Digital Control Systems (CCE 341)

By

Dr : Asmaa Aly Hagar

Lecture2

Course Academic Calendar

Week Num	Date	Content
Week1	25/9/2025	Introduction
Week2	02/10/2025	Sampled Data Systems and the Z-Transform
Week3	09/10/2025	Properties of z-transform, Inverse z-Transforms
Week4	16/10/2025	Difference equations
Week5	23/10/2025	Open Loop Discreate Time System Analysis
Week6	30/10/2025	Closed Loop Discreate Time System Analysis
Week7	06/11/2025	Midterm
Week8	13/11/2025	System Time Response
Week9	20/11/2025	Mapping from S-plane to Z-plane
Week10	27/11/2025	Damping Ratio and undamped natural frequency in the Z-plane
Week11	04/12/2025	Stability of Sampled Systems
Week12	11/12/2025	Root Locus for Sampled Data Systems
Week13	18/12/2025	Nyquist Criterion and Bode Diagrams
Week14	25/12/2025	Digital Controller Design

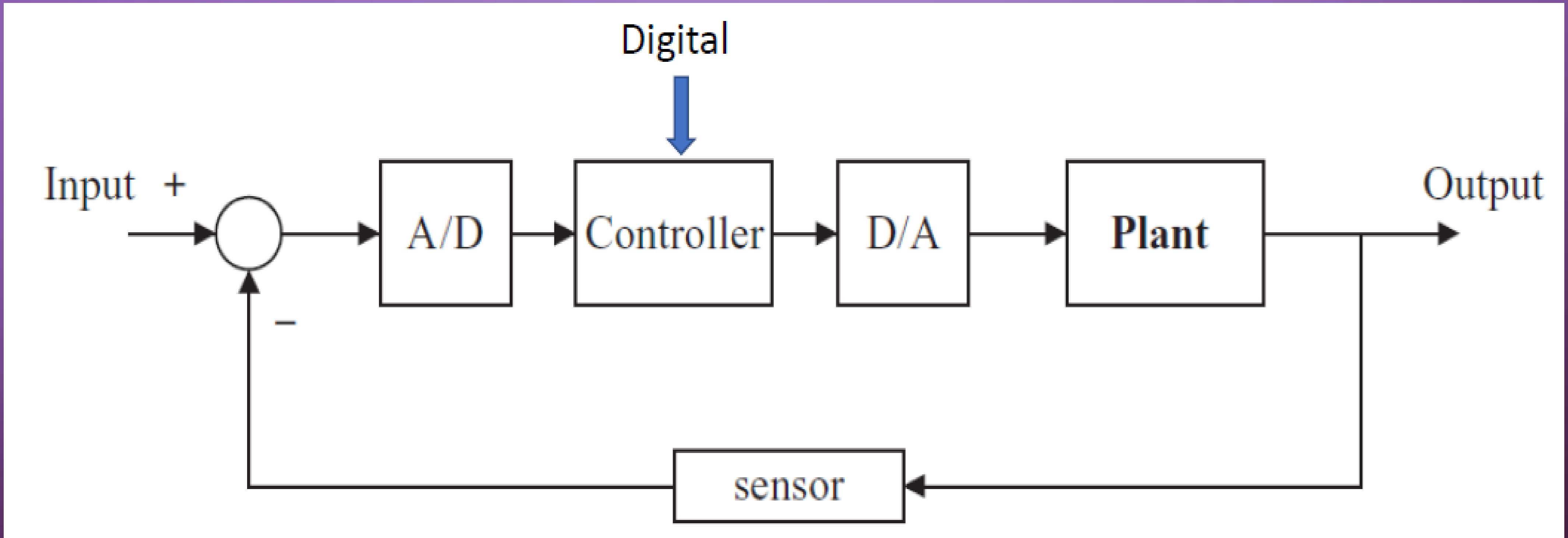
Contents of the Lecture

I. Sampling

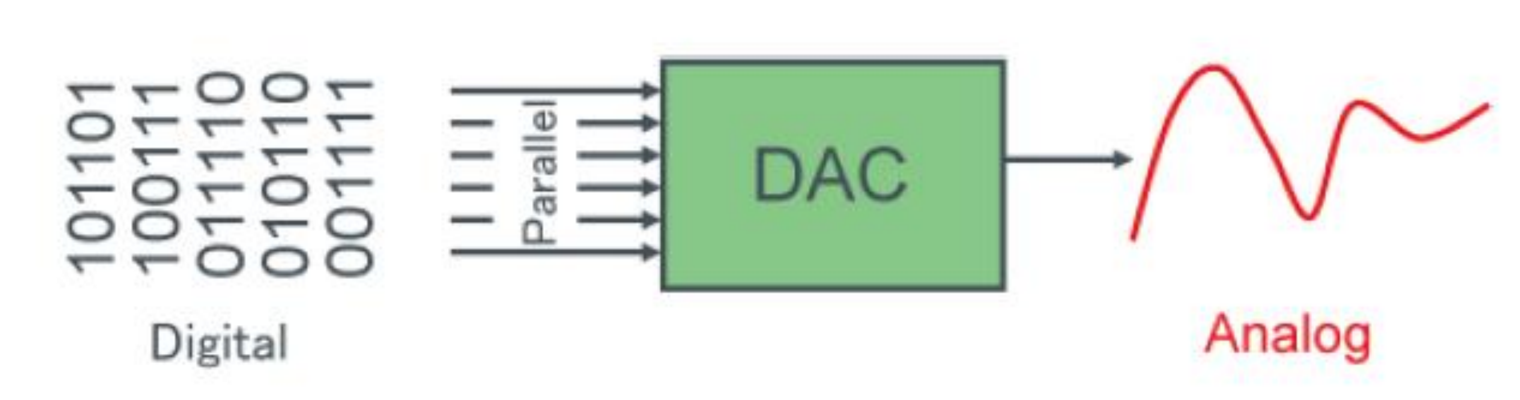
II. Z-Transform

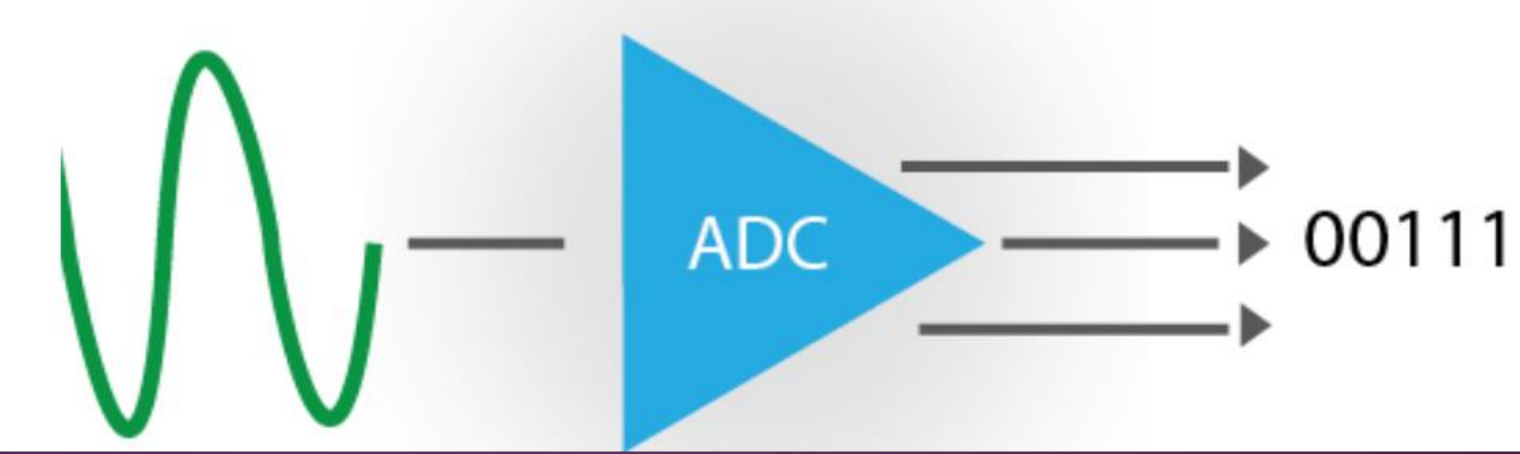
I. Sampling

Digital control system



D/A and A/D


101101
100111
011110
010110
001111
Digital
Parallel
DAC
Analog

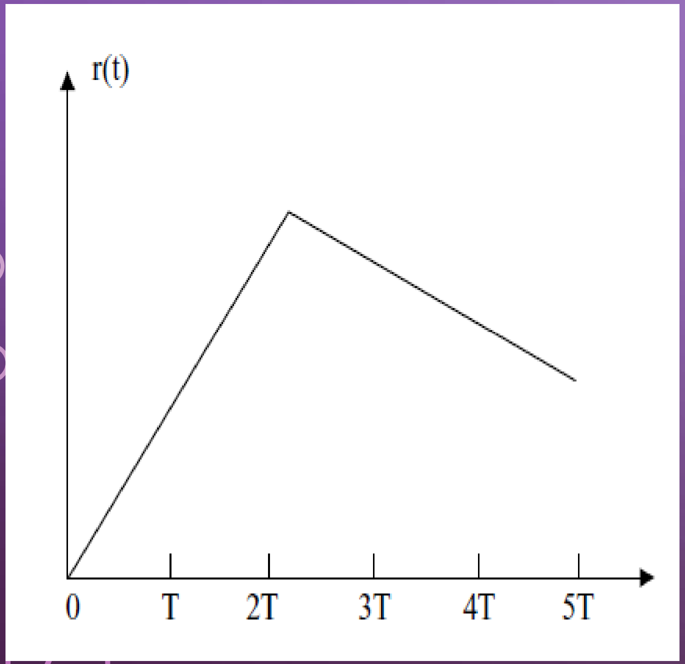

ADC
00111

