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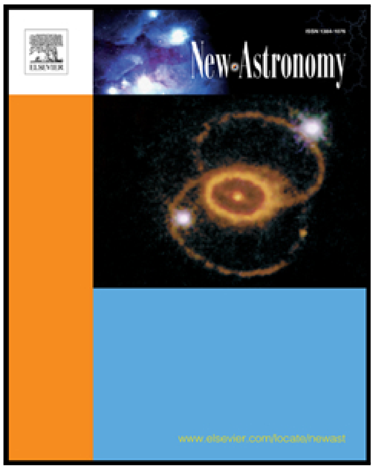

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PII: S1384-1076(20)30167-6
DOI: https://doi.org/10.1016/j.newast.2020.101547
Reference: NEASPA 101547

To appear in: *New Astronomy*

Received date: 4 May 2020
Revised date: 4 November 2020
Accepted date: 8 November 2020

Please cite this article as: Ahmed Estiak, S.B. Faruque, Quantum gravitomagnetic clock effect in Kerr gravitational field, *New Astronomy* (2020), doi: https://doi.org/10.1016/j.newast.2020.101547

Quantum gravitomagnetic clock effect in Kerr gravitational field

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Abstract

We have found an approximate solution of Dirac equation using Foldy-Wouthuysen-Tani Hamiltonian of a Dirac particle in the Kerr gravitational field. We have solved the equation approximately using time-independent perturbation theory for the positive energy states. We have found frequencies by which these states oscillate. Difference of the periods of any of these two states has an identical form of the classical gravitomagnetic clock effect where the terms are quantized. So that, we have found a quantum version of the gravitomagnetic clock effect of a Dirac fermion in the Kerr gravitational field.

Keywords: Gravitomagnetism; Gravitomagnetic clock effect; Dirac equation; Quantum; Kerr gravitational field

1 Introduction

Gravitational effect on quantum particles is a very active research area of theoretical physics [**1**, **2**]. Clock effect in gravitational field is another active research area [**3**, **4**]. This clock effect was first discovered by J.M. Cohen and B. Mashoon [**5**] in 1993 and known as the gravitomagnetic clock effect. Gravitomagnetic clock effect is the difference of periods in prograde and retrograde orbital motion of a particle in the equatorial plane of a central massive body like the Kerr black hole [**5**]. Let $T_+(T_-)$ is the period for prograde (retrograde) orbital motion around a central body, then for $r >> \frac{2GM}{c^2}$, this gravitomagnetic clock effect is given by $T_+ - T_- = \frac{4\pi J}{Mc^2} = \frac{4\pi a}{c}$ [**6**]. Here, the Kerr parameter, $a = \frac{J}{Mc}$. J is the spin angular momentum of the central body of mass M and c is the speed of light. Spin of the orbiting test particle of mass m lowers the gravitomagnetic clock effect by an amount of $\frac{6\pi S}{mc^2}$ [**7**, **8**].

The behaviors of spin zero particles in gravitational field are studied quantum mechanically in [**2**, **9**], and dynamics of spin 1/2 particles are studied in [**10**]. A quantum treatment of the classical gravitomagnetic clock effect for spin 1/2 particles in Schwarzschild field was shown by S.B. Faruque in [**11**].

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We are going to treat spin 1/2 particles again, but this time in Kerr gravitational field as a more generalized case. We shall treat the problem in a simplified picture. Let us assume the spin of a Dirac fermion contributing relativistically in the non-relativistic FWT (Foldy-Wouthuysen-Tani) Hamiltonian in Kerr gravitational field. The positive and the negative energy states of a Dirac fermion are uncoupled by the FWT transformation. It does not violate the covariance of the Dirac theory [**12**].

