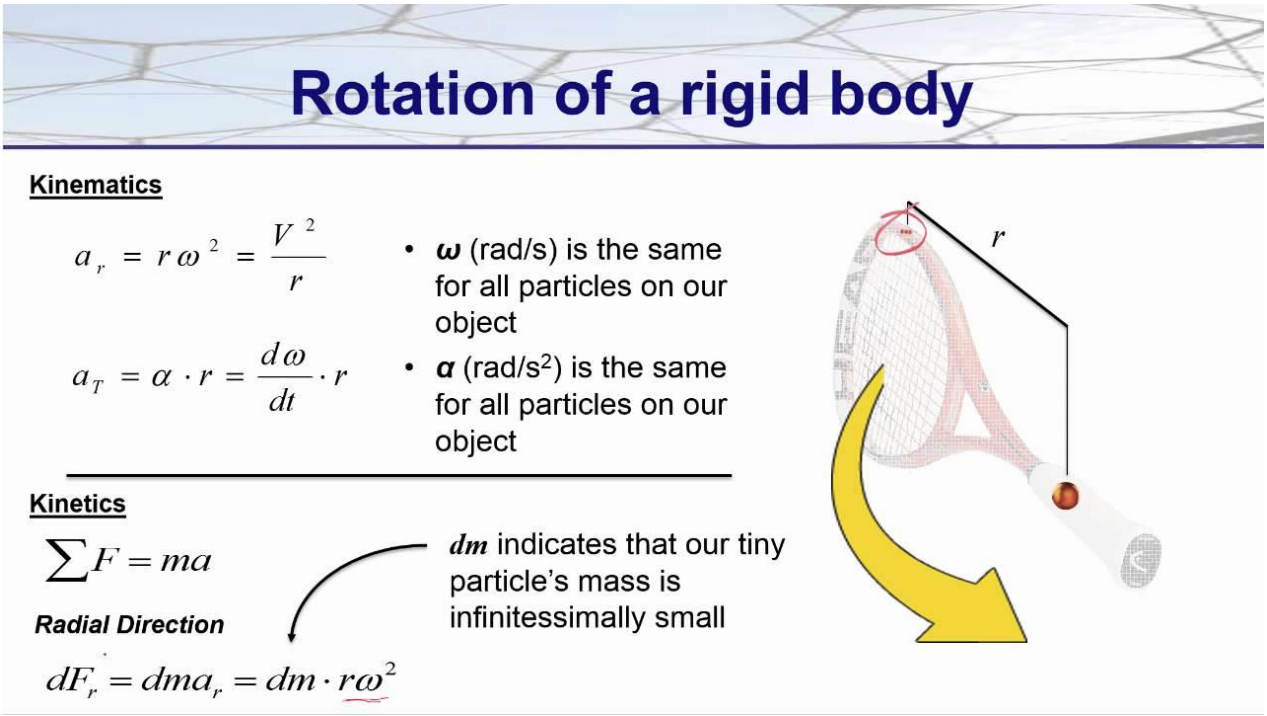
CUAV Pixhack v3 (FMUv3)

mRo Pixhawk (FMUv2)

.

the general rigid body dynamics (RBD) equations in 3D space.

The general rigid body dynamics (RBD) equations in 3D space describe the motion of a rigid body under the influence of external forces and moments. These equations consist of translational motion (Newton's Second Law) and rotational motion (Euler's Equations of Motion).



1. Translational Motion (Newton's Second Law)

$$m\frac{d\mathbf{V}}{dt} = \sum \mathbf{F}$$

where:

- m is the mass of the rigid body,
- $\mathbf{V}$ is the velocity of the center of mass in an inertial frame,
- $\sum \mathbf{F}$ represents the total external force acting on the body,
- $\frac{d\mathbf{V}}{dt}$ is the acceleration of the center of mass.

2. Rotational Motion (Euler's Equations of Motion)

$$\mathbf{I}\frac{d\boldsymbol{\omega}}{dt} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega}) = \sum \mathbf{M}$$

where:

- $\mathbf{I}$ is the **inertia tensor** of the rigid body,
- $\boldsymbol{\omega}$ is the **angular velocity vector** in the body-fixed frame,
- $\sum \mathbf{M}$ is the total **external moment (torque)** acting on the body,
- $\frac{d\boldsymbol{\omega}}{dt}$ represents the **angular acceleration**.

Classification of The 12 equations of motion

The **12 equations of motion** for a **rigid body in 3D space** consist of **kinematics** and **kinetics** equations. These are classified as follows:

Category	Variable	Description	Frame of Reference
Position (Global Location)	x,y,z	Aircraft's position	Earth-fixed (Inertial)
Linear Velocities	u,v,w	Forward, sideward, vertical velocity	Body-fixed
Orientation (Attitude)	φ,θ,ψ	Roll, pitch, yaw angles	Earth-fixed (Euler Angles)
Angular Velocities	p,q,r	Roll, pitch, yaw rates	Body-fixed

1. Kinematics Equations (6 equations)

Kinematics describes the motion of the body **without considering forces or moments**. These equations define the relationship between position, velocity, angular velocity, and orientation.

- **Translational Kinematics (3 equations):**
  These equations define the time evolution of the position vector $\mathbf{r}$ of the center of mass.

$$\frac{d\mathbf{r}}{dt} = \mathbf{V}$$

  where $\mathbf{V}$ is the velocity of the center of mass.

- **Rotational Kinematics (3 equations):**
  These equations describe how the orientation of the body changes over time, often represented using **Euler angles**, **quaternions**, or **direction cosines**.

$$\frac{d\mathbf{\Theta}}{dt} = \mathbf{T}(\boldsymbol{\omega})$$

2. Kinetics Equations (6 equations)

Kinetics involves the forces and moments that cause motion.

- **Translational Kinetics (Newton's Second Law, 3 equations):**
  These equations describe the **force balance** in each direction.

$$m\frac{d\mathbf{V}}{dt} = \sum \mathbf{F}$$

  where:

  - m is the mass,
  - $\mathbf{V}$ is the velocity of the center of mass,
  - $\sum \mathbf{F}$ represents the total external force.

- **Rotational Kinetics (Euler's Equations of Motion, 3 equations):**
  These describe the **moment balance** in each direction.

$$\mathbf{I}\frac{d\boldsymbol{\omega}}{dt} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega}) = \sum \mathbf{M}$$

  where:

  - $\mathbf{I}$ is the inertia tensor,
  - $\boldsymbol{\omega}$ is the angular velocity,
  - $\sum \mathbf{M}$ represents the total external moment.

Derivation of the Equations of Motion (EOM) for Fixed-Wing Airplanes

To derive the **Equations of Motion (EOM) for Fixed-Wing Airplanes**, additional **aerodynamic, propulsion, and gravitational forces** must be incorporated into the **Rigid Body Dynamics (RBD) equations**.

1. Aerodynamic Forces and Moments

These forces arise from the interaction between the aircraft and the surrounding airflow, typically decomposed into **lift, drag, and side force**.

- **Aerodynamic Force Components:**

$$\mathbf{F}_{\text{aero}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

where:

- X (Drag): Force opposite to the aircraft's velocity.
- Y (Side Force): Lateral force due to asymmetry or sideslip.
- Z (Lift): Force perpendicular to the velocity vector.

