MTE 408 s | ROBOTICS

TRAJECTORY GENERATION

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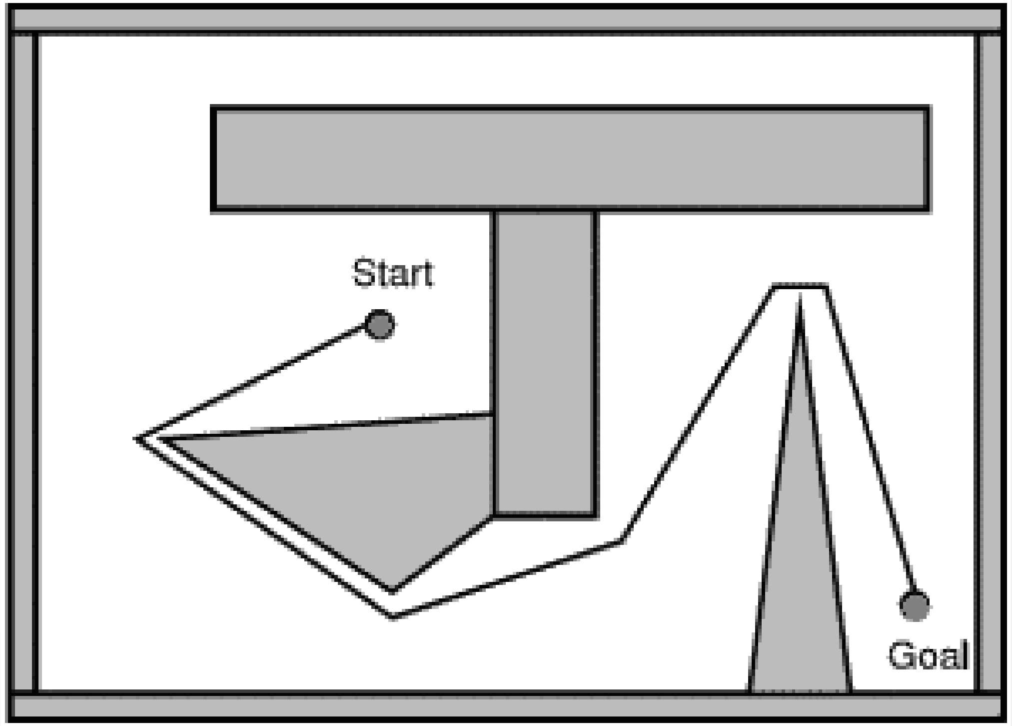
Concepts

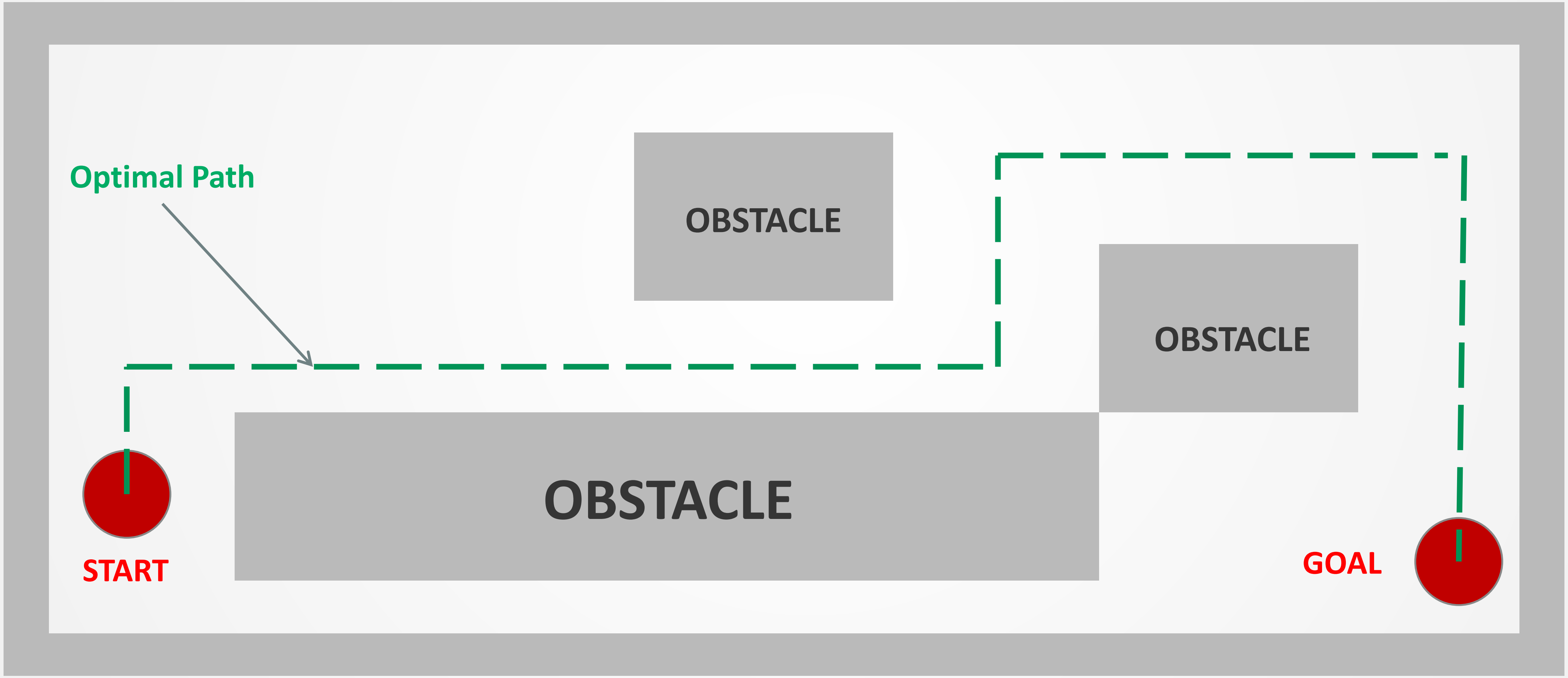
*The difference between **motion planning** and **trajectory planning** (generation)*

Motion Planning

- intended to build a **collision-free path** with respect to ***cost function*** (minimum travel time or minimum power consumption).
- The **scene** or **horizon** is analyzed for minimizing the cost function to achieve the final goal

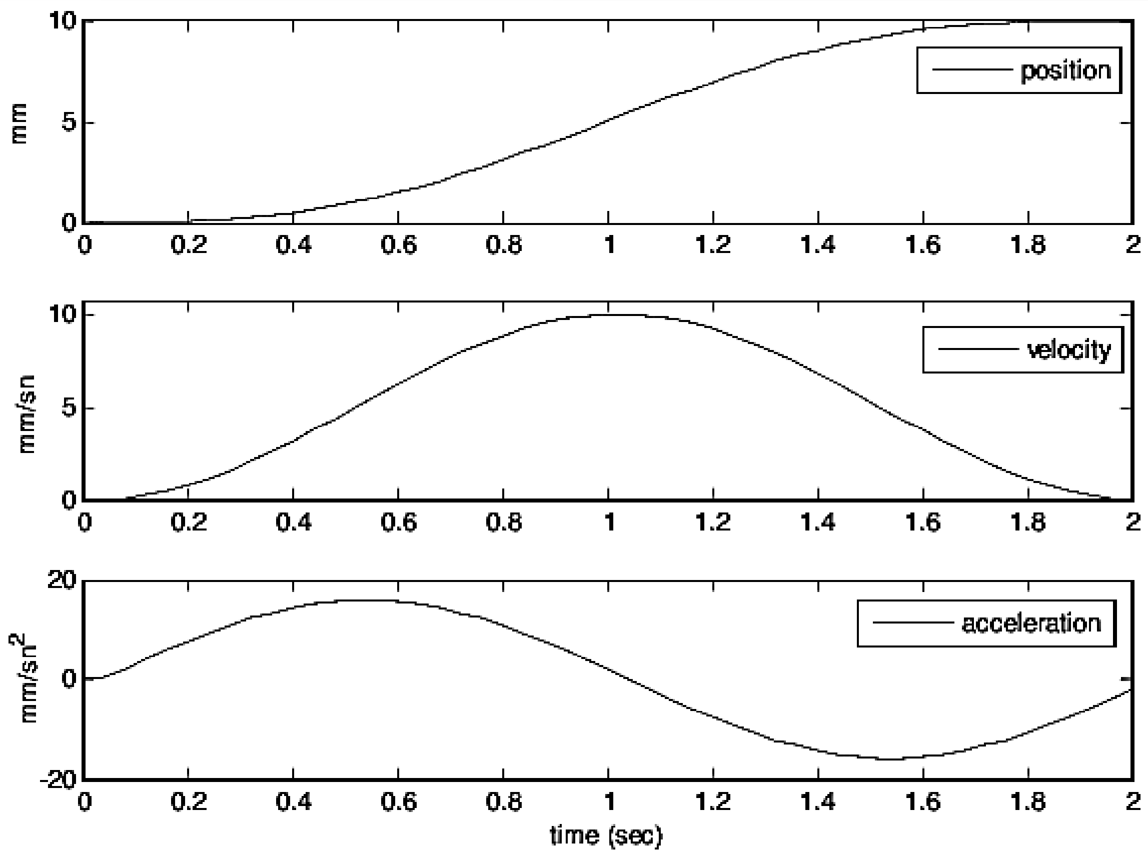


Motion Planning

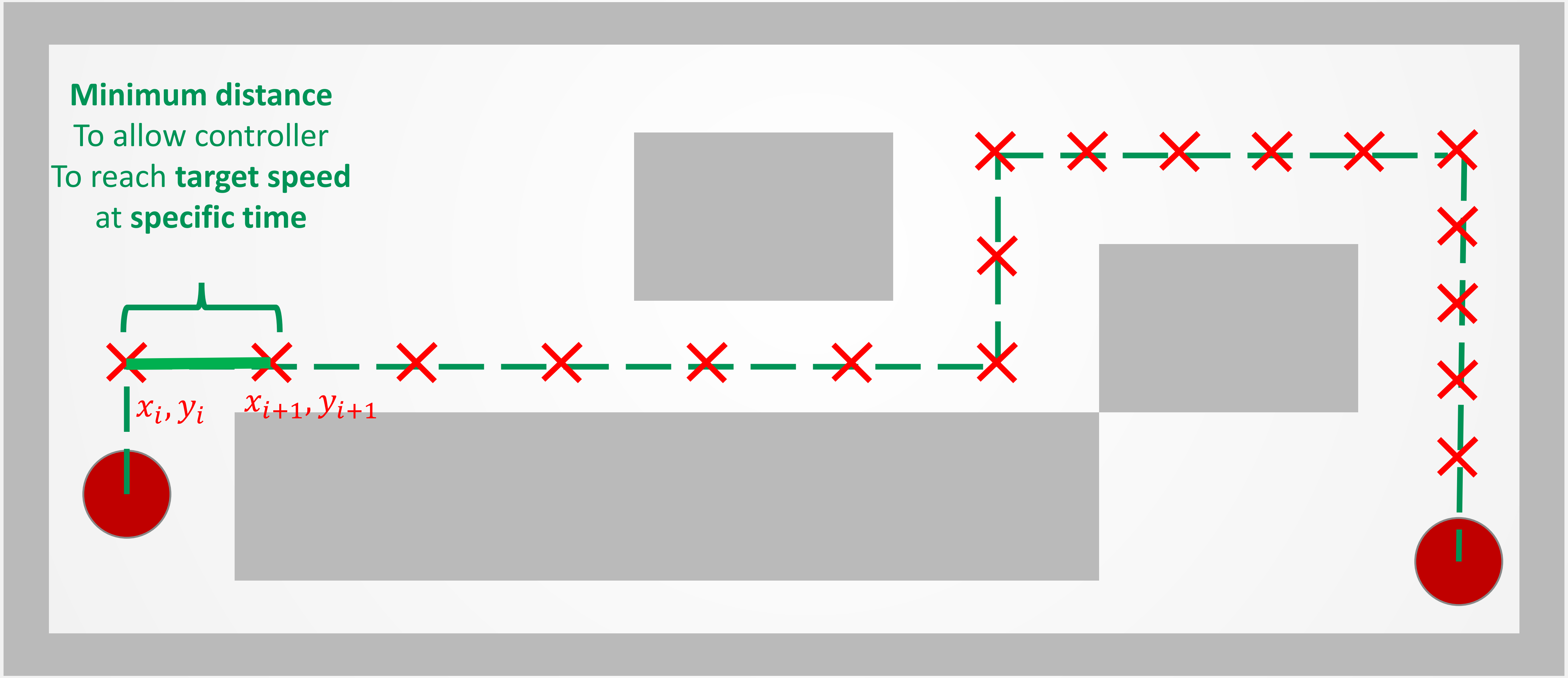


Trajectory Planning (Generation)

- Once path is identified, motion dynamics (***speed***, ***acceleration*** and ***jerk***) profile is identified using polynomial function.
- **Polynomial coefficients** are computed from problem **boundary conditions** (**initial** and **final** states of position, velocity and acceleration).

