Motion Planning for Autonomous Vehicles

Hamiltonian (Optimal control theory)

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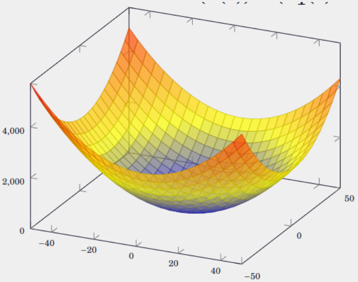
March 4, 2023



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- Boundary conditions for optimal control: with the free final time and the final state specified, free, lies on the moving point $x_f = \theta(t_f)$, or lies on a moving surface $m(x(t))$)

Find the point on the line $y_1 + y_2 = 5$ that is nearest the origin.

Elimination method (direct method)

- $$\begin{aligned} \underset{y_1, y_2 \in \mathbb{R}}{\text{minimize}} \quad & f(y_1, y_2) = y_1^2 + y_2^2, \quad \text{square distance} \\ \text{subject to} \quad & y_1 + y_2 = 5 \end{aligned}$$

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- The differential

$$df(y_1, y_2) = \left(\frac{\partial f(\cdot)}{\partial y_1}\right)\Delta y_1 + \left(\frac{\partial f(\cdot)}{\partial y_2}\right)\Delta y_2 \tag{1}$$

where $f(\cdot) = f(y_1, y_2)$.

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- If $f(y_1^*, y_2^*)$ is the extreme point,

$$df(y_1^*, y_2^*) = \left(\frac{\partial f(y_1^*, y_2^*))}{\partial y_1}\right)\Delta y_1 + \left(\frac{\partial f(y_1^*, y_2^*))}{\partial y_2}\right)\Delta y_2 \tag{2}$$

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Elimination method (direct method)

- If and only if y_1 and y_2 are **independent** Δy_1 and Δy_2 can be **selected arbitrarily**.
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- However, in **this example**, y_1 and y_2 are **dependent**.
- Hence, considering $f(y_1, y_2)$ only function of y_2

$$\begin{aligned} df(y_2^*) = \left(-10 + 4y_2^*\right)\Delta y_2 = 0 \\ \Rightarrow y_2^* = 2.5,\ y_1^* = 2.5 \end{aligned} \tag{3}$$

- $$f_a(y_1, y_2, p) = y_1^2 + y_2^2 + p(y_1 + y_2 - 5), \tag{4}$$

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$$df(y_1, y_2, p) = \Big(\frac{\partial f(\cdot)}{\partial y_1}\Big)\Delta y_1 + \Big(\frac{\partial f(\cdot)}{\partial y_2}\Big)\Delta y_2 + \Big((y_1 + y_2 - 5)\Big)\Delta p \tag{5}$$

  where $f(\cdot) = f(y_1, y_2, p)$. If $f(y_1^*, y_2^*, p)$ is the extreme point

- $$df(y_1^*, y_2^*, p) = \left(2y_1^* + p\right)\Delta y_1 + \left(2y_2^* + p\right)\Delta y_2 + \left((y_1^* + y_2^* - 5)\right)\Delta p = 0 \quad (6)$$

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- Sometime **Lagrange multiplier** is defined in this form as well: $f(x, y, \ldots) - pg(x, y, \ldots)$

The Lagrange multiplier method: general formulation

- Consider $f(y_1, y_2, ..., y_{n+m})$, **subject to n constraints**:

$$
\begin{aligned}
a_1 \cdot \begin{bmatrix} y_1 & y_2 & \dots & y_{n+m} \end{bmatrix} = 0 \\
\vdots \\
a_n \cdot \begin{bmatrix} y_1 & y_2 & \dots & y_{n+m} \end{bmatrix} = 0
\end{aligned}
\tag{7}
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- Hence, there are (m+n) - n =**m number of independent variables**.

