Motion Planning for Autonomous Vehicles

Model Predictive Control (MPC)

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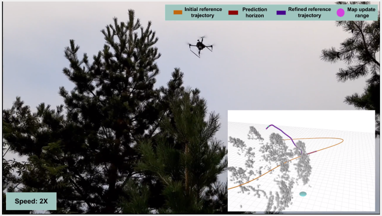
April 16, 2023



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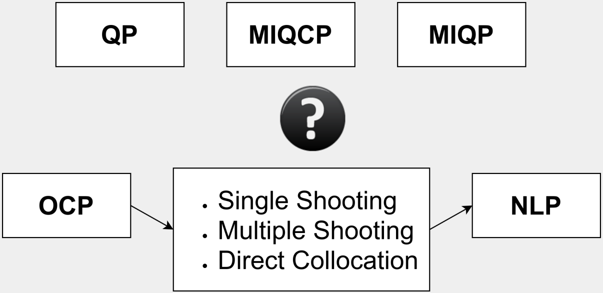
Contents

Ways to solve Optimal Control (OCP) Problems

An OCP problem can be transformed into an NLP problem. A problem is always solved better in a **nonlinear manner** as opposed to a **linearizing motion model** at every time since the motion model is nonlinear.



A **OCP** problem can **transform** into **NLP** in various ways, including MS (**Multiple-Shooting**) and DC (**Direct-Collocation**).

OCP Using Nonlinear Programming Problem (NLP)

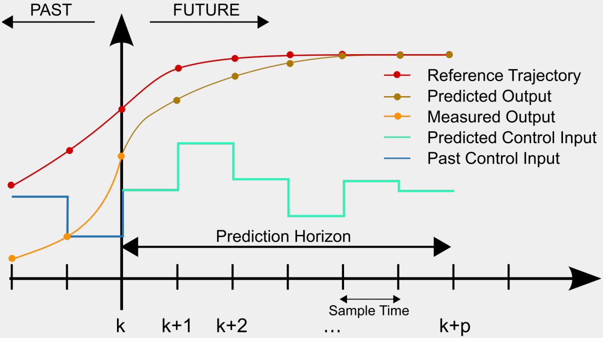
OCP

$$
\begin{aligned}
\min_{u} \quad & J_n(x_0,u) = \sum_{k=0}^{n-1} c(x(k),u(k)) \\
\text{s.t.} \quad & x_{k+1} = f(x(k),u(k)) \\
& x(0) = x_0 \\
& u(k) \in U \, \forall k \in [0,n-1],\; u \in U \subseteq \mathbf{R}^{n_u} \\
& x(k) \in X \, \forall k \in [0,n],\; x \in X \subseteq \mathbf{R}^{n_x}
\end{aligned}
$$

NLP

$$
\begin{aligned}
\min_{w} \quad & \varphi(F(w,x_0,t_k),w) \\
\text{s.t.} \quad & x_{k+1} = f(x(k),u(k)) \\
& g_1(F(w,x_0,t_k),w) \leq 0 \\
& g_2(F(w,x_0,t_k),w) = 0
\end{aligned}
$$

In general, use the specified **model** to **predict** the motion of the system, **generate** a **locally optimal or feasible trajectory**, and **repeat** the procedure



Simon, D. (2014). Model Predictive Control in Flight Control Design: Stability and Reference Tracking (Doctoral dissertation, Linköping University Electronic Press).

Prediction Simulate states forward in time up to a defined horizon, **prediction horizon**, N_e from the **current state**

$$\underbrace{\mathbf{u}_k = \begin{bmatrix} u_{0|k} \\ u_{1|k} \\ \vdots \\ u_{N_e-1|k} \end{bmatrix}}_{\text{control inputs}} \xrightarrow{\text{estimate or calculate}} \underbrace{\mathbf{x}_k = \begin{bmatrix} x_{1|k} \\ x_{2|k} \\ \vdots \\ x_{N_e|k} \end{bmatrix}}_{\text{state vector}} \tag{1}$$

where $x_{i|k}$ denoted, current state x_k and $x_{i|k}$ denoted, i steps into the future, same for control as well

The **prediction** by **minimizing** a **stage cost**

$$J_{N_e}(x_k, \mathbf{u}_k) = \sum_{h=0}^{N_e} \left\| \mathbf{x}_{k+h} - \mathbf{x}_{k+h}^{ref} \right\|_Q^2 + \left\| \mathbf{u}_{k+h} - \mathbf{u}_{k+h}^{ref} \right\|_R^2$$