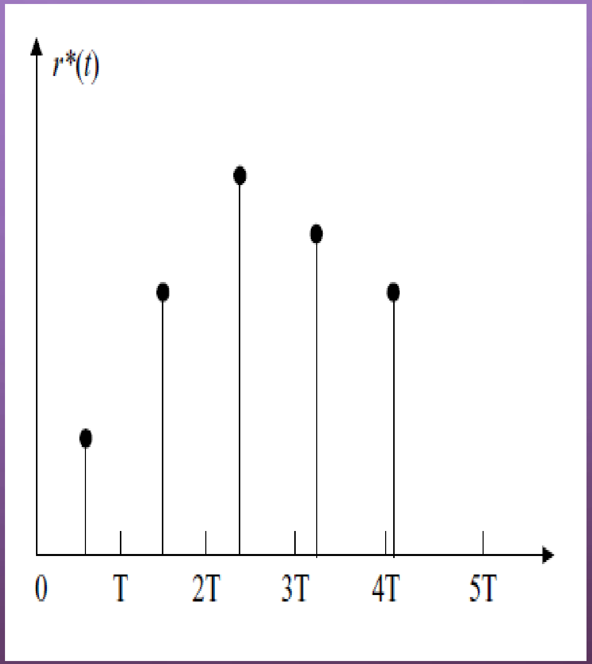
Sampling

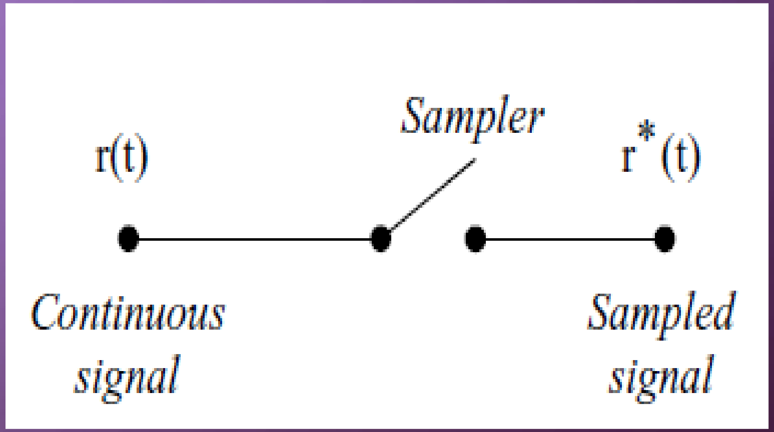
- Sampled data system operates on discrete-time rather than continuous-time signals.
- A digital computer is used as the controller in such a system.
- A D/A converter is usually connected to the output of the computer to drive the plant.
- We will assume that all the signals enter and leave the computer at the same fixed times, known as the sampling times.
- The digital computer performs the controller or the compensation function within the system.
- The A/D converter converts the error signal, which is a continuous signal, into digital form so that it can be processed by the computer.
- At the computer output the D/A converter converts the digital output of the computer into a form which can be used to drive the plant.

THE SAMPLING PROCESS

- A sampler is basically a switch that closes every T seconds.
- When a continuous signal r (t) is sampled at regular intervals T , the resulting discrete-time signal







THE SAMPLING PROCESS

The ideal sampling process can be considered as the multiplication of a pulse train with a continuous signal, i.e.

