Compositions and Partitions

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Compositions

a) There are 2^{n-1} compositions of n.

Proof. Consider n stars in a line. Between every pair of stars, we can choose to put a divider in between the two, or leave the two stars in the same section. Every distribution of dividers among our stars will yield a unique partition if we consider the size of section k to be the k^{th} part of our composition.

For example, if $n = 6$, we could have one distribution of dividers be: $* \mid * * * \mid * \mid *$, which would correspond to the composition $1 + 3 + 1 + 1$.

We can also map every composition of n to a distribution of dividers among n stars, by placing in dividers such that the number of stars in each section corresponds to our composition.

Since we have a bijection between compositions of n and placements of dividers among n stars, we can simply count the number of ways to place the dividers to get the total number of compositions of n. Since there are $n-1$ gaps where we can choose to put, or not put, a divider, there are 2^{n-1} distributions of our dividers, and thus 2^{n-1} compositions of n. □

b) There are $\binom{n-1}{k-1}$ compositions of n into k parts.

Proof. Now, let our k parts be k different bins, and let our integer n be decomposed into n parts of 1. The composition of n is a bijection to "tossing" our n parts of 1, into the k bins, and counting the total number of combinations of the number of parts in each bin. However, our only restriction is that our parts in the bins cannot be 0, as a composition of n cannot contain 0.

Let's arrange our n parts in a straight line. Suppose we have $k-1$ dividers. Now consider the following arrangement of $n = 5$: $\mid * * * * * \mid$.

Note that whenever we insert a divider, it separates the 5 parts into one additional group. Thus, when we insert $k-1$ dividers, it will separate our parts into k groups, or bins. Thus, this problem reduces to the number of ways to insert the $k-1$ dividers into our parts of 1.

Note that we can only insert our dividers in between the parts and not on the outside, or else they will form bins of 0, which is not allowed. In total, we have $n-1$ spaces for some $n \in \mathbb{N}$ and $k-1$ dividers where $k \leq n$. Thus, our total number of compositions of n with k parts are the $\binom{n-1}{k-1}$ ways to insert $k-1$ dividers into $n-1$ spaces.

□

First, we will define the indexing of the Fibonacci Sequence as follows: $F_0 = 0$, $F_1 = 1$, and for all $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

a) The answer is F_{n+1}.

Proof. Let a_n be the number of compositions of n with parts 1 and 2. We will use strong induction. As the base cases, note that 1 is the only composition of 1 with parts 1 and 2, and $1+1$ and 2 are the only compositions of 2 with parts 1 and 2. Hence, $a_1 = F_2$ and $a_2 = F_3$.

Now, assume that $a_n = F_{n+1}$ for all $n \leq k$. Observe that all compositions of $k+1$ can be split into two groups: compositions ending in 1, and compositions ending in 2. Any composition of $k+1$ with all parts 1 and 2 ending in 1 can be transformed into an arbitrary composition of k with all parts 1 and 2 by simply deleting the 1 at the end. Similarly, any composition of $k+1$ with all parts 1 and 2 ending in 2 can be transformed into an arbitrary composition of $k-1$ with all parts 1 and 2 by simply deleting the 2 at the end. Thus, there are $a_k + a_{k-1}$ compositions of $k+1$ with parts 1 and 2. In particular, using our inductive assumption,

$$a_{k+1} = a_k + a_{k-1} = F_{k+1} + F_k = f_{k+2}$$

as desired. □

b) The answer is F_n.

Proof. We start with a preliminary lemma:
Lemma: If $n \geq 2$ is even, then

$$F_n = F_{n-1} + F_{n-3} + \cdots + F_3 + F_1$$

and if n is odd, then

$$F_n = F_{n-1} + F_{n-3} + \cdots + F_2 + 1.$$

Proof. We will use strong induction. For the base cases, note that this is trivially true for $n = 2$ since $F_2 = 1 = F_1$ and for $n = 1$ since $F_1 = 2 = 1$.

