Digital Control Systems (CCE 341)

By

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Lecture3

Course Academic Calendar

Week Num	Date	Content
Week1	25/9/2025	Introduction
Week2	02/10/2025	Sampled Data Systems and the Z-Transform
Week3	09/10/2025	Properties of z-transform, Inverse z-Transforms
Week4	16/10/2025	Difference equations
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Week14	25/12/2025	Digital Controller Design

Contents of the Lecture

I. Properties of Z-Transform

II. Inverse of Z-Transforms

I. Properties of Z-Transform

Properties of Z-Transform

1. Linearity property

Suppose that the z-transform of $f(nT)$ is $F(z)$ and the z-transform of $g(nT)$ is $G(z)$. Then

$$Z[f(nT) \pm g(nT)] = Z[f(nT)] \pm Z[g(nT)] = F(z) \pm G(z) \qquad (6.20)$$

and for any scalar a

$$Z[af(nT)] = aZ[f(nT)] = aF(z) \qquad (6.21)$$

2. Left-shifting property

Suppose that the z-transform of $f(nT)$ is $F(z)$ and let $y(nT) = f(nT + mT)$. Then

$$Y(z) = z^m F(z) - \sum_{i=0}^{m-1} f(iT) z^{m-i}. \qquad (6.22)$$

If the initial conditions are all zero, i.e. $f(iT) = 0$, $i = 0, 1, 2, \ldots, m-1$, then,

$$Z[f(nT + mT)] = z^m F(z). \qquad (6.23)$$

$$Z\{y_{n-m}\} = z^m Y(z) - z^m y_0 - z^{m-1} y_1 - \ldots - z y_{m-1}$$

Properties of Z-Transform

3. Right-shifting property

   Suppose that the z-transform of $f(nT)$ is $F(z)$ and let $y(nT) = f(nT - mT)$. Then

$$Y(z) = z^{-m}F(z) + \sum_{i=0}^{m-1} f(iT - mT)z^{-i}. \tag{6.24}$$

   If $f(nT) = 0$ for $k < 0$, then the theorem simplifies to

$$Z[f(nT - mT)] = z^{-m}F(z). \tag{6.25}$$

$$\mathbb{Z}\{y_{n-m}\} = y_{-m} + y_{-m+1}z^{-1} + \ldots + y_{-1}z^{-m+1} + z^{-m}Y(z)$$

4. Attenuation property

   Suppose that the z-transform of $f(nT)$ is $F(z)$. Then,

$$Z[e^{-anT}f(nT)] = F[ze^{aT}]. \tag{6.26}$$

   This result states that if a function is multiplied by the exponential e^{-anT} then in the z-transform of this function z is replaced by ze^{aT}.

Properties of Z-Transform

5. Initial value theorem

   Suppose that the z-transform of $f(nT)$ is $F(z)$. Then the initial value of the time response is given by

$$\lim_{n\to 0} f(nT) = \lim_{z\to\infty} F(z). \tag{6.27}$$

6. Final value theorem

   Suppose that the z-transform of $f(nT)$ is $F(z)$. Then the final value of the time response is given by

$$\lim_{n\to\infty} f(nT) = \lim_{z\to 1}(1 - z^{-1})F(z). \tag{6.28}$$

Note that this theorem is valid if the poles of $(1 - z^{-1})F(z)$ are inside the unit circle or at $z = 1$.

Properties of Z-Transform

7. Differentiation Property

If:

$$X(z) = \mathcal{Z}\{x[n]\}$$

then:

$$\mathcal{Z}\{n\,x[n]\} = -z\frac{dX(z)}{dz}$$

Let:

$$x[n] = a^n u[n]$$

Then:

$$X(z) = \frac{z}{z-a}$$

Now, find $\mathcal{Z}\{na^n u[n]\}$:

Using the property:

$$\mathcal{Z}\{nx[n]\} = -z\frac{dX(z)}{dz}$$

Compute:

$$\frac{dX(z)}{dz} = \frac{(z-a) - z(1)}{(z-a)^2} = \frac{-a}{(z-a)^2}$$

Then:

$$-z\frac{dX(z)}{dz} = \frac{az}{(z-a)^2}$$

Properties of Z-Transform: Examples

Example1:

The z-transform of a unit ramp function $r(nT)$ is

$$R(z) = \frac{Tz}{(z-1)^2}.$$

Find the z-transform of the function $5r(nT)$.