2 FWT Hamiltonian of a Dirac fermion in Kerr gravitational field

The FWT Hamiltonian for a Dirac particle in the Kerr geometry in a slowly rotating, weak-field limit is given in [**13**, **14**]. For non-relativistic limit and rotating central object with uniform mass density, the Hamiltonian for the Dirac fermion in Kerr gravitational field becomes:

$$\begin{aligned} H = \; & mc^2 + \frac{p^2}{2m} + m\phi - \overrightarrow{\omega_p}.(\vec{L} + \vec{S}) + \frac{1}{c^2}(\frac{4GMR^2}{5r^3}\overrightarrow{\omega_s}.(\vec{L} + \vec{S}) - \frac{p^4}{8m^3} + \frac{1}{2}m\phi^2 + \\ & \frac{3}{2m}\vec{p}.\phi\vec{p} + \frac{3GM}{2mr^3}\vec{L}.\vec{S} + \frac{6GMR^2}{5r^5}\vec{S}.[\vec{r} \times (\vec{r} \times \overrightarrow{\omega_s})]), \end{aligned} \tag{1}$$

where m is the mass of the Dirac fermion under consideration, $\vec{p}$ is the linear momentum of the Dirac fermion, $\phi = \frac{-GM}{r}$ is the gravitational potential due to gravity of the central body of mass M, $\overrightarrow{\omega_p}$ is the angular velocity of the particle, $\vec{L} = \vec{r} \times \vec{p}$ is the orbital angular momentum of the particle, $\vec{S} = \frac{\hbar\vec{\sigma}}{2}$ is the intrinsic spin of the particle with the Pauli spin matrices $\vec{\sigma}$, $\overrightarrow{\omega_s}$ is the angular velocity of the central body, R is the radius of the central body.

The Kerr parameter of the central rotating object, $\vec{a} = \frac{\vec{J}}{Mc} = \frac{2}{5c}R^2\overrightarrow{\omega_s}$, where, $\vec{J}$ is the angular momentum of the central body. Let, the direction of $\overrightarrow{\omega_s}$ or the axis of rotation is in the direction of z-axis.

Now, in equation (1) the term $\frac{1}{c^2}(\frac{4GMR^2}{5r^3}\overrightarrow{\omega_s}.(\vec{L} + \vec{S}))$ becomes:

$$\frac{1}{c^2}(\frac{4GMR^2}{5r^3}\overrightarrow{\omega_s}.(\vec{L} + \vec{S})) = \frac{1}{c^2}\frac{2G\vec{J}}{r^3}.(\vec{L} + \vec{S}). \tag{2}$$

The last term of the equation (1) becomes:

$$\frac{1}{c^2}(\frac{6GMR^2}{5r^5}\vec{S}.[\vec{r} \times (\vec{r} \times \overrightarrow{\omega_s})]) = -\frac{1}{c^2}(\frac{6GMR^2\omega_s}{5r^3})\vec{S}.\vec{n_z} = -\frac{1}{c^2}(\frac{3GJ}{r^3}S_z), \tag{3}$$

as we assume that, the Dirac fermion is in orbital motion in the equatorial plane of the central object. Here, $\vec{n_z}$ is the unit vector along the z-axis and S_z is the z component of the intrinsic spin of the particle. In our case, the central object is the Kerr black hole. From equation (1), (2),and (3), we can write the Hamiltonian for the Dirac fermion as

$$\begin{aligned} H = \; & mc^2 + \frac{p^2}{2m} + m\phi - \overrightarrow{\omega_p}.(\vec{L} + \vec{S}) + \frac{1}{c^2}(\frac{2G\vec{J}}{r^3}.(\vec{L} + \vec{S}) - \frac{p^4}{8m^3} + \frac{1}{2}m\phi^2 + \\ & \frac{3}{2m}\vec{p}.\phi\vec{p} + \frac{3GM}{2mr^3}\vec{L}.\vec{S} - \frac{3GJ}{r^3}S_z) \end{aligned} \tag{4}$$