This can be solved numerically to estimate optimal $\mathbf{u}_k^*$

$$\mathbf{u}_k^* = \min_{\mathbf{u}} \quad J_{N_e}(x_k, \mathbf{u}_k), \quad Q \in \mathbb{R}^{n_x \times n_x} \succeq 0, \quad R \in \mathbb{R}^{n_u \times n_u} \succ 0$$

$$\begin{aligned} \text{s.t.} \quad & g_1(\mathbf{u}) = 0, \quad g_2(\mathbf{u}) \leq 0 \\ & p^{lower} \leq \mathbf{x}_{k+h} \leq p^{upper} \quad \forall 0 \leq h \leq N_e \\ & u^{lower} \leq \mathbf{u}_{k+h} \leq u^{upper} \quad \forall 0 \leq h \leq N_e - 1, \end{aligned}$$

Apply the **first** element of $\mathbf{u}_k^*$ on the **system** and **repeat** the optimization

- The **model** can be defined in **various ways**, multivariable, linear, or nonlinear, deterministic, stochastic or fuzzy
- Can handle **different types of constraints**, e.g., linear, quadratic, and nonlinear
- **Near-optimal** control inputs
- , however, requires **online optimization** that may be costly

- If the plant **model** is linear, the model's **state depends** linearly on **control** inputs, i.e., $x_{k+1} = f(x_k, u_k)$. Hence, **cost**, in general, is quadratic in u_k subject to linear **constraints**. Such problems can be formulated as a convex quadratic program and **guaranteed** to have **global optimal solution** all the time.

$$\min_{\mathbf{u}} \mathbf{u}^\top R \mathbf{u} + 2r^\top \mathbf{u} \quad s.t \; A\mathbf{u} \leq b$$

$$\underset{x}{\text{minimize}} \quad f(x)$$

where $f(x) = \frac{1}{2}x^\top Qx + b^\top x + c$, where $c \in \mathbb{R}, x \in \mathbb{R}^2$, and Q is 2×2 matrix. First order necessary condition $\Delta f(x) = 0$

$$\begin{aligned} df &= \frac{1}{2}x^\top Q^\top dx + \frac{1}{2}x^\top Q dx + b^\top dx \\ &= \underbrace{\left(x^\top \frac{Q^\top + Q}{2} + b^\top\right)}_{d\dot{f}(x) = \Delta f(x)} dx \end{aligned} \tag{2}$$

Since $Q^\top = Q$, $\Delta f(x) = Qx + b$. Hence, the critical point: $Qx = -b$
Second order necessary condition $\Delta^2 f(x) = Q$. It can be either **minimum**, **maximum**, saddle point, or **singular** point, i.e., **at least one eigenvalue becomes zero**.

- If the plant **model** is nonlinear, the model's **state depends** non-linearly on **control** inputs, i.e., $x_{k+1} = Ax_k + Bu_k$. Hence, **cost**, in general, is nonconvex in u_k subject to convex and nonconvex **constraints**. Such problems are formulated as a nonlinear program and **does not guarantee** to have **global optimal solution** all the time. Therefore, the solution can have local minima, locally optimal, and may not be solved efficiently or reliably

$$\min_{\mathbf{u}} J(x_k, \mathbf{u}) \quad s.t\; g(x_k, \mathbf{u}) \leq 0$$

Model Predictive Control: Prediction model

- **Discrete-time** necessary to have sampling interval δ, piecewise optimization is carried out
- **Continuous time** not necessary to have sampling interval δ, nor piece wise optimization is carried out. Can be linearized, good for nonlinear continuous-time systems

- **Hard** constraints are **satisfied** all the time, it is **not possible** to satisfy, the **problem** is **infeasible**
  - Box constraints

$$
\begin{aligned}
p^{lower} \leq \mathbf{x}_{k+h} \leq p^{upper} \quad & \forall 0 \leq h \leq N_e \\
u^{lower} \leq \mathbf{u}_{k+h} \leq u^{upper} \quad & \forall 0 \leq h \leq N_e - 1,
\end{aligned} \tag{3}
$$

  - System dynamics constraints

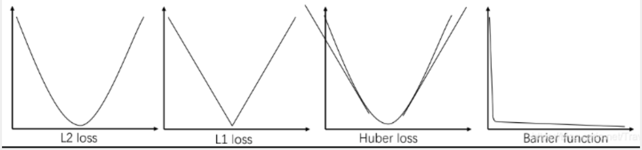
$$
g_1(\mathbf{w}) = \begin{bmatrix} \bar{\mathbf{x}}_k - \mathbf{x}_k \\ \vdots \end{bmatrix}. \tag{4}
$$