Now, assume that these formulas hold for all $n \leq k$. If $k+1$ is even, then, using the inductive assumption,

$$\begin{aligned} F_{k+1} &= F_k + F_{k-1} \\ &= F_k + (F_{k-2} + F_{k-4} + \cdots + F_3 + F_1) \end{aligned}$$

as desired. Otherwise, $k+1$ is odd, and using the inductive assumption,

$$\begin{aligned} F_{k+1} &= F_k + F_{k-1} \\ &= F_k + (F_{k-2} + F_{k-4} + \cdots + F_2 + 1) \end{aligned}$$

as desired. □

With this, we prove our main assertion with strong induction. Let b_n be the number of compositions of n with only odd parts. Note that 1 is the only composition of 1 with odd parts and $1+1$ is the only composition of 2 with odd parts, so $b_1 = b_2 = 1$. In particular, $b_1 = F_1$ and $b_2 = F_2$.

Now, assume that $b_n = F_n$ for all $n \leq k$. Either $k+1$ is odd or even. If $k+1$ is odd, then any composition of $k+1$ with odd parts must end with some number in the set $\{1, 3, 5, \ldots, k+1\}$. For any member $i < k+1$ of this set, a composition of $k+1$ with all odd parts ending in i can be transformed into an arbitrary composition of $k+1-i$ with only odd parts by simply deleting the i at the end. Finally, there is also the (one) composition of $k+1$ with all odd parts consisting of only itself, since $k+1$ is odd. Thus,

$$\begin{aligned} b_{k+1} &= b_k + b_{k-2} + b_{k-4} + \cdots + b_2 + 1 \\ &= F_k + F_{k-2} + F_{k-4} + \cdots + F_2 + 1 \\ &= F_{k+1} \text{ by the lemma.} \end{aligned}$$

Otherwise, $k+1$ is even, and any composition of $k+1$ with odd parts must end with some number in the set $\{1, 3, 5, \ldots, k\}$. For any member i of this set, a composition of $k+1$ with all odd parts ending in i can be transformed into an arbitrary composition of $k+1-i$ with only odd parts by simply deleting the i at the end. Thus,

$$\begin{aligned} b_{k+1} &= b_k + b_{k-2} + b_{k-4} + \cdots + b_3 + b_1 \\ &= F_k + F_{k-2} + F_{k-4} + \cdots + F_3 + F_1 \\ &= F_{k+1} \text{ by the lemma.} \end{aligned}$$

Hence, in any case, $b_n = F_n$ as desired. □

c) The answer is F_{n-1}.

Proof. We start with a preliminary lemma:
Lemma: If $n \geq 1$, then

$$F_n = F_{n-2} + F_{n-3} + F_{n-4} + \cdots + F_1 + 1.$$

Proof. We will use strong induction. For our base cases, note that $F_1 = 1$ and $F_2 = 1$, so the identity holds.

Now, assume this formula holds for all $n \leq k$. Then,

$$\begin{aligned} F_{k+1} &= F_k + F_{k-1} \\ &= (F_{k-2} + F_{k-3} + F_{k-4} + \cdots + F_1 + 1) + F_{k-1} \\ &= F_{k-1} + F_{k-2} + F_{k-3} + \cdots + F_1 + 1 \end{aligned}$$

as desired. □

With this, we prove our main assertion using strong induction. Let c_n be the number of compositions of n with all parts at least 2. Note that $c_1 = 0$ since no composition of 1 can have all parts at least

2, and $c_2 = 1$ since the only composition of 2 with all parts at least 2 is 2. In particular, $b_1 = F_0$ and $b_2 = F_1$.

Now, assume that $b_n = F_{n-1}$ for all $n \leq k$ and $k \geq 2$. Notice that any composition of $k+1$ with all parts at least 2 must end with some number in the set $\{2, 3, 4, \ldots, k+1\}$. For any member $i < k+1$ of this set, a composition of $k+1$ with all parts at least 2 ending in i can be transformed into an arbitrary composition of $k+1-i$ with all parts at least 2 by simply deleting the i at the end. Finally, there is also the (one) composition of $k+1$ with all parts at least 2 consisting of only itself (since $k \geq 2$). Thus,

$$\begin{aligned} c_{k+1} &= c_{k-1} + c_{k-2} + c_{k-3} + \cdots + c_1 + 1 \\ &= F_{k-2} + F_{k-3} + F_{k-4} + \cdots + F_0 + 1 \\ &= F_k \text{ by the lemma.} \end{aligned}$$