Now, we can drop the first term mc^2 of the right-hand side of the equation (4) as it just adds a constant to the energy eigenvalue. This term has no physical relevance to our analysis. We can neglect the fourth term $\overrightarrow{\omega_p}.(\vec{L} + \vec{S})$ because in the non-relativistic limit $\overrightarrow{\omega_p}$ is very very small. We can also neglect the sixth term $\frac{1}{c^2}(\frac{p^4}{8m^3})$ as the particle is slowly rotating. We can also neglect the seventh term $\frac{1}{c^2}(\frac{1}{2}m\phi^2)$ as it is very small compared to the third term $m\phi$ where the potential ϕ is very small. We drop the eighth term $\frac{3}{2mc^2}\vec{p}.\phi\vec{p} = -\frac{3\hbar^2 GM}{2mc^2}\vec{\nabla}.\frac{\vec{r}}{r^3}$ as we analyze the situation with fairly constant gravity field. So, in a slowly rotating weak field limit, the Hamiltonian of the Dirac particle in Kerr black hole becomes:

$$\begin{aligned} H &= \frac{p^2}{2m} + m\phi + \frac{1}{c^2}(\frac{2G\vec{J}}{r^3}.(\vec{L}+\vec{S}) + \frac{3GM}{2mr^3}\vec{S}.\vec{L} - \frac{3GJ}{r^3}S_z) \\ &= \frac{p^2}{2m} - \frac{GMm}{r} + \frac{1}{c^2}(\frac{2GJ}{r^3}(L_z + \frac{\hbar}{2}\sigma_z) + \frac{3GM\hbar}{4mr^3}\vec{\sigma}.\vec{L} - \frac{3}{2}\frac{GJ\hbar}{r^3}\sigma_z) \\ &= \frac{p^2}{2m} - \frac{GMm}{r} + \frac{1}{c^2}(\frac{3GM\hbar}{4mr^3}\vec{\sigma}.\vec{L} + \frac{2GJ}{r^3}L_z - \frac{1}{2}\frac{GJ\hbar}{r^3}\sigma_z) \end{aligned} \tag{5}$$

This Hamiltonian is exactly reduced to the Hamiltonian of the Schwarzschild case [**11**] if there is no rotation of the black hole or $J = 0$. The two Kerr black hole terms of our Hamiltonian with J is almost similar to the two Kerr black hole terms of the Hamiltonian for the Dirac particle in a slowly rotating weak field limit in Kerr black hole derived by B R Iyer and Arvind Kumar in their paper "Dirac equation in Kerr space-time" [**15**]. The fourth term of equation (6) is the exact match with [**15**] and the last term is also similar without the coefficient. However, the coefficients have a similar order of magnitude.

3 Quantum gravitomagnetic clock effect

The Hamiltonian in equation (6) is a time-independent Hamiltonian. The solution of this Hamiltonian can be written as

$$\psi(\vec{x}, t) = \psi(\vec{x})exp(\frac{-iEt}{\hbar}). \tag{6}$$

We fix $\psi(\vec{x})$ as four component spinor as follows:

$$\psi(\vec{x}) = \begin{pmatrix} \psi_1(\vec{x}) \\ \psi_2(\vec{x}) \end{pmatrix}, \tag{7}$$

where $\psi_1(\vec{x})$ and $\psi_2(\vec{x})$ are two component spinors. The equations satisfied by these are

$$(\frac{p^2}{2m} - \frac{GMm}{r} + \frac{1}{c^2}(\frac{3GM\hbar}{4mr^3}\vec{\sigma}.\vec{L} + \frac{2GJ}{r^3}L_z - \frac{1}{2}\frac{GJ\hbar}{r^3}\sigma_z))\psi_1(\vec{x}) = E\psi_1(\vec{x}) \tag{8}$$

and

$$(-\frac{p^2}{2m} + \frac{GMm}{r} - \frac{1}{c^2}(\frac{3GM\hbar}{4mr^3}\vec{\sigma}.\vec{L} + \frac{2GJ}{r^3}L_z - \frac{1}{2}\frac{GJ\hbar}{r^3}\sigma_z))\psi_2(\vec{x}) = E\psi_2(\vec{x}). \tag{9}$$

The positive and negative energy states are uncoupled by the FWT transformation. $\psi_1(\vec{x})$ and $\psi_2(\vec{x})$ are simultaneous eigenstate of H, J^2, L^2, S^2 because of the term $\vec{\sigma}.\vec{L}$. Here, $\vec{J} = \vec{L} + \vec{S}$.