- **Soft** constraints may be **violated** to **avoid infeasibility**
  Consider the following **hard-constraints** optimization problem

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{s.t.} \quad & g_1(x) = c, \quad g_2(x) \leq d \end{aligned}$$

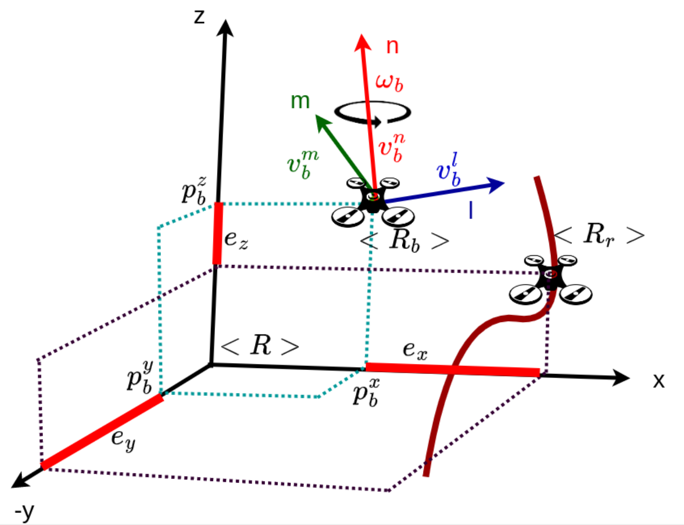
  Can be **converted** into a **soft constraints optimization problem**

$$\min_{x} \quad f(x) + \lambda_1 g_1(x) + \lambda_2 g_2(x)$$



  by using penalty terms or loss functions

Reference trajectory tracking



Simplified Motion Model

The system states $\mathbf{x}_k = [p_k^x, p_k^y, p_k^z, \alpha_k^z]^T \in \mathbb{R}^{n_x}$ and control inputs $\mathbf{u}_k = [v_k^x, v_k^y, v_k^z, \omega_k^z]^T \in \mathbb{R}^{n_u}$, where p_k^i and $v_k^i, i \in \{x, y, z\}$ denotes the quadrotor center position(m) and velocity (m/s) in each direction, i.e., x,y,z, at time t = k in the world coordinate frame; α_k^z and ω_k^z denote the yaw angle (rad) and yaw rate (rad/s) around the z-axis, respectively.
The simplified motion model is expressed by $\dot{\mathbf{x}}_k = \mathbf{f}_c(\mathbf{x}_k, \mathbf{u}_k)$

$$\dot{\mathbf{x}}_k = \begin{bmatrix} \dot{p}_k^x \\ \dot{p}_k^y \\ \dot{p}_k^z \\ \dot{\alpha}_k^z \end{bmatrix} = \begin{bmatrix} v_k^x cos(\alpha_k^z) - v_k^y sin(\alpha_k^z) \\ v_k^x sin(\alpha_k^z) + v_k^y cos(\alpha_k^z) \\ v_k^z \\ \omega_k^z \end{bmatrix}, \tag{5}$$

where $\mathbf{f}_c(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ and $n_x = n_u = 4$

Simplified Motion Model

Forward Euler discretization, $\mathbf{x}_{k+1} = \mathbf{f}_d(\mathbf{x}_k, \mathbf{u}_k)$ is introduced for a given sampling period in seconds, $\delta \in \mathbb{R} > 0$, e.g., $\delta = 0.1s$

$$\mathbf{x}_{k+1} = \begin{bmatrix} p_k^x \\ p_k^y \\ p_k^z \\ \alpha_k^z \end{bmatrix} + \delta \begin{bmatrix} v_k^x cos(\alpha_k^z) - v_k^y sin(\alpha_k^z) \\ v_k^x sin(\alpha_k^z) + v_k^y cos(\alpha_k^z) \\ v_k^z \\ \omega_k^z \end{bmatrix}, \tag{6}$$

where $\mathbf{f}_d(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_x}$.

With Multiple Shooting

$$
\begin{aligned}
J_{N_e}(\mathbf{x}_k, \mathbf{u}_k) &= \sum_{h=0}^{N_e} \left\| \mathbf{x}_{k+h} - \mathbf{x}_{k+h}^{ref} \right\|_Q^2 + \left\| \mathbf{u}_{k+h} - \mathbf{u}_{k+h}^{ref} \right\|_R^2 \\
\min_{\mathbf{w}} \quad & J_{N_e}(\mathbf{x}_k, \mathbf{u}_k), \quad Q \in \mathbb{R}^{n_x \times n_x} \succeq 0, \quad R \in \mathbb{R}^{n_u \times n_u} \succ 0 \\
\text{s.t.} \quad & g_1(\mathbf{w}) = 0, \quad g_2(\mathbf{w}) \leq 0 \\
& p^{lower} \leq \mathbf{x}_{k+h} \leq p^{upper} \quad \forall 0 \leq h \leq N_e \\
& u^{lower} \leq \mathbf{u}_{k+h} \leq u^{upper} \quad \forall 0 \leq h \leq N_e - 1,
\end{aligned}
\tag{7}
$$