Hence, $c_n = F_{n-1}$ as desired. □

d) Let $a_{n,k}$ be the number of compositions of n with 1 and 2 and a total of k parts. Let $b_{n,k}$ be the number of compositions of n with all odd parts and a total of k parts. Let $c_{n,k}$ be the number of compositions of n with all parts at least 2 and a total of k parts.

a) The answer is

$$a_{n,k} = \begin{cases} \binom{k}{n-k} & \text{if } n/2 \leq k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First of all, for $a_{n,k}$ to be nonzero, k must be between $n/2$ and n, since if $k < n/2$ then the maximum sum of a composition with k parts all 1 or 2 is $2k < n$, and if $k > n$ then the minimum sum of a composition with k parts is $k > n$.

Now, assume $n/2 < k < n$. Then, there must be $n-k$ 2s and $2k-n$ 1s in any composition of n with k parts all 1 or 2 (from solving a simple system of linear equations). So, out of the k positions in any such composition, $n-k$ must be 2s; this yields $\binom{k}{n-k}$ compositions. □

b) The answer is

$$b_{n,k} = \begin{cases} \binom{\left(\frac{n+k-2}{2}\right)}{k-1} & \text{if } 1 \leq k \leq n \text{ and } k \equiv n \pmod 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Clearly, if $k \not\equiv n \pmod 2$, then the sum of k odd numbers is congruent to k modulo 2, which can never equal n. So, we must have $1 \leq k \leq n$ and $k \equiv \pmod 2$ for $b_{n,k}$ to be nonzero.

Now, assume $k \equiv n \pmod 2$. Let $p_1 = 2a_1 + 1$, $p_2 = 2a_2 + 1$, ..., and $p_k = 2a_k + 1$ be the k odd parts of any composition of n into k odd parts. Then, all a_i are nonnegative. We have

$$\begin{aligned} n &= p_1 + p_2 + \cdots + p_k \\ &= 2a_1 + 1 + 2a_2 + 1 + \cdots + 2a_k + 1 \\ &= 2(a_1 + a_2 + \cdots + a_k) + k \\ \implies \frac{n-k}{2} &= a_1 + a_2 + \cdots + a_k. \end{aligned}$$

By Stars and Bars, this equation has $\binom{\left(\frac{n+k-2}{2}\right)}{k-1}$. □

c) The answer is

$$c_{n,k} = \begin{cases} \binom{n-k-1}{k-1} & \text{if } 1 \le k \le n/2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Clearly k cannot exceed $n/2$, since if it did, the minimum sum of a composition with k parts at least 2 would be $2k > n$. So, for $c_{n,k}$ to be nonzero, $1 \le k \le n/2$.

Now, assume $1 \le k \le n/2$. Let $p_1 = a_2 + 2$, $p_2 = a_2 + 2$, ..., and $p_k = a_k + 2$ be the k parts of any partition of n into k parts which are all at least 2. Then, all a_i are nonnegative. We have

$$\begin{aligned} n &= p_1 + p_2 + \cdots + p_k \\ &= a_1 + 2 + a_2 + 2 + \cdots + a_k + 2 \\ \implies n - 2k &= a_1 + a_2 + \cdots + a_k. \end{aligned}$$

By Stars and Bars, this has $\binom{n-k-1}{k-1}$ solutions. □

The number of compositions of n with parts a and b is equivalent to the number of compositions of $n+a$ with parts $a+bk$, for k a non-negative integer.

Proof. To start, note that we can biject compositions of n with parts a and b to compositions of $(n+a)$ with parts a and b that start with a part a, by simply adding or removing a first part of a.

From there, note that for any b in our composition of $(n+a)$ starting with a, there exists a closest a to its left. We can group all of our bs with their respective closest a on the left. For example, if our composition is $a+a+b+b+b+a+b+a+b+b$, we can group our bs to their respective as to get: $(a)+(a+b+b+b)+(a+b)+(a+b+b)$. Note that this since our sum did not change, (we simply inserted parentheses), this is a valid composition of $(n+a)$. Since all the parts of our composition are $a+b+b+\ldots+b$, for some non-negative integer k number of bs, we can write each part of our composition as $a+bk$.