So, we expect the solutions to contain spin-angular functions y_l^{jm} [**16**]. This y_l^{jm} is the combination of spin functions and spherical harmonics. There is nothing to concern about the radial functions entering in the states $\psi_1(\vec{x})$ and $\psi_2(\vec{x})$. These two radial functions going with either of these states satisfy the same radial equation. So that, for each of these states, we can advance with only one radial function and we need not solve both equations (8) and (9). Solution of any one equation is enough to get the same for the other equation. Therefore, we write the solution to $\psi_1(\vec{x})$ explicitly,

$$\psi_1(\vec{x}) = R_1(r)y_l^{jm}, \tag{10}$$

and observe that

$$(\vec{\sigma}.\vec{L} + L_z - \sigma_z)y_l^{jm} = (\kappa + \hbar m_l - 2m_s)y_l^{jm}, \tag{11}$$

where $\kappa = l\hbar$ for $j = l + \frac{1}{2}$ and $\kappa = -(l+1)\hbar$ for $j = l - \frac{1}{2}$, where j is the total angular momentum quantum number and l is the orbital angular momentum quantum number. Here, $m_l = -l, -l+1, ..., l-1, l$ and $m_s = \frac{1}{2}, -\frac{1}{2}$. Using equation (10) and (11) in equation (8), we get

$$\frac{-\hbar^2}{2mr^2}\frac{d}{dr}(r^2\frac{dR_1}{dr}) + \frac{\hbar^2 l(l+1)}{2mr^2}R_1 - \frac{GMm}{r}R_1 + \frac{1}{c^2}(\frac{3GM\kappa\hbar}{4mr^3} + \frac{2GJ\hbar m_l}{r^3} - \frac{GJ\hbar m_s}{r^3})R_1 = ER_1. \tag{12}$$

We now deal the problem using the time-independent perturbation theory. Let, $H = H_0 + \lambda H_1$, where,

$$H_0 = \frac{-\hbar^2}{2mr^2}\frac{d}{dr}(r^2\frac{d}{dr}) + \frac{\hbar^2 l(l+1)}{2mr^2} - \frac{GMm}{r} \tag{13}$$

and

$$H_1 = \frac{1}{c^2}(\frac{3GM\kappa\hbar}{4mr^3} + \frac{2GJ\hbar m_l}{r^3} - \frac{GJ\hbar m_s}{r^3}) \tag{14}$$

The solution to the problem $H_0 R_1^0 = E_0 R_1^0$ is given in [**11**]. From there,

$$E_0 = -\frac{G^2M^2m^3}{2\hbar^2}\frac{1}{n^2}, \tag{15}$$

where, $n = 1, 2, ...$ is the principal quantum number. The expectation value of H_1 in the unperturbed states,

$$E_1 = \left\langle R_1^0 \middle| H_1 \middle| R_1^0 \right\rangle \tag{16}$$

From literature [**17**],

$$\left\langle R_1^0 \middle| \frac{1}{r^3} \middle| R_1^0 \right\rangle = \frac{1}{l(l+\frac{1}{2})(l+1)}\frac{1}{a^3n^3}, \tag{17}$$

where, $a = \frac{\hbar^2}{GMm^2}$. This is analogous to the Bohr radius, but with the coefficients of the gravitational potential term. From equation (14),(16),and (17),

$$E_1 = \frac{1}{c^2}(\frac{3GM\kappa\hbar}{4mr^3 l(l+\frac{1}{2})(l+1)a^3n^3} + \frac{2GJ\hbar m_l}{l(l+\frac{1}{2})(l+1)a^3n^3} - \frac{GJ\hbar m_s}{l(l+\frac{1}{2})(l+1)a^3n^3}). \tag{18}$$

Using equation (15) and (18) we can write,

$$\begin{aligned} E &= E_0 + E_1 \\ &= -\frac{G^2M^2m^3}{2\hbar^2}\frac{1}{n^2} + \frac{1}{c^2}(\frac{3GM\kappa\hbar}{4mr^3 l(l+\frac{1}{2})(l+1)a^3n^3} + \frac{2GJ\hbar m_l}{l(l+\frac{1}{2})(l+1)a^3n^3} - \frac{GJ\hbar m_s}{l(l+\frac{1}{2})(l+1)a^3n^3}) \end{aligned} \tag{19}$$

Here, firstly we will consider only two allowed states of m_l, which are $m_l = -l, +l$, the lower and upper limit of m_l respectively. There are two allowed states of j, which are $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$. That means, we are choosing only two allowed energy states $E_{j=l-\frac{1}{2}, m_l=l}$ and $E_{j=l+\frac{1}{2}, m_l=-l}$ for simplicity. The other possibilities for different m_l values will be discussed later.