where $\mathbf{w} = [\mathbf{u}_k, \ldots, \mathbf{u}_{k+N_e-1}, \mathbf{x}_k, \ldots, \mathbf{x}_{k+N_e}]$ denotes the decision variables set to be minimized.

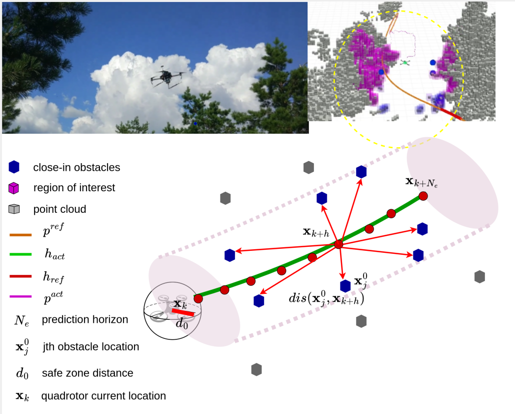
Notations u^{lower} and u^{upper} define the minimum and maximum linear and angular velocities allowed

Term $g_1(\mathbf{w})$ depicts the constraints that system dynamics imposes as follows:

$$g_1(\mathbf{w}) = \begin{bmatrix} \bar{\mathbf{x}}_k - \mathbf{x}_k \\ \vdots \\ f_d(\mathbf{x}_{k+h}, \mathbf{u}_{k+h}) - \mathbf{x}_{k+h+1} \\ \vdots \\ f_d(\mathbf{x}_{k+N_e-1}, \mathbf{u}_{k+N_e-1}) - \mathbf{x}_{k+N_e} \end{bmatrix}. \tag{8}$$

With Multiple Shooting

Reconstructing obstacle constraints in each iteration is necessary to **incorporate the dynamic environment changes** into the trajectory tracker



With Multiple Shooting

Term $g_2(\mathbf{w})$ describes the constraints imposed by obstacles.

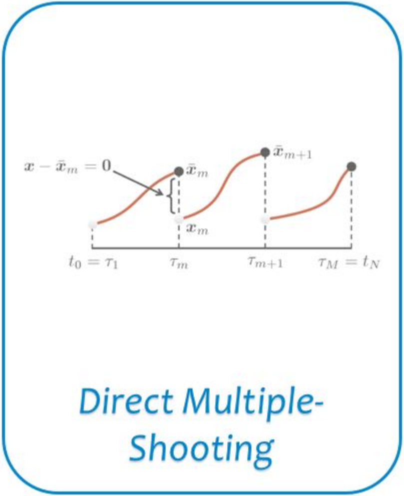
$$g_2(\mathbf{w}) = \begin{bmatrix} dis(\mathbf{x}_j^o, \mathbf{x}_k) \\ \vdots \\ dis(\mathbf{x}_j^o, \mathbf{x}_{k+h}) \\ \vdots \\ dis(\mathbf{x}_j^o, \mathbf{x}_{k+N_e}) \end{bmatrix}, \; j = 1, \ldots, N_o, \tag{9}$$

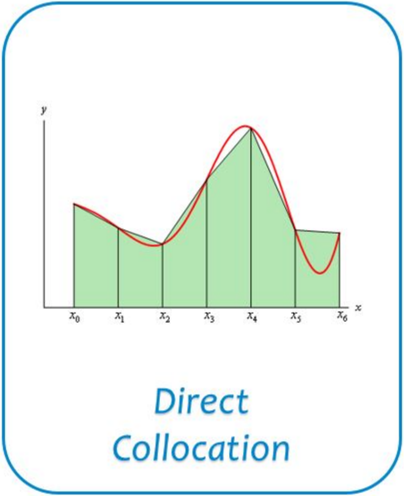
where $\bar{\mathbf{x}}_k$ is the initial position and N_o is the number of obstacles, and $dis(\mathbf{x}_j^o, \mathbf{x}_{k+h})$ can be calculated as follows:

$$-\sqrt{(x_j^o - x_{k+h})^2 + (y_j^o - y_{k+h})^2 + (z_j^o - z_{k+h})^2} + d^o$$

