Partitions

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PROMYS

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What is a Partition?

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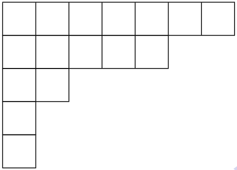
- Partitions can be illustrated visually using a *Ferrer's Diagram* (also called a *Young Diagram*), which is a grid (usually of boxes or dots) representing the parts of the partition. Rows represent the value of each part, and they customarily appear in decreasing order.
- Ex) The partition $7+5+2+1+1$ of 16 is represented like so:



Recurrence And Counting Arguments

Partitions with parts

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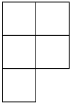
- For example, in this partition of 5,



both 2 and 1 are considered parts.

Restrictions on Parts

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- Since order doesn't matter, lets arrange all partitions such that the largest part k is always at the very end of the sum.

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- Going back to our example of $p_2(5)$, we can see that in our case with largest part of 2, we have $1+1+1+2$ and $1+2+2$.
- However, since we always end with 2 in the three above examples, we can draw a bijection between $p_2(5-2)$ and $p_2(5)$ with largest part 5. This is due to the fact that by adding 2 to every partition in $p_2(3)$, we guarantee that there will be a largest part of 2 in the new partitions that add up to $3+2=5$.

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- Going back to our example of $p_2(5)$, we can see that in our case with largest part of 2, we have $1+1+1+2$ and $1+2+2$.
- However, since we always end with 2 in the three above examples, we can draw a bijection between $p_2(5-2)$ and $p_2(5)$ with largest part 5. This is due to the fact that by adding 2 to every partition in $p_2(3)$, we guarantee that there will be a largest part of 2 in the new partitions that add up to $3+2=5$.
- Thus, to generalize, for $p_k(n)$ where all partitions have the largest part k, it is equivalent to $p_k(n-k)$.

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- In that case, it's simply equivalent to $p_{k-1}(n)$, as this covers all partitions of n with parts less than or equal to $k-1$, which makes up the complete set of numbers that are less than k.
- So summing the two cases up, we have $p_k(n) = p_k(n-k) + p_{k-1}(n)$.

Power of Young Diagrams

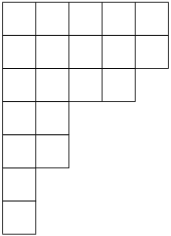
Numerical 2

There are 7 partitions of 6 *with largest part at most* 3*:* $1+1+1+1+1+1$, $1+1+1+1+2$, $1+1+2+2$, $2+2+2$, $1+1+1+3$, $1+2+3$, *and* $3+3$. *There are also* 7 *partitions of* 6 *with at most* 3 *parts:* 6, $5+1$, $4+2$, $4+1+1$, $3+3$, $3+2+1$, *and* $2+2+2$.

Theorem:
The number of partitions of n with at most k parts is equal to the number of partitions of n with no part exceeding k.

Interpreting This Visually

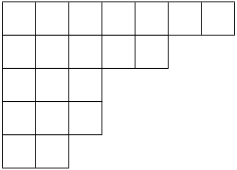
Take any arbitrary partition of n with at most k parts:



Then, all rows of this Young Diagram have length at most k.

Interpreting This Visually

Now, imagine reflecting this diagram over its diagonal, swapping the rows with the columns:



This diagram can have at most k columns, so this new Young Diagram represents a partition of n with at most k parts.

Analyzing Triangles with Partitions

Scalene Triangle Curiosities

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- t_n counts the triples of integers (a, b, c) such that $a + b + c = n$ $a > b > c > 0$, and $a < b + c$.
- Ex) $t_{15} = 3$, since we have 3 such triples: $(6, 5, 4)$, $(7, 5, 3)$, $(7, 6, 2)$

Motivational Numerical Patterns

Numerical 3

t_{17} is 4, our four triples are: $(8,7,2)$, $(8,6,3)$, $(8,5,4)$, $(7,6,4)$. *There are also four partitions of* 17 *into parts* 2, 3, *and* 4, *such that each part is used at least once:* $2+2+2+2+2+3+4$, $2+2+2+3+4+4$, $2+2+3+3+3+4$, $2+3+4+4+4$

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t_{17} is 4, our four triples are: $(8,7,2)$, $(8,6,3)$, $(8,5,4)$, $(7,6,4)$. *There are also four partitions of* 17 *into parts* 2, 3, *and* 4, *such that each part is used at least once:* $2+2+2+2+2+3+4$, $2+2+2+3+4+4$, $2+2+3+3+3+4$, $2+3+4+4+4$

