

Digital Control Systems (CCE 341)

By

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Lecture4

Course Academic Calendar

Week Num	Date	Content
Week1	25/9/2025	Introduction
Week2	02/10/2025	Sampled Data Systems and the Z-Transform
Week3	09/10/2025	Properties of z-transform, Inverse z-Transforms
Week4	16/10/2025	Difference equations
Week5	23/10/2025	Open Loop Discrete Time System Analysis
Week6	30/10/2025	Closed Loop Discrete Time System Analysis
Week7	06/11/2025	Midterm
Week8	13/11/2025	System Time Response
Week9	20/11/2025	Mapping from S-plane to Z-plane
Week10	27/11/2025	Damping Ratio and undamped natural frequency in the Z-plane
Week11	04/12/2025	Stability of Sampled Systems
Week12	11/12/2025	Root Locus for Sampled Data Systems
Week13	18/12/2025	Nyquist Criterion and Bode Diagrams
Week14	25/12/2025	Digital Controller Design

Contents of the Lecture

- I. Difference equations**
- II. Converting from difference equations to Z-transform**
- III. Converting from z-transform to difference equation**

Differential Equation in control (Continuous-Time Model)

- In analog or continuous-time control systems, the plant (the system being controlled) is described by a differential equation.

Example:

$$\frac{dy(t)}{dt} + 3y(t) = 2u(t)$$

- $y(t)$: output (continuous)
- $u(t)$: input (continuous)
- $\frac{dy(t)}{dt}$: derivative, represents rate of change

- This equation models how the system output changes continuously with time.

Difference Equation in Control (Discrete-Time Model)

- ❑ In digital control systems, the controller works in discrete-time — it reads samples and updates the control signal at discrete time intervals T (called the sampling period).
- ❑ Therefore, we must represent the system by a difference equation, which relates current and past samples.

Example:

$$y[k + 1] = a_1y[k] + a_2y[k - 1] + b_1u[k] + b_2u[k - 1]$$

- $y[k]$: output at sample k
- $u[k]$: input at sample k
- Coefficients a_i, b_i are obtained from the discrete equivalent of the continuous model

Difference equations

Difference equations arise in problems where the independent variable, usually time, is assumed to have a discrete set of possible values.

$$\begin{aligned}y(k+n) + a_{n-1}y(k+n-1) + \dots + a_1y(k+1) + a_0y(k) \\= b_nu(k+n) + b_{n-1}u(k+n-1) + \dots + b_1u(k+1) + b_0u(k)\end{aligned}$$

Difference equations

Example:

Consider the difference equation given below

$$y(k+3) + 2y(k+2) + 4y(k+1) - y(k) = u(k+1) + 3u(k)$$

What is the order of this difference equation? Is this equation linear or nonlinear, time variant or time invariant, homogenous or non-homogenous?

The above equation is 3rd order due to the (k+3) term,

The above equation is Linear and Time invariant, because all coefficients are constants (1, 2, 4 and -1),

The above equation is non-homogenous, as the coefficients regarding the input (u) are nonzero

Difference equations

EXAMPLE 2.2

For each of the following difference equations, determine the order of the equation. Is the equation (a) linear, (b) time invariant, or (c) homogeneous?

1. $y(k+2) + 0.8y(k+1) + 0.07y(k)u(k)$
2. $y(k+4) + \sin(0.4k)y(k+1) + 0.3y(k) = 0$
3. $y(k+1) = -0.1y^2(k)$

Solution

1. The equation is second order. All terms enter the equation linearly and have constant coefficients. The equation is therefore LTI. A forcing function appears in the equation, so it is nonhomogeneous.
2. The equation is fourth order. The second coefficient is time dependent, but all the terms are linear and there is no forcing function. The equation is therefore linear time varying and homogeneous.
3. The equation is first order. The right-hand side (RHS) is a nonlinear function of $y(k)$, but does not include a forcing function or terms that depend on time explicitly. The equation is therefore nonlinear, time invariant, and homogeneous.

Converting from difference equations to Z-transform

$$y_{n+1} - 3y_n = 4 \quad n = 0, 1, 2, \dots$$

with initial condition $y_0 = 1$.