So that, if $j = l - \frac{1}{2}$ (for which $\kappa = -(l+1)$), $m_l = l$ and $m_s = \frac{1}{2}$ then we can write,

$$\begin{aligned} E_{j=l-\frac{1}{2}, m_l=l} &= -\frac{G^2M^2m^3}{2\hbar^2}\frac{1}{n^2} - \frac{3GM\hbar^2}{4mc^2a^3n^3 l(l+\frac{1}{2})} + \frac{2GJ\hbar l}{c^2a^3n^3 l(l+\frac{1}{2})(l+1)} + \frac{GJ\hbar}{2c^2a^3n^3 l(l+\frac{1}{2})(l+1)} \\ &= -\frac{G^2M^2m^3}{2\hbar^2}\frac{1}{n^2} - \frac{3GM\hbar^2}{4mc^2a^3n^3 l(l+\frac{1}{2})} + \frac{GJ\hbar(4l+1)}{2c^2a^3n^3 l(l+\frac{1}{2})(l+1)}. \end{aligned} \tag{20}$$

If $j = l + \frac{1}{2}$ (for which $\kappa = l$), $m_l = -l$ and $m_s = -\frac{1}{2}$ then similarly we can write,

$$E_{j=l+\frac{1}{2},m_l=-l} = -\frac{G^2M^2m^3}{2\hbar^2}\frac{1}{n^2} + \frac{3GM\hbar^2}{4mc^2a^3n^3(l+\frac{1}{2})(l+1)} - \frac{GJ\hbar(4l+1)}{2c^2a^3n^3l(l+\frac{1}{2})(l+1)}. \tag{21}$$

Now, the frequency, $\omega = \frac{E}{\hbar}$, and the corresponding period of oscillation of the states $T = \frac{2\pi}{\omega}$. It is possible to calculate the corresponding period of oscillations for the two states from equation (20) and (21). To calculate this, we are treating the second and third term of equation (20) and (21) as very small compared to the first term. Here,

$$\begin{aligned}
T_{j=l-\frac{1}{2},m_l=l} &= \frac{2\pi\hbar}{-\frac{G^2M^2m^3}{2\hbar^2}\frac{1}{n^2} - \frac{3GM\hbar^2}{4mc^2a^3n^3l(l+\frac{1}{2})} + \frac{GJ\hbar(4l+1)}{2c^2a^3n^3l(l+\frac{1}{2})(l+1)}} \\
&= \frac{2\pi\hbar}{-\frac{G^2M^2m^3}{2\hbar^2}\frac{1}{n^2}(1 + \frac{3\hbar^4}{2m^4c^2a^3nGMl(l+\frac{1}{2})} - \frac{J\hbar^3(4l+1)}{c^2a^3nM^2m^3Gl(l+\frac{1}{2})(l+1)})} \\
&= -\frac{4\pi\hbar^3n^2}{G^2M^2m^3}(1 + \frac{3\hbar^4}{2m^4c^2a^3nGMl(l+\frac{1}{2})} - \frac{J\hbar^3(4l+1)}{c^2a^3nM^2m^3Gl(l+\frac{1}{2})(l+1)})^{-1} \\
&= -\frac{4\pi\hbar^3n^2}{G^2M^2m^3}(1 - \frac{3}{2}\frac{\hbar^4}{m^4c^2a^3nGMl(l+\frac{1}{2})} + \frac{J\hbar^3(4l+1)}{c^2a^3nM^2m^3Gl(l+\frac{1}{2})(l+1)}).
\end{aligned} \tag{22}$$