$$r^*(t) = P(t)r(t), \tag{6.1}$$

where $P(t)$ is the delta pulse train as shown in Figure 6.6, expressed as

$$P(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT); \tag{6.2}$$

thus,

$$r^*(t) = r(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{6.3}$$

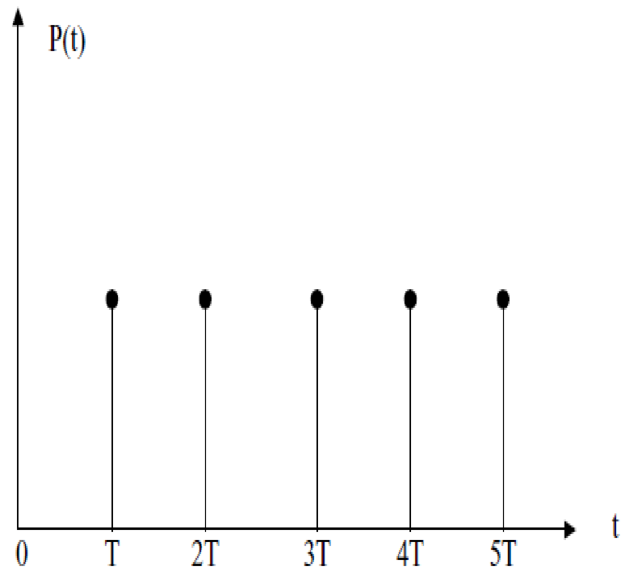


Figure 6.6 Delta pulse train

THE SAMPLING PROCESS

or

$$r^*(t) = \sum_{n=-\infty}^{\infty} r(nT)\delta(t - nT). \tag{6.4}$$

Now

$$r(t) = 0, \quad \text{for } t < 0, \tag{6.5}$$

and

$$r^*(t) = \sum_{n=0}^{\infty} r(nT)\delta(t - nT). \tag{6.6}$$

Taking the Laplace transform of (6.6) gives

$$R^*(s) = \sum_{n=0}^{\infty} r(nT)e^{-snT}. \tag{6.7}$$

Equation (6.7) represents the Laplace transform of a sampled continuous signal $r(t)$.

Zero-order hold (ZOH)

A D/A converter converts the sampled signal $r^*(t)$ into a continuous signal $y(t)$. The D/A can be approximated by a zero-order hold (ZOH) circuit as shown in Figure 6.7. This circuit remembers the last information until a new sample is obtained, i.e. the zero-order hold takes the value $r(nT)$ and holds it constant for $nT \le t < (n+1)T$, and the value $r(nT)$ is used during the sampling period.