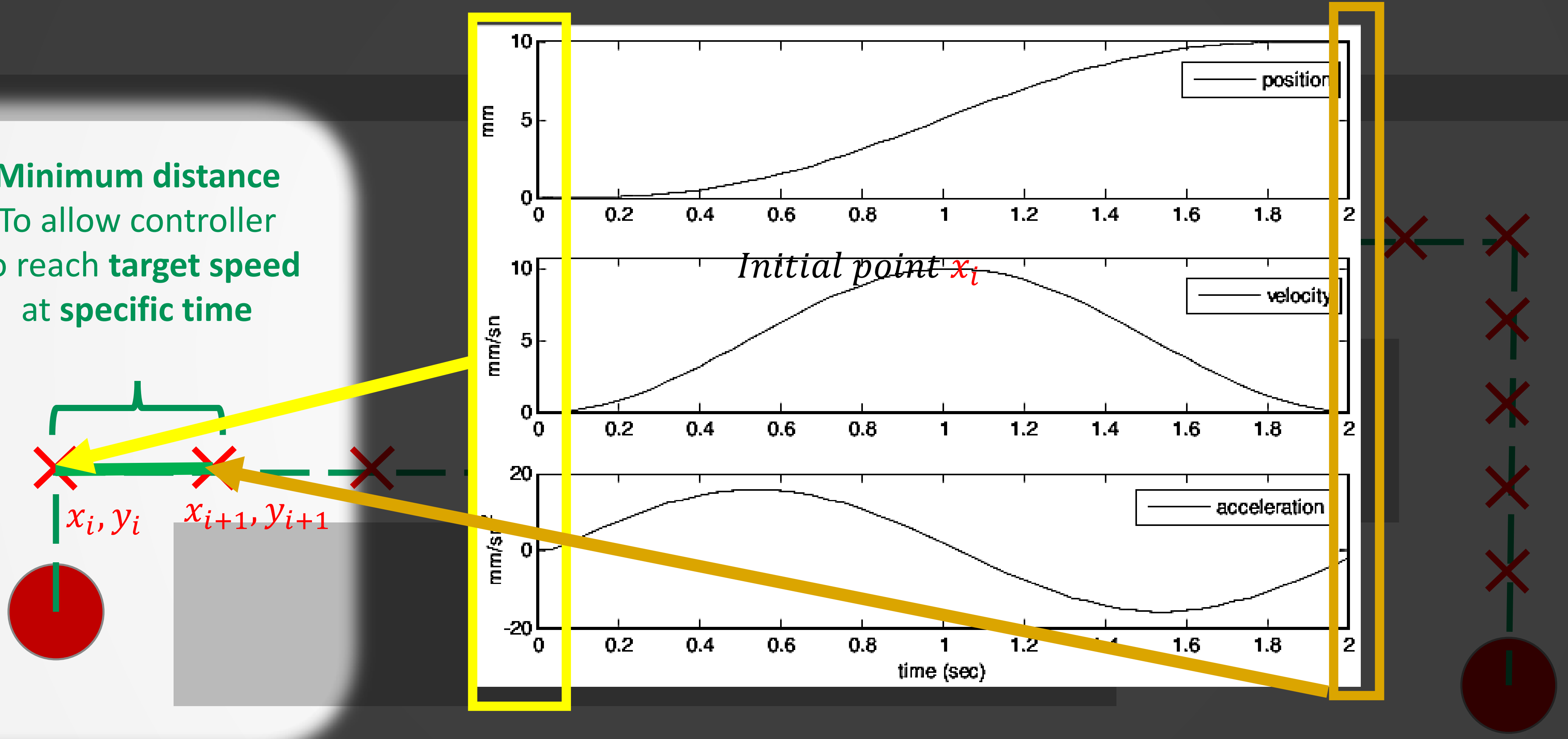


Trajectory Planning (Generation)



Trajectory Planning

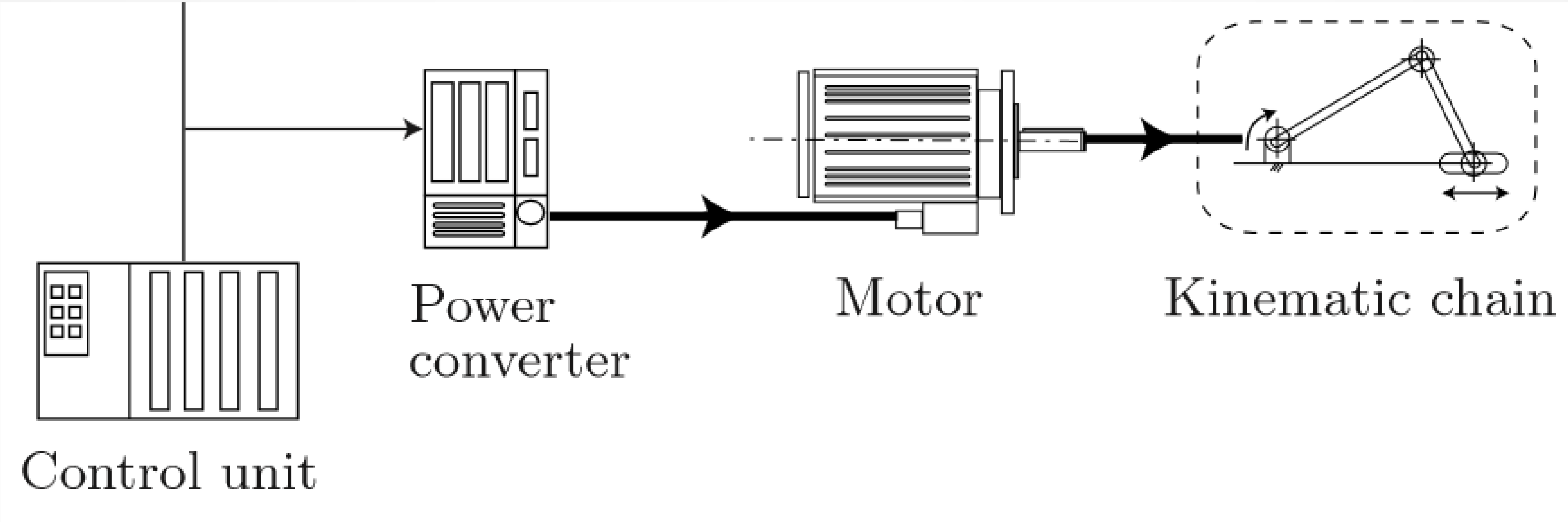
Minimum distance
To allow controller
To reach **target speed**
at **specific time**



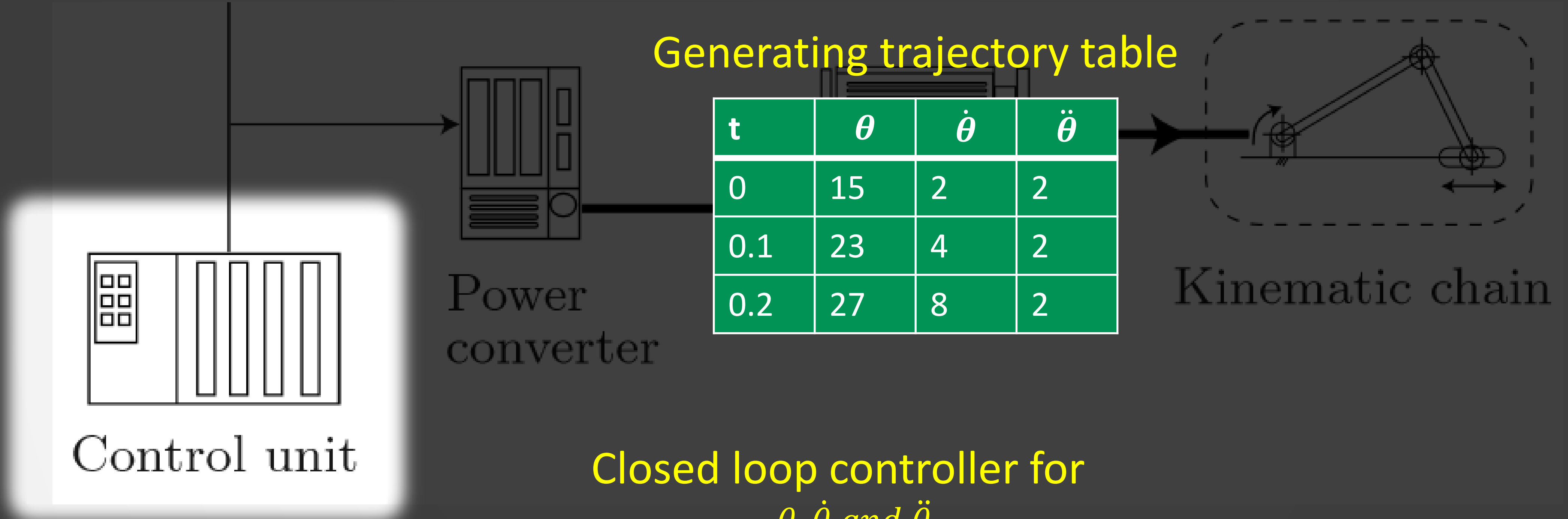
Initial point x_i

Final point x_{i+1}

Trajectory Planning (Single Axis)



Trajectory Planning (Single Axis)



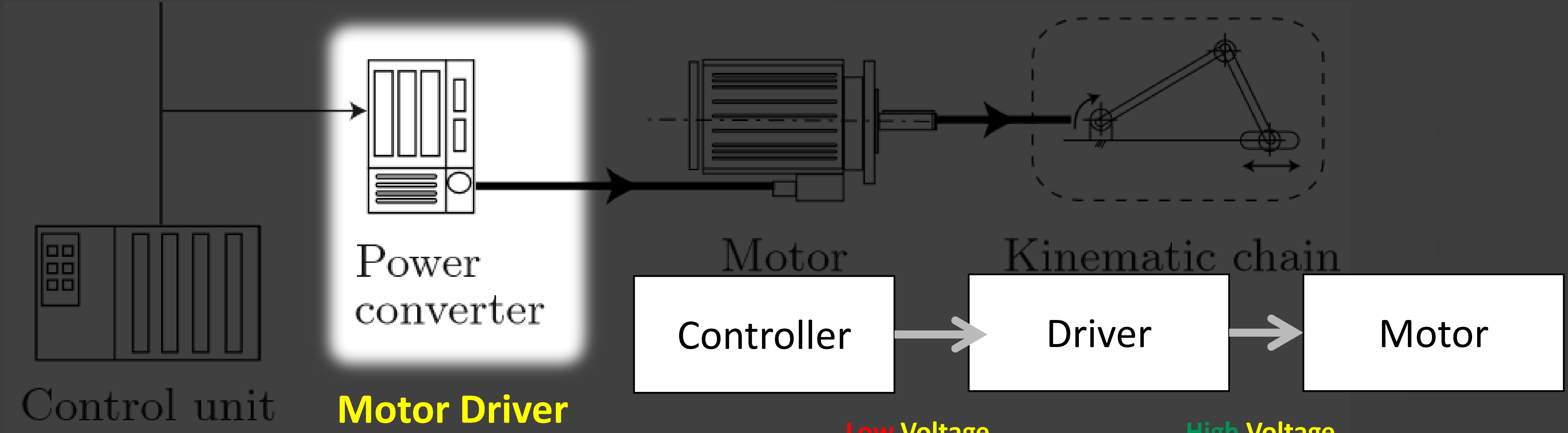
Generating trajectory table

t	$\boldsymbol{\theta}$	$\dot{\boldsymbol{\theta}}$	$\ddot{\boldsymbol{\theta}}$
0	15	2	2
0.1	23	4	2
0.2	27	8	2