$$\begin{aligned} & f_a(y_1, y_2, ..., y_{n+m}, p_1, ..., p_n) \\ = & f_a(y_1, y_2, ..., y_{n+m}) + p_1(a_1 \cdot \begin{bmatrix} y_1 & y_2 & ... & y_{n+m} \end{bmatrix}) \\ & + ... + p_n(a_n \cdot \begin{bmatrix} y_1 & y_2 & ... & y_{n+m} \end{bmatrix}) \end{aligned} \quad (8)$$

The Lagrange multiplier method: general formulation

By taking differential

$$\underbrace{\frac{\partial f_a(\cdot)}{\partial y_1}\Delta y_1 + \ldots + \frac{\partial f_a(\cdot)}{\partial y_{n+m}}\Delta y_{n+m}}_{\text{n+m number of equations}} + \underbrace{\frac{\partial f_a(\cdot)}{\partial p_1}\Delta p_1 + \ldots + \frac{\partial f_a(\cdot)}{\partial p_n}\Delta p_n}_{\text{n number of equations}} \tag{9}$$

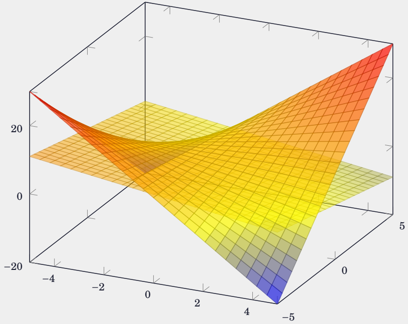
$$\Rightarrow \frac{\partial f_a(\cdot)}{\partial y_1}\Delta y_1 + \ldots + \frac{\partial f_a(\cdot)}{\partial y_{n+m}}\Delta y_{n+m} + a_1 \Delta p_1 + \ldots + a_n \Delta p_n$$

where $\forall\, a_i \in \mathbb{R}^{m+n} = 0, i \in [1,..,n]$. **Each p_i is selected such that corresponding** Δy_i is zero. The **coefficients** of the remaining m **independent** variables $\Delta_j, j \in [1,m]$ **must vanish** to obtain $df_a(\cdot) = 0$.

Consider a surface and plane in the $\mathbb{R}^3$ are defined in the following way.

$$\begin{aligned} y_3 &= y_1 y_2 + 5 \\ y_1 + y_2 + y_3 &= 1 \end{aligned} \tag{10}$$

Find the closest distance from the origin, such that the plane and surface are intercepted by each other.



The Lagrange multiplier method

- $$f_a(y_1, y_2, y_3, p_1, p_2) = y_1^2 + y_2^2 + y_3^2 + p_1(y_1 y_2 + 5 - y_3) + p_2(y_1 + y_2 + y_3 - 1) \tag{11}$$

- $$f_a(y_1, y_2, y_3, p_1, p_2) = y_1^2 + y_2^2 + y_3^2 + p_1(y_1 y_2 + 5 - y_3) + p_2(y_1 + y_2 + y_3 - 1) \quad (11)$$

- Using the Lagrange multiplier method eq.9, the optimal values can be found by solving follows equations:

$$\begin{aligned} y_1^* + y_2^* + y_2^* - 1 &= 0 \\ y_1^* \cdot y_2^* + 5 - y_3^* &= 0 \\ 2y_1^* + p_1^* y_2^* + p_2^* &= 0 \\ 2y_2^* + p_1^* y_1^* + p_2^* &= 0 \\ 2y_3^* - p_1^* + p_2^* &= 0 \\ \Rightarrow y_1^*, y_2^*, y_3^* &= \begin{cases} (2, -2, 1) \\ (-2, 2, 1) \end{cases} \end{aligned} \quad (12)$$

$f_a(y_1^*, y_2^*, y_3^*) = 9$ and distance = $\sqrt{{y_1^*}^2 + {y_2^*}^2 + {y_3^*}^2}$ = 3

Constrained Minimization of Functional: Point constraints

Necessary conditions for a function w^* to be an extremal for a functional of the form

$$J(w) = \int_{t_0}^{t_f} g(w(t), \dot{w}(t), t) dt, \tag{13}$$

where w is an $(n+m)$ vector of functions. If there are n number of constraints to be satisfied:

$$f_i(w(t), t) = 0, i = 0, .., n \tag{14}$$

are called **point constraints**.