Thus we have mapped every composition of $(n+a)$ with parts a and b that start with a part a to a composition of $(n+a)$ with parts $a+bk$. Note that this composition of $(n+a)$ with parts $a+bk$ is unique for each composition of $(n+a)$ with parts a and b that start with a part a, since we can easily unpack a part of $a+bk$ to $a+\underbrace{b+b+\ldots+b}_{\text{k b's}}$, and thus if two compositions of $(n+a)$ with parts a and b that start with a part a were to map to the same composition of $(n+a)$ with parts $a+kb$, they would have to be the exact same composition. Since we have a bijection between compositions of n with parts a and b to compositions of $(n+a)$ with parts a and b that start with a part a, we have established a mapping from each composition of n with parts a and b to a unique composition of $(n+a)$ with parts $a+bk$.

Now we show the reverse direction, mapping from compositions of $(n+a)$ with parts to compositions of $a+bk$ to a unique composition of n with parts a and b. Consider a composition of $(n+a)$ with parts $a+bk$. For each part $a+bk$, we can write out $a+bk$ as $a+\underbrace{b+b+\ldots+b}_{\text{k b's}}$. Thus we can write each composition of $(n+a)$ with parts $a+bk$ as a unique composition of $(n+a)$ with parts a and b. For example, we can write $(a+2b)+a+(a+3b)$ as $a+b+b+a+a+b+b+b$. Note that this partition will start with a part a, and thus each composition of $(n+a)$ with parts $a+bk$ maps to a unique composition of $(n+a)$ with parts a and b starting with an a, which maps to a unique composition of n with parts a and b by our earlier bijection. This establishes our reverse direction.

Taking our reverse and forwards directions, we have a bijection between compositions of n with parts a and b and compositions of $(n+a)$ with parts $a+bk$, and so the number of compositions of n with parts a and b is equivalent to the number of compositions of $n+a$ with parts $a+bk$, for k a non-negative integer. $\square$

a) Evaluate

$$\sum a_1 a_2 ... a_k$$

where the sum is over all compositions of n with k parts.

Proof. The composition of n over k parts can be expressed as a n stars and $k-1$ bars. The number of these stars are given in a group of a_i.

We now have $n+k-1$ elements in our compositions, and we need to choose $k-1$ from these $n+k-1$ to give us the compositions of groups. If we choose an element from the i^{th} group, we will add k to what we are choosing from to express the multiplication of all of the a_i groups. So,

$$\sum a_1 a_2 \dots a_k = \binom{n+k-1}{k-1+k} = \binom{n+k-1}{2k-1}$$

$\square$

b) express the value of the sum in (a) over all compositions of n (with any number of parts) in terms of Fibonacci numbers.

The answer is

$$\sum_{k=1}^{n} \binom{n+k-1}{2k-1} = F_{2n}$$

Lemma: For all natural x,

$$\sum_{k=0}^{x} \binom{x+k}{2k} = F_{2x+1}$$

and

$$\sum_{k=0}^{x} \binom{x+k+1}{2k+1} = F_{2x+2}.$$

Proof. We will use strong induction. The base cases of $x=1$ and $x=2$ are easy to check on both cases and hold true.

Now, assume both formulas hold until $x=n$. Then,

$$\begin{aligned}
\sum_{k=0}^{n+1} \binom{n+k+1}{2k} &= \sum_{k=0}^{n+1} \left(\binom{n+k}{2k} + \binom{n+k}{2k-1} \right) \\
&= \left(\sum_{k=0}^{n} \binom{n+k}{2k} \right) + \binom{2n+1}{2n+2} + \binom{n}{-1} + \left(\sum_{k=1}^{n+1} \binom{n+k}{2k-1} \right) \\
&= \sum_{k=0}^{n} \binom{n+k}{2k} + \sum_{k=0}^{n} \binom{n+k+1}{2k+1} \\
&= F_{2n+1} + F_{2n+2} \\
&= F_{2n+3}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{n+1}\binom{n+k+2}{2k+1} &= \sum_{k=0}^{n+1}\left(\binom{n+k+1}{2k+1}+\binom{n+k+1}{2k}\right) \\
&= \left(\sum_{k=0}^{n}\binom{n+k+1}{2k+1}\right)+\binom{2n+1}{2n+2}+\left(\sum_{k=0}^{n+1}\binom{n+k+1}{2k}\right) \\
&= F_{2n+2}+0+F_{2n+3} \\
&= F_{2n+4}.
\end{aligned}$$

