Linear Quadratic Gaussian

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Introduction

- Definition of LQG.
- Advantages of LQG control.

Definition of LQG

- **LQG** is a control strategy that uses feedback to adjust system behavior.
- It combines the **Kalman filter** and **LQR** to generate optimal control signals.

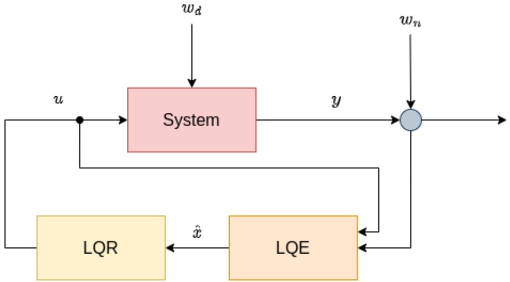

Figure: LQG block diagram

Advantages of LQG Control

- **Optimal Control:** It provides optimal control signals that minimize a quadratic cost function. (LQR)
- **Robustness:** It can handle systems with measurement noise and disturbances. (Kalman Filter)
- **Adaptability:** It can be easily extended to handle nonlinear systems.

Mathematical Foundations

- Kalman filter.
- Linear quadratic regulator.
- Linear quadratic Gaussian

Kalman Filter: is a mathematical algorithm that uses a series of measurements to estimate the state of a system. It provides an optimal solution for the linear Gaussian problem, where the system is linear and noise is Gaussian.

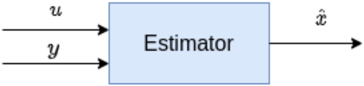

Figure: State Estimator

Kalman Filter

Given the motion model, measurement model, estimated error, and estimated measurement by:

$$\dot{x} = Ax + Bu + w_d \tag{1}$$

$$y = Cx + w_n \tag{2}$$

$$\varepsilon = x - \hat{x} \tag{3}$$

$$\hat{y} = C\hat{x} \tag{4}$$

we can express the derivative of the estimated state as:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + K_f(y - \hat{y} + w_n) \\ \dot{\hat{x}} &= A\hat{x} + Bu + K_f y - K_f C\hat{x} + K_f w_n \\ \dot{\hat{x}} &= (A - K_f C)\hat{x} + \begin{bmatrix} B & K_f \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} + K_f w_n \end{aligned} \tag{5}$$

Kalman Filter

from (3)

$$
\begin{aligned}
\dot{\varepsilon} &= \dot{x} - \dot{\hat{x}} \\
\dot{\varepsilon} &= Ax + Bu + w_d - A\hat{x} - Bu - K_f Cx + K_f C\hat{x} - K_f w_n \\
\dot{\varepsilon} &= A(x - \hat{x}) + K_f C(\hat{x} - x) - K_f w_n + w_d \\
\dot{\varepsilon} &= A(x - \hat{x}) - K_f C(x - \hat{x}) - K_f w_n + w_d \\
\dot{\varepsilon} &= (A - K_f C)\varepsilon - K_f w_n + w_d
\end{aligned}
\tag{6}
$$

Linear Quadratic Regulator

LQR: is a mathematical technique used to find the control signals that minimize a quadratic cost function. The cost function penalizes deviations from the desired system behavior and control effort.

$$J = \int_0^\infty \mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{u}^T \mathbf{R} \mathbf{u} \, dt \tag{7}$$

where **Q** and **R** are weighting matrices.

The optimal control law that minimizes the cost function is given by:

$$\mathbf{u}^* = -\mathbf{K}\mathbf{x} \tag{8}$$

where **K** is the feedback gain matrix.

The optimal feedback gain matrix can be obtained by solving the algebraic Riccati equation:

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T + \mathbf{Q} = 0 \tag{9}$$

where **P** is a positive semidefinite matrix, called the solution to the Riccati equation.

Linear Quadratic Gaussian

Motion model:

$$\dot{x} = Ax + Bu + w_d \tag{10}$$

Measurement model:

$$y = Cx + w_n \tag{11}$$

Estimated error:

$$\varepsilon = x - \hat{x} \tag{12}$$

Control input:

$$u = -k_r x \tag{13}$$

Linear Quadratic Gaussian

By substituting (13) in (10)

$$\dot{x} = Ax - BK_r\hat{x} + w_d \tag{14}$$

let

$$\hat{x} = x - (x - \hat{x}) \tag{15}$$

By substituting (15) in (14)

$$\begin{aligned} \dot{x} &= Ax - BK_r x + BK_r(x - \hat{x}) + w_d \\ \dot{x} &= (A - BK_r)x + BK_r\varepsilon + w_d \end{aligned} \tag{16}$$

Linear Quadratic Gaussian

Recall eq.(6)

$$\dot{\varepsilon} = (A - K_f C)\varepsilon + w_d - K_f w_n \tag{17}$$

$$\begin{bmatrix} \dot{x} \\ \dot{\varepsilon} \end{bmatrix} = \begin{bmatrix} A - BK_r & BK_r \\ 0 & A - K_f C \end{bmatrix} \begin{bmatrix} x \\ \varepsilon \end{bmatrix} + \begin{bmatrix} I & 0 \\ I & -K_f \end{bmatrix} \begin{bmatrix} w_d \\ w_n \end{bmatrix} \tag{18}$$

Eq.(18) represents the **separation principle** in which **LQR** and **LQE** are independent; however, by obtaining their eigenvalues separately they can maintain a **stable** system.

Table: Comparison of LQR, LQG, and MPC

Feature	LQR	LQG	MPC
Control Objective	Quadratic cost	Quadratic cost	Finite-horizon cost
Observation Method	Open-loop	Closed-loop	Closed-loop
Controller Type	State-feedback	Optimal state-feedback	Optimal feedback and feedforward
Model Dependency	Accurate model	Accurate model	Model predictive control
Robustness	Not robust	Robust	Robust
Computational Costs	Low	High	High
Online Adaptability	No	Yes	Yes
Nonlinear Systems	Not applicable	Not applicable	Applicable
Gaussian Noise Model	Yes	Yes	No

References I

Steve Brunton
Control Bootcamp: Linear Quadratic Gaussian (LQG)
https://www.youtube.com/watch?v=H4_hFazBGxU&list=PLMrJAkhIeNNR20Mz-VpzgfQs5zrYi085m&index=23.

Thanks!