Numerical 4

t_{19} is 5, our five triples are: $(9,8,2)$, $(9,7,3)$, $(9,6,4)$ $(8,7,4)$, $(8,6,5)$ *There are also five partitions of* 19 *into parts* 2, 3, *and* 4, *such that each part is used at least once:* $2+2+2+2+2+2+3+4$, $2+2+2+2+3+4+4$, $2+2+3+4+4+4$, $2+2+2+3+3+3+4$, $2+3+3+3+4+4$

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t_{17} is 4, our four triples are: $(8, 7, 2)$, $(8, 6, 3)$, $(8, 5, 4)$, $(7, 6, 4)$. *There are also four partitions of* 17 *into parts* 2, 3, *and* 4, *such that each part is used at least once:* $2+2+2+2+2+3+4$, $2+2+2+3+4+4$, $2+2+3+3+3+4$, $2+3+4+4+4$

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Theorem:
t_n is equivalent to the number of partitions of n into parts 2, 3, *and* 4, *such that each part is used at least once*

An Intuition for Plops

- Consider our side lengths as buckets A, B, and C, holding a, b, and c drops respectively for the triple (a, b, c).

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- 2, 3, and 4 totals are reminiscent of our numerical motivation, and we need at least one of each plop to form a working triangle triple!

Plop Partitioning Patterns

- For t_{17} we have four triples: $(8, 7, 2)$, $(8, 6, 3)$, $(8, 5, 4)$, $(7, 6, 4)$, and four partitions of 17 into parts 2, 3, and 4, such that each part is used at least once: $2+2+2+2+2+3+4$, $2+2+2+3+4+4$, $2+2+3+3+3+4$, $2+3+4+4+4$

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- For each of these partitions, let the number of 2s be the number of two-plops, the number of 3s be the number of three-plops, and the number of 4s be the number of four-plops

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- For each of these partitions, let the number of 2s be the number of two-plops, the number of 3s be the number of three-plops, and the number of 4s be the number of four-plops
- Our partitions yield the following triples (x, y, z) of x two-plops, y three-plops, and z four-plops: $(5, 1, 1)$, $(3, 1, 2)$, $(2, 3, 1)$, $(1, 1, 3)$

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- After applying our plop-operations, each of these triples yields a unique triple from t_{17}: $(5,1,1) \to (8,7,2)$, $(3,1,2) \to (8,6,3)$, $(2,3,1) \to (7,6,4)$, $(1,1,3) \to (8,5,4)$

A Conjecture with a Gameplan

Theorem:
The number of ways to distribute n drops into our 3 buckets using two-plops, three-plops, and four-plops, where every plop is used at least once, is the same as the number of ways to distribute n drops into our three buckets A, B, and C such that $a > b > c > 0$, and $a < b + c$

- To show t_n is the number of partitions of n with parts 2, 3, and 4, where each part is used at least once, we show:

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- (1) a bijection between the number of partitions of n we can form with our plops, using each plop at least once, and the partitions of n with parts 2, 3, and 4, where each part is used at least once

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- (1) a bijection between the number of partitions of n we can form with our plops, using each plop at least once, and the partitions of n with parts 2, 3, and 4, where each part is used at least once
- (2) a bijection between the number of partitions of n we can form with our plops, using each plop at least once, and t_n, the number of ways to distribute n drops into our three buckets A, B, and C such that $a > b > c > 0$, and $a < b + c$

Bijection 1: Plops to Partitions

Claim:
the number of ways to distribute n *drops into our* 3 *buckets using two, three, and four plops, where every plop is used at least once is exactly the same as the number of partitions of* n *into parts of* 2, 3, *and* 4 *where every part is used at least once.*

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- For a distribution of n drops into the 3 buckets using x two-plops, y three-plops, and z four-plops, where $x, y, z \geq 1$, we have that $2x + 3y + 4z = n$.

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- Thus we can partition n into x 2s, y 3s, and z 4s such that $x, y, z \geq 1$.

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- Thus we can partition n into x 2s, y 3s, and z 4s such that $x, y, z \geq 1$.
- For the backwards direction, we just apply the same logic in reverse

Bijection 2: Plops to Scalene Triangles

Claim:

the number of ways to distribute n drops into our 3 buckets using two-plops, three-plops, and four-plops, where every plop is used at least once, is the same as the number of ways to distribute n drops into our three buckets A, B, and C such that $a > b > c > 0$, and $a < b + c$

Bijection 2: Plops to Scalene Triangles ($\Leftarrow$)

($\Leftarrow$) We start by showing each integer triple (a, b, c) such that $a+b+c = n$, $a > b > c > 0$, and $a < b + c$ maps to a unique way to distribute our n drops into the three buckets using two-plops, three-plops, and four-plops, where every plop is used at least once:

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- For any such triple (a, b, c), we then have: $(b - c), (b + c - a), (a - b) \geq 1$

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- For any such triple (a, b, c), we then have: $(b - c), (b + c - a), (a - b) \geq 1$
- Note that taking $(b - c)$ two-plops $\langle 1, 1, 0 \rangle$, $(b + c - a)$ three-plops $\langle 1, 1, 1 \rangle$, and $(a - b)$ four-plops $\langle 2, 1, 1 \rangle$ yields a total of: $2(b - c) + 3(b + c - a) + 4(a - b) = a + b + c = n$ drops

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- Thus each scalene triple (a, b, c) corresponds to a unique triple $((b - c), (b + c - a), (a - b))$, and each such triple maps to a unique way to distribute our n drops using each plop at least once

Bijection 2: Plops to Scalene Triangles ($\Rightarrow$)

($\Rightarrow$) Now we show that each distribution of n drops into our 3 buckets using x two-plops, y three-plops, and z four-plops such that $x, y, z \geq 1$ corresponds to a unique triangle triple (a, b, c) such that $a + b + c = n$, $a > b > c > 0$, and $a < b + c$:

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- Considering our buckets A, B, C, we have: $a = x + y + 2z$, $b = x + y + z$, and $c = y + z$.

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- Considering our buckets A, B, C, we have: $a = x + y + 2z$, $b = x + y + z$, and $c = y + z$.
- Each unique triple (x, y, z) maps to a unique triple $((x + y + 2z), (x + y + z), (y + z)) = (a, b, c)$

Bijection 2: Plops to Scalene Triangles ($\Rightarrow$)

($\Rightarrow$) Now we show that each distribution of n drops into our 3 buckets using x two-plops, y three-plops, and z four-plops such that $x, y, z \geq 1$ corresponds to a unique triangle triple (a, b, c) such that $a + b + c = n$, $a > b > c > 0$, and $a < b + c$:

- Considering our buckets A, B, C, we have: $a = x + y + 2z$, $b = x + y + z$, and $c = y + z$.
- Each unique triple (x, y, z) maps to a unique triple $((x + y + 2z), (x + y + z), (y + z)) = (a, b, c)$
- $a > b > c > 0$, since $x + y + 2z > x + y + z > y + z > 0$

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- $a + b + c = 2x + 3y + 4z = n$ total drops
- From each unique distribution of n drops using plops, with each plop used at least once, we have generated a unique triple of integers (a, b, c) such that $a > b > c > 0$, $a < b + c$, and $a + b + c = n$

Finale!

Partitions of n using parts of 2, 3, and 4, such that each part is used at least once

$$\Updownarrow$$

Distributions of n drops using two-plops, three-plops, and four-plops such that each plop is used at least once

$$\Updownarrow$$

Distributions of n drops into the three buckets A, B, C, such that $a > b > c > 0$ and $a < b + c$, or t_n

Even More Applications in Geometry

Different integer triangles with largest side n

Let x, y, and n be the sides of the triangle and n is the largest sides. So,

$$x + y > n \quad \text{from triangle inequality.} \tag{1a}$$

$$n > x \quad \text{from the problem assumption} \tag{1b}$$

$$n > y \tag{1c}$$

By graphing all these inequalities, we get the following graph. To consider all different triangle, we should exclude the values when $x = y$. This equation is symmetric around the line $x = y$.

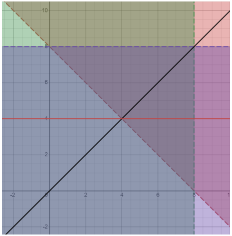
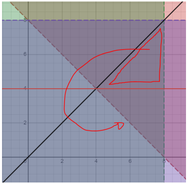


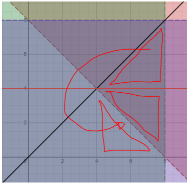
Figure: All inequalities

- By rotating the upper triangle by 270 degrees, it would form a square with side length of $\frac{n}{2}$. We should also exclude the lattice point on the side of the this formed square.
- So, we would get a square with side length of $\frac{n-1}{2}$. Thus, the number of different triangle with integer sides and largest side in is

$$\lfloor \frac{(n-1)^2}{4} \rfloor$$



(a) Upper triangle



(b) Triangle rotation

Figure

Generating Functions

How do Generating Functions Work?

- Generating functions are a way of representing sequences of numbers using a power series. Generating functions are useful because they can be used to solve combinatorial problems using algebra.