- $$\begin{aligned} J_a(w,p) &= \int_{t_0}^{t_f} \Big(g(w(t),\dot{w}(t),t) + p_1(t)(f_1(w(t),t)) + \ldots + p_n(t)(f_n(w(t),t)) \\ &= \int_{t_0}^{t_f} \Big(g(w(t),\dot{w}(t),t) + P^\top(t) f(w(t),t) \Big) dt \end{aligned}$$

  where $P(t) \in n \times 1$ and $f(w(t),t) \in 1 \times n$ vectors.

- $$J_a(w,p) = \int_{t_0}^{t_f} \Big(g(w(t),\dot{w}(t),t) + p_1(t)(f_1(w(t),t)) + \ldots + p_n(t)(f_n(w(t),t))$$
  $$= \int_{t_0}^{t_f} \Big(g(w(t),\dot{w}(t),t) + P^\top(t) f(w(t),t)\Big) dt$$
  where $P(t) \in n \times 1$ and $f(w(t),t) \in 1 \times n$ vectors.
- By taking differential
  $$\delta J_a(w,\delta w,P,\delta P) = \int_{t_0}^{t_f} \Big\{\Big(\frac{\partial g^\top(\cdot)}{\delta w} + P^\top(t)\Big(\frac{\partial f(\cdot)}{\partial w}\Big)\Big)\delta w(t) + \Big(\frac{\partial g^\top(\cdot)}{\delta \dot{w}}\Big)\delta \dot{w}(t) + \Big(\frac{\partial f^\top(\cdot)}{\partial P}\Big)\delta P(t)\Big\} dt \quad (15)$$
  where $\frac{\partial f(\cdot)}{\partial w} = \begin{bmatrix} \frac{\partial f_1(\cdot)}{\partial w_1} & \cdots & \frac{\partial f_1(\cdot)}{\partial w_{n+m}} \\ \vdots & & \\ \frac{\partial f_n(\cdot)}{\partial w_1} & \cdots & \frac{\partial f_n(\cdot)}{\partial w_{n+m}} \end{bmatrix} \in \mathbb{R}^{n\times(n+m)}$, $g(\cdot) = g(w(t),\dot{w}(t),t)$, and $f(\cdot) = f(w(t),t)$.

- To deduce $\delta\dot{w}$, using integration by parts, eq.(15) can be rewritten as follows:

$$\delta J_a(w,\delta w,P,\delta P) = \int_{t_0}^{t_f} \Big\{\Big(\frac{\partial g^\top(\cdot)}{\delta w} + P^\top(t)\Big(\frac{\partial f(\cdot)}{\partial w}\Big) - \frac{d}{dt}\Big(\frac{\partial g^\top(\cdot)}{\delta \dot{w}}\Big)\Big)\delta w(t) + \Big(f^\top(\cdot)\Big)\delta P(t)\Big\} dt \tag{16}$$

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- For an extremum, $\delta J_a(w,\delta w,P,\delta P) = 0$ and the point constraints must be satisfied, i.e., $f(w^*(t),t) = 0,[t_0,t_f]$. Therefore,

$$\frac{\partial g^\top(\cdot)}{\delta w} + P^\top(t)\left(\frac{\partial f(\cdot)}{\partial w}\right) - \frac{d}{dt}\left(\frac{\partial g^\top(\cdot)}{\delta \dot{w}}\right) = 0 \tag{17}$$

at $w(t) \Rightarrow w^*(t),[t_0,t_f]$.

By considering $g_a(w(t), \dot{w}(t), P(t), t) = g_a(\cdot) := g(\cdot) + P^\top(t)(f(\cdot))$, eq.(17) can be written as Eular equation form

$$\frac{\partial g_a^\top(\cdot)}{\delta w} - \frac{d}{dt}\left(\frac{\partial g_a^\top(\cdot)}{\delta \dot{w}}\right) = 0 \tag{18}$$

at $w(t) \Rightarrow w^*(t), [t_0, t_f]$.