- **Aerodynamic Moment Components:**

$$\mathbf{M}_{\text{aero}} = \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

where:

- L (Roll Moment): About the **x-axis** (longitudinal).
- M (Pitch Moment): About the **y-axis** (lateral).
- N (Yaw Moment): About the **z-axis** (vertical).

These aerodynamic forces and moments depend on parameters such as:

- **Angle of attack α**
- **Sideslip angle β**
- **Control surface deflections (elevator, rudder, ailerons)**
- **Dynamic pressure**

2. Propulsive (Thrust) Forces

The engine generates thrust, which is added as an external force.

$$\mathbf{F}_{\text{thrust}} = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}$$

For **jet aircraft**, thrust is mainly along the **x-axis** (T_x).

For **propeller-driven aircraft**, additional moments may result due to **propwash** or **torque effects**.

3. Gravitational Forces

Gravity acts on the aircraft's center of mass, adding another external force:

$$\mathbf{F}_{\text{gravity}} = mg \begin{bmatrix} -\sin\theta \\ \cos\theta \sin\phi \\ \cos\theta \cos\phi \end{bmatrix}$$

where:

- g is the acceleration due to gravity,
- ϕ (Roll angle), θ (Pitch angle) define aircraft orientation relative to Earth.

4. Transformation Between Reference Frames

Aircraft motion is usually described in the **body-fixed frame**, but forces and moments are often analyzed in the **inertial (Earth) frame**. Transformation equations (e.g., **direction cosine matrix (DCM)** or **quaternions**) are added to switch between coordinate systems.

For small angles, the transformation from **body to Earth frame** uses:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \mathbf{T}(\boldsymbol{\omega}) \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

where:

- p, q, r are body angular velocity components,
- $\dot{\phi}, \dot{\theta}, \dot{\psi}$ are Euler angle rates.

Final Form: Fixed-Wing Aircraft Equations of Motion

With these additions, the full **6-degree-of-freedom (6-DOF) equations of motion** for a fixed-wing aircraft become:

1. **Translational Motion (Newton's Second Law in Body Frame)**

$$m\frac{d\mathbf{V}}{dt} = \sum(\mathbf{F}_{\text{aero}} + \mathbf{F}_{\text{thrust}} + \mathbf{F}_{\text{gravity}})$$

2. **Rotational Motion (Euler's Equations in Body Frame)**

$$\mathbf{I}\frac{d\boldsymbol{\omega}}{dt} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega}) = \sum \mathbf{M}_{\text{aero}} + \mathbf{M}_{\text{thrust}}$$

3. **Kinematic Equations (Attitude Evolution)**

$$\frac{d\boldsymbol{\Theta}}{dt} = \mathbf{T}(\boldsymbol{\omega})$$

4. **Force and Moment Coefficients**

   Aerodynamic forces and moments are often rewritten in coefficient form:

$$C_X, C_Y, C_Z, C_L, C_M, C_N$$

   using:

$$F = C \cdot qS, \quad M = C \cdot qSc$$

   where S is wing area, c is mean aerodynamic chord.

the assumptions introduced while deriving the airplane equations of motion

Assumption	Purpose	When It Fails
Rigid Body	Ignores structural deformation	Flexible aircraft, wing bending
Flat Earth	Simplifies gravity calculations	High-altitude, long-range flights
Small Angles	Linearization for easier analysis	Large maneuvers, aerobatics

Symmetric Airplane	Removes unnecessary complexity	Asymmetric damage, non-uniform loads
Constant Mass & Inertia	Avoids time-dependent equations	Fuel burn, payload changes
Linear Aerodynamics	Makes stability analysis easier	Stall, high AoA, supersonic flow
Quasi-Steady Aerodynamics	Removes wake and vortex effects	Helicopters, UAV hovering
No Control Surface Dynamics	Simplifies control modeling	High-speed jets, rate-limited actuators
Standard Atmosphere	Avoids weather effects	Gusty winds, turbulence

Classification of the airplanes (EOM) equations mathematically

Category	Classification	Explanation
Order	First-order	The EOMs describe the time rate of change of velocity and angular velocity, making them **first-order differential equations**. However, when written in terms of displacement, they become **second-order**.
Type	Ordinary Differential Equations (ODEs)	The EOMs depend on a finite number of variables (position, velocity, angles) and their derivatives concerning time, making them **ODEs** rather than PDEs (which involve spatial derivatives).
Linearity	Nonlinear	The original equations contain **quadratic terms** (e.g., products of angular velocities), making them **nonlinear**. However, they can be **linearized** for small perturbations.
Coupling	Coupled	Translational and rotational motions influence each other (e.g., roll affects yaw), making the equations **coupled**. However, they can be **decoupled** for specific flight conditions like small disturbances.

the (Body axes) versus the (earth or inertial axes)

Feature	Body Axes (Aircraft Frame)	Earth Axes (Inertial Frame)

Fixed to	Aircraft center of gravity	Earth's surface (assumed fixed)
Orientation	Rotates with the aircraft	Fixed relative to Earth
X-Axis Direction	Forward along fuselage	Geographic North (or another fixed direction)
Y-Axis Direction	Rightward (wing direction)	Geographic East
Z-Axis Direction	Downward (toward ground in aircraft frame)	Downward (toward Earth's center)
Used for	Forces, moments, control inputs	Navigation, trajectory, wind effects

the pitch angle (θ) versus the angle of attack (α)

Term	Pitch Angle (θ)	Angle of Attack (α)
Definition	The angle between the **body X-axis** (fuselage direction) and the **Earth (inertial) horizontal**.	The angle between the **body X-axis** (fuselage direction) and the **relative wind (free stream airflow)**.
Reference Frame	Measured relative to the **Earth (inertial frame)**.	Measured relative to the **airflow (aerodynamic frame)**.
Effect on Flight	Determines the **aircraft's nose-up or nose-down attitude** relative to the horizon.	Directly affects **lift generation** and stall behavior.
Relation to Flight Path	$\theta=\alpha+\gamma$, where γ is the **flight path angle** (angle between velocity vector and horizontal).	Affects the pressure distribution around the wing, impacting lift and drag.

Example	An aircraft flying level with a **5° nose-up pitch** has θ=5∘	If the airflow comes at an **8° angle relative to the fuselage**, then α=8∘

the sideslip angle (β) versus the heading angle (ψ)

Term	Sideslip Angle (β)	Heading Angle (ψ)
Definition	The angle between the **body Y-axis** (lateral axis) and the **oncoming airflow (relative wind)**.	The angle between the **Earth X-axis** (North) and the aircraft's **velocity vector in the horizontal plane**.
Reference Frame	Measured in the **body frame**.	Measured in the **Earth (inertial) frame**.
Effect on Flight	Indicates **yaw misalignment** between the aircraft's motion and fuselage direction (important for stability).	Describes the **direction of travel** relative to geographic North.
Relation to Yaw	A nonzero βmeans the aircraft is experiencing **aerodynamic yawing forces** (e.g., from wind or asymmetric thrust).	ψ is affected by **wind, yaw maneuvers, and navigation corrections**.
Example	If an aircraft is moving forward but slightly **crabbed to the left** due to wind, it has β>0∘	If an aircraft is flying **eastward**, its heading is ψ=90∘

Term	Frame of Reference	Reference Direction	Indicates
Pitch Angle (θ)	Earth (inertial frame)	Earth's horizontal	Nose-up or nose-down attitude

Angle of Attack (α)	Aerodynamic frame	Relative airflow	How air meets the wing, affecting lift
Sideslip Angle (β)	Body frame	Relative wind	Lateral motion (slipping/skidding)
Heading Angle (ψ)	Earth (inertial frame)	Geographic North	Direction of motion

attitude representations

When representing an aircraft's orientation in space, we can use different mathematical approaches:

- **Euler Angles (Roll φ, Pitch θ, Yaw ψ)**
- **Direction Cosine Matrix (DCM)**
- **Quaternions (q0,q1,q2,q3q_0, q_1, q_2, q_3q0,q1,q2,q3)**
- **Axis-Angle Representation (r,θ)**

1. Euler Angles (ϕ,θ,ψ)

Euler angles define orientation using **three sequential rotations** about different axes.