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- Generating functions are a way of representing sequences of numbers using a power series. Generating functions are useful because they can be used to solve combinatorial problems using algebra.
- Generating functions and partitions go hand in hand because finding all partitions of positive integers summing to n is equivalent to finding the coefficient of the x^n term in some generating function.

Numerical 5

There are five partitions of 7 into all odd parts: $1+1+1+1+1+1+1$, $1+1+1+1+3$, $1+1+5$, $1+3+3$, *and* 7. *There are also five partitions of* 7 *into all distinct parts:* 7, $6+1$, $5+2$, $4+3$, *and* $4+2+1$.

Patterns in Numericals

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Numerical 6

There are eight partitions of 10 into all odd parts: $1+1+1+1+1+1+1+1+1+1$, $1+1+1+1+1+1+1+3$, $1+1+1+1+1+5$, $1+1+3+5$, $1+1+1+1+3+3$, $1+3+3+3$, $1+1+1+7$, $1+9$, $3+7$, *and* $5+5$. *There are also eight partitions of 10 into all distinct parts:* 10, $9+1$, $8+2$, $7+3$, $7+2+1$, $6+4$, $6+3+1$, $5+4+1$, $5+3+2$, *and* $4+3+2+1$.

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Numerical 6

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Conjecture:

Is the number of partitions of n into odd parts the same as the number of partitions of n into distinct parts?

Generalizing This Pattern

Numerical 7

There are five partitions of 5 into parts not divisible by 3*:* $1+1+1+1+1$, $1+1+1+2$, $1+2+2$, $1+4$, *and* 5. *There are also five partitions of* 5 *into parts not appearing more than two times:* 5, $4+1$, $3+2$, $2+2+1$, *and* $3+1+1$.

Generalizing This Pattern

Numerical 7

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Theorem:

The number of partitions of n into parts not divisible by k is the same as the number of partitions of n into parts which appear less than k times.

Proving Our Conjecture With Generating Functions

We can use generating functions to prove the conjecture.

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The generating function for a general partition is

$$\prod_{i=1}^{\infty}\sum_{j=0}^{\infty} x^{ij} = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)\cdots.$$

(Why?)

Proving Our Conjecture With Generating Functions

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(Why?)

The coefficient of x^n in this product is the number of partitions of n. We can modify this generating function to count only the partitions of n which have no parts divisible by k (call this generating function $A(x)$), as well as the partitions of n which have no parts appearing k or more times (call this generating function $B(x)$).

Finding $A(x)$

The generating function for general partitions contains a factor of $1 + x^i + x^{2i} + x^{3i} + \cdots$ for every natural number i because there are no restrictions on any partition's parts.

Finding $A(x)$

The generating function for general partitions contains a factor of $1 + x^i + x^{2i} + x^{3i} + \cdots$ for every natural number i because there are no restrictions on any partition's parts. So, the generating function for partitions with no parts divisible by k is

$$A(x) = \prod_{\substack{i=1 \\ k \nmid i}}^{\infty} \sum_{j=0}^{\infty} x^{ij}$$

$$= (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots) \cdots$$
$$(1 + x^{k-1} + x^{2k-2} + \cdots)(1 + x^{k+1} + x^{2k+2} + \cdots) \cdots.$$

Finding $B(x)$

The generating function for general partitions contains infinitely many terms in each factor $1 + x^i + x^{2i} + x^{3i} + \cdots$ because there are no restrictions on any partition's parts.

Finding $B(x)$

The generating function for general partitions contains infinitely many terms in each factor $1 + x^i + x^{2i} + x^{3i} + \cdots$ because there are no restrictions on any partition's parts. So, the generating function for partitions with no parts appearing k or more times is

$$B(x) = \prod_{i=1}^{\infty} \sum_{j=0}^{k-1} x^{ij}$$

$$(1+x+x^2 \cdots + x^{k-1})(1+x^2+x^4+\cdots+x^{2k-2})(1+x^3+x^6+\cdots+x^{3k-3})\cdots.$$

Proving $A(x) = B(x)$

To finish the proof, we need to show that $A(x) = B(x)$. Note that

$$B(x) = \prod_{i=1}^{\infty} \sum_{j=0}^{k-1} x^{ij}$$

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&= \prod_{\substack{i=1 \\ i \nmid k}}^{\infty} \frac{1}{1 - x^{i}} \\
&= (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots) \cdots \\
&= A(x).
\end{aligned}
$$

Finishing Touches

$$A(x) = B(x)$$

$$A(x) = B(x)$$

$$\Updownarrow$$

The number of partitions of n with no part divisible by k is the same as the number of partitions of n with no part appearing k or more times.

Thank You David Fried and PROMYS!