Closed loop controller for

$\theta, \dot{\theta}$ *and* $\ddot{\theta}$

Trajectory Planning (Single Axis)

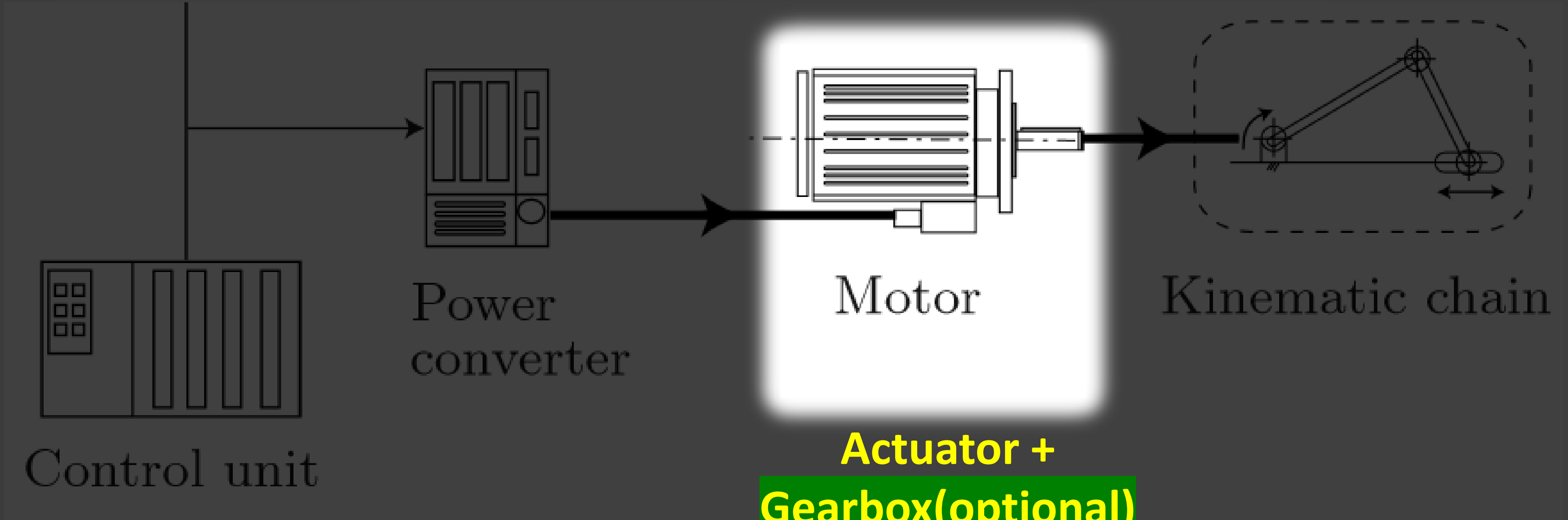


Motor Driver
(Current/Voltage Step Up)

Low Voltage
Low Current

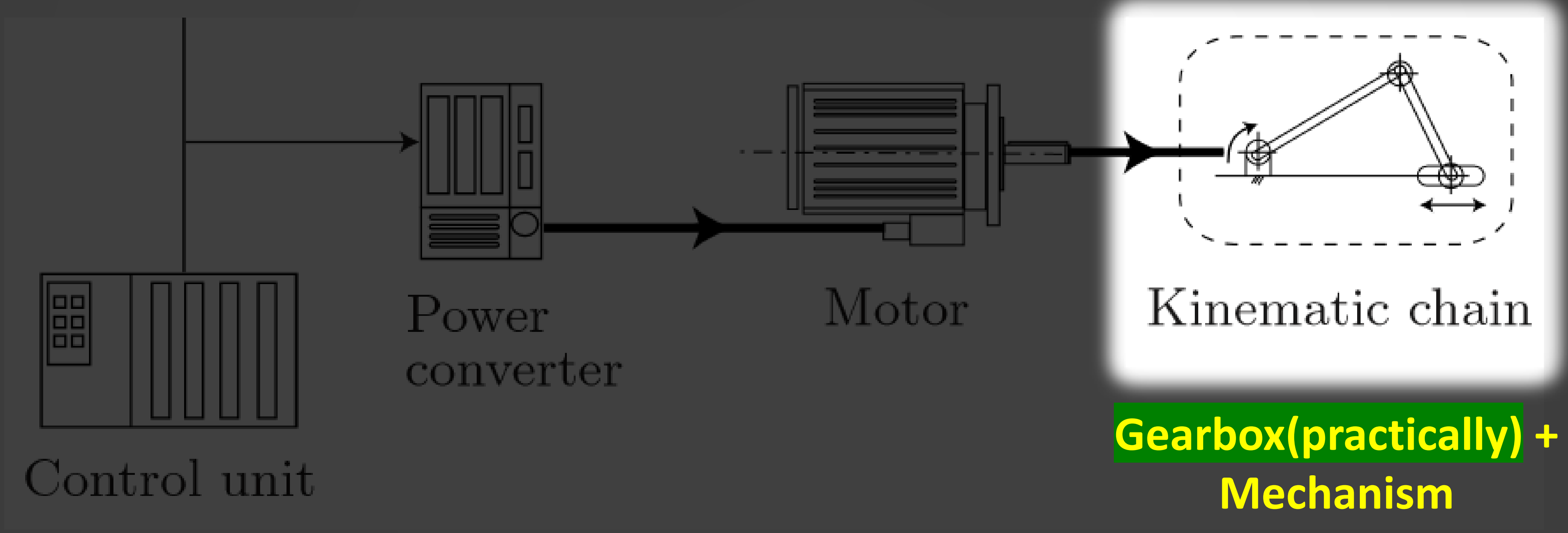
High Voltage
High Current

Trajectory Planning (Single Axis)



Actuator +
Gearbox(optional)
(*Final Control Element*)

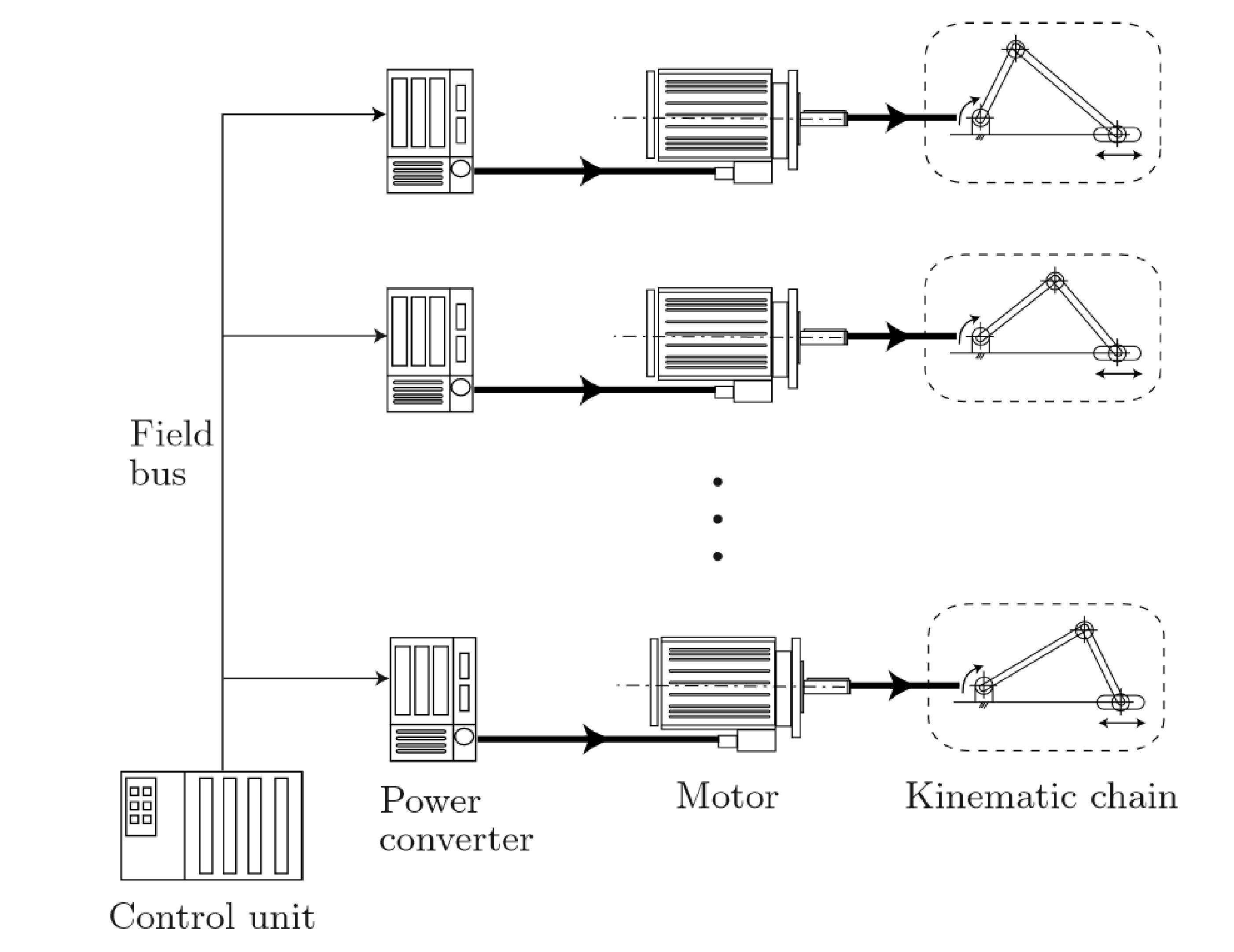
Trajectory Planning (Single Axis)