We multiply both sides of (1) by z^{-n} and sum each side over all positive integer values of n and zero. We obtain

$$\sum_{n=0}^{\infty} (y_{n+1} - 3y_n)z^{-n} = \sum_{n=0}^{\infty} 4z^{-n}$$

or

$$\sum_{n=0}^{\infty} y_{n+1}z^{-n} - 3 \sum_{n=0}^{\infty} y_n z^{-n} = 4 \sum_{n=0}^{\infty} z^{-n} \quad (2)$$

Converting from difference equations to Z-transform

The right-hand side is the z-transform of the constant sequence $\{4, 4, \dots\}$ which is $\frac{4z}{z-1}$.

If $Y(z) = \sum_{n=0}^{\infty} y_n z^{-n}$ denotes the z-transform of the sequence $\{y_n\}$ that we are seeking then

$$\sum_{n=0}^{\infty} y_{n+1} z^{-n} = z Y(z) - z y_0 \text{ (by the left shift theorem).}$$

Consequently (2) can be written

$$z Y(z) - z y_0 - 3 Y(z) = \frac{4z}{z-1} \quad (3)$$

Converting from difference equations to Z-transform

$$(z - 3)Y(z) - z = \frac{4z}{(z - 1)}$$

$$(z - 3)Y(z) = \frac{4z}{z - 1} + z = \frac{z^2 + 3z}{z - 1}$$

so

$$Y(z) = \frac{z^2 + 3z}{(z - 1)(z - 3)}$$

Converting from z-transform to difference equation

$$\frac{Y(z)}{X(z)} = \frac{1 + z^{-1}}{2(1 - z^{-1})}$$

$$2(1 - z^{-1})Y(z) = (1 + z^{-1})X(z)$$

$$Y(z) - Y(z)z^{-1} = \frac{1}{2}X(z) + \frac{1}{2}X(z)z^{-1}$$

$$y[n] - y[n - 1] = \frac{1}{2}x[n] + \frac{1}{2}x[n - 1]$$

$$y[n] = y[n - 1] + \frac{1}{2}x[n] + \frac{1}{2}x[n - 1]$$

Table 6.1 Some commonly used z -transforms

$f(kT)$	$F(z)$
$\delta(t)$	1
1	$\frac{z}{z - 1}$
kT	$\frac{Tz}{(z - 1)^2}$
e^{-akT}	$\frac{z}{z - e^{-aT}}$
kTe^{-akT}	$\frac{Tze^{-aT}}{(z - e^{-aT})^2}$
a^k	$\frac{z}{z - a}$
$1 - e^{-akT}$	$\frac{z(1 - e^{-aT})}{(z - 1)(z - e^{-aT})}$
$\sin akT$	$\frac{z \sin aT}{z^2 - 2z \cos aT + 1}$
$\cos akT$	$\frac{z(z - \cos aT)}{z^2 - 2z \cos aT + 1}$

Laplace transform	Corresponding z -transform
$\frac{1}{s}$	$\frac{z}{z - 1}$
$\frac{1}{s^2}$	$\frac{Tz}{(z - 1)^2}$
$\frac{1}{s^3}$	$\frac{T^2z(z + 1)}{2(z - 1)^3}$
$\frac{1}{s + a}$	$\frac{z}{z - e^{-aT}}$
$\frac{1}{(s + a)^2}$	$\frac{Tze^{-aT}}{(z - e^{-aT})^2}$
$\frac{a}{s(s + a)}$	$\frac{z(1 - e^{-aT})}{(z - 1)(z - e^{-aT})}$
$\frac{b - a}{(s + a)(s + b)}$	$\frac{z(e^{-aT} - e^{-bT})}{(z - e^{-aT})(z - e^{-bT})}$
$\frac{(b - a)s}{(s + a)(s + b)}$	$\frac{(b - a)z^2 - (be^{-aT} - ae^{-bT})z}{(z - e^{-aT})(z - e^{-bT})}$
$\frac{a}{s^2 + a^2}$	$\frac{z \sin aT}{z^2 - 2z \cos aT + 1}$
$\frac{s}{s^2 + a^2}$	$\frac{z^2 - z \cos aT}{z^2 - 2z \cos aT + 1}$
$\frac{s}{(s + a)^2}$	$\frac{z[z - e^{-aT}(1 + aT)]}{(z - e^{-aT})^2}$

Thank you