Obtain the necessary condition that must be satisfied by the curve of the **smallest length which lies on the surface** $w_1^2(t) + w_2^2(t) + t^2 = r^2 \;\; \forall\, t \in [t_0, t_f]$, where initial and final points are specified, w_0, t_0 and w_f, t_f, respectively, by minimizing the following objective:

$$J(w) = \int_{t_0}^{t_f} \sqrt{1 + \dot{w}_1^2(t + \dot{w}_2^2(t))} dt \tag{19}$$

- The argumeted function $g_a(w(t), \dot{w}(t), P(t), t) = g_a(\cdot) := \sqrt{1 + \dot{w}_1^2(t + \dot{w}_2^2(t))} + P(t)(w_1^2(t) + w_2^2(t) + t^2 - r^2)$. To find an extremal, need to solve the eq.(18) at $w(t) \Rightarrow w^*(t), [t_0, t_f]$.

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$$\begin{aligned} \frac{\partial g_a^\top(\cdot)}{\delta w} - \frac{d}{dt}\Big(\frac{\partial g_a^\top(\cdot)}{\delta \dot{w}}\Big) = 0 \\ \Rightarrow 2w_1^*(t)P^*(t) - \frac{d}{dt}\frac{\dot{w}_1^*(t)}{\sqrt{1 + \dot{w^*}_1^2(t) + \dot{w^*}_2^2(t)}} = 0 \\ \Rightarrow 2w_2^*(t)P^*(t) - \frac{d}{dt}\frac{\dot{w}_2^*(t)}{\sqrt{1 + \dot{w^*}_1^2(t) + \dot{w^*}_2^2(t)}} = 0 \end{aligned} \tag{20}$$

The Lagrange multiplier method: differential equation constraints

- If $w_1(t)$ and $w_2(t)$ are related as $\dot{w}_1(t) = w_2(t)$, where initial and final points are specified, w_0, t_0 and w_f, t_f, respectively, by minimizing the following objective:

$$J(w) = \frac{1}{2}\int_{t_0}^{t_f} w_1^2(t) + w_2^2(t) dt \tag{21}$$

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- The argumeted function becomes $g_a(w(t), \dot{w}(t), P(t), t) = g_a(\cdot) := \frac{1}{2}(w_1^2(t) + w_2^2(t)) + P(t)(\dot{w}_1(t) - w_2(t))$. To find the **necessary conditions** at an extremal, need to solve the eq.(18) at $w(t) \Rightarrow w^*(t), [t_0, t_f]$.

$$\frac{\partial g_a^\top(\cdot)}{\delta w} - \frac{d}{dt}\Big(\frac{\partial g_a^\top(\cdot)}{\delta \dot{w}}\Big) = 0$$
$$\Rightarrow w_1^*(t) - \dot{P}^*(t) = 0, \quad \Rightarrow w_2^*(t) - P^*(t) = 0, \quad \Rightarrow \dot{w}_1^*(t) = w_2^*(t) \tag{22}$$

- Suppose that the system

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) - x_1(t)\\ \dot{x}_2(t) &= -2x_1(t) - 3x_2(t) + u(t)\end{aligned} \tag{23}$$

is to control minimizing the following objective

$$J(x,u) = \int_{t_0}^{t_f} \frac{1}{2}\Big(x_1^2(t) + x_2^2(t) + u^2(t)\Big)dt \tag{24}$$

Find the necessary conditions for obtaining the optimal control.

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  Find the necessary conditions for obtaining the optimal control.

- Consider the system state and control input are denoted as $x = [x_1\ x_2] \in \mathbb{R}^2$ and $u \in \mathbb{R}$, respectively, where $w = [x; u] \in \mathbb{R}^3$. Therefore, the constraints set have the following form:

$$\begin{aligned} 0 &= w_2(t) - w_1(t) - \dot{w}_1(t) \\ 0 &= -2w_1(t) - 3w_2(t) + w_3(t) - \dot{w}_2(t) \end{aligned} \tag{25}$$