The impulse response of a zero-order hold is shown in Figure 6.8. The transfer function of a zero-order hold is given by

$$G(t) = H(t) - H(t - T), \tag{6.8}$$

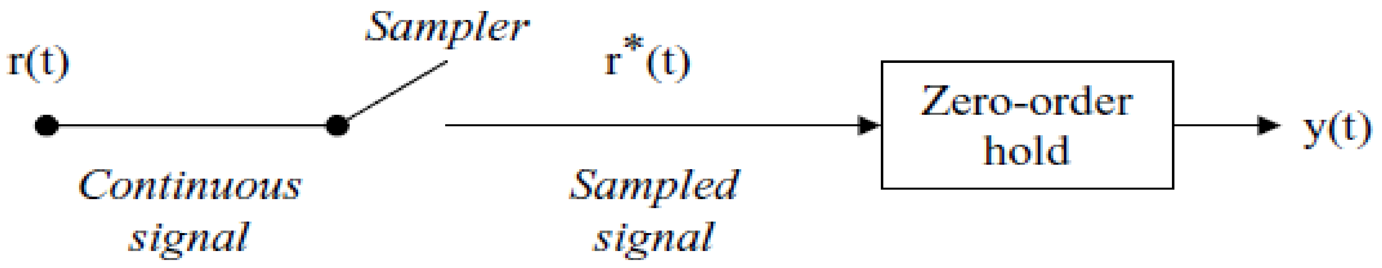


Figure 6.7 A sampler and zero-order hold

Zero-order hold (ZOH)

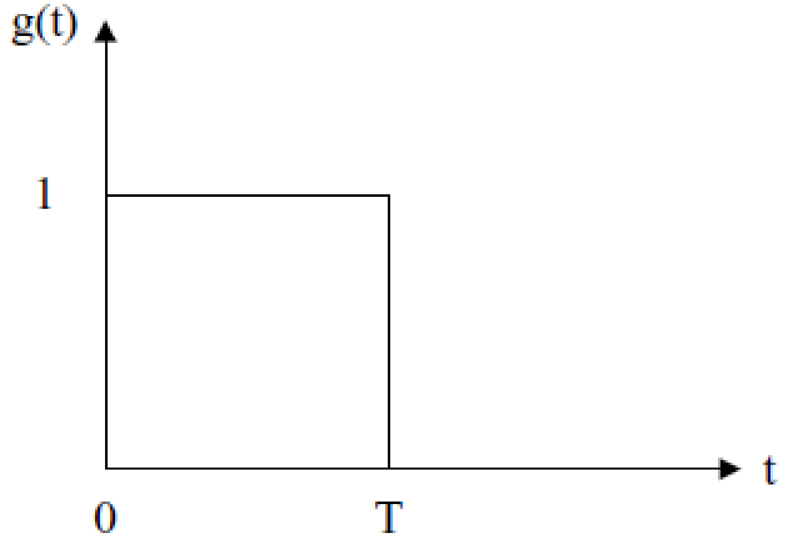


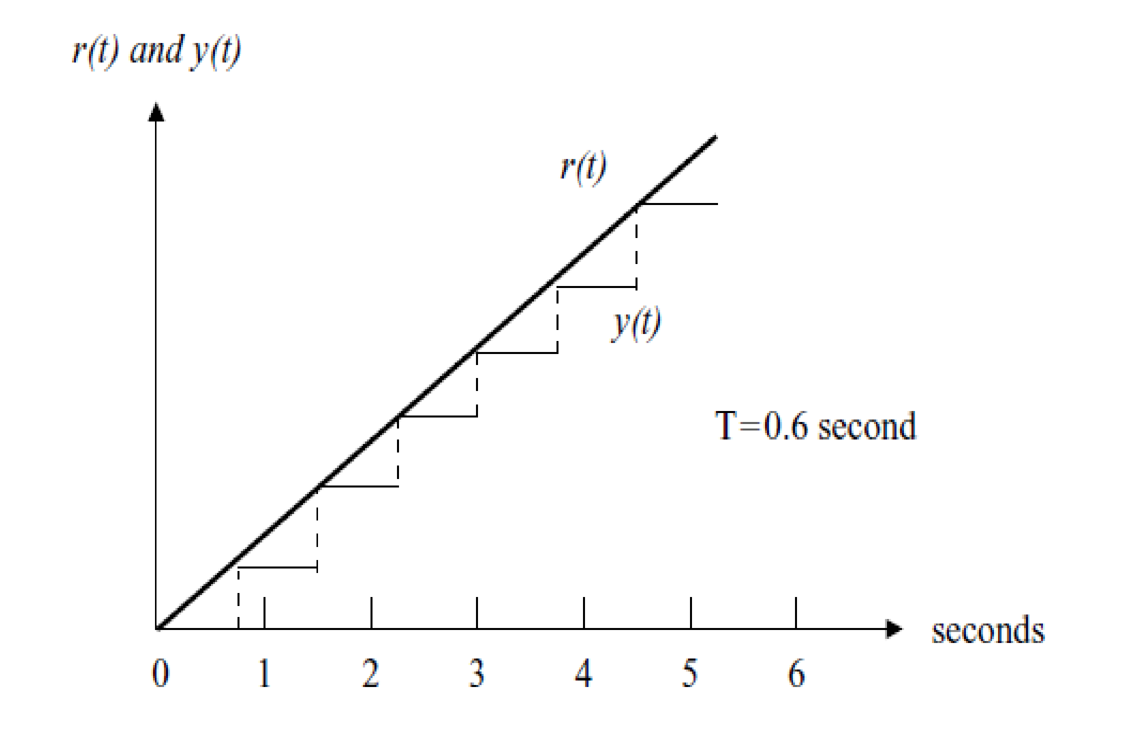
Figure 6.8 Impulse response of a zero-order hold