By inserting $a = \frac{\hbar^2}{GMm^2}$ in equation (22), we get,

$$T_{j=l-\frac{1}{2},m_l=l} = -\frac{4\pi\hbar^3n^2}{G^2M^2m^3} + \frac{6\pi\hbar n}{mc^2l(l+\frac{1}{2})} - \frac{4\pi Jn(4l+1)}{Mc^2l(l+\frac{1}{2})(l+1)} \tag{23}$$

Similarly,

$$T_{j=l+\frac{1}{2},m_l=-l} = -\frac{4\pi\hbar^3n^2}{G^2M^2m^3} - \frac{6\pi\hbar n}{mc^2l(l+\frac{1}{2})} + \frac{4\pi Jn(4l+1)}{Mc^2l(l+\frac{1}{2})(l+1)} \tag{24}$$

So that,

$$T_{j=l+\frac{1}{2},m_l=-l} - T_{j=l-\frac{1}{2},m_l=l} = \frac{8\pi J}{Mc^2}\frac{n(4l+1)}{l(l+\frac{1}{2})(l+1)} - \frac{6\pi\hbar}{mc^2}\frac{n(2l+1)}{l(l+\frac{1}{2})(l+1)} \tag{25}$$

Here we are declaring the terms $\frac{n(4l+1)}{l(l+\frac{1}{2})(l+1)}$ and $\frac{n(2l+1)}{l(l+\frac{1}{2})(l+1)}$ as quantization factor. We can get the difference of periods of oscillation of any two states $m_l = m_{l1}$ and $m_l = m_{l2}$ in similar way. That is:

$$T_{j=l+\frac{1}{2},m_l=m_{l2}} - T_{j=l-\frac{1}{2},m_l=m_{l1}} = \frac{8\pi J}{Mc^2}\frac{n(2(m_{l1}-m_{l2})+1)}{l(l+\frac{1}{2})(l+1)} - \frac{6\pi\hbar}{mc^2}\frac{n(2l+1)}{l(l+\frac{1}{2})(l+1)} \tag{26}$$

We can see, for different m_l values, only the quantization factor of the first term in the right-hand side of equation (25) will change accordingly. All the other terms will remain the same as equation (25).

4 Discussion

We have to interpret this result now. Here, firstly we consider two states of a Dirac fermion with the same l. In one state $j = l + \frac{1}{2}$ and $m_l = -l$. In another state $j = l - \frac{1}{2}$ and $m_l = l$. The difference of the period of oscillation of these two states is given by equation (25). In the classical gravitomagnetic clock effect, the periods discussed are of the prograde and retrograde orbital motion of a particle. In the case of the classical gravitomagnetic clock effect, the motion of the particle in prograde orbit is slower than the motion in a retrograde orbit. But in this quantum situation, no reference to orbit is reasonable, rather, we have different states with different total angular momentum and different magnetic quantum number. Formula (25) gives the oscillation-period difference in previously mentioned two states. This can be considered as an observable in the framework of quantum mechanics. Equation (26) is the more generalized form of equation (25) with any m_l values.

The formula of classical gravitomagnetic clock effect given in [**7**] in the orbit of a spinning particle orbiting the Kerr black hole is:

$$T_+ - T_- = \frac{4\pi J}{Mc^2} - \frac{6\pi S}{mc^2} \tag{27}$$

The structure of the equation (26) is the same and analogous with the equation (27). The only difference is in the terms in the right-hand side which are quantized in our equation (26). So, we can consider the formula (26) as a quantum analogue of the classical gravitomagnetic clock effect. We are declaring this quantum effect in the equation (26) as quantum gravitomagnetic clock effect in the Kerr field upon a Dirac fermion.

5 Conclusion

In this study, we find a formula for a Dirac particle in the Kerr gravitational field which is a quantized version of the classical gravitomagnetic clock effect. Further study is needed in the area of the quantum nature of the gravitomagnetic clock effect. In this study, we consider mainly the quantum effect. But in the classical gravitomagnetic effect, we only consider the general relativistic effect. However, the results are analogous. So, maybe equation (26) and equation (27) have some connection which can reveal some relations between quantum mechanics and general relativity.

6 Funding

This study was financially supported by the National Science and Technology Fellowship of Science and Technology Ministry, Government of the People's Republic of Bangladesh.

7 Conflict of interest

We have no conflict of interest.

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.