Advantages:

- **Intuitive & easy to interpret** (directly relate to aircraft roll, pitch, and yaw).
- Common in flight dynamics, autopilots, and human interfaces.
- Requires only **three parameters**, making it **compact**.

Disadvantages:

- **Gimbal Lock:** Loss of one degree of freedom when pitch (θ) is ±90°.
- Complex trigonometric functions required for conversions.

- Singularities in transformations make certain maneuvers problematic.

*Best for:***Small-angle approximations, human-readable attitude representation.**

2. Direction Cosine Matrix (DCM)

The **DCM** (or **rotation matrix**) is a **3×3 matrix** that transforms vectors between reference frames.

Advantages:

- **No singularities** (avoids gimbal lock).
- Directly used in **force and moment transformations**.
- Can represent **any rotation uniquely**.

Disadvantages:

- Requires **9 elements** (redundant since only 3 are independent).
- Computationally expensive **(matrix multiplications required)**.
- Requires **orthonormalization** to avoid numerical drift over time.

*Best for:***Precise numerical calculations in navigation & control systems.**

3. Quaternions (q0,q1,q2,q3)

A quaternion is a **4D complex number** that represents rotations without gimbal lock.

Advantages:

- **No gimbal lock or singularities**.
- **More computationally efficient** than DCM for rotations.
- **Compact (4 elements)** while avoiding redundancy.

- **Stable in iterative computations** (used in simulations and spacecraft).

Disadvantages:

- Less intuitive than Euler angles.
- Requires **normalization** to prevent drift over time.
- More complex mathematical operations (e.g., quaternion multiplication).

*Best for:***Flight simulations, spacecraft, and real-time attitude control.**

4. Axis-Angle Representation (r,θ)

Defines rotation by an **axis of rotation** $r=(r_x, r_y, r_z)$ and a **rotation angle** θ

Advantages:

- **Minimal representation** for single-axis rotations.
- Directly used in **mechanical systems and robotics**.

Disadvantages:

- **Not ideal for continuous rotations** (e.g., flight dynamics).
- Requires **conversion** for matrix-based calculations.

*Best for:***Describing single-axis rotations or manual transformations.**

General comparison

Representation	Parameters	Gimbal Lock?	Computational Cost	Intuitive?	Best Use Case
Euler Angles	3	Yes	High (trigonometry)	Yes	Aircraft control, displays
DCM (Rotation Matrix)	9	No	Moderate (matrix math)	No	Precise transformations
DCM (Rotation Matrix)	4	No	Low (efficient math)	No	Flight sims, spacecraft

Axis-Angle	4	No	High (for multiple transformations)	Yes	Single-axis rotations

Numerical solution of ODEs

the numerical solving algorithms for ODEs

1. Euler's Method: This is one of the simplest numerical methods for solving ODEs. It is an explicit method which means the solution can be advanced step by step, using the value at the current step to estimate the value at the next step.

2. Heun's Method: Also known as the improved Euler method, this technique is a predictor-corrector method that improves upon the basic Euler method by using an average of the slopes at the beginning and the end of the interval to estimate the next value.

3. Midpoint Method: This method is a second-order method which uses the slope at the midpoint of the interval to estimate the next value, providing a better approximation than Euler's method.

4. Runge-Kutta Methods: These are a family of iterative methods which include the well-known fourth-order Runge-Kutta method (RK4). RK4 is widely used due to its balance between computational effort and accuracy.

5. Adams-Bashforth Methods: These are multi-step methods that use the values of the solution at several previous steps to compute the next value. They are explicit methods and are often used in conjunction with other methods to start the computation.

6. Adams-Moulton Methods: These are implicit multi-step methods that use the value at the current step as well as previous steps to estimate the next value. They are generally more accurate and stable than explicit methods but require solving an equation at each step.

7. Backward Differentiation Formulas (BDF): These are implicit multi-step methods that are particularly useful for solving stiff equations. They use past values of the solution and its derivative to estimate the next value.

8. Verlet Integration: Commonly used in molecular dynamics simulations, this method is particularly good for solving Newton's equations of motion and is known for its stability and energy conservation properties.

9. Leapfrog Integration: This is a simple method that is particularly useful for problems where the velocity and position are updated in an alternating fashion, often used in the context of Hamiltonian mechanics.

10. Symplectic Integrators: These are designed for solving Hamiltonian systems and are particularly good at conserving the symplectic structure of the system, which is important for long-term simulations.

Choosing an algorithm for solving the Airplanes EOM,

for solving the Airplanes EOM we usually use thefourth-order Runge-Kutta method (RK4).

Initial conditions needed : At t = 0 we need the unknowns (u, v , w , p , q , r , ϕ , θ , ψ , x , y , z)

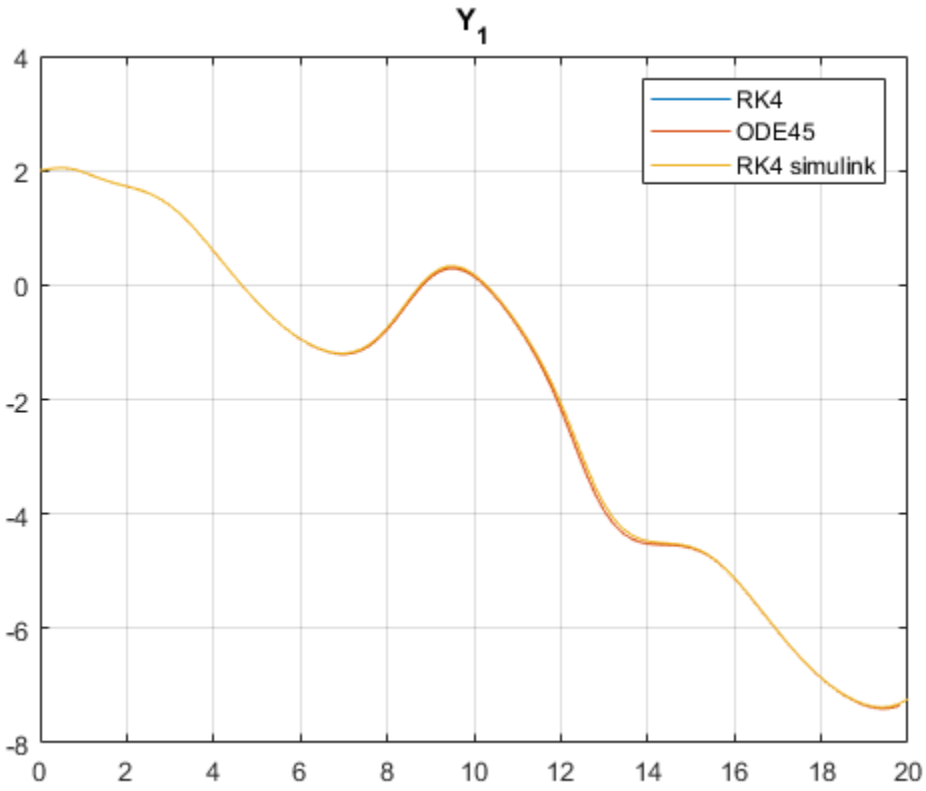
Inputs needed in each iteration:F_x , F_y , F_z , L , M , N