where d^o is the safe zone distance between the robot and close-in obstacles

Multiple shooting vs Direct Collocation



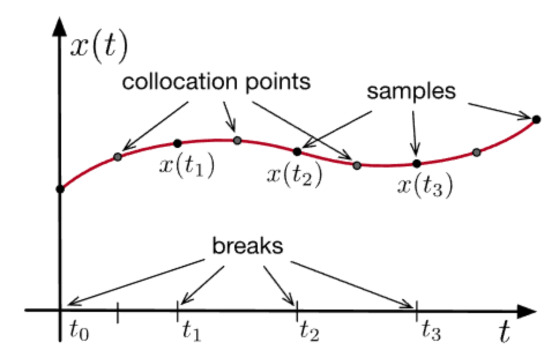


http://www.ee.ic.ac.uk/ICLOCS/Overview.html

With Direct Collocation (DC)

- **Multiple shooting**: **nonlinearity** with a sparsity structure to reduce the nonlinearity
- **Direct collocation**: add more degrees of freedom. Thus, exploits even more, but computation power increases dramatically
- **Collocation points** with respect to a **chosen polynomial**: Lagrangian 3rd order (N_d) polynomial, B-spline or Bézier
- **Fixed time interval** in multiple-shooting, but in DC, it has more **freedom** to determine how should **define points between two consecutive time interval**

- Kept the same discretization as in the multiple-shooting, i.e., $u(t) = u_k$, for $t \in [t_k, t_{k+1}], \quad k = 0, .., N_e - 1$, where N_e is the prediction horizon length

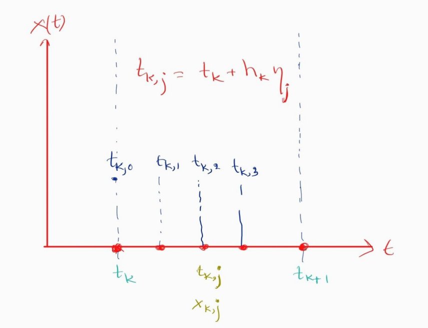


https://underactuated.csail.mit.edu/trajopt.html

- Consecutive time interval (t_k and t_{k+1}) is divided into small sub-intervals

$$t_{k,j} := t_k + h_k \eta_j, \quad k = 0, \dots, N_e - 1,\ j = 0, \dots, N_d$$

where Legendre points of order $N_d = 3$
$\eta = [0, 0.112, 0.500, 0.888]$ and $h_k = t_{k+1} - t_k$ and $x_{k,j}$ denote the states at $t_{k,j}$



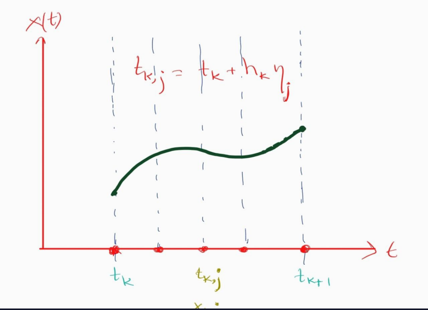
With Direct Collocation (DC)

- In each control interval, the Langrangian polynomial is defined as

$$L_j(\eta) = \prod_{r=0, r\neq j}^{N_d} \frac{\eta - \eta_r}{\eta_j - \eta_r}$$

with property

$$L_j(\eta) = \begin{cases} 1, & \text{if j =r} \\ 0, & \text{otherwise} \end{cases}$$



- State trajectory can be approximated using these basis functions

$$\bar{x}_k(t) = \Sigma_{r=0}^{N_d} L_r\left(\frac{t - t_k}{h_k}\right)x_{k,r}$$

Also, state at the end of the control interval

$$\bar{x}_{k+1}(t) = \Sigma_{r=0}^{N_d} L_r\left(1\right)x_{k,r}$$

And state time derivative at each collocation point except η_0

$$\dot{\bar{x}}_k(t) = \frac{1}{h_k}\Sigma_{r=0}^{N_d} \dot{L}_r\left(\eta_j\right)x_{k,r} := \frac{1}{h_k}\Sigma_{r=0}^{N_d} C_{r,j}x_{k,r}$$

- Hence, these collocation equations that necessary to satisfy every state at every collocation point

$$h_k f_c(x_{k,j}, u_k) - \Sigma_{r=0}^{N_d} C_{r,j} x_{k,r} = 0, \quad k = 0, ..., N_e - 1, \quad j = 0, ..., N_d$$

And the approximation of the end state

$$x_{k+1}(t) - \Sigma_{r=0}^{N_d} L_r(1) x_{k,r} = 0 \quad k = 0, ..., N_e - 1$$