Solution

Using the linearity property of z-transforms,

$$Z[5r(nT)] = 5R(z) = \frac{5Tz}{(z-1)^2}.$$

Properties of Z-Transform: Examples

Example2:

The z-transform of trigonometric function $r(nT) = \sin nwT$ is

$$R(z) = \frac{z \sin wT}{z^2 - 2z \cos wT + 1}.$$

find the z-transform of the function $y(nT) = e^{-2T} \sin nWT$.

Solution

Using property 4 of the z-transforms,

$$Z[y(nT)] = Z[e^{-2T} r(nT)] = R[ze^{2T}].$$

Thus,

$$Z[y(nT)] = \frac{ze^{2T} \sin wT}{(ze^{2T})^2 - 2ze^{2T} \cos wT + 1} = \frac{ze^{2T} \sin wT}{z^2 e^{4T} - 2ze^{2T} \cos wT + 1}$$

or, multiplying numerator and denominator by e^{-4T},

$$Z[y(nT)] = \frac{ze^{-2T} \sin wT}{z^2 - 2ze^{-2T} + e^{-4T}}.$$

Properties of Z-Transform: Examples

Example3:

Given the function

$$G(z) = \frac{0.792z}{(z-1)(z^2 - 0.416z + 0.208)},$$

find the final value of $g(nT)$.

Solution

Using the final value theorem,

$$\lim_{n\to\infty} g(nT) = \lim_{z\to 1}(1 - z^{-1})\frac{0.792z}{(z-1)(z^2 - 0.416z + 0.208)}$$

$$= \lim_{z\to 1} \frac{0.792}{z^2 - 0.416z + 0.208}$$

$$= \frac{0.792}{1 - 0.416 + 0.208} = 1.$$

II. Inverse Z-transforms

Inverse Z-transforms

- The inverse z-transform is obtained in a similar way to the inverse Laplace transforms.
- polynomial being of order no higher than the denominator.
- By finding the inverse z-transform we find the sequence associated with the given z-transform polynomial. As in the case of inverse Laplace transforms, we are interested in the output time response of a system.
- Therefore, we use an inverse transform to obtain y(t) from Y (z).

Inverse Z-transforms

- There are several methods to find the inverse z-transform of a given function.

- The following methods will be described here:

1. Power series (long division)
2. Expanding Y (z) into partial fractions and using z-transform tables to find the inverse transforms.
3. Obtaining the inverse z-transform using an inversion integral.

Inverse Z-transforms

Given a z-transform function $Y(z)$, we can find the coefficients of the associated sequence $y(nT)$at the sampling instants by using the inverse z-transform. The time function $y(t)$ is then determined as

$$y(t) = \sum_{n=0}^{\infty} y(nT)\delta(t - nT).$$

Method 1: Power series. This method involves dividing the denominator of $Y(z)$ into the numerator such that a power series of the form

$$Y(z) = y_0 + y_1 z^{-1} + y_2 z^{-2} + y_3 z^{-3} + \dots$$

is obtained. Notice that the values of $y(n)$ are the coefficients in the power series.

Inverse Z-transforms

Example4:

Find the inverse z-transform for the polynomial

$$Y(z) = \frac{z^2 + z}{z^2 - 3z + 4}.$$

Solution

Dividing the denominator into the numerator gives

$$\begin{array}{r|l}
 & 1 + 4z^{-1} + 8z^{-2} + 8z^{-3} \\ \hline
z^2 - 3z + 4 & z^2 + z \\
 & \underline{z^2 - 3z + 4} \\
 & \quad 4z - 4 \\
 & \quad \underline{4z - 12 + 16z^{-1}} \\
 & \qquad 8 - 16z^{-1} \\
 & \qquad \underline{8 - 24z^{-1} + 32z^{-2}} \\
 & \qquad\quad 8z^{-1} - 32z^{-2} \\
 & \qquad\quad 8z^{-1} - 24z^{-2} + 32z^{-3} \\
 & \qquad\quad \cdots
\end{array}$$

and the coefficients of the power series are

$$\begin{aligned}
y(0) &= 1, \\
y(T) &= 4, \\
y(2T) &= 8, \\
y(3T) &= 8,
\end{aligned}$$

Inverse Z-transforms

Example4:

Find the inverse z-transform for the polynomial

$$Y(z) = \frac{z^2 + z}{z^2 - 3z + 4}.$$

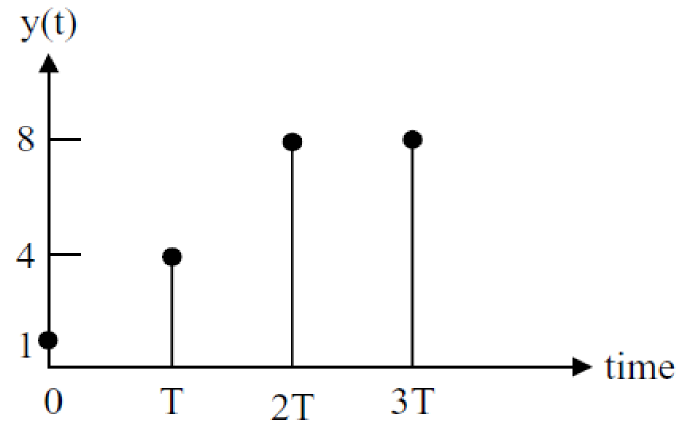

Figure 6.15 First few samples of $y(t)$

The required sequence is

$$y(t) = \delta(t) + 4\delta(t - T) + 8\delta(t - 2T) + 8\delta(t - 3T) + \dots$$

Figure 6.15 shows the first few samples of the time sequence $y(nT)$.

Inverse Z-transforms

Example4:

Find the inverse z-transform for the polynomial

$$Y(z) = \frac{z^2 + z}{z^2 - 3z + 4}.$$

```
y = filter(b, a, x)
```

Where:

- b → numerator coefficients of $H(z)$
- a → denominator coefficients of $H(z)$
- x → input sequence (e.g., δ[n] for impulse response)
- y → output (time-domain signal)

In MATLAB:

```
num=[1 1 0]
den=[1 -3 4]
x=[1 zeros(1,4)]
yk=filter(num,den,x)
```

```
num =

     1     1     0

den =

     1    -3     4

x =

     1     0     0     0     0

yk =

     1     4     8     8    -8
```

Inverse Z-transforms

Example5:

Find the inverse z-transform for $Y(z)$ given by the polynomial

$$Y(z) = \frac{z}{z^2 - 3z + 2}.$$

Solution

Dividing the denominator into the numerator gives

$$\begin{array}{r|l}
 & z^{-1} + 3z^{-2} + 7z^{-3} + 15z^{-4} \\
\hline
z^2 - 3z + 2 & z \\
 & \underline{z - 3 + 2z^{-1}} \\
 & 3 - 2z^{-1} \\
 & \underline{3 - 9z^{-1} + 6z^{-2}} \\
 & 7z^{-1} - 6z^{-2} \\
 & \underline{7z^{-1} - 21z^{-2} + 14z^{-3}} \\
 & 15z^{-2} - 14z^{-3} \\
 & 15z^{-2} - 45z^{-3} + 30z^{-4} \\
 & \ldots
\end{array}$$

and the coefficients of the power series are

$$\begin{aligned}
y(0) &= 0 \\
y(T) &= 1 \\
y(2T) &= 3 \\
y(3T) &= 7 \\
y(4T) &= 15
\end{aligned}$$

Inverse Z-transforms

Example5:

Find the inverse z-transform for $Y(z)$ given by the polynomial

$$Y(z) = \frac{z}{z^2 - 3z + 2}.$$

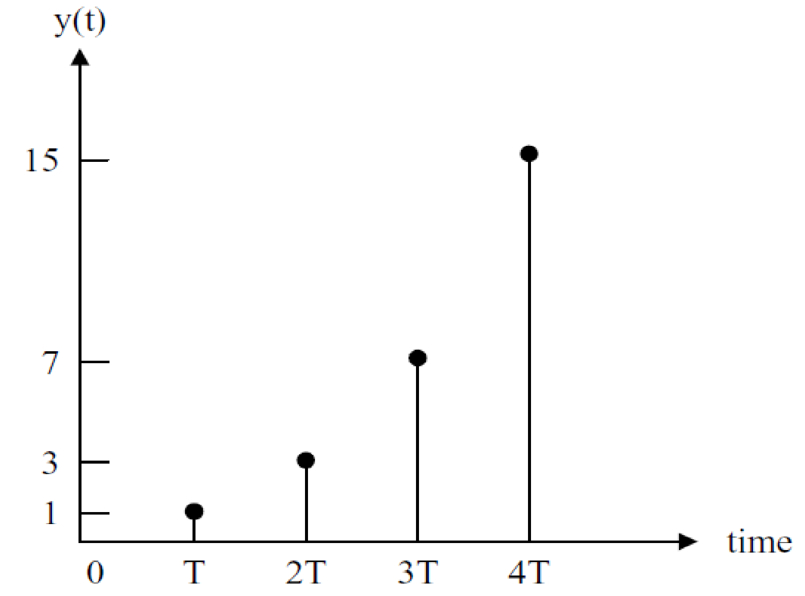

Figure 6.16 First few samples of $y(t)$

The required sequence is thus

$$y(t) = \delta(t - T) + 3\delta(t - 2T) + 7\delta(t - 3T) + 15\delta(t - 4T) + \dots .$$

Inverse Z-transforms

Example5:

Find the inverse z-transform for $Y(z)$ given by the polynomial

$$Y(z) = \frac{z}{z^2 - 3z + 2}.$$

```
y = filter(b, a, x)
```

Where:

- `b` → numerator coefficients of $H(z)$
- `a` → denominator coefficients of $H(z)$
- `x` → input sequence (e.g., δ[n] for impulse response)
- `y` → output (time-domain signal)

In MATLAB:

```
num=[0 1 0]
den=[1 -3 2]
x=[1 zeros(1,4)]
yk=filter(num,den,x)
```

Output

```
num =

    0    1    0

den =

    1   -3    2

x =

    1    0    0    0    0

yk =

     0     1     3     7    15
```

- **disadvantage** of power series method: it does not give a **closed form** of the resulting sequence.

Inverse Z-transforms

Method 2: Partial fractions. Similar to the inverse Laplace transform techniques, a partial fraction expansion of the function $Y(z)$ can be found, and then tables of known z-transforms can be used to determine the inverse z-transform.

Looking at the z-transform tables, we see that there is usually a z term in the numerator. It is therefore more convenient to find the partial fractions of the function $Y(z)/z$ and then multiply the partial fractions by z to obtain a z term in the numerator.

$f(kT)$	$F(z)$
$\delta(t)$	1
1	$\frac{z}{z-1}$
kT	$\frac{Tz}{(z-1)^2}$
$(kT)^2$	$\frac{T^2z(z+1)}{2(z-1)^3}$
$(kT)^3$	$\frac{T^3z(z^2+4z+1)}{(z-1)^4}$
e^{-akT}	$\frac{z}{z-e^{-aT}}$
kTe^{-akT}	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
a^k	$\frac{z}{z-a}$
$1-e^{-akT}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\sin akT$	$\frac{z\sin aT}{z^2-2z\cos aT+1}$
$\cos akT$	$\frac{z(z-\cos aT)}{z^2-2z\cos aT+1}$
$e^{-akT}\sin bkT$	$\frac{e^{-aT}z\sin bT}{z^2-2e^{-aT}z\cos bT+e^{-2aT}}$
$e^{-akT}\cos bkT$	$\frac{z^2-e^{-aT}z\cos bT}{z^2-2e^{-aT}z\cos bT+e^{-2aT}}$

Inverse Z-transforms

Example6:

Find the inverse z-transform of the function

$$Y(z) = \frac{z}{(z-1)(z-2)}$$

Solution

The above expression can be written as

$$\frac{Y(z)}{z} = \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}.$$

The values of A and B can be found by equating like powers in the numerator, i.e.

$$A(z-2) + B(z-1) \equiv 1.$$

We find $A = -1$, $B = 1$, giving

$$\frac{Y(z)}{z} = \frac{-1}{z-1} + \frac{1}{z-2}$$

or

$$Y(z) = \frac{-z}{z-1} + \frac{z}{z-2}$$

Inverse Z-transforms

Example6:

From the z-transform tables we find that

$$y(nT) = -1 + 2^n$$

and the coefficients of the power series are

$$\begin{aligned} y(0) &= 0, \\ y(T) &= 1, \\ y(2T) &= 3, \\ y(3T) &= 7, \\ y(4T) &= 15, \\ &\dots \end{aligned}$$

so that the required sequence is

$$y(t) = \delta(t - T) + 3\delta(t - 2T) + 7\delta(t - 3T) + 15\delta(t - 4T) + \dots.$$

Inverse Z-transforms

Example7:

Find the inverse z-transform of the function

$$Y(z) = \frac{1}{(z-1)(z-2)}.$$

Solution

The above expression can be written as

$$\frac{Y(z)}{z} = \frac{1}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}.$$