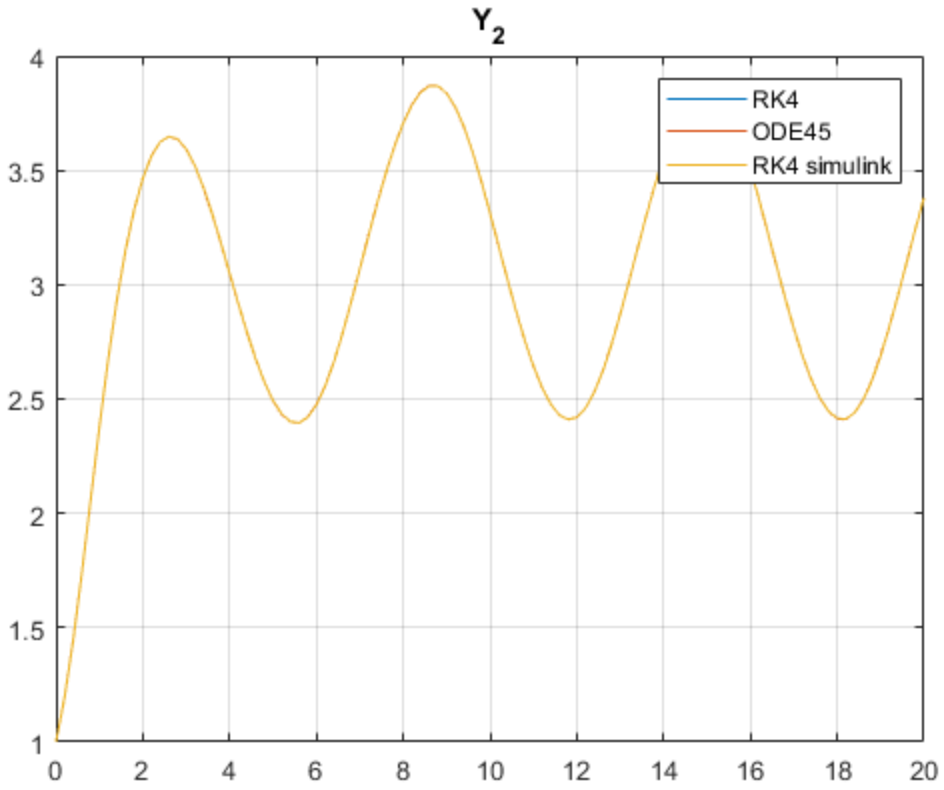
I & m are constants

Outputs calculated in each iteration:u, v , w , p , q , r , ϕ , θ , ψ , x , y , z

Solving the system

MATLABresults





Published with MATLAB® R2023b

As shwon above, the RK4 function RK4 simulink produced the same results as the MATLAB function “ODE45”

- # RBD equations solution

.1. Problem definition

After exploring various numerical methods for solving ODEs, this task requires implementing the 4th-order Runge-Kutta (RK4) algorithm to solve the 12 rigid body dynamics (RBD) equations. The solution will be computed using fixed input forces and moments, along with given initial conditions for the 12 state variables.

.2. Givens

$$t_{final} = 25\ sec$$

$$Forces\ [F_x\ ;F_y\ ;F_z] = [2\ ;\ 8\ ;\ 3]\ N$$

$$Moments\ [p\ ;q\ ;r] = [14;\ 20;\ 7]\ N.m.$$

$$mass = 11\ kg$$

$$I = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 5 & -4 \\ -1 & -4 & 0.2 \end{bmatrix} kg.m^2$$

.3. Initial conditions ($S.I.\ units$)

$At\ t = 0$

$$[u, v, w] = [10, 2, 0]$$

$$[p, q, r] = [2 * \frac{\pi}{180}, 1 * \frac{\pi}{180}, 0 * \frac{\pi}{180}]$$

$$[\phi, \theta, \psi] = [20 * \frac{\pi}{180}, 15 * \frac{\pi}{180}, 30 * \frac{\pi}{180}]$$

$$[x, y, z] = [2, 4, 7]$$

.4. Mathematical modeling

- As mentioned in the first task, the rigid body dynamics (RBD) equations consist of 12 coupled, nonlinear, first-order differential equations (ODEs). These equations can be categorized into six kinetic equations describing forces and moments and six kinematic equations governing motion.

Kinetics equations:

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = m \times \left(\begin{bmatrix} u^{\cdot} \\ v^{\cdot} \\ w^{\cdot} \end{bmatrix} + \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \end{bmatrix} \right) \qquad (4.1)$$

$$\begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} + \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \qquad (4.2)$$

Kinematics equations:

$$\begin{bmatrix} \dot{\Phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin(\Phi)\tan\theta & \cos(\Phi)\tan\theta \\ 0 & \cos(\Phi) & -\sin(\Phi) \\ 0 & \sin(\Phi)/\cos(\theta) & \cos(\Phi)/\cos(\theta) \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \qquad (4.3)$$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix} = \begin{bmatrix} cos(\theta)cos(\psi) & sin(\varphi)sin(\theta)cos(\psi) - cos(\varphi)sin(\psi) & cos(\varphi)sin(\theta)cos(\psi) + sin(\varphi)sin(\psi) \\ cos(\theta)sin(\psi) & sin(\varphi)sin(\theta)sin(\psi) + cos(\varphi)cos(\psi) & cos(\varphi)sin(\theta)sin(\psi) - sin(\varphi)cos(\psi) \\ -sin(\theta) & sin(\varphi)cos(\theta) & cos(\varphi)cos(\theta) \end{bmatrix} \begin{bmatrix}u\\ v\\ w\end{bmatrix}$$

(4.4)

.1. Problem solution

- We will utilize the vector form of the rigid body dynamics (RBD) equations to compute the 12 state variables after 25 seconds, using the given parameters and initial conditions. To derive the required 12-state formulation from the RBD equations, mathematical manipulation and rearrangement will be performed to systematically organize the kinetic and kinematic equations.