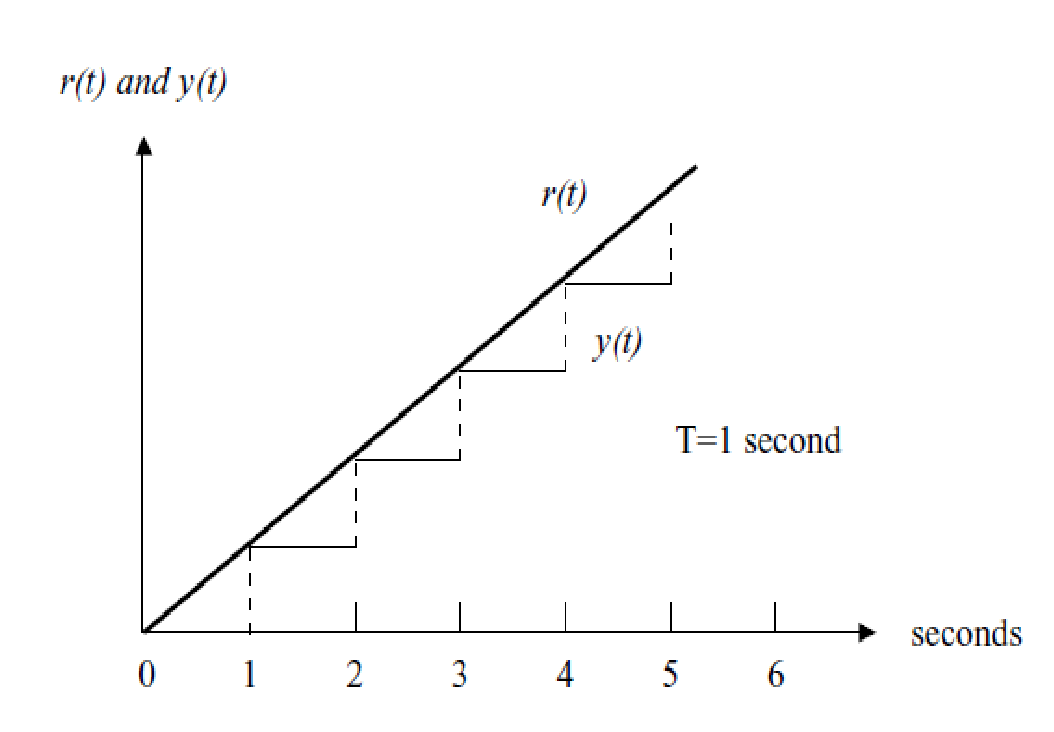
where $H(t)$ is the step function, and taking the Laplace transform yields

$$G(s) = \frac{1}{s} - \frac{e^{-Ts}}{s} = \frac{1 - e^{-Ts}}{s}. \tag{6.9}$$

Zero-order hold (ZOH)

- A sampler and zero order hold can accurately follow the input signal if the sampling time T is small compared to the transient changes in the signal.
- The response of a sampler and a zero-order hold to a ramp input is shown in the following Figures for two different values of sampling period.





Zero-order hold (ZOH) : Example

Figure 6.10 shows an ideal sampler followed by a zero-order hold.

Assuming the input signal $r(t)$ is as shown in the figure, show the waveforms after the sampler and also after the zero-order hold.

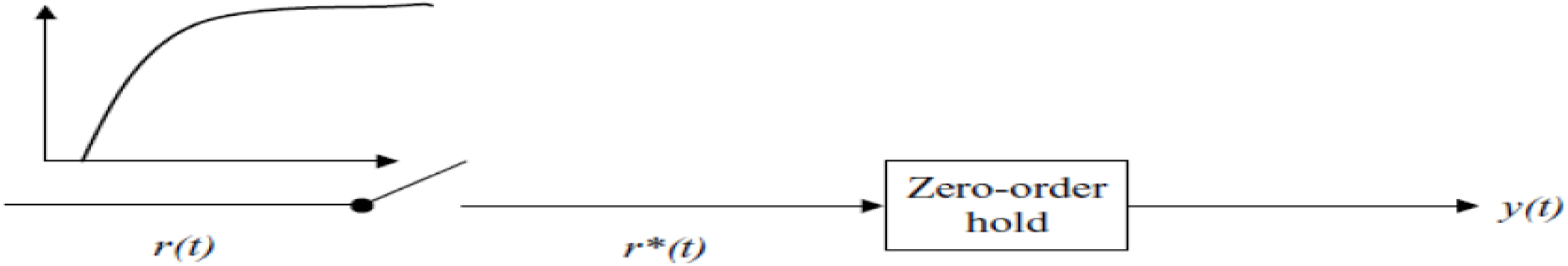
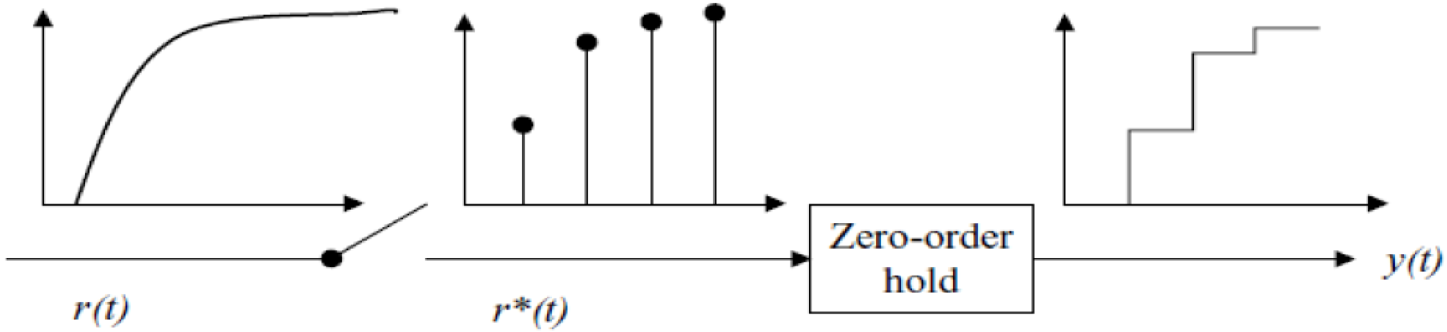