The values of A, B and C can be found by equating like powers in the numerator, i.e.

$$A(z-1)(z-2) + Bz(z-2) + Cz(z-1) \equiv 1$$

or

$$A(z^2 - 3z + 2) + Bz^2 - 2Bz + Cz^2 - Cz \equiv 1,$$

giving

$$\begin{aligned} A + \quad B + C &= 0, \\ -3A - 2B - C &= 0, \\ 2A \qquad\qquad &= 1. \end{aligned}$$

The values of the coefficients are found to be $A = 0.5$, $B = -1$ and $C = 0.5$. Thus,

$$\frac{Y(z)}{z} = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}$$

Inverse Z-transforms

Example7:

Find the inverse z-transform of the function

$$Y(z) = \frac{1}{(z-1)(z-2)}.$$

$$Y(z) = \frac{1}{2} - \frac{z}{z-1} + \frac{z}{2(z-2)}.$$

Using the inverse z-transform tables, we find

$$y(nT) = a - 1 + \frac{2^n}{2} = a - 1 + 2^{n-1}$$

where

$$a = \begin{cases} 1/2, & n = 0, \\ 0, & n \neq 0, \end{cases}$$

the coefficients of the power series are

$$\begin{aligned} y(0) &= 0 \\ y(T) &= 0 \\ y(2T) &= 1 \\ y(3T) &= 3 \\ y(4T) &= 7 \\ y(5T) &= 15 \\ &\ldots, \end{aligned}$$

and the required sequence is

$$y(t) = \delta(t-2T) + 3\delta(t-3T) + 7\delta(t-4T) + 15\delta(t-5T) + \ldots.$$

Inverse Z-transforms

The process of finding inverse z-transforms is aided by considering what form is taken by the roots of $Y(z)$. It is useful to distinguish the case of distinct real roots and that of multiple order roots.

Case I: Distinct real roots. When $Y(z)$ has distinct real roots in the form

$$Y(z) = \frac{N(z)}{(z-p_1)(z-p_2)(z-p_3)\ldots(z-p_n)},$$

then the partial fraction expansion can be written as

$$Y(z) = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \frac{A_3}{z-p_3} + \ldots + \frac{A_n}{z-p_n}$$

and the coefficients A_i can easily be found as

$$A_i = (z-p_i)\,Y(z)\big|_{z=p_i} \quad \text{for } i = 1,2,3,\ldots,n.$$

Inverse Z-transforms

Example8:

Using the partial expansion method described above, find the inverse z-transform of

$$Y(z) = \frac{z^2 + z}{(z - 0.5)(z - 0.8)(z - 1)}.$$

Solution

Rewriting the function as

$$\frac{Y(z)}{z} = \frac{A}{z - 0.5} + \frac{B}{z - 0.8} + \frac{C}{z - 1}$$

we find that

$$A = (z - 0.5)\left.\frac{z + 1}{(z - 0.5)(z - 0.8)(z - 1)}\right|_{z=0.5} = 10,$$

$$B = (z - 0.8)\left.\frac{z + 1}{(z - 0.5)(z - 0.8)(z - 1)}\right|_{z=0.8} = -30,$$

$$C = (z - 1)\left.\frac{z + 1}{(z - 0.5)(z - 0.8)(z - 1)}\right|_{z=1} = 20.$$

Thus,

$$Y(z) = \frac{10z}{z - 0.5} - \frac{30z}{z - 0.8} + \frac{20z}{z - 1}$$

The inverse transform is found from the tables as

$$y(nT) = 10(0.5)^n - 30(0.8)^n + 20$$

Inverse Z-transforms

Example8:

Using the partial expansion method described above, find the inverse z-transform of

$$Y(z) = \frac{z^2 + z}{(z - 0.5)(z - 0.8)(z - 1)}.$$

The coefficients of the power series are

$$\begin{aligned} y(0) &= 0 \\ y(T) &= 1 \\ y(2T) &= 3.3 \\ y(3T) &= 5.89 \\ &\dots \end{aligned}$$

and the required sequence is

$$y(t) = \delta(t - T) + 3.3\delta(t - 2T) + 5.89\delta(t - 3T) + \dots.$$

Inverse Z-transforms

Case II: Multiple order roots. When $Y(z)$ has multiple order roots of the form

$$Y(z) = \frac{N(z)}{(z-p_1)(z-p_1)^2(z-p_1)^3\ldots(z-p_1)^r},$$

then the partial fraction expansion can be written as

$$Y(z) = \frac{\lambda_1}{z-p_1} + \frac{\lambda_2}{(z-p_1)^2} + \frac{\lambda_3}{(z-p_1)^3} + \ldots + \frac{\lambda_r}{(z-p_1)^r}$$

and the coefficients λ_i can easily be found as

$$\lambda_{r-k} = \frac{1}{k!}\,\frac{d^k}{dz^k[(z-p_i)^r(X(z)/z)]}\bigg|_{z=p_i}. \tag{6.29}$$