From kinetics forces equations:

$$\begin{bmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\end{bmatrix} = \frac{1}{m}\begin{bmatrix}F_x\\ F_y\\ F_z\end{bmatrix} - \begin{bmatrix}\mathrm{p}\\ q\\ r\end{bmatrix} \times \begin{bmatrix}u\\ v\\ w\end{bmatrix} \qquad (4.5)$$

From kinetics forces equations:

$$\begin{bmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\end{bmatrix} = \frac{1}{m}\begin{bmatrix}F_x\\ F_y\\ F_z\end{bmatrix} - \begin{bmatrix}\mathrm{p}\\ q\\ r\end{bmatrix} \times \begin{bmatrix}u\\ v\\ w\end{bmatrix} \qquad (4.5)$$

From kinetics moments equations:

$$\begin{bmatrix}\dot{p}\\ \dot{q}\\ \dot{r}\end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}^{-1} \left(\begin{bmatrix}L\\ M\\ N\end{bmatrix} - \begin{bmatrix}p\\ q\\ r\end{bmatrix} \times \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix}p\\ q\\ r\end{bmatrix} \right) \qquad (4.6)$$

From kinematics equations:

$$\begin{bmatrix}\dot{\varphi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} = \begin{bmatrix} 1 & \sin(\Phi)\tan\theta & \cos(\Phi)\tan\theta \\ 0 & \cos(\Phi) & -\sin(\Phi) \\ 0 & \sin(\Phi)/\cos(\theta) & \cos(\Phi)/\cos(\theta) \end{bmatrix} \begin{bmatrix}p\\ q\\ r\end{bmatrix} \qquad (4.7)$$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix} = \begin{bmatrix} cos(\theta)cos(\psi) & sin(\varphi)sin(\theta)cos(\psi) - cos(\varphi)sin(\psi) & cos(\varphi)sin(\theta)cos(\psi) + sin(\varphi)sin(\psi) \\ cos(\theta)sin(\psi) & sin(\varphi)sin(\theta)sin(\psi) + cos(\varphi)cos(\psi) & cos(\varphi)sin(\theta)sin(\psi) - sin(\varphi)cos(\psi) \\ -sin(\theta) & sin(\varphi)cos(\theta) & cos(\varphi)cos(\theta) \end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix}$$

(4.8)

.1. Runge – Kutta method – 4th Order

This method determines the new state x_{n+1} at each time step based on the previous state xnx_n using the following formula:

$$y_{n+1} = y_n + \frac{1}{6} \times dt \times (k_1 + 2k_2 + 2k_3 + k_4) \qquad (4.9)$$

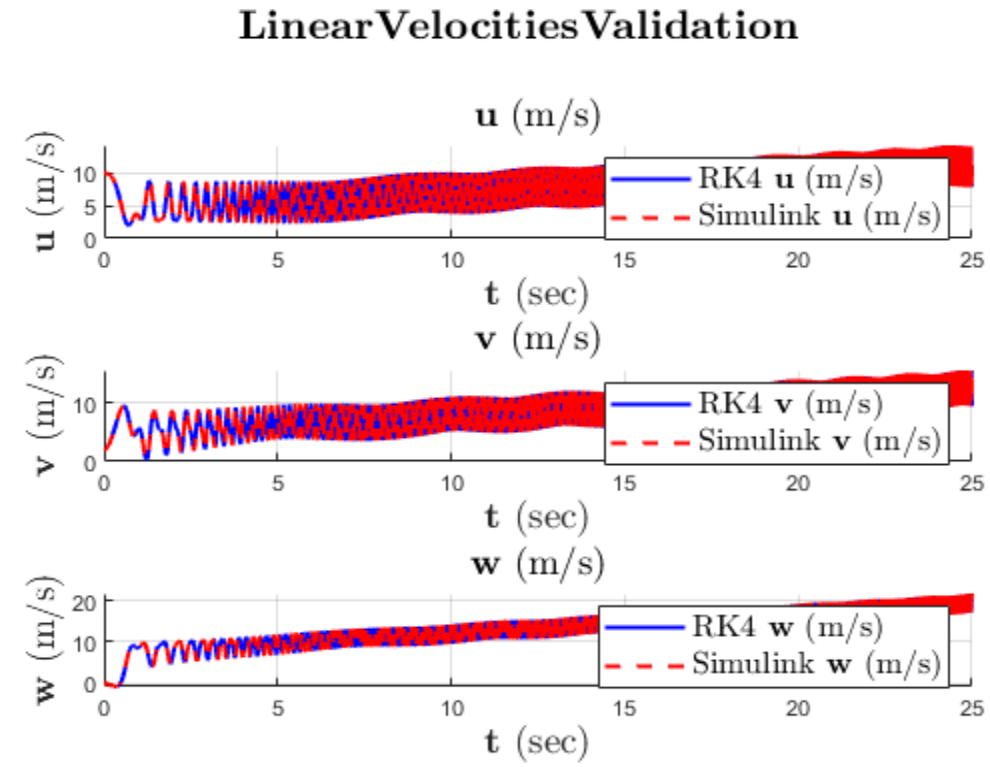
$$t_{n+1} = t_n + dt$$

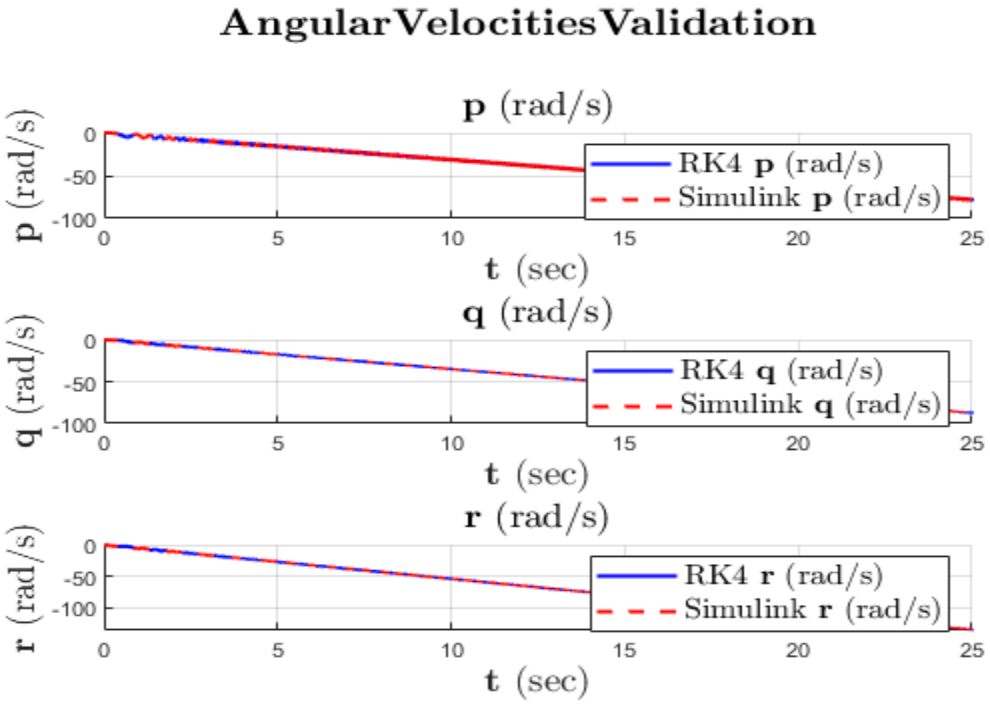
Where;

$$k_1 = f(t_n, y_n)$$
$$k_2 = f(t_n + \frac{1}{2}dt, y_n + \frac{1}{2}dt \times k_1)$$
$$k_3 = f(t_n + \frac{1}{2}dt, y_n + \frac{1}{2}dt \times k_2)$$
$$k_4 = f(t_n + dt, y_n + dt \times k_3)$$