- The argumeted function

$$g_a(w(t),\dot{w}(t),P(t),t) = g_a(\cdot) := \frac{1}{2}(w_1^2(t) + w_2^2(t) + w_3^2(t))$$
$$+ p_1(t)(w_2(t) - w_1(t) - \dot{w}_1(t)) + p_2(t)(-2w_1(t) - 3w_2(t) + w_3(t) - \dot{w}_2(t)) \tag{26}$$

- The argumeted function

$$g_a(w(t),\dot{w}(t),P(t),t) = g_a(\cdot) := \frac{1}{2}(w_1^2(t)+w_2^2(t)+w_3^2(t))$$
$$+p_1(t)(w_2(t)-w_1(t)-\dot{w}_1(t))+p_2(t)(-2w_1(t)-3w_2(t)+w_3(t)-\dot{w}_2(t)) \tag{26}$$

- To find an extremal, need to solve the eq.(18) at $w(t) \Rightarrow w^*(t), [t_0, t_f]$.

$$\frac{\partial g_a^\top(\cdot)}{\delta w} - \frac{d}{dt}\left(\frac{\partial g_a^\top(\cdot)}{\delta \dot{w}}\right) = 0$$
$$\Rightarrow \dot{p}_1^*(t) = -w_1^*(t)+p_1^*(t)+2p_2^*(t), \quad \Rightarrow \dot{p}_2^*(t) = -w_2^*(t)-p_1^*(t)+3p_3^*(t)$$
$$\Rightarrow w_3^*(t)+p_2^*(t) = 0$$
$$\Rightarrow w_2^*(t)-w_1^*(t)-\dot{w}_1^*(t), \quad \Rightarrow -2w_1^*(t)-3w_2^*(t)+w_3^*(t)-\dot{w}_2^*(t) \tag{27}$$

Necessary condition for optimal control

- Consider that the system

$$\dot{x}(t) = f(x(t), u(t), t) \tag{28}$$

where $x(t) \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ to follow an **admissible trajectory** x^* that minimize the following objective function

$$J(u) = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), u(t), t) dt \tag{29}$$

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- The initial condition $x(t_0) = x_0$ is given. If $h(x(t_f), t_f)$ is a differentiable function
$$h(x(t_f), t_f) = \int_{t_0}^{t_f} \frac{dh(x(t), t)}{dt} dt + h(x(t_0), t_0) \tag{30}$$

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- With that, the objective function becomes

$$J(u) = \int_{t_0}^{t_f} \left(g(x(t), u(t), t) + \frac{dh(x(t), t)}{dt} \right) dt + h(x(t_0), t_0) \tag{31}$$

Since the initial condition is given

$$\begin{aligned} J(u) &= \int_{t_0}^{t_f} \left(g(x(t), u(t), t) + \frac{dh(x(t), t)}{dt} \right) dt \\ &= \int_{t_0}^{t_f} \left(g(x(t), u(t), t) + \left(\frac{\partial h(x(t), t)}{\partial x} \right)^\top \dot{x}(t) + \frac{\partial h(x(t), t)}{\partial t} \right) dt \end{aligned} \tag{32}$$

Necessary condition for optimal control

- In order to include differential equation constraints in the objective function

$$
\begin{aligned}
J(u) = \int_{t_0}^{t_f} \Big(g(x(t), u(t), t) + \left(\frac{\partial h(x(t), t)}{\partial x}\right)^\top \dot{x}(t) + \frac{\partial h(x(t), t)}{\partial t} \\
+ P(t)^\top \big(f(x(t), u(t), t) - \dot{x}(t) \big) \Big) dt
\end{aligned} \tag{33}
$$

where $P(t) = [p_1(t), \ldots, p_n(t)]^\top$ (Lagrange multipliers).

- In order to include differential equation constraints in the objective function

$$J(u) = \int_{t_0}^{t_f} \Big(g(x(t),u(t),t) + \Big(\frac{\partial h(x(t),t)}{\partial x}\Big)^\top \dot{x}(t) + \frac{\partial h(x(t),t)}{\partial t} + P(t)^\top \Big(f(x(t),u(t),t) - \dot{x}(t)\Big)\Big)dt \tag{33}$$

where $P(t) = [p_1(t), \ldots, p_n(t)]^\top$ (Lagrange multipliers).