Inverse Z-transforms

Example9:

Using (6.29), find the inverse z-transform of

$$Y(z) = \frac{z^2 + 3z - 2}{(z+5)(z-0.8)(z-2)^2}.$$

Solution

Rewriting the function as

$$\frac{Y(z)}{z} = \frac{z^2 + 3z - 2}{z(z+5)(z-0.8)(z-2)^2} = \frac{A}{z} + \frac{B}{z+5} + \frac{C}{z-0.8} + \frac{D}{(z-2)} + \frac{E}{(z-2)^2}$$

we obtain

$$A = z\,\frac{z^2 + 3z - 2}{z(z+5)(z-0.8)(z-2)^2}\bigg|_{z=0} = \frac{-2}{5 \times (-0.8) \times 4} = 0.125,$$

$$B = (z+5)\,\frac{z^2 + 3z - 2}{z(z+5)(z-0.8)(z-2)^2}\bigg|_{z=-5} = \frac{8}{-5 \times (-5.8) \times 49} = 0.0056,$$

$$C = (z-0.8)\,\frac{z^2 + 3z - 2}{z(z+5)(z-0.8)(z-2)^2}\bigg|_{z=0.8} = \frac{1.04}{0.8 \times 5.8 \times 1.14} = 0.16,$$

Inverse Z-transforms

Example9:

$$E = (z-2)^2 \left.\frac{z^2+3z-2}{z(z+5)(z-0.8)(z-2)^2}\right|_{z=2} = \frac{8}{2\times 7\times 1.2} = 0.48,$$

$$D = \frac{d}{dz}\left[\frac{z^2+3z-2}{z(z+5)(z-0.8)}\right]\Bigg|_{z=2}$$

$$= \left.\frac{[z(z+5)(z-0.8(2z+3)-(z^2+3z-2)(3z^2+8.4z-4)]}{(z^3+4.2z^2-4z)^2}\right|_{z=2} = -0.29.$$

We can now write $Y(z)$ as

$$Y(z) = 0.125 + \frac{0.0056z}{z+5} + \frac{0.016z}{z-0.8} - \frac{0.29z}{(z-2)} + \frac{0.48z}{(z-2)^2}$$

The inverse transform is found from the tables as

$$y(nT) = 0.125a + 0.0056(-5)^n + 0.016(0.8)^n - 0.29(2)^n + 0.24n(2)^n,$$

where

$$a = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

- Note: for last term, we used the multiplication by k property which is equivalent to a z-differentiation.

In MATLAB,

`residue()` finds the **partial fraction expansion** of a **rational function**, such as:

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \ldots}{a_0 + a_1 z^{-1} + \ldots}$$

```
[r, p, k] = residue(b, a)
```

where:

Symbol	Meaning
`b`	Numerator coefficients (vector)
`a`	Denominator coefficients (vector)
`r`	Residues
`p`	Poles
`k`	Direct polynomial terms (if any)

- in MATLAB, you can find the partial fraction expansion of a ratio of two polynomials $F(z)$ with:

$$F(z) = \frac{2z^3 + z^2}{z^3 + z + 1}$$

```
num = [2 1 0 0];
den = [1 0 1 1];
[r,p,k] = residue(num,den)

r =
0.5354 + 1.0390i
0.5354 - 1.0390i
-0.0708 + 0.0000i

p =
0.3412 + 1.1615i
0.3412 - 1.1615i
-0.6823 + 0.0000i

k =
2
```

- residue returns the complex roots and poles, and a constant term in k,
- representing the partial fraction expansion

$$F(z) = \frac{0.5354 + 1.0390i}{z - (0.3412 + 1.1615j)} + \frac{0.5354 - 1.0390i}{z - (0.3412 - 1.1615j)} + \frac{-0.0708}{z + 0.6823} + 2$$


Thank you