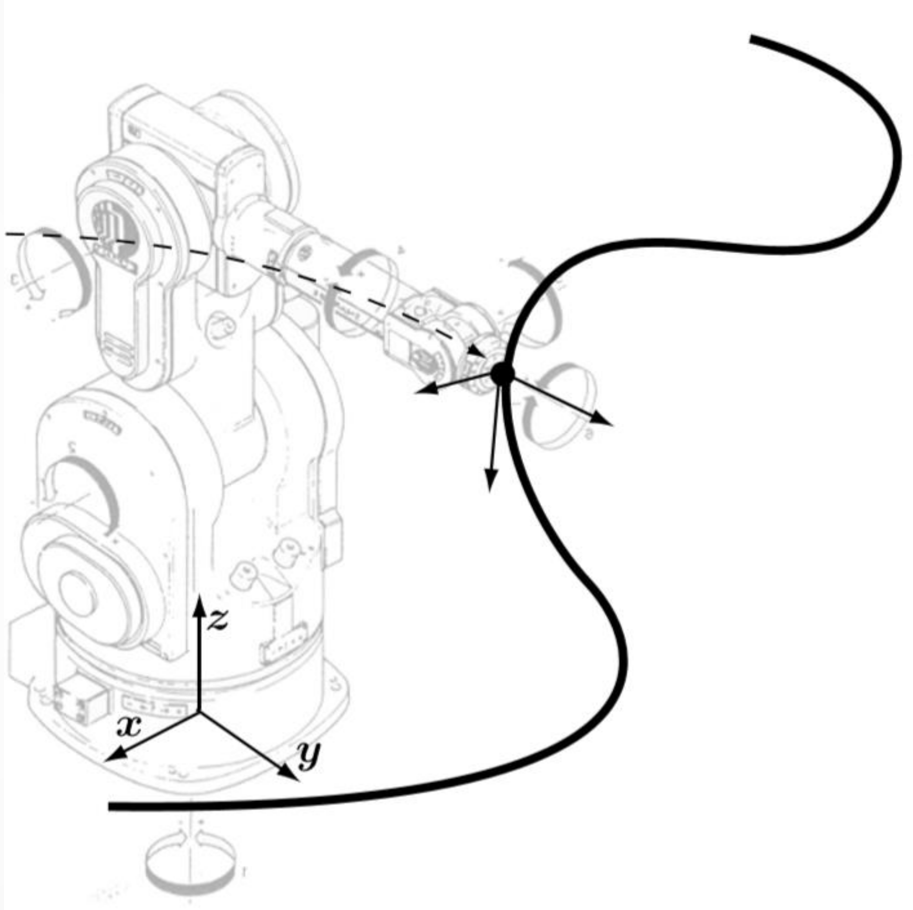


Gearbox(practically) + Mechanism

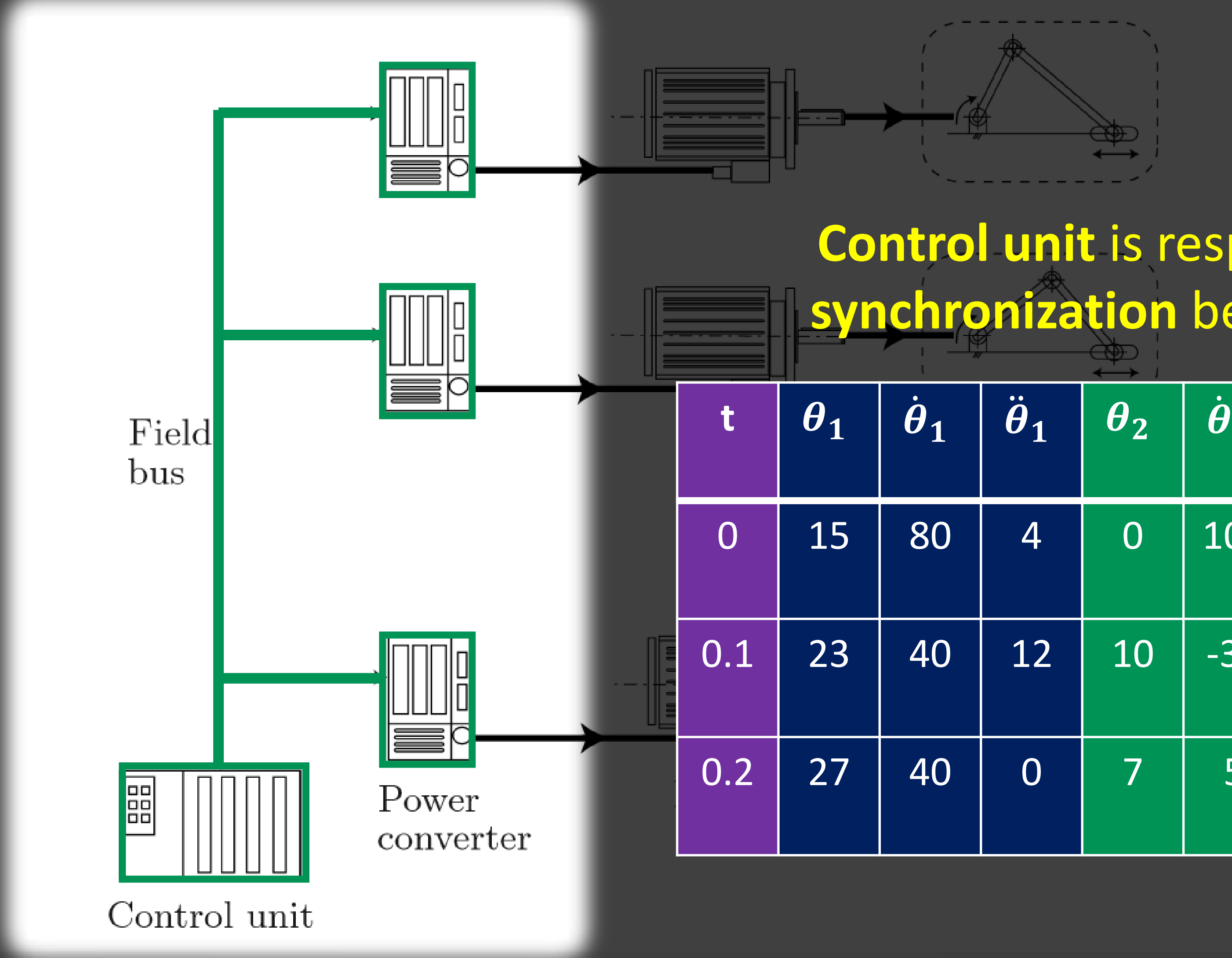
(*Plant*)

Trajectory Planning (Multiple Axes)





Trajectory Planning (Multiple Axes)



Control unit is responsible for **synchronization** between axes

t	θ_1	$\dot{\theta}_1$	$\ddot{\theta}_1$	θ_2	$\dot{\theta}_2$	$\ddot{\theta}_2$	θ_3	$\dot{\theta}_3$	$\ddot{\theta}_3$
0	15	80	4	0	100	2	10	2	2
0.1	23	40	12	10	-30	2	43	4	2
0.2	27	40	0	7	5	2	27	8	2

Polynomial Trajectories

General form

$$s(t) = a_0 + a_1 t + a_2 t^2 + \cdots a_n t^n$$

$$\dot{s}(t) = \frac{ds}{dt}$$

$$\ddot{s}(t) = \frac{d^2 s}{dt^2}$$

Boundary conditions
for finding coefficients
$\boldsymbol{a_0 \ldots a_n}$

$s(t) \ldots displacement$

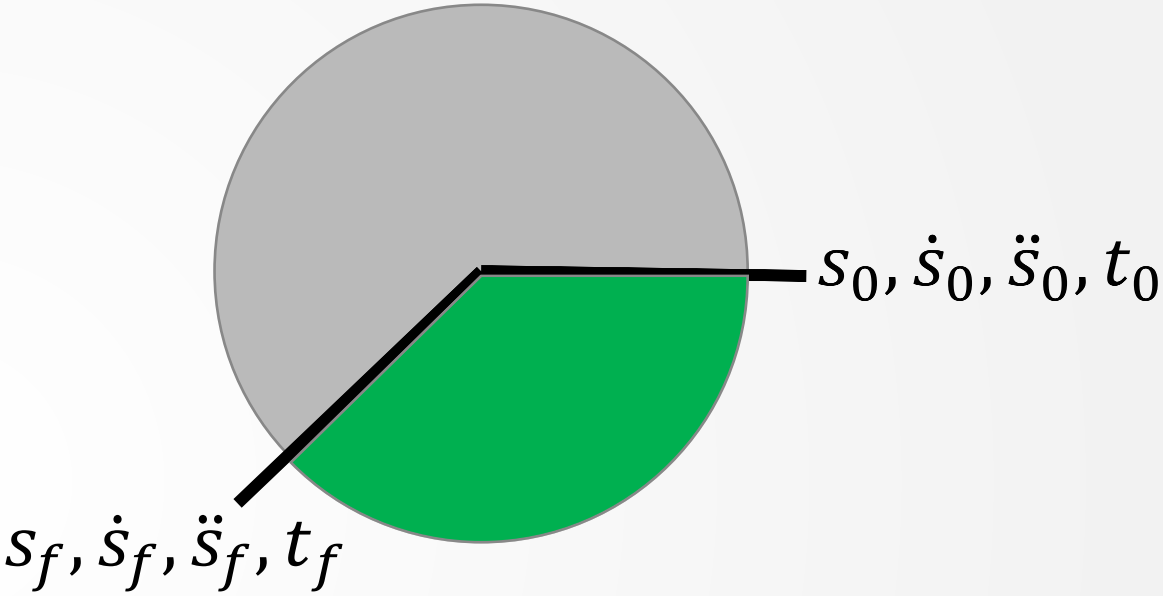
$\dot{s}(t) \ldots velocity$

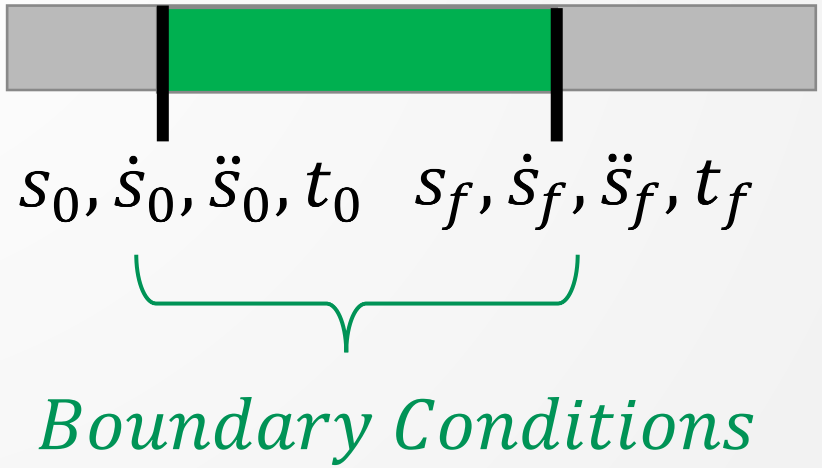
$\ddot{s}(t) \ldots acceleration$

$t \in [t_0, t_f]$

$t_f \ldots final\ time$

$t_o \ldots initial\ time$





Polynomial Trajectories

$Linear\ Trajectory$

$$s(t) = a_0 + a_1(t_f - t_0)$$

$$\dot{s}(t) = \frac{ds}{dt} = a_1$$

$$\ddot{s}(t) = 0$$

$To\ find\ the\ two\ coefficients\ , a_0\ and\ a_1, we\ need$ $\boldsymbol{two\ boundary\ conditions}$ $s(t_0)\ and\ s(t_f)$ ($\boldsymbol{initial}$ and $\boldsymbol{final}$ $displacement$)

$$\begin{aligned} s(t_0) &= a_0 \ \rightarrow \{1\} \\ s(t_f) &= a_0 + a_1(t_f - t_0) \rightarrow \{2\} \end{aligned} \Rightarrow \begin{bmatrix} s(t_0) \\ s(t_f) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & T \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}, \qquad T = t_f - t_0$$

Polynomial Trajectories

$Linear\ Trajectory$

$$\begin{bmatrix} s(t_0) \\ s(t_f) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & T \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}, \qquad T = t_f - t_0$$

$To\ find\ \boldsymbol{a_0}\ and\ \boldsymbol{a_1}, it\ is\ easy\ to\ find\ it\ using\ the\ inverse\ method$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & T \end{bmatrix}^{-1} \begin{bmatrix} s(t_0) \\ s(t_f) \end{bmatrix}, \qquad s(t_0)\ and\ s\left(t_f\right)\ are\ known\ values$$

$$\begin{bmatrix} 1 & 0 \\ 1 & T \end{bmatrix}^{-1} = \frac{1}{(1)(T) - (0)(1)} \begin{bmatrix} T & -0 \\ -1 & 1 \end{bmatrix} = \frac{1}{T} \begin{bmatrix} T & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{T} & \frac{1}{T} \end{bmatrix}$$

Polynomial Trajectories

Linear Trajectory

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & T \end{bmatrix}^{-1} \begin{bmatrix} s(t_0) \\ s(t_f) \end{bmatrix}, \qquad s(t_0) \text{ and } s(t_f) \text{ are known values}$$