- The eq.(33) can be written by considering $g_a(x(t),\dot{x}(t),u(t),P(t),t) = g(x(t),u(t),t) + \left(\frac{\partial h(x(t),t)}{\partial x}\right)^\top \dot{x}(t) + \frac{\partial h(x(t),t)}{\partial t} + P(t)^\top \Big(f(x(t),u(t),t) - \dot{x}(t)\Big)\Big)$

$$J(u) = \int_{t_0}^{t_f} \Big(g_a(x(t),\dot{x}(t),u(t),P(t),t)\Big)dt \tag{34}$$

- To obtain an optimal solution $\delta J(u^*) = 0$

$$\delta J(u^*) = \Big(\frac{\partial g_a(\cdot)}{\partial \dot{x}}\Big)^\top \delta x_f + \Big(g_a(\cdot) - \Big(\frac{\partial g_a(\cdot)}{\partial \dot{x}}\Big)^\top \dot{x}^*(t_f)\Big)\delta t_f$$
$$+ \int_{t_0}^{t_f} \Big(\Big(\Big(\frac{g_a(\cdot)}{\delta x}\Big)^\top - \frac{d}{dt}\frac{g_a(\cdot)}{\delta \dot{x}}\Big)^\top\Big)\delta x(t) + \Big(\frac{g_a(\cdot)}{\delta u}\Big)^\top \delta u + \Big(\frac{g_a(\cdot)}{\delta P}\Big)^\top \delta P(t)\Big) dt \tag{35}$$

where $g_a(\cdot) = g_a(x^*(t_f), \dot{x}^*(t_f), u^*(t_f), P^*(t_f), t_f)$ and
$\delta x(t_f) = \delta x_f - \dot{x}^*(t_f)\delta t_f$

Necessary condition for optimal control

- To obtain an optimal solution $\delta J(u^*) = 0$

$$\delta J(u^*) = \left(\frac{\partial g_a(\cdot)}{\partial \dot{x}}\right)^\top \delta x_f + \left(g_a(\cdot) - \left(\frac{\partial g_a(\cdot)}{\partial \dot{x}}\right)^\top \dot{x}^*(t_f)\right)\delta t_f$$
$$+ \int_{t_0}^{t_f} \left(\left(\left(\frac{g_a(\cdot)}{\delta x}\right)^\top - \frac{d}{dt}\frac{g_a(\cdot)}{\delta \dot{x}}\right)^\top\right)\delta x(t) + \left(\frac{g_a(\cdot)}{\delta u}\right)^\top \delta u + \left(\frac{g_a(\cdot)}{\delta P}\right)^\top \delta P(t)\Big)dt \tag{35}$$

  where $g_a(\cdot) = g_a(x^*(t_f), \dot{x}^*(t_f), u^*(t_f), P^*(t_f), t_f)$ and $\delta x(t_f) = \delta x_f - \dot{x}^*(t_f)\delta t_f$
- The **solution** to **this** is govern by Hamiltonian

$$H(x(t),u(t),P(t),t) := g(x(t),u(t),t) + P^{\top}(t)f(x(t),u(t),t) \tag{36}$$

Necessary conditions

$$\begin{aligned}
\dot{x}^*(t) &= \frac{H(\cdot)}{\partial P} \\
\dot{P}^*(t) = -\frac{H(\cdot)}{\partial x} &= -\left(\frac{\partial f(\cdot)}{\partial x}\right)^{\top} P^*(t) - \frac{\partial g(\cdot)}{\partial x} \\
0 = \frac{H(\cdot)}{\partial u} &= \left(\frac{\partial f(\cdot)}{\partial u}\right)^{\top} P^*(t) + \frac{\partial g(\cdot)}{\partial u} \\
\left(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\right)^{\top} \delta x_f + \left(H(\cdot) + \frac{\partial h(\cdot)}{\partial t}\right)\delta t_f &= 0
\end{aligned} \tag{37}$$

where $H(\cdot) = H(x^*(t),u^*(t),P^*(t),t)$ and $\forall\ t \in [t_0, t_f]$

Hamiltonian: necessary conditions

- system dynamics constraints

$$\dot{x}^*(t) = f(x^*(t), u^*(t), t) \tag{38}$$

- costate equations

$$P^*(t) = -\left(\frac{\partial f(\cdot)}{\partial x}\right)^\top P^*(t) - \frac{\partial g(\cdot)}{\partial x} \tag{39}$$