Figure 6.10 Ideal sampler and zero-order hold for Example 6.1



II. Z-Transform

THE Z-TRANSFORM

The z-transformation is used in sampled data systems just as the Laplace transformation is used in continuous-time systems.

The z-transform is defined so that: $Z = e^{sT}$

the z-transform of the function $r(t)$ is $Z[r(t)] = R(z)$ which, from (6.7), is given by

$$R(z) = \sum_{n=0}^{\infty} r(nT)z^{-n}.$$

Notice that the z-transform consists of an infinite series in the complex variable z, and

$$R(z) = r(0) + r(T)z^{-1} + r(2T)z^{-2} + r(3T)z^{-3} + \ldots,$$

i.e. the $r(nT)$ are the coefficients of this power series at different sampling instants.

THE Z-TRANSFORM

- The response of a sampled data system can be determined easily by finding the z-transform of the output and then calculating the inverse z-transform.

- Just like the Laplace transform techniques used in continuous-time systems.

THE Z-TRANSFORM

1. Unit step function

Consider a unit step function as shown in Figure 6.12, defined as

$$r(nT) = \begin{cases} 0, & n < 0, \\ 1, & n \geq 0. \end{cases}$$

From (6.11),

$$R(z) = \sum_{n=0}^{\infty} r(nT) z^{-n} = \sum_{n=0}^{\infty} z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + \ldots$$

or

$$R(z) = \frac{z}{z-1}, \quad \text{for } |z| > 1.$$

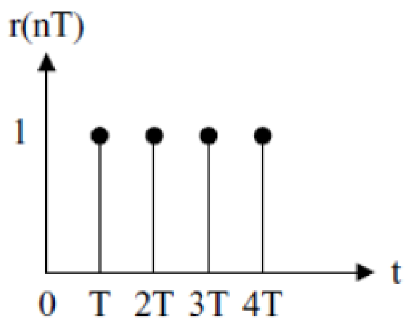


Figure 6.12 Unit step function

THE Z-TRANSFORM

2. Unit ramp function

Consider a unit ramp function as shown in Figure 6.13, defined by

$$r(nT) = \begin{cases} 0, & n < 0, \\ nT, & n \geq 0. \end{cases}$$

From (6.11),

$$R(z) = \sum_{n=0}^{\infty} r(nT)z^{-n} = \sum_{n=0}^{\infty} nTz^{-n} = Tz^{-1} + 2Tz^{-2} + 3Tz^{-3} + 4Tz^{-4} + \ldots$$

or

$$R(z) = \frac{Tz}{(z-1)^2}, \qquad \text{for } |z| > 1.$$

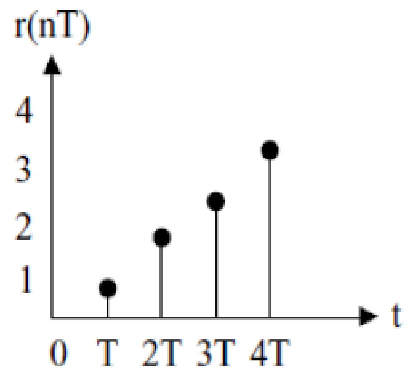


Figure 6.13 Unit ramp function

THE Z-TRANSFORM

3. Exponential function

Consider the exponential function shown in Figure 6.14, defined as

$$r(nT) = \begin{cases} 0, & n < 0, \\ e^{-anT}, & n \geq 0. \end{cases}$$

From (6.11),

$$R(z) = \sum_{n=0}^{\infty} r(nT) z^{-n} = \sum_{n=0}^{\infty} e^{-anT} z^{-n} = 1 + e^{-aT} z^{-1} + e^{-2aT} z^{-2} + e^{-3aT} z^{-3} + \ldots$$

or

$$R(z) = \frac{1}{1 - e^{-aT} z^{-1}} = \frac{z}{z - e^{-aT}}, \quad \text{for } |z| < e^{-aT}. \tag{6.12}$$