So,

$$\sum_{k=1}^{n}\binom{n+k-1}{2k-1}=F_{2n}$$

□

Partitions

0.1 Reccurence Arguments

If we have the function $p_k(n)$, then the largest element is either equal to k, or the largest element is less than k. If the largest element is k, we can obtain our partitions by simply adding k to all partitions of $n-k$ with largest part k. Then we will have partitions of $n-k+k=n$ with k as the largest part, which is equivalent to $p_k(n)$.

The only other case is when the largest part is less than k. Then, it implies the largest part is less than or equal to $k-1$. Thus, it is then equivalent to $p_{\text{k-1}}(n)$.

Summing our two cases, we have that $p_k(n) = p_k(n-k) + p_{\text{k-1}}(n)$

Claim: We can show a bijection between t_n (the number of partitions of n into triples (a, b, c) of integers such that $a > b > c > 0$ and $a < b + c$), and the number of partitions of n into parts of 2, 3, and 4, such that each part shows up at least once.

To start, lets think of "buckets" A, B, and C, containing a, b, and c "drops" respectively, such that a, b, and c are integers which satisfy $a > b > c > 0$ and $a < b + c$. Given n drops, the number of ways to distribute these n drops among our 3 buckets is t_n.

Now we define three operations, known colloquially as "plops", that we can perform on our buckets:

- "two-plop", where we put one drop in A, one drop in B, and no drops in C
- "three-plop", where we put one drop in A, one drop in B, and one drop in C
- "four-plop", where we put two drops in A, one drop in B, and one drop in C

Note that the number of ways to distribute n drops into our 3 buckets using our plops, where every plop is used at least once is exactly the same as the number of partitions of n into parts of 2, 3, and 4 where every part is used at least once. We show this via the following bijection:

For any distribution of n drops into the 3 buckets using x two-plops, y three-plops, and z four-plops, where $x, y, z \geq 1$, we have that $2x + 3y + 4z = n$, since each two-plop uses 2 drops, each three-plop uses 3 drops, and each four-plop uses up 4 drops. Thus we can partition n into x 2s, y 3s, and z 4s such that $x, y, z \geq 1$. So every way to distribute our n drops into our buckets using each plop at least once maps to a unique partition of n into 2s, 3s, and 4s, such that each part is used at least once. For the backwards direction of our bijection, we just state that any partition n into x 2s, y 3s, and z 4s such that $x, y, z \geq 1$ can be mapped to a unique way to distribute n drops into 3 buckets using x two-plops, y three-plops, and z four-plops such that each plop is used at least once (since $x, y, z \geq 1$), and thus we have established a bijection.

Since we have a bijection between the number of ways to distribute n drops into our 3 buckets using two-plops, three-plops, and four-plops, where every plop is used at least once, and the number of partitions of n into parts of 2, 3, and 4 where every part is used at least once, we can prove our original claim by showing that: the number of ways to distribute n drops into our 3 buckets using two-plops, three-plops, and four-plops, where every plop is used at least once, is the same as the number of ways to distribute n drops into our three buckets A, B, and C such that $a > b > c > 0$, and $a < b + c$.

To do this we will form a bijection between these two ways of distributing our n drops. To start off, suppose we have a distribution of n drops into our three buckets A, B, and C such that $a > b > c > 0$, and $a < b+c$. Note that we then have $(b-c), (b+c-a), (a-b) \geq 1$. Given the values of $(b-c), (b+c-a), (a-b)$, we can solve for a unique solution (a, b, c), and thus we can map each triple (a, b, c) to a unique triple $((b - c), (b + c - a), (a - b))$. From here, note that taking $(b - c)$ two-plops, $(b + c - a)$ three-plops, and $(a - b)$ four-plops yields a total of $2(b - c) + 3(b + c - a) + 4(b - c) = a + b + c = n$ drops, and thus since $(b - c), (b + c - a), (a - b) \geq 1$, we have a distribution of n drops into our 3 buckets using two-plops, three-plops, and four-plops, where every plop is used at least once.