- $\delta u(t)$ is independent, hence corresponding coefficients must be zero

$$0 = \left(\frac{\partial f(\cdot)}{\partial u}\right)^\top P^*(t) + \frac{\partial g(\cdot)}{\partial u} \tag{40}$$

- if t_f and $x(t_f)$ are not fixed,

$$\left(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\right)^\top \delta x_f + \left(g(\cdot) + \frac{\partial h(\cdot)}{\partial t} + P^*(t_f)(f(\cdot))\right)\delta t_f = 0 \tag{41}$$

Optimal control

Consider that the system

$$\dot{x}(t) = f(x(t), u(t), t) \tag{42}$$

where $x(t) \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ to follow an admissible trajectory x^* that minimize the following objective function

$$J(u) = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), u(t), t)dt \tag{43}$$

The initial condition $x(t_0) = x_0$ is given.

Necessary conditions for optimal control

$$H(x(t),u(t),P(t),t) := g(x(t),u(t),t) + P^\top(t) f(x(t),u(t),t) \tag{44}$$

$$\begin{aligned}
\dot{x}^*(t) &= \frac{H(\cdot)}{\partial P} \\
\dot{P}^*(t) = -\frac{H(\cdot)}{\partial x} &= -\left(\frac{\partial f(\cdot)}{\partial x}\right)^\top P^*(t) - \frac{\partial g(\cdot)}{\partial x} \\
0 = \frac{H(\cdot)}{\partial u} &= \left(\frac{\partial f(\cdot)}{\partial u}\right)^\top P^*(t) + \frac{\partial g(\cdot)}{\partial u} \\
\left(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\right)^\top \delta x_f + \left(H(\cdot) + \frac{\partial h(\cdot)}{\partial t}\right)\delta t_f &= 0
\end{aligned} \tag{45}$$

where $H(\cdot) = H(x^*(t),u^*(t),P^*(t),t)$, $h(\cdot) = h(x^*(t),t)$, $g(\cdot) = g(x(t),u(t),t)$, and $\forall\, t \in [t_0,t_f]$

Boundary conditions for optimal control: with fixed final time

Term $x(t_f)$ can be either free, fixed or lie on a surface. However, t_f is fixed.

- Final state specified: $\delta x_f = 0$ and $\delta t_f = 0 \Rightarrow x^*(t_f) = x_f$
- Final state free: $\delta t_f = 0$ and δx_f is arbitrary

$$\begin{aligned} \left(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\right)^\top \delta x_f + \left(H(\cdot) + \frac{\partial h(\cdot)}{\partial t}\right)\delta t_f = 0 \\ \frac{\partial h(\cdot)}{\partial x} - P^*(t_f) = 0 \end{aligned} \tag{46}$$

Boundary conditions for optimal control: with fixed final time

- Final state free: $\delta t_f = 0$ and δx_f is dependant

$$\begin{aligned} \Big(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\Big)^\top \delta x_f + \Big(H(\cdot) + \frac{\partial h(\cdot)}{\partial t}\Big)\delta t_f = 0 \\ \Big(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\Big)^\top \delta x_f = 0 \end{aligned} \tag{47}$$

Consider the final state of a provided system that is required to lie on the circle $h(x(t)) = (x_1(t) - 3)^2 + (x_2(t) - 4)^2 - 4$.

Boundary conditions for optimal control: with fixed final time

- Admissible changes in $x(t_f)$ are tangent to the $h(x(t))$ at point $x^*(t_f), t_f$. The tangent lies normal to the gradient vector

$$\frac{h(x)(t)}{\partial x}\Big|_{x^*(t_f)} = \begin{bmatrix} 2(x_1^*(t_f) - 3) \\ 2(x_2^*(t_f) - 4) \end{bmatrix} \tag{48}$$

at point $(x^*(t_f), t_f)$.