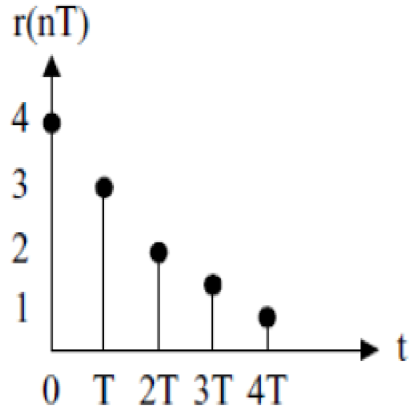


Figure 6.14 Exponential function

THE Z-TRANSFORM

4. General Exponential Function

Consider the general exponential function

$$r(n) = \begin{cases} 0, & n < 0, \\ p^n, & n \geq 0. \end{cases}$$

From (6.11),

$$R(z) = \sum_{n=0}^{\infty} r(nT)z^{-n} = \sum_{n=0}^{\infty} p^n z^{-n} = 1 + pz^{-1} + p^2z^{-2} + p^3z^{-3} + \ldots$$

or

$$R(z) = \frac{z}{z-p}, \qquad \text{for } |z| < |p|.$$

Similarly, we can show that

$$R(p^{-k}) = \frac{z}{z-p^{-1}}.$$

THE Z-TRANSFORM

5. Sine Function

Consider the sine function, defined as

$$r(nT) = \begin{cases} 0, & n < 0, \\ \sin n\omega T. & n \geq 0. \end{cases}$$

Recall that

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j},$$

so that

$$r(nT) = \frac{e^{jn\omega T} - e^{-jn\omega T}}{2j} = \frac{e^{jn\omega T}}{2j} - \frac{e^{-jn\omega T}}{2j}. \tag{6.13}$$

But we already know from (6.12) that the z-transform of an exponential function is

$$R(e^{-anT}) = R(z) = \frac{z}{z - e^{-aT}}.$$

Therefore, substituting in (6.13) gives

$$R(z) = \frac{1}{2j}\left(\frac{z}{z - e^{j\omega T}} - \frac{z}{z - e^{-j\omega T}}\right) = \frac{1}{2j}\left(\frac{z(e^{j\omega T} - e^{-j\omega T})}{z^2 - z(e^{j\omega T} + e^{-j\omega T}) + 1}\right)$$

or

$$R(z) = \frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1}.$$

THE Z-TRANSFORM

6. Cosine Function

Consider the cosine function, defined as

$$r(nT) = \begin{cases} 0, & n < 0, \\ \cos n\omega T, & n \geq 0. \end{cases}$$

Recall that

$$\cos x = \frac{e^{jx} + e^{-jx}}{2},$$

so that

$$r(nT) = \frac{e^{jn\omega T} + e^{-jn\omega T}}{2} = \frac{e^{jn\omega T}}{2} + \frac{e^{-jn\omega T}}{2}. \tag{6.14}$$

But we already know from (6.12) that the z-transform of an exponential function is

$$R(e^{-anT}) = R(z) = \frac{z}{z - e^{-aT}}.$$

Therefore, substituting in (6.14) gives

$$R(z) = \frac{1}{2}\left(\frac{z}{z - e^{j\omega T}} + \frac{z}{z - e^{-j\omega T}}\right)$$

or

$$R(z) = \frac{z(z - \cos \omega T)}{z^2 - 2z \cos \omega T + 1}.$$

THE Z-TRANSFORM

7. Discrete impulse Function

Consider the discrete impulse function defined as

$$\delta(n) = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

From (6.11),

$$R(z) = \sum_{n=0}^{\infty} r(nT)z^{-n} = \sum_{n=0}^{\infty} z^{-n} = 1.$$

THE Z-TRANSFORM

8. Delayed Discrete impulse Function

The delayed discrete impulse function is defined as

$$\delta(n-k) = \begin{cases} 1, & n = k > 0, \\ 0, & n \neq k. \end{cases}$$

From (6.11),

$$R(z) = \sum_{n=0}^{\infty} r(nT) z^{-n} = \sum_{n=0}^{\infty} z^{-n} = z^{-n}.$$

Table of Z-Transform

$f(kT)$	$F(z)$
$\delta(t)$	1
1	$\dfrac{z}{z-1}$
kT	$\dfrac{Tz}{(z-1)^2}$
$(kT)^2$	$\dfrac{T^2z(z+1)}{2(z-1)^3}$
$(kT)^3$	$\dfrac{T^3z(z^2+4z+1)}{(z-1)^4}$
e^{-akT}	$\dfrac{z}{z-e^{-aT}}$
kTe^{-akT}	$\dfrac{Tze^{-aT}}{(z-e^{-aT})^2}$
a^k	$\dfrac{z}{z-a}$
$1-e^{-akT}$	$\dfrac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\sin akT$	$\dfrac{z\sin aT}{z^2-2z\cos aT+1}$
$\cos akT$	$\dfrac{z(z-\cos aT)}{z^2-2z\cos aT+1}$
$e^{-akT}\sin bkT$	$\dfrac{e^{-aT}z\sin bT}{z^2-2e^{-aT}z\cos bT+e^{-2aT}}$
$e^{-akT}\cos bkT$	$\dfrac{z^2-e^{-aT}z\cos bT}{z^2-2e^{-aT}z\cos bT+e^{-2aT}}$

Conversion between Laplace and z-Transforms

Given a function $G(s)$, find $G(z)$ which denotes the z-transform equivalent of $G(s)$.

It is important to realize that G(z) is not obtained by simply substituting z for s in G(s)!

Method 1: inverse Laplace transform then apply z-transform to the time function.

Method 2: using Laplace to z-transform table

Method 3:

- Given the Laplace transform $G(s)$, express it in the form $G(s) = N(s)/D(s)$ and then use the following formula to find the z-transform $G(z)$:

$$G(z) = \sum_{n=1}^{p} \frac{N(x_n)}{D'(x_n)} \frac{1}{1 - e^{x_n T} z^{-1}}, \tag{6.15}$$

where $D' = \partial D / \partial s$ and the x_n, $n = 1, 2, \ldots, p$, are the roots of the equation $D(s) = 0$.