Since every distribution of n drops into our buckets such that $a > b > c > 0$, and $a < b + c$ maps to a unique triple $(b-c), (b+c-a), (a-b)$, and each of these triples maps to a unique distribution of n drops into the buckets using two-plops, three-plops, and four-plops, where every plop is used at least once, we have a mapping from every distribution of n drops into our three buckets A, B, and C such that $a > b > c > 0$, and $a < b+c$ to a unique distribution of n drops into our 3 buckets using two-plops, three-plops, and four-plops.

To complete the bijection, we just need to show the other direction. Suppose we have a distribution of n drops using x two-plops, y three-plops, and z four-plops such that $x, y, z \geq 1$. Note that we then have $2x + 3y + 4z = n$. We have $x + y + 2z$ drops in bucket A, $x + y + z$ drops in bucket B, and $y + z$ drops in bucket C. Thus we have $a = x + y + 2z$, $b = x + y + z$, and $c = y + z$. Given the values $x + y + 2z$, $x + y + z$, and $y + z$, we can solve for a unique solution (x, y, z), and thus each triple (x, y, z) must map to a unique triple $(x+y+2z, x+y+z, y+z) = (a, b, c)$. From here, note that a, b, and c are integers which satisfy:

- $a > b$, since $x + y + 2z > x + y + z$
- $b > c$, since $x + y + z > y + z$
- $c > 0$, since $y + z > 0$
- $b + c > a$, since $x + 2y + 2z > x + y + 2z$

Thus we have generated integers (a, b, c) such that $a > b > c > 0$, $a < b + c$, and $a + b + c = n$, and so we have a distribution of n drops into our three buckets A, B, and C such that $a > b > c > 0$, and $a < b + c$. So we have a mapping from any distribution of n drops into our 3 buckets using two-plops, three-plops, and four-plops such that each plop is used at least once to a unique triple (a, b, c), which in turn maps to a unique distribution of n drops into our buckets such that $a > b > c > 0$, and $a < b + c$, completing the other direction of our bijection.

Since we have shown both directions of our bijection, we have that the number of ways to distribute n drops into our 3 buckets using two-plops, three-plops, and four-plops, where every plop is used at least once, is the same as the number of ways to distribute n drops into our three buckets A, B, and C such that $a > b > c > 0$, and $a < b + c$, and thus the number of ways to distribute n drops into our three buckets A, B, and C such that $a > b > c > 0$, and $a < b + c$ is the same as the number of partitions of n into parts of 2, 3, and 4 such that each part is used at least once. And so we have proven our claim!

Now that we have a neat bijection, we can find interesting and powerful ways to express t_n. Note that the number of partitions of n into parts of $2, 3$, and 4 such that each part is used at least once is the same as the number of partitions of $n - 9$ into parts of $2, 3$, and 4: we can map from our partitions of n to our partitions of $n - 9$ by taking off a part of 2, 3, and 4 (there is at least one part of each, so we know that there will be a part we can take off), and we can map back from our partitions of $n - 9$ to our partitions of n by adding back a part of 2, 3, and 4, where we then have a guarantee that our partitions of n will contain at least one part of 2, 3, and 4. This establishes our desired bijection.

From here, we can continue to simplify! We can equate the number of partitions of $n-9$ into parts of 2, 3, and 4 to the quantity $p_4(n-9) - p_4(n-10)$, which counts the number of partitions of $n-9$ into parts of 1, 2, 3, and 4 minus the number of partitions of $n-10$ into parts of 1, 2, 3, and 4:

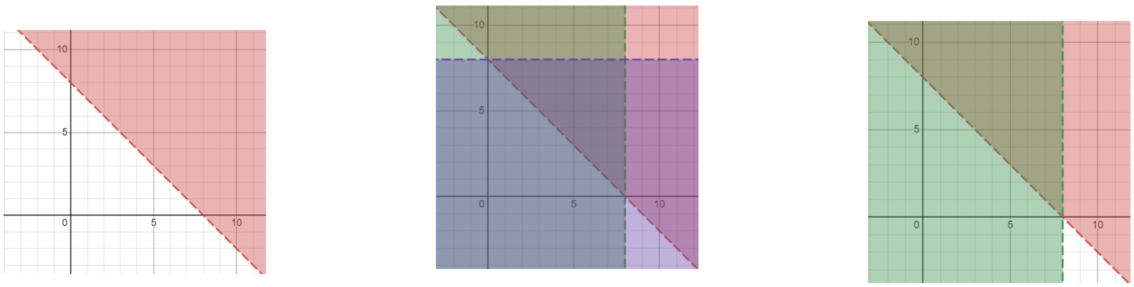
We can split each partition of $n-9$ into parts of 1, 2, 3, and 4 into one of two categories: if it contains a part of 1, and if it does not contain a part of 1. We would like to count the partitions which do not contain a part of 1. Note that the number of partitions of $n-9$ into parts of 1, 2, 3, and 4 which contain a part of 1 is simply $p_4(n-10)$, since we can biject between our partitions of $(n-10)$ with parts 1, 2, 3, and 4 and our partitions of $n-9$ into parts of 1, 2, 3, and 4 which contain a part of 1 by simply adding or taking a way a part of 1. Thus the number of partitions of $n-9$ into parts 1, 2, 3, and 4 which do not contain a part of 1, or in other words the partitions of $n-9$ into parts of 2, 3, and 4, is simply $p_4(n-9) - p_4(n-10)$.

So we have $t_n = p_4(n-9) - p_4(n-10)$, as desired.

Let x, y, and n be the sides of the a triangle and n is the largest sides. So,

$$x + y > n \quad \text{from triangle inequality.} \tag{1a}$$
$$n > x \quad \text{from the problem assumption} \tag{1b}$$
$$n > y \tag{1c}$$

By graphing these equations, we get the following graphs. The number of triangles with integer sides will be the number of lattice points bounded by the intersection between these three regions as in Figure 1(b).



(a) $x + y > n$

(b) intersection between three inequalities.

(c) $n > x$

Figure 1

To consider all different triangle, we should exclude the values when $x = y$. This equation is symmetric around the line $x = y$.

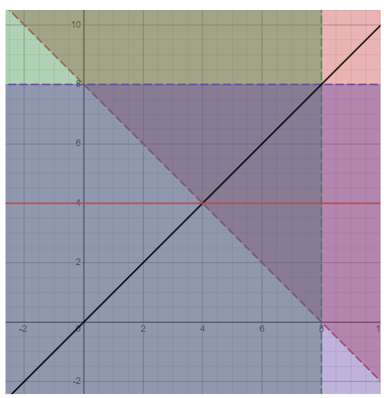
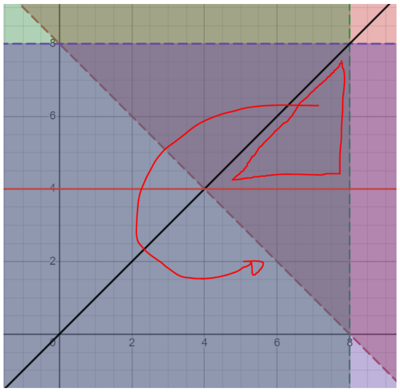


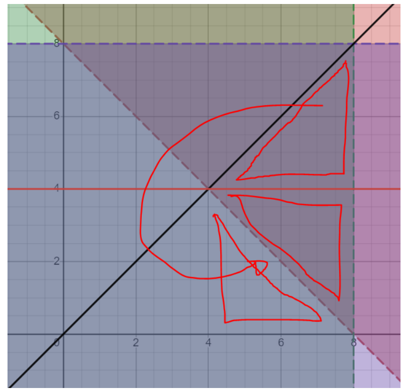
Figure 2: Lines of symmetry

Consider the right triangle formed by splitting the intersection area. This triangle is symmetric around $y = \frac{n}{2}$. By rotating the upper triangle by 270 degrees, it would form a square with side length of $\frac{n}{2}$. We should also exclude the lattice point on the side of the this formed square. So, we would get a square with side length of $\frac{n-1}{2}$. Thus, the number of different triangle with integer sides and largest side in is