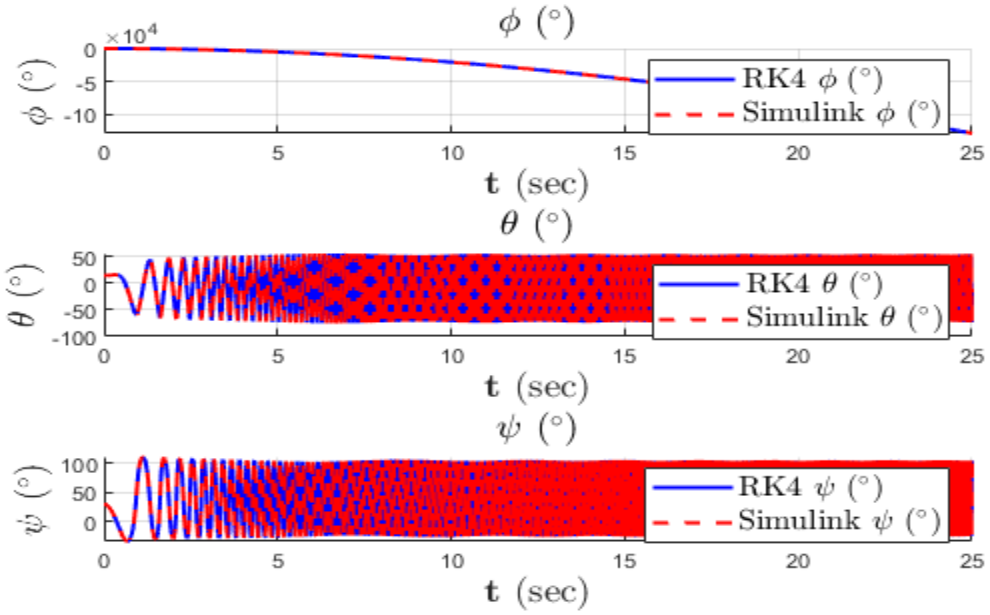
This set of equations is also applicable to multi-dimensional variables in vector form. By implementing a MATLAB code and starting from the given initial conditions, each of the 12 state variables will be computed iteratively using the Runge-Kutta method, ultimately reaching their final values by the end of the analysis period.

Figures

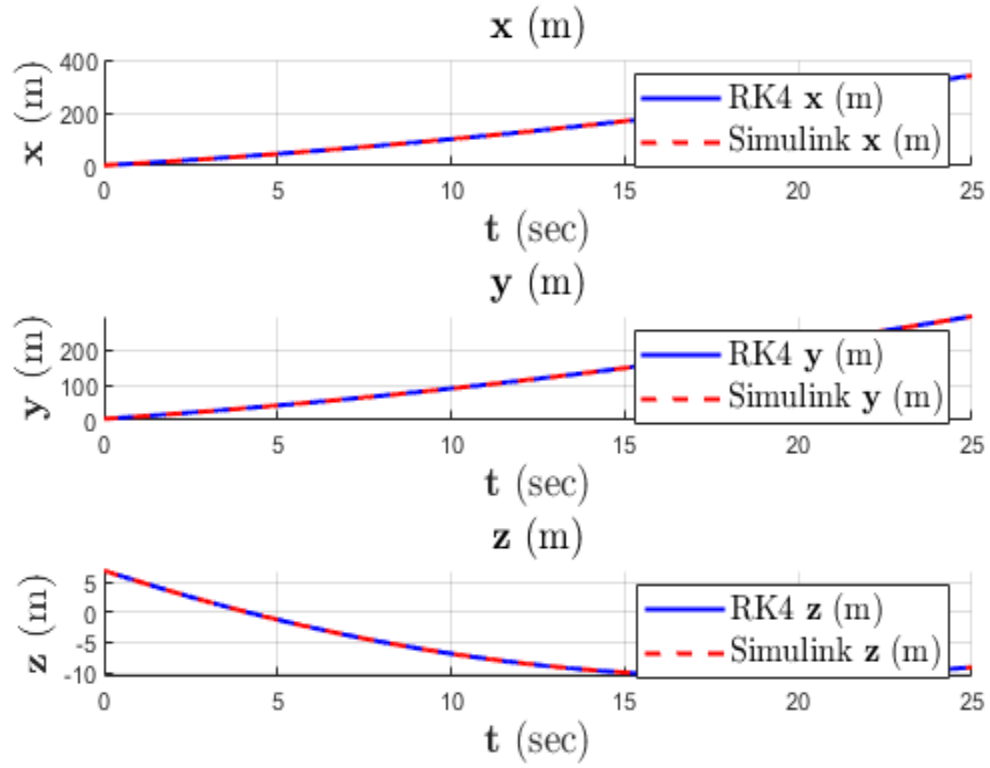

Linear Velocities Validation
u (m/s)
u (m/s)
RK4 u (m/s)
Simulink u (m/s)
t (sec)
v (m/s)
v (m/s)
RK4 v (m/s)
Simulink v (m/s)
t (sec)
w (m/s)
w (m/s)
RK4 w (m/s)
Simulink w (m/s)
t (sec)


Angular Velocities Validation
p (rad/s)
p (rad/s)
RK4 p (rad/s)
Simulink p (rad/s)
t (sec)
q (rad/s)
q (rad/s)
RK4 q (rad/s)
Simulink q (rad/s)
t (sec)
r (rad/s)
r (rad/s)
RK4 r (rad/s)
Simulink r (rad/s)
t (sec)

Euler Angles Validation



Position Validation



Since MATLAB code and Simulink are solving the same equations with the same solver, we got identical graphs.

Bonus:

To compare two signals (vectors) and quantify the error between them, the following mathematical expressions can be used:

1. Mean Squared Error (MSE)

$$MSE = \frac{1}{N}\sum_{i=1}^{N}(x_i - y_i)^2$$

- Measures the average squared difference between corresponding elements of two vectors.
- Sensitive to large errors due to squaring.

2. Root Mean Squared Error (RMSE)

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - y_i)^2}$$

- Provides an error measurement in the same unit as the original signals.
- Useful for understanding deviations in real-world terms.

3. Mean Absolute Error (MAE)

$$MAE = \frac{1}{N}\sum_{i=1}^{N}|x_i - y_i|$$

- Measures the average absolute differences.
- Less sensitive to outliers compared to MSE.

4. Normalized Root Mean Squared Error (NRMSE)

$$NRMSE = \frac{RMSE}{\max(y) - \min(y)}$$

- Normalizes RMSE using the range of the reference signal.
- Useful for comparing errors across different datasets.

5. Pearson Correlation Coefficient (r)

$$r = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum(x_i - \bar{x})^2 \sum(y_i - \bar{y})^2}}$$

- Measures the linear relationship between two signals.
- Values close to 1 indicate strong correlation.

6. Signal-to-Noise Ratio (SNR)

$$SNR = 10 \log_{10} \left(\frac{\sum y_i^2}{\sum (x_i - y_i)^2} \right)$$

- Evaluates how much the signal is corrupted by error.
- Higher values indicate better similarity.

7. Cross-Correlation

$$R_{xy}(k) = \sum_i x_i y_{i+k}$$

- Measures similarity as a function of signal displacement.
- Useful for time-shifted comparisons.