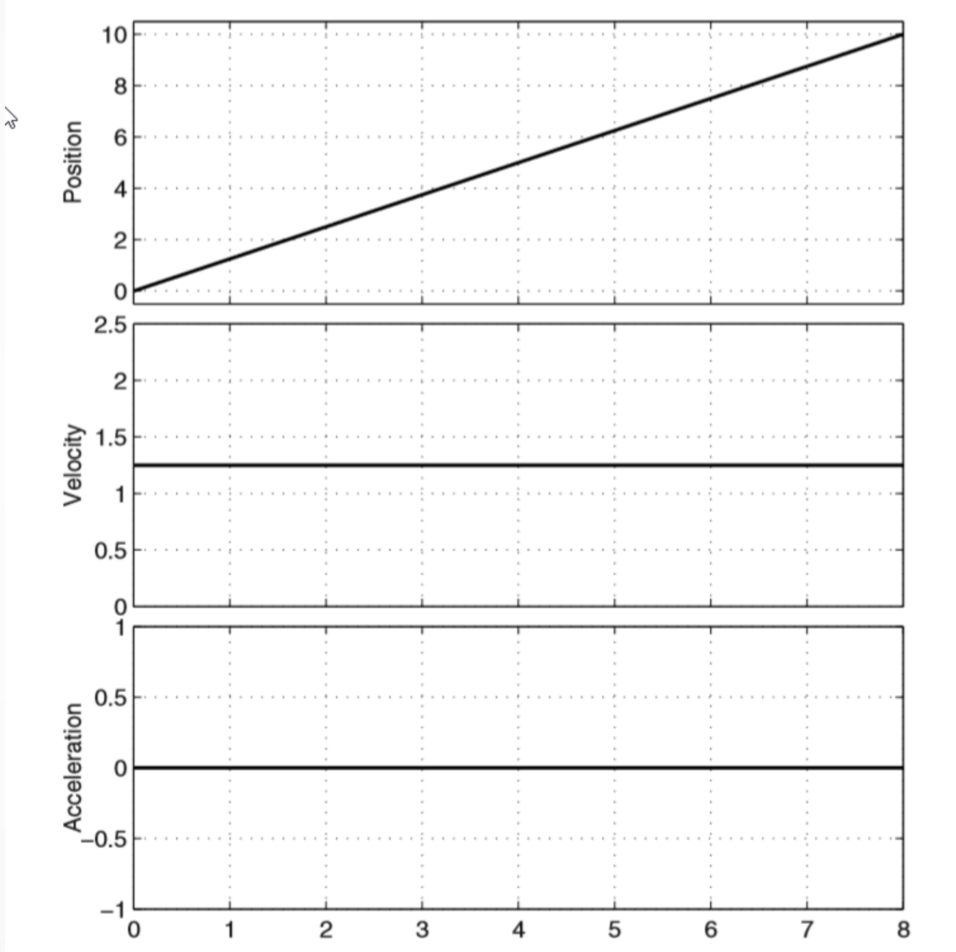
$$\begin{bmatrix} 1 & 0 \\ 1 & T \end{bmatrix}^{-1} = \frac{1}{(1)(T) - (0)(1)} \begin{bmatrix} T & -0 \\ -1 & 1 \end{bmatrix} = \frac{1}{T} \begin{bmatrix} T & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{T} & \frac{1}{T} \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{T} & \frac{1}{T} \end{bmatrix} \begin{bmatrix} s(t_0) \\ s(t_f) \end{bmatrix} = \begin{bmatrix} s(t_0) \\ -\frac{s(t_0)}{T} + -\frac{s(t_f)}{T} \end{bmatrix}$$

Polynomial Trajectories

Linear Trajectory

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{T} & \frac{1}{T} \end{bmatrix} \begin{bmatrix} s(t_0) \\ s(t_f) \end{bmatrix} = \begin{bmatrix} s(t_0) \\ -\frac{s(t_0)}{T} + -\frac{s(t_f)}{T} \end{bmatrix}$$



$$a_0 = s(t_0)$$

$$a_1 = -\frac{s(t_0)}{T} + \frac{s(t_f)}{T} = \frac{s(t_f) - s(t_0)}{t_f - t_0} = \frac{h}{T}$$

$$h = s(t_f) - s(t_0) = q_f - q_0$$

Polynomial Trajectories

Example

Determine the linear trajectory equations given

$s(t_0) = q_0 = 0\,, s(t_f) = q_f = 10\,, t_0 = 0\,, t_f = 8$

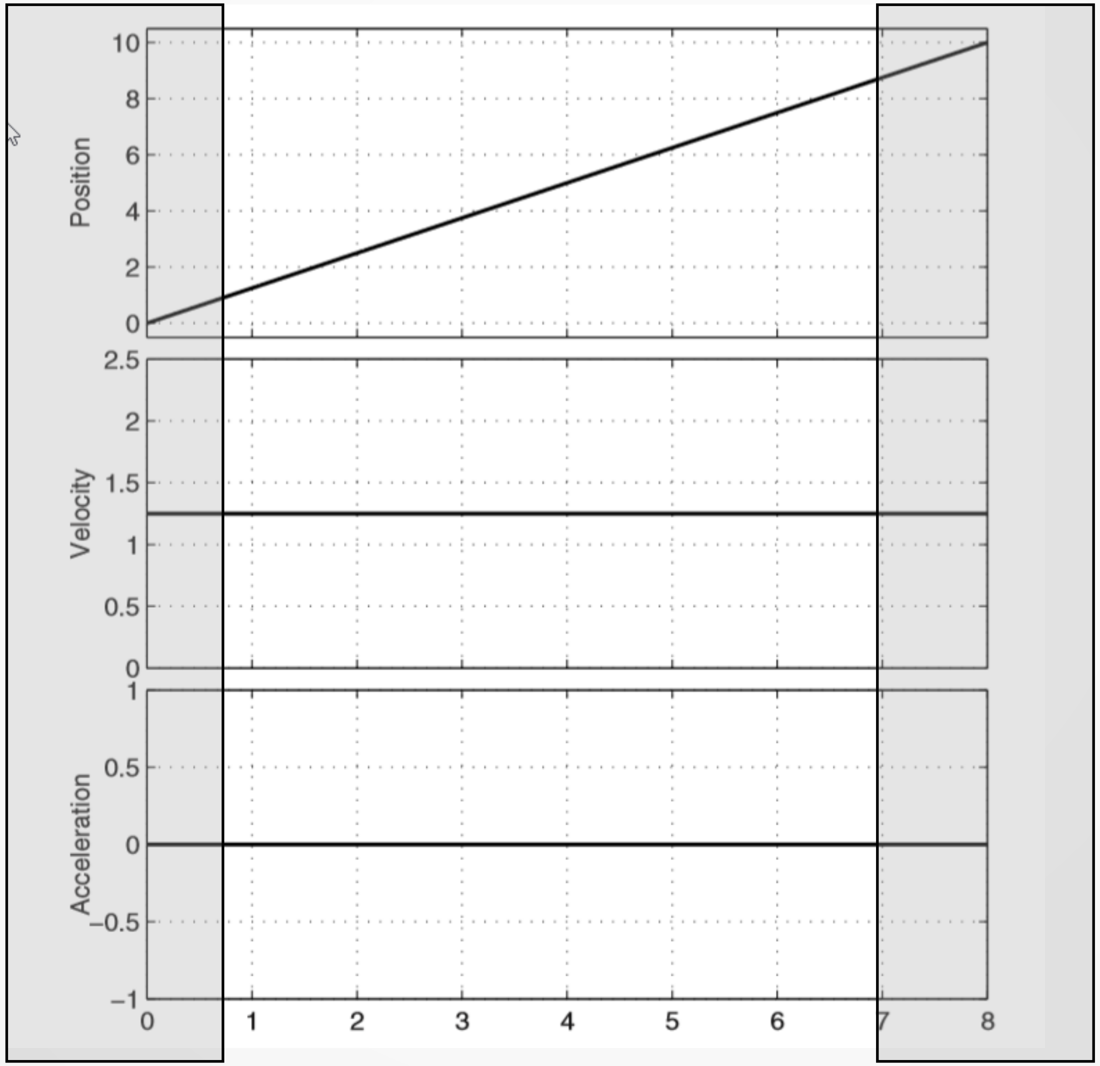
Solution

$$a_0 = s(t_0) = 0$$

$$a_1 = \frac{s(t_f) - s(t_0)}{t_f - t_0} = \frac{10 - 0}{8 - 0} = \frac{10}{8}$$

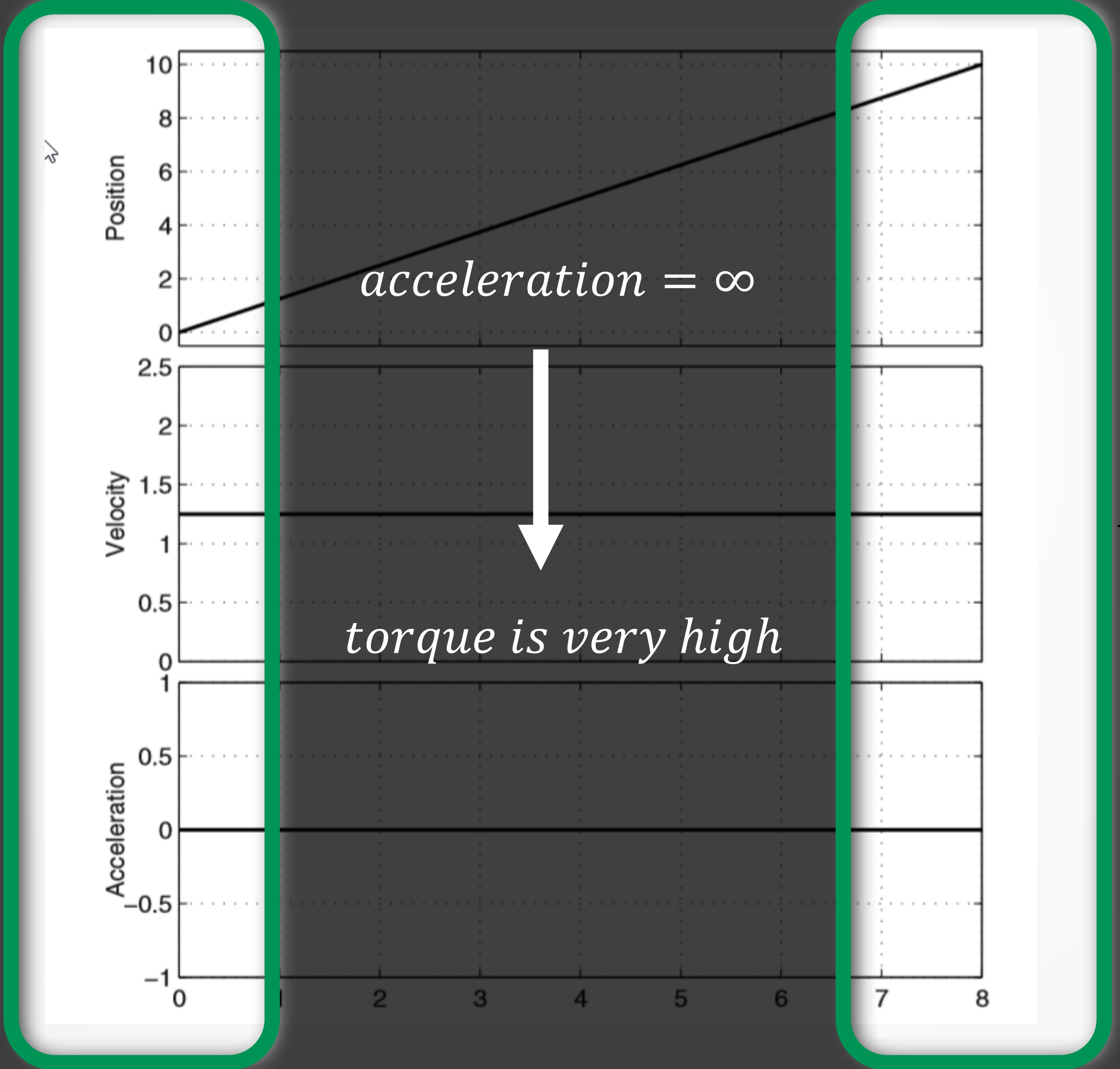
$$s(t) = a_0 + a_1 t = \frac{10}{8} t \quad and \quad \dot{s}(t) = a_1 = \frac{10}{8}$$