Conversion between Laplace and z-Transforms

Example:

Let

$$G(s) = \frac{1}{s^2 + 5s + 6}.$$

Determine $G(z)$ by the methods described above.

Solution

Method 1: By finding the inverse Laplace transform. We can express $G(s)$ as a sum of its partial fractions:

$$G(s) = \frac{1}{(s+3)(s+2)} = \frac{1}{s+2} - \frac{1}{s+3}. \tag{6.16}$$

The inverse Laplace transform of (6.16) is

$$g(t) = L^{-1}[G(s)] = e^{-2t} - e^{-3t}. \tag{6.17}$$

From the definition of the z-transforms we can write (6.17) as

$$\begin{aligned} G(z) &= \sum_{n=0}^{\infty} (e^{-2nT} - e^{-3nT}) z^{-n} \\ &= (1 + e^{-2T} z^{-1} + e^{-4T} z^{-2} + \dots) - (1 + e^{-3T} z^{-1} + e^{-6T} z^{-2} + \dots) \\ &= \frac{z}{z - e^{-2T}} - \frac{z}{z - e^{-3T}} \end{aligned}$$

Conversion between Laplace and z-Transforms

Example:

Let

$$G(s) = \frac{1}{s^2 + 5s + 6}.$$

Determine $G(z)$ by the methods described above.

Method 2: By using the z-transform transform tables for the partial product. From Table 6.1, the z-transform of $1/(s + a)$ is $z/(z - e^{-aT})$. Therefore the z-transform of (6.16) is

$$G(z) = \frac{z}{z - e^{-2T}} - \frac{z}{z - e^{-3T}}$$

or

$$G(z) = \frac{z(e^{-2T} - e^{-3T})}{(z - e^{-2T})(z - e^{-3T})}.$$

Laplace transform	Corresponding z-transform
$\frac{1}{s}$	$\frac{z}{z-1}$
$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
$\frac{1}{s^3}$	$\frac{T^2 z(z+1)}{2(z-1)^3}$
$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
$\frac{1}{(s+a)^2}$	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
$\frac{a}{s(s+a)}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\frac{b-a}{(s+a)(s+b)}$	$\frac{z(e^{-aT}-e^{-bT})}{(z-e^{-aT})(z-e^{-bT})}$
$\frac{(b-a)s}{(s+a)(s+b)}$	$\frac{(b-a)z^2-(be^{-aT}-ae^{-bT})z}{(z-e^{-aT})(z-e^{-bT})}$
$\frac{a}{s^2+a^2}$	$\frac{z \sin aT}{z^2-2z \cos aT+1}$
$\frac{s}{s^2+a^2}$	$\frac{z^2-z \cos aT}{z^2-2z \cos aT+1}$
$\frac{s}{(s+a)^2}$	$\frac{z[z-e^{-aT}(1+aT)]}{(z-e^{-aT})^2}$

Conversion between Laplace and z-Transforms

Example:

Let

$$G(s) = \frac{1}{s^2 + 5s + 6}.$$

Determine $G(z)$ by the methods described above.

Method 3: By using the z-transform tables for G(s). From Table 6.1, the z-transform of

$$G(s) = \frac{b-a}{(s+a)(s+b)} \tag{6.18}$$

is

$$G(z) = \frac{z(e^{-aT} - e^{-bT})}{(z - e^{-aT})(z - e^{-bT})}. \tag{6.19}$$

Comparing (6.18) with (6.16) we have, $a = 2$, $b = 3$. Thus, in (6.19) we get

$$G(z) = \frac{z(e^{-2T} - e^{-3T})}{(z - e^{-2T})(z - e^{-3T})}.$$

Conversion between Laplace and z-Transforms

Example:

Let

$$G(s) = \frac{1}{s^2 + 5s + 6}.$$

Determine $G(z)$ by the methods described above.

Method 4: By using equation (6.15). Comparing our expression

$$G(s) = \frac{1}{s^2 + 5s + 6}$$

with (6.15), we have $N(s) = 1$, $D(s) = s^2 + 5s + 6$ and $D'(s) = 2s + 5$, and the roots of $D(s) = 0$ are $x_1 = -2$ and $x_2 = -3$. Using (6.15),

$$G(z) = \sum_{n=1}^{2} \frac{N(x_n)}{D'(x_n)} \frac{1}{1 - e^{x_n T} z^{-1}}$$

or, when $x_1 = -2$,

$$G_1(z) = \frac{1}{1} \frac{1}{1 - e^{-2T} z^{-1}}$$

and when $x_1 = -3$,

$$G_2(z) = \frac{1}{-1} \frac{1}{1 - e^{-3T} z^{-1}}.$$

Thus,

$$G(z) = \frac{1}{1 - e^{-2T} z^{-1}} - \frac{1}{1 - e^{-3T} z^{-1}} = \frac{z}{z - e^{-2T}} - \frac{z}{z - e^{-3T}}$$

Conversion between Laplace and z-Transforms

Example

write a MATLAB commands to convert $G(s) = \dfrac{1}{s^2 + 5s + 6}$ into discrete with a sample period $T = 1$.

```
>> G = tf([1],[1 5 6]);     % continuous time transfer function
>> T = 1;
>> Gd = c2d(G,T,'impulse') % discrete time transfer function

Gd =
0.08555 z - 8.162e-19
-------------------------
z^2 - 0.1851 z + 0.006738

Sample time: 1 seconds
Discrete-time transfer function.
```


Thank you