Appendices

Appendix A

MATLAB code

```
clc;
close all;
clearvars;
```

Time Vector Initialization

```
t0 = 0;             % Initial time (s)
tf = 25;            % Final time (s)
dt = 0.001;         % Time step (s)
t_vec = t0:dt:tf;   % Time vector
```

External Forces and Moments

```
Forces  = [2; 8; 3];       % Forces (N)
Moments = [14; 20; 7];     % Moments (N.m)
```

Aircraft Mass and Inertia

```
g = 9.81; % Gravity (m/s^2)
m = 11;   % Mass of the aircraft (kg)

% Inertia Matrix (kg.m^2)
I_mat = [1 -2 -1;
         -2 5 -4;
         -1 -4 0.2];
```

Initial Conditions

Initial state vector: [u, v, w, p, q, r, ϕ, θ, ψ, x, y, z]

```
states_vec(:,1) = [10, 2, 0, 2*pi/180, pi/180, 0, 20*pi/180, 15*pi/180, 30*pi/180, 2, 4, 7];

% Assign initial values
[u0,v0,w0,p0,q0,r0,phi0,theta0,psi0,x0,y_0,z0] =
deal(10,2,0,2*pi/180,pi/180,0,20*pi/180,15*pi/180,30*pi/180,2,4,7);
```

Runge-Kutta 4th Order (RK4)

```
[t_vec_RK4, states_vec_RK4] = raunge_kutta_4(t_vec, states_vec(:,1), Forces, Moments, m, I_mat);
```

Simulink RK4

```
simOut = sim("RBD_simulink_model.slx");
```

Convert Euler Angles to Degrees

Convert RK4 results

```
states_vec_RK4(7:9, :) = rad2deg(states_vec_RK4(7:9, :));

% Convert Simulink results
simOut.phi_simulink.data   = rad2deg(simOut.phi_simulink.data);
simOut.theta_simulink.data = rad2deg(simOut.theta_simulink.data);
simOut.psi_simulink.data   = rad2deg(simOut.psi_simulink.data);
```

Plot Results

```
plotValidation(t_vec_RK4, states_vec_RK4, simOut);
```

Appendix B

Raunge kutta4th order function

```
function [t_vec, states_vec] = raunge_kutta_4(t_vec, y0, Forces, Moments, m, I_mat)

    n = length(t_vec);
    dt = t_vec(2) - t_vec(1);
    states_vec = zeros(12,n);
    states_vec(:,1) =y0;

    for i =1:n-1

        K1 = get_states_dot(t_vec(i),states_vec(:,i), Forces, Moments, m, I_mat);
        K2 = get_states_dot(t_vec(i)+.5*dt,states_vec(:,i)+K1*0.5*dt, Forces, Moments,
m, I_mat);
        K3 = get_states_dot(t_vec(i)+.5*dt,states_vec(:,i)+K2*0.5*dt, Forces, Moments,
m, I_mat);
        K4 = get_states_dot(t_vec(i)+dt,states_vec(:,i)+K3*dt, Forces, Moments, m,
I_mat);

        states_vec(:,i+1) = states_vec(:,i) + dt/6 * (K1+2.*K2+2.*K3+K4);
    end
end
```

get_states_dot Function

```
function states_dot = get_states_dot(~, states_vec, Forces, Moments, m, I_mat)
% get_states_dot Computes the time derivatives of the state variables
%
% Inputs:
%   - states_vec: State vector [u, v, w, p, q, r, phi, theta, psi, x, y, z] (12x1)
%   - Forces: External forces in body frame [Fx, Fy, Fz] (3x1)
%   - Moments: External moments in body frame [Mx, My, Mz] (3x1)
%   - m: Mass of the vehicle (scalar)
%   - I_mat: Inertia matrix (3x3)
%
% Outputs:
%   - states_dot: Time derivative of states (12x1)

% Extract states
u     = states_vec(1);
v     = states_vec(2);
w     = states_vec(3);
p     = states_vec(4);
q     = states_vec(5);
r     = states_vec(6);
phi   = states_vec(7);
theta = states_vec(8);
psi   = states_vec(9);
x     = states_vec(10);
y     = states_vec(11);
z     = states_vec(12);

% Rotation matrix for Euler angles (ZYX convention)
J = [ 1,  sin(phi)*tan(theta),  cos(phi)*tan(theta);
      0,  cos(phi),            -sin(phi);
      0,  sin(phi)/cos(theta),  cos(phi)/cos(theta)];

% Compute translational dynamics
```

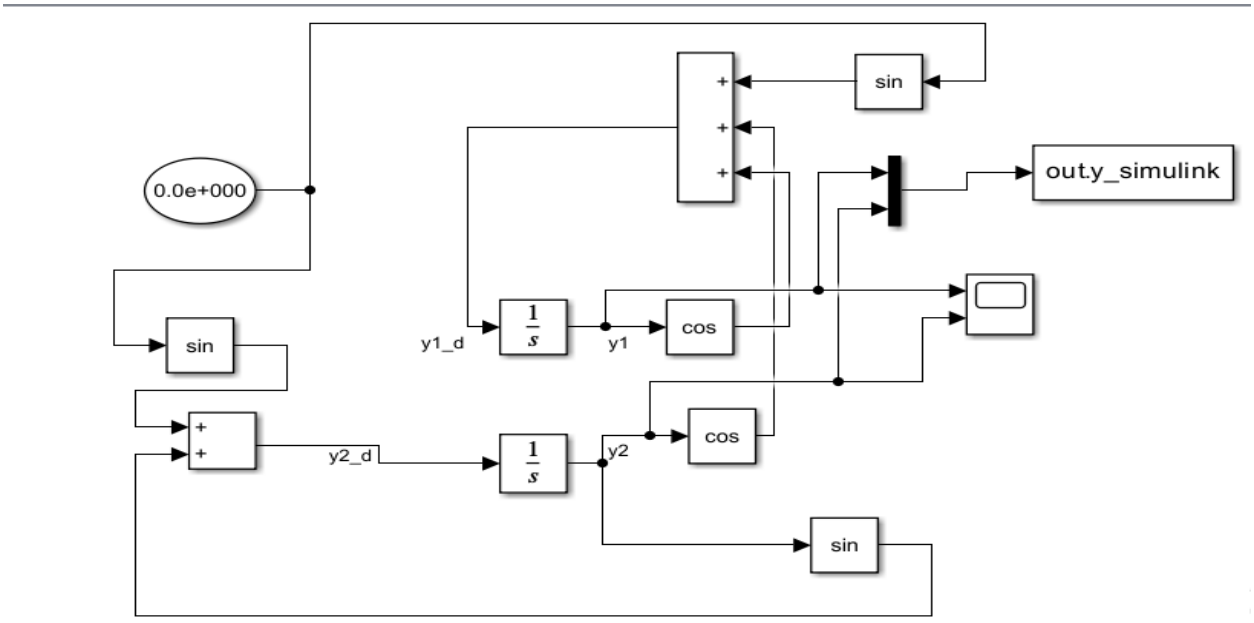
```
vel_dot = (1/m) * Forces - cross([p; q; r], [u; v; w]);

% Compute rotational dynamics
omega_dot = I_mat \ (Moments - cross([p; q; r], I_mat * [p; q; r]));

% Compute Euler angle rates
euler_dot = J * [p; q; r];

% Compute position rates in the inertial frame
pos_dot = eul2rotm([psi, theta, phi], 'ZYX') * [u; v; w];

% Concatenate state derivatives
states_dot = [vel_dot; omega_dot; euler_dot; pos_dot];

end
```