Polynomial Trajectories



$Linear\ Trajectory$

Polynomial Trajectories



$acceleration = \infty$

torque is very high

Linear Trajectory

Major disadvantage of linear trajectory is ***the infinite acceleration*** *at* ***start*** *and* ***stop***

Polynomial Trajectories

$Cubic\ Trajectory$

$s(t) = a_0 + a_1T + a_2(T)^2 + a_3(T)^3$

$\dot{s}(t) = a_1 + 2a_2(T) + 3a_3(T)^2$

$\ddot{s}(t) = 2a_2 + 6a_3(T)$

$To\ find\ the\ four\ coefficients\ , a_0\ , a_1, a_2\ and\ a_3 we\ need\ \boldsymbol{four\ boundary\ conditions}\ s(t_0)\ , s(t_f)\ , \dot{s}(t_0)\ and\ \dot{s}(t_f)$

Polynomial Trajectories

$Cubic\ Trajectory$

$To\ find\ the\ four\ coefficients\ , a_0\ , a_1, a_2\ and\ a_3 we\ need$ $\boldsymbol{four\ boundary\ conditions}$ $s(t_0)\ , s(t_f)\ , \dot{s}(t_0)\ and\ \dot{s}(t_f)$

$$s(t_0) = a_0$$

$$s(t_f) = a_0 + a_1 T + a_2 (T)^2 + a_3 (T)^3$$

$$\dot{s}(t_0) = a_1$$

$$\dot{s}(t_f) = a_1 + 2a_2(T) + 3a_3(T)^2$$

$$\Rightarrow \begin{bmatrix} s(t_0) \\ s(t_f) \\ \dot{s}(t_0) \\ \dot{s}(t_f) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & T & T^2 & T^3 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2T & 3T^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$\boldsymbol{s(t_0)\ , s(t_f)\ , \dot{s}(t_0)}\ and\ \boldsymbol{\dot{s}(t_f)\ are\ known}$

Polynomial Trajectories

$Cubic\ Trajectory$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & T & T^2 & T^3 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2T & 3T^2 \end{bmatrix}^{-1} \begin{bmatrix} s(t_0) \\ s(t_f) \\ \dot{s}(t_0) \\ \dot{s}(t_f) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{3}{T^2} & \frac{3}{T^2} & -\frac{2}{T} & -\frac{1}{T} \\ \frac{2}{T^3} & -\frac{2}{T^3} & \frac{1}{T^2} & \frac{1}{T^2} \end{bmatrix} \begin{bmatrix} q_0 \\ q_f \\ \dot{q}_0 \\ \dot{q}_f \end{bmatrix}$$

$\boldsymbol{s(t_0)}$, $\boldsymbol{s(t_f)}$, $\boldsymbol{\dot{s}(t_0)}$ and $\boldsymbol{\dot{s}(t_f)}$ $\boldsymbol{are\ known}$

Polynomial Trajectories

Cubic Trajectory

$$\boldsymbol{a_0 = q_0 = s(t_0)}$$

$$\boldsymbol{a_1 = \dot{q}_0 = \dot{s}(t_0)}$$

$$\boldsymbol{a_2 = \frac{3h - (2\dot{q}_0 + \dot{q}_f)T}{T^2}}$$

$$\boldsymbol{a_3 = \frac{-2h - (\dot{q}_0 + \dot{q}_f)T}{T^3}}$$

Rewrite q , $\dot{q}$ *and* $\ddot{q}$

$$q = a_0 + a_1 T + a_2 (T)^2 + a_3 (T)^3$$

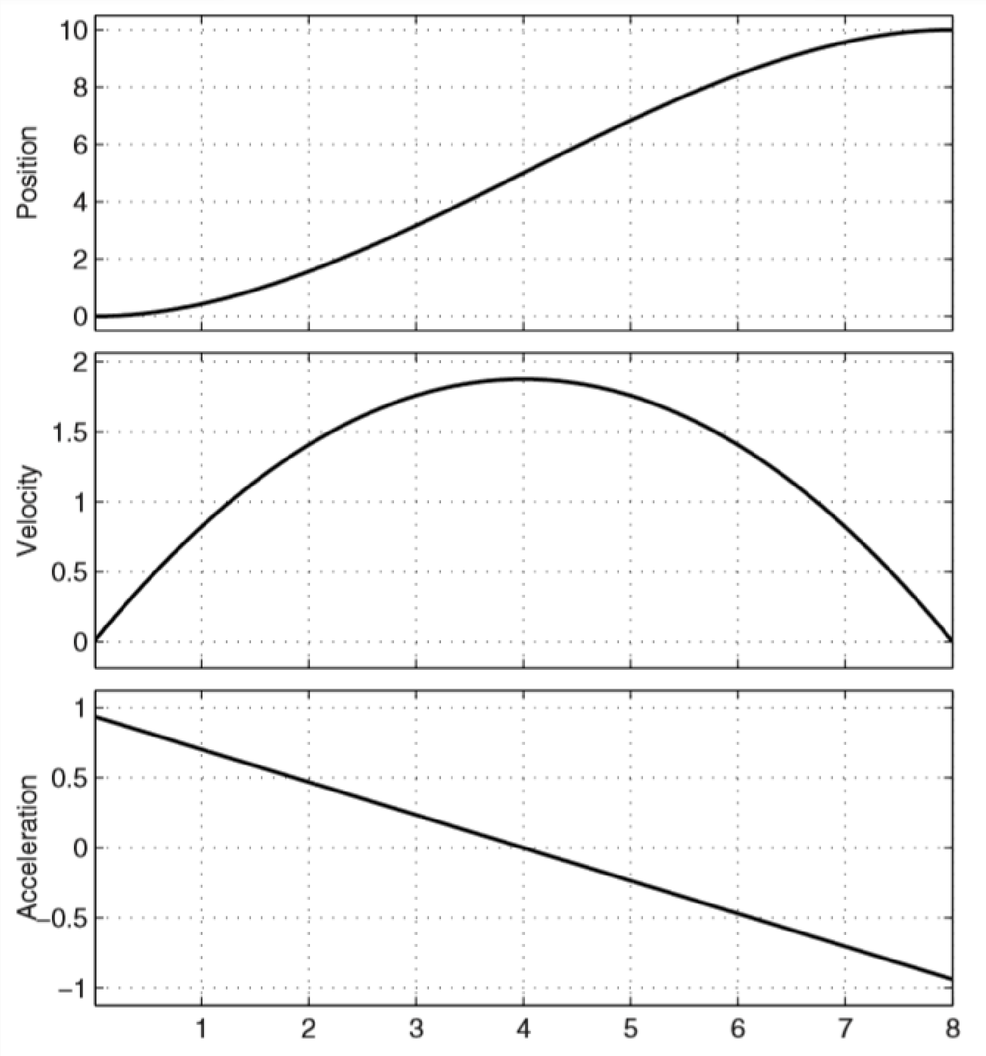
$$\dot{q} = a_1 + 2a_2 (T) + 3a_3 (T)^2$$

$$\ddot{q} = 2a_2 + 6a_3 (T)$$

Polynomial Trajectories

$Example$

Parameter	Value
q_0	0
q_f	10
$\dot{q}_0$	0
$\dot{q}_f$	0
t_0	0
t_f	8



Polynomial Trajectories

Example

Parameter	Value
q_0	0
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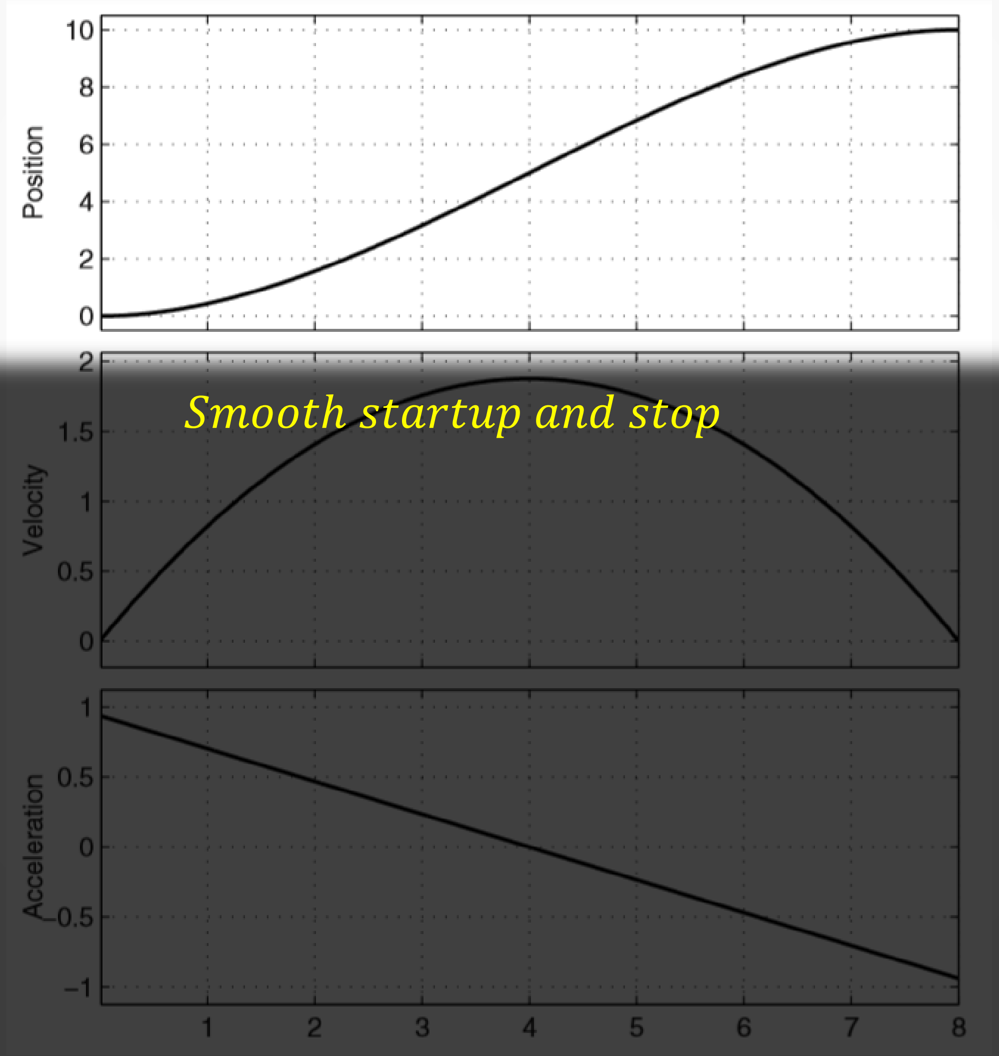