$$[\frac{(n-1)^2}{4}]$$



(a) Upper triangle



(b) Triangle rotation

Figure 3

The generating function for the number of partitions of n such that no part is divisible by d is

$$(1+x+x^2+\cdots)(1+x^2+x^4+\cdots)\cdots(1+x^{d-1}+x^{2(d-1)}+\cdots)(1+x^{d+1}+x^{2(d+1)}+\cdots)\cdots$$

$$=\prod_{\substack{i=1\\ d\nmid i}}^{\infty}\frac{1}{1-x^i}.$$

The generating function for the number of partitions of n such that no part is appears d or more times is

$$(1+x+x^2+\cdots+x^{d-1})(1+x^2+x^4+\cdots+x^{2(d-1)})(1+x^3+x^6+\cdots+x^{3(d-1)})\cdots$$

$$=\prod_{i=1}^{\infty}\frac{1-x^{di}}{1-x^i}.$$

Observe that

$$\begin{aligned}\prod_{i=1}^{\infty}\frac{1-x^{di}}{1-x^i} &= \left(\prod_{\substack{i=1\\ d\nmid i}}^{\infty}\frac{1}{1-x^i}\right)\left(\prod_{\substack{i=1\\ d\mid i}}^{\infty}\frac{1}{1-x^i}\right)\left(\prod_{i=1}^{\infty}1-x^{di}\right)\\ &= \left(\prod_{\substack{i=1\\ d\nmid i}}^{\infty}\frac{1}{1-x^i}\right)\left(\prod_{\substack{i=1\\ d\mid i}}^{\infty}\frac{1}{1-x^i}\right)\left(\prod_{\substack{i=1\\ d\mid i}}^{\infty}1-x^{i}\right)\\ &= \prod_{\substack{i=1\\ d\nmid i}}^{\infty}\frac{1}{1-x^i}.\end{aligned}$$

In particular, both generating functions are identical. Thus, equating coefficients for the x^n term on both sides, the number of partitions of n such that no part is divisible by d is equal to the number of partitions of n such that no part appears d or more times.

For the sum of consecutive numbers, we consider the following formula: if $m < n$, then

$$m + (m+1) + (m+2) + \cdots + (n-1) + n = \frac{1}{2}(m+n)(n-m+1).$$

Observe that $(m+n)+(n-m+1) \equiv 2n+1 \equiv 1 \pmod 2$, so one of $m+n$ and $n-m+1$ must be odd. Since $1 < m < n$, $m+n \geq 3$, so if $m+n$ was odd, then some odd prime would divide $\frac{1}{2}(m+n)(n-m+1)$, making it impossible for it to be a power of 2. Otherwise, $n-m+1$ is odd. Again, if $n-m+1$ was greater than 1, then $\frac{1}{2}(m+n)(n-m+1)$ couldn't be a power of 2. The only remaining case is if $n-m+1=1$. But this would mean that $n=m$, which contradicts $m<n$. Hence, $\sum_{i=m}^{n} i$ cannot be a power of 2 if $n > m$.

For the reverse direction, if n isn't a power of two, then there exists an odd prime in its prime factorization. Let $n = pm$, where p is odd and at least 3 (taking p as the smallest odd prime factor of n suffices). Let $p = 2k+1$. Then, observe that

$$\begin{aligned} n &= \underbrace{m + m + \cdots + m}_{p} \\ &= \underbrace{(m-k) + (m-k+1) + \cdots + (m-1) + m + (m+1) + \cdots + (m+k-1) + (m+k)}_{2k+1 = p \text{ terms}}. \end{aligned}$$

Lemma: If a positive integer n can be expressed as the sum of consecutive integers, then it can be expressed as the sum of consecutive positive integers.

Proof. Let

$$n = (a) + (a+1) + (a+2) + \cdots + (a+k-2) + (a+k-1)$$

be a representation of n as the sum of k consecutive integers (not necessarily positive). The terms of this sum are increasing, so if $a > 0$ then we have found a sum of consecutive positive integers equal to n, and we are done. Otherwise, $a \leq 0$. □