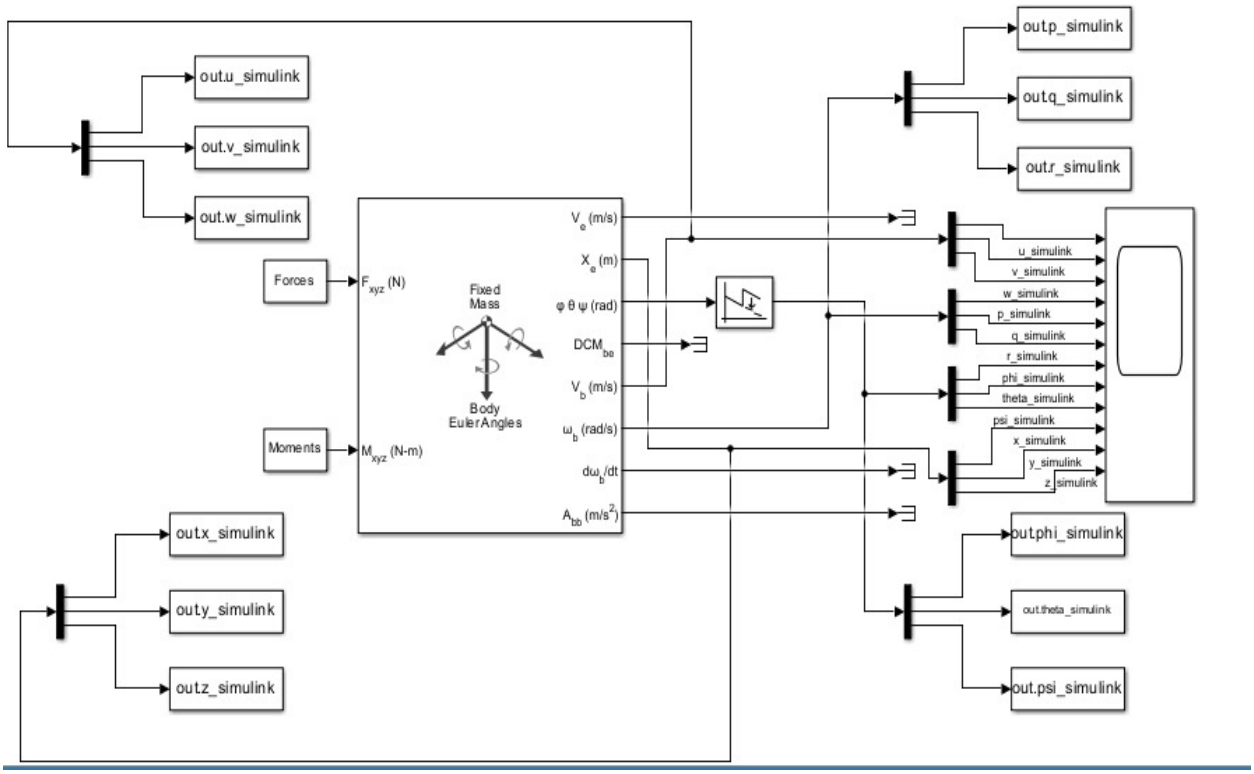
plotting Function

```
function plotValidation(t_vec_RK4, states_vec_RK4, simOut)
    % Function to compare simulation results from RK4 and Simulink
    figure_titles = {'Linear Velocities Validation', ...
                     'Angular Velocities Validation', ...
                     'Euler Angles Validation', ...
                     'Position Validation'};

    labels = {'$\bf{u}$ (m/s)', '$\bf{v}$ (m/s)', '$\bf{w}$ (m/s)', ...
              '$\bf{p}$ (rad/s)', '$\bf{q}$ (rad/s)', '$\bf{r}$ (rad/s)', ...
              '$\bf{\phi}$ ($^\circ$)', '$\bf{\theta}$ ($^\circ$)', '$\bf{\psi}$
($^\circ$)', ...
              '$\bf{x}$ (m)', '$\bf{y}$ (m)', '$\bf{z}$ (m)'};

    simData = {simOut.u_simulink, simOut.v_simulink, simOut.w_simulink, ...
               simOut.p_simulink, simOut.q_simulink, simOut.r_simulink, ...
               simOut.phi_simulink, simOut.theta_simulink, simOut.psi_simulink, ...
               simOut.x_simulink, simOut.y_simulink, simOut.z_simulink};

    for j = 1:4  % Loop for 4 figures
        figure;
        sgtitle(['$\bf{' figure_titles{j} '}$'], 'Interpreter', 'latex', 'FontSize',
16);

        for i = 1:3
            idx = (j-1)*3 + i;
            subplot(3,1,i); hold on;

            plot(t_vec_RK4, states_vec_RK4(idx, :), 'b', 'LineWidth', 1.5);
            plot(simData{idx}.time, simData{idx}.data, '--r', 'LineWidth', 1.5);

            title(labels{idx}, 'Interpreter', 'latex', 'FontSize', 14);
            xlabel('$\bf{t}$ (sec)', 'Interpreter', 'latex', 'FontSize', 14);
            ylabel(labels{idx}, 'Interpreter', 'latex', 'FontSize', 14);

            legend(['RK4 ' labels{idx}], ['Simulink ' labels{idx}], ...
                   'Interpreter', 'latex', 'FontSize', 12);
            grid on;
        end
    end
end
```

Appendix C

Simulink model





References :

- **Stevens, B. L., & Lewis, F. L. (2003).***Aircraft Control and Simulation: Dynamics, Controls Design, and Autonomous Systems* (3rd ed.). Wiley.

  Covers equations of motion, stability, and attitude representations in-depth.

- **Etkin, B., & Reid, L. D. (1996).***Dynamics of Flight: Stability and Control* (3rd ed.). Wiley.

  Discusses rigid body dynamics, stability, and control of fixed-wing aircraft.

- **Zipfel, P. H. (2007).***Modeling and Simulation of Aerospace Vehicle Dynamics* (2nd ed.). AIAA.

  Provides an in-depth discussion of quaternions, DCM, and attitude control.

•NASA Technical Reports: https://ntrs.nasa.gov/

•Free lectures and notes on aircraft stability, control, and motion equations Previous years

•Aircraft Control and Simulation" by Brian L. Stevens & Frank L. Lewis – Covers sensors used in autopilots and flight control.

•https://worldaviationato.com/en/aircraft-autopilot/

•https://www.airbus.com/en/node/51851

•https://www.unmannedsystemstechnology.com/expo/uav-autopilot-systems/

•https://controltheorymaster.files.wordpress.com/2017/11/farid-golnaraghi-benjamin-c-kuo-automatic-control-systems.pd

- Bishop, C. M. (2006). *Pattern Recognition and Machine Learning*. Springer.
- Kay, S. M. (1993). *Fundamentals of Statistical Signal Processing*. Prentice Hall.
- Ljung, L. (1999). *System Identification: Theory for the User*. Prentice Hall.