Polynomial Trajectories

Example

Parameter	Value
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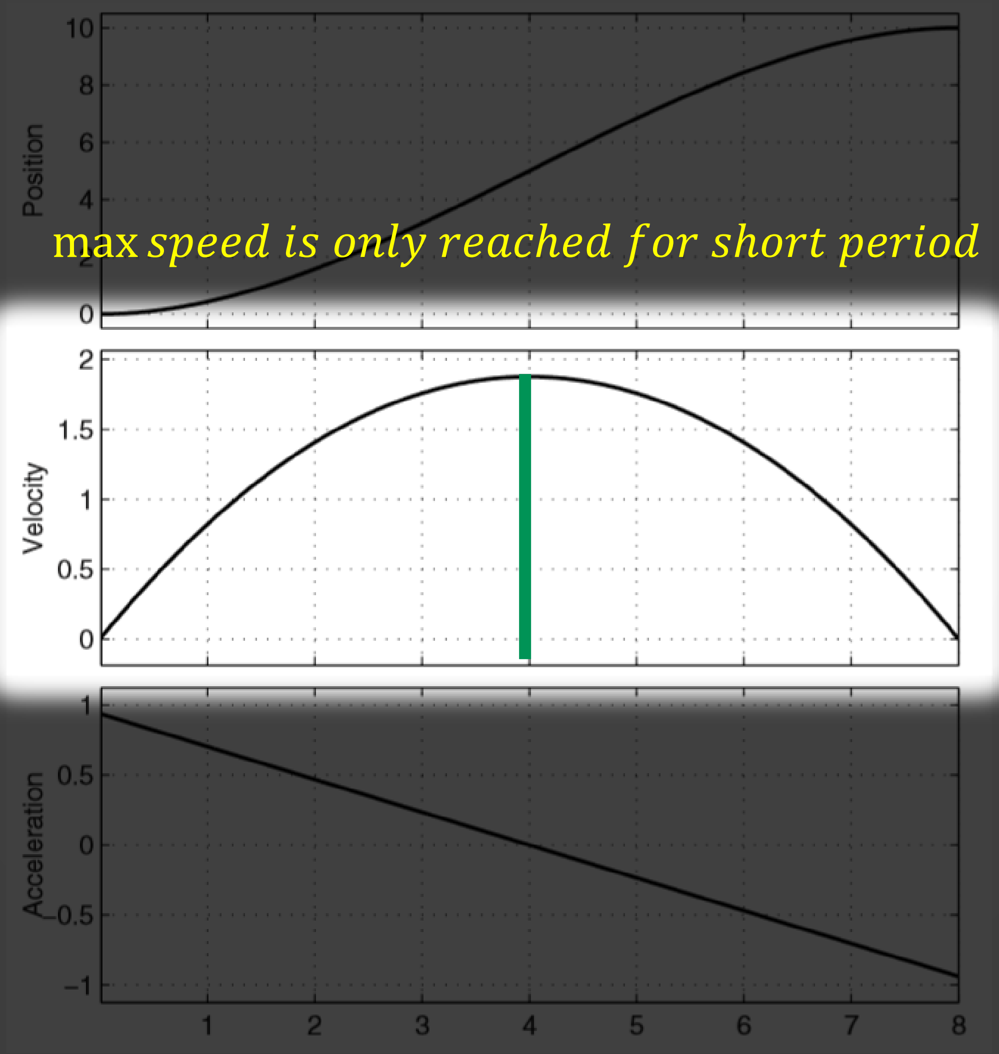

Polynomial Trajectories

Example

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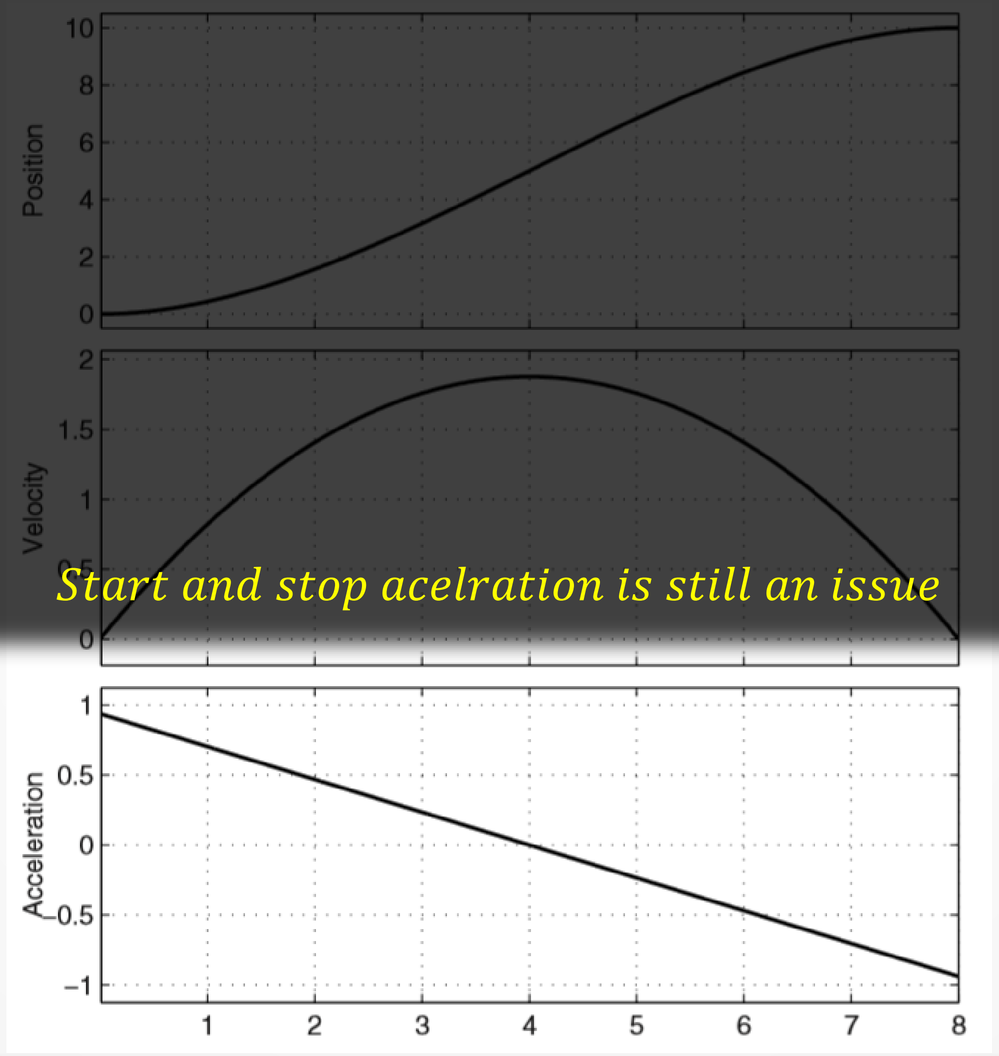


Polynomial Trajectories

$Quintic\ Trajectory$

$$s(t) = a_0 + a_1 T + a_2 (T)^2 + a_3 (T)^3 + a_4 (T)^4 + a_5 (T)^5$$

$$\dot{s}(t) = a_1 + 2a_2 T + 3a_3 (T)^2 + 4a_4 (T)^3 + 5a_5 (T)^4$$

$$\ddot{s}(t) = 2a_2 + 6a_3 T + 12a_4 (T)^2 + 20a_5 (T)^3$$

$To\ find\ the\ six\ coefficients\ , a_0\ , a_1, a_2\ ,\ a_3\ , a_4\ and\ a_5\ we\ need$
$\boldsymbol{six\ boundary\ conditions}\ s(t_0)\ , s\left(t_f\right), \dot{s}(t_0)\ , \dot{s}\left(t_f\right), \ddot{s}(t_0)\ and\ \ddot{s}\left(t_f\right)$

Polynomial Trajectories

$Quintic\ Trajectory$

$$s(t) = a_0 + a_1 T + a_2 (T)^2 + a_3 (T)^3 + a_4 (T)^4 + a_5 (T)^5$$

$$\dot{s}(t) = a_1 + 2a_2 T + 3a_3 (T)^2 + 4a_4 (T)^3 + 5a_5 (T)^4$$

$$\ddot{s}(t) = 2a_2 + 6a_3 T + 12a_4 (T)^2 + 20a_5 (T)^3$$

To find the six coefficients , a_0 , a_1, a_2 , a_3 , a_4 *and* a_5 *we need* ***six boundary conditions*** $s(t_0)$, $s(t_f)$, $\dot{s}(t_0)$, $\dot{s}(t_f)$, $\ddot{s}(t_0)$ *and* $\ddot{s}(t_f)$

Exercise : *Write it down into matrix form and find the inverse using MATLAB*

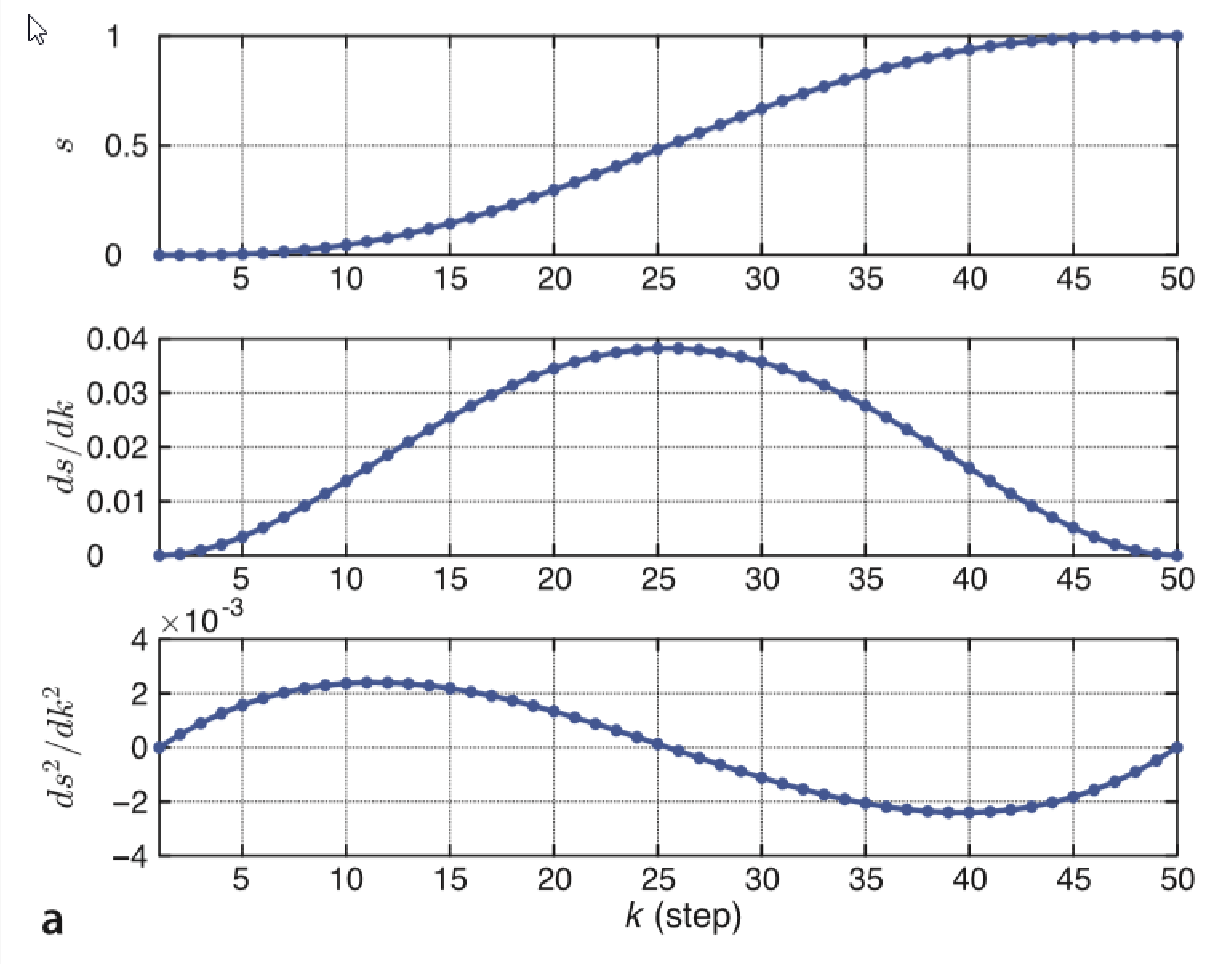
Polynomial Trajectories

$Quintic\ Trajectory$

$$a_0 = q_0$$
$$a_1 = \dot{q}_0$$
$$a_2 = \frac{1}{2} a_0$$
$$a_3 = \frac{1}{2T^3}\left[20h - \left(8\dot{q}_f + 12\dot{q}_0\right)T - (3a_0 - a_1)T^2\right]$$
$$a_4 = \frac{1}{2T^4}\left[-30h - \left(14\dot{q}_f + 16\dot{q}_0\right)T - (3a_0 - 2a_1)T^2\right]$$
$$a_5 = \frac{1}{2T^5}\left[12h - 6\left(\dot{q}_f + \dot{q}_0\right)T - (a_1 - a_0)T^2\right]$$

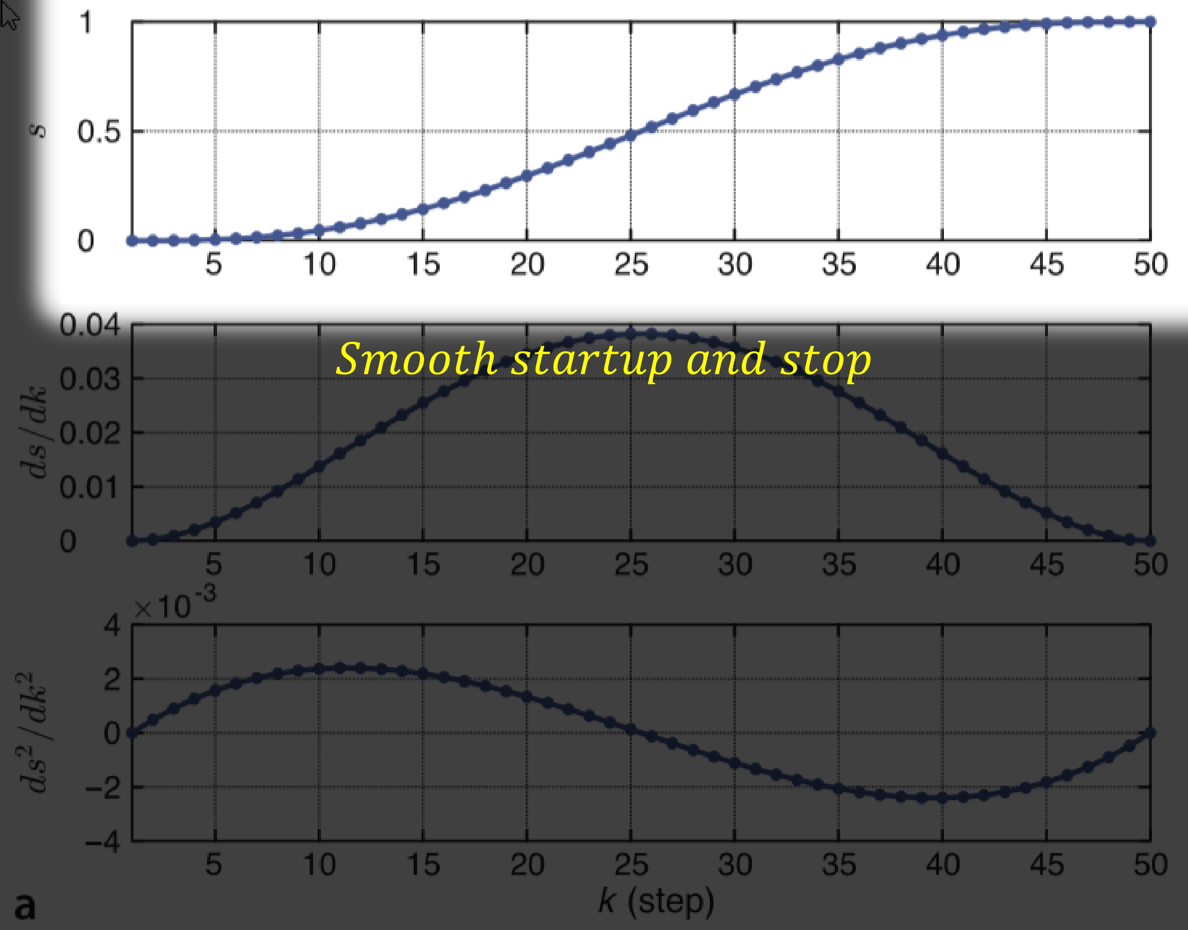
Polynomial Trajectories

Quintic Trajectory



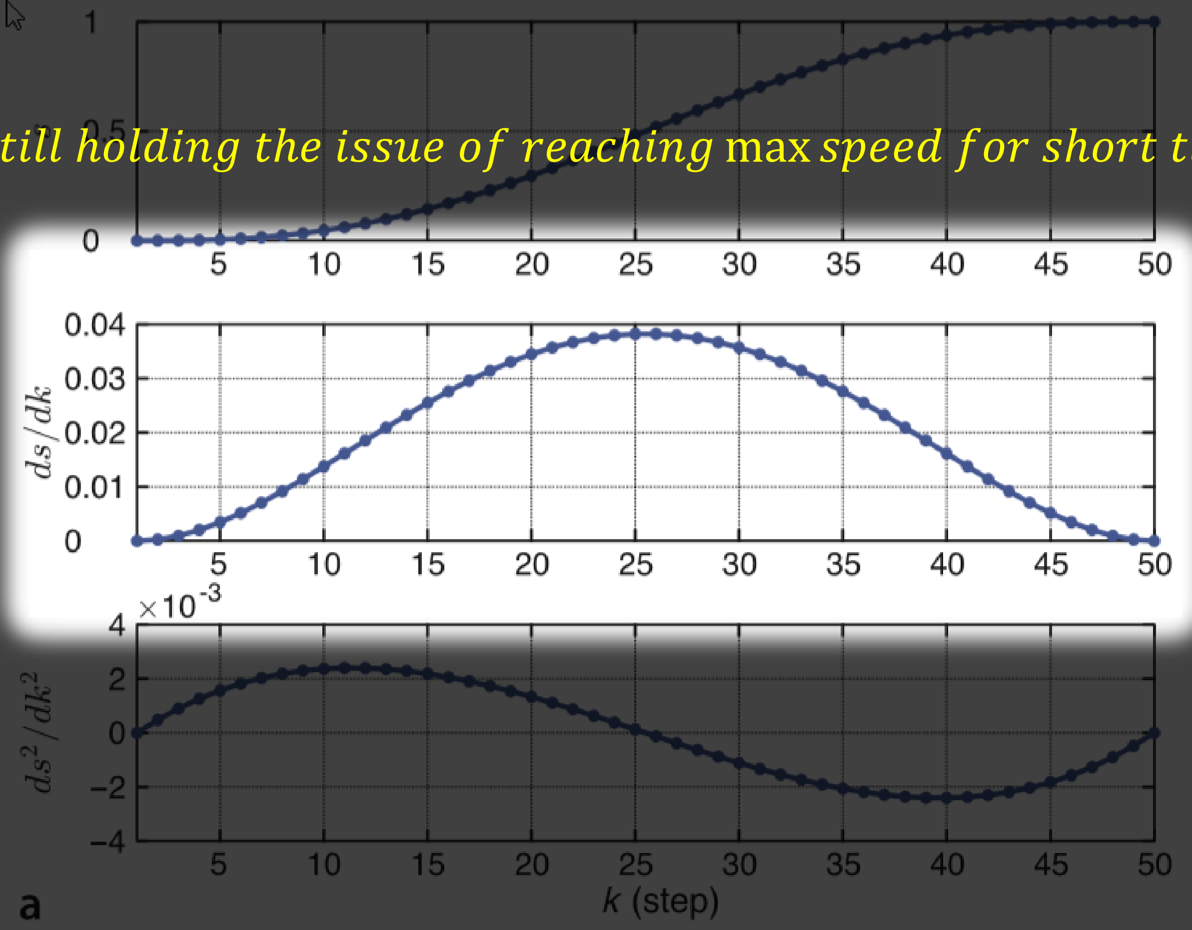
Polynomial Trajectories

Quintic Trajectory



Polynomial Trajectories

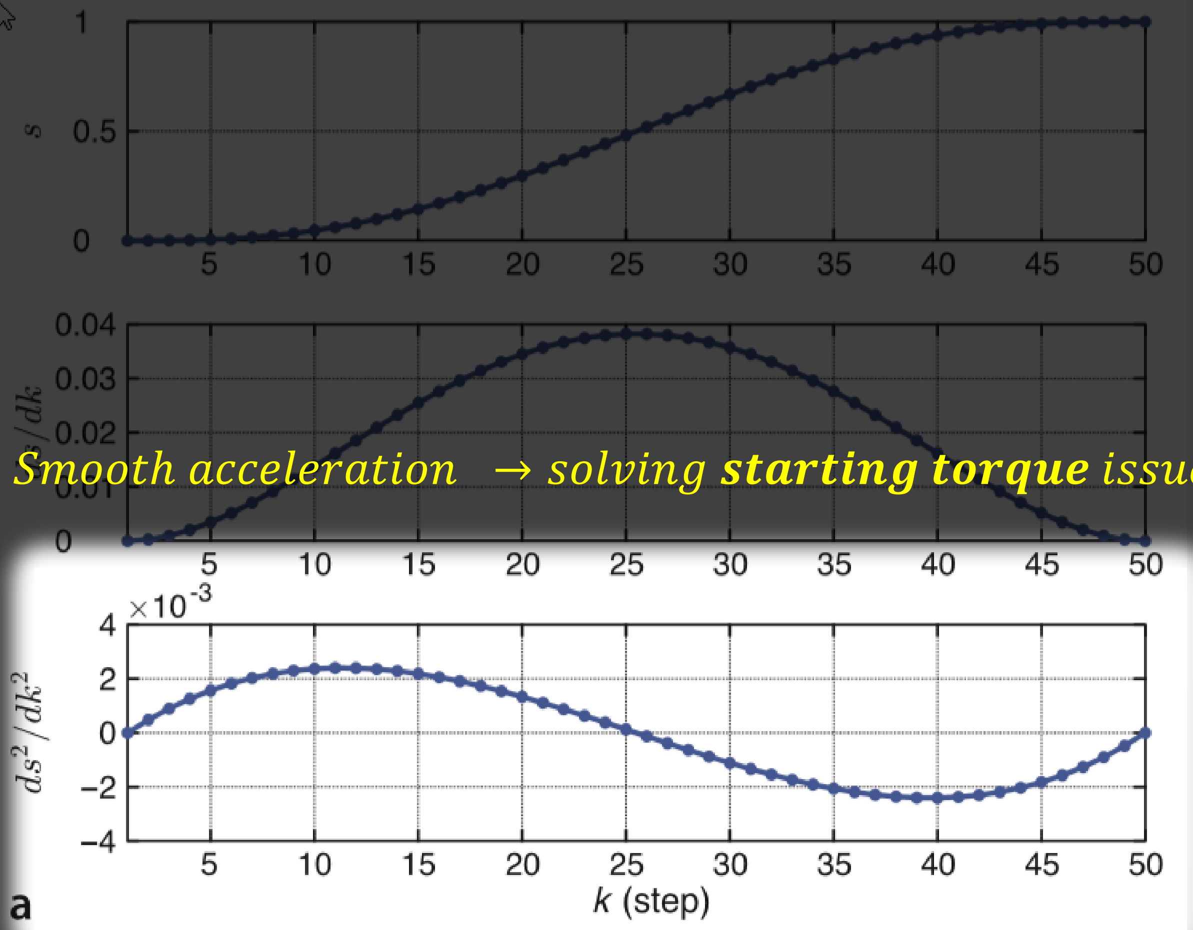
$Quintic\ Trajectory$



$Still\ holding\ the\ issue\ of\ reaching\ \max speed\ for\ short\ time$

Polynomial Trajectories

Quintic Trajectory



*Smooth acceleration → solving **starting torque** issue*

End of Lecture

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