Boundary conditions for optimal control: with fixed final time

- Admissible changes in $x(t_f)$ are tangent to the $h(x(t))$ at point $x^*(t_f), t_f$. The tangent lies normal to the gradient vector

$$\frac{h(x)(t)}{\partial x}|_{x^*(t_f)} = \begin{bmatrix} 2(x_1^*(t_f) - 3) \\ 2(x_2^*(t_f) - 4) \end{bmatrix} \tag{48}$$

  at point $(x^*(t_f), t_f)$.
- Term $\delta x(t_f)$ must be normal to the gradient, so that

$$\begin{aligned} \left(\frac{h(x)(t)}{\partial x}|_{x^*(t_f)}\right)^\top \delta x(t_f) = \begin{bmatrix} 2(x_1^*(t_f) - 3) \\ 2(x_2^*(t_f) - 4) \end{bmatrix}^\top \delta x(t_f) = 0 \\ \Rightarrow \delta x_2(t_f) = \frac{-(x_1^*(t_f) - 3)}{(x_2^*(t_f) - 4)} \delta x_1(t_f) \end{aligned} \tag{49}$$

Boundary conditions for optimal control: with fixed final time

Therefore, eq.47 becomes

$$
\begin{aligned}
\Big(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\Big)^{\top} \delta x_f &= 0 \\
\Big(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\Big)^{\top} \begin{bmatrix} 1 \\ \frac{-(x_1^*(t_f)-3)}{(x_2^*(t_f)-4)} \end{bmatrix} &= 0
\end{aligned} \tag{50}
$$

In this way, boundary conditions can be calculated. Moreover, final state $h(x(t_f))$ at t_f must satisfy the
$h(x^*(t_f)) = (x_1^*(t_f) - 3)^2 + (x_2^*(t_f) - 4)^2 - 4$

Boundary conditions for optimal control: with free final time

- The final state specified: $\delta x_f = 0$ and δt_f is arbitrary

$$\begin{aligned}\Big(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\Big)^\top \delta x_f + \Big(H(\cdot) + \frac{\partial h(\cdot)}{\partial t}\Big)\delta t_f &= 0 \\ H(\cdot) + \frac{\partial h(\cdot)}{\partial t} &= 0\end{aligned} \tag{51}$$

Boundary conditions for optimal control: with free final time

- The final state free
  Both δt_f and $\delta x(t_f)$ are arbitrary and independent, therefore their coefficients must be zero; that is

$$\begin{aligned}\Big(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\Big)^\top \delta x_f + \Big(H(\cdot) + \frac{\partial h(\cdot)}{\partial t}\Big)\delta t_f &= 0 \\ \frac{\partial h(\cdot)}{\partial x} - P^*(t_f) &= 0 \\ H(\cdot) + \frac{\partial h(\cdot)}{\partial t} &= 0\end{aligned} \tag{52}$$

  where n+1 equations has be solved, i.e, $\frac{\partial h(\cdot)}{\partial x} - P^*(t_f) = 0$ (n equations), and $H(\cdot) + \frac{\partial h(\cdot)}{\partial t}$ (1 equation)

Boundary conditions for optimal control: with free final time

- The final state lies on the moving point $x_f = \theta(t_f)$
  Term $\delta x(t_f)$ lies on the moving point $\theta(t_f) \Rightarrow \delta x_f = \frac{d\theta(t_f)}{dt}\delta t_f$

$$
\begin{aligned}
\Big(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\Big)^\top \delta x_f + \Big(H(\cdot) + \frac{\partial h(\cdot)}{\partial t}\Big)\delta t_f = 0 \\
\Big(\Big(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\Big)^\top \frac{d\theta(t_f)}{dt} + \Big(H(\cdot) + \frac{\partial h(\cdot)}{\partial t}\Big)\Big)\delta t_f = 0
\end{aligned}
\tag{53}
$$

where $x^*(t_f) = \theta(t_f)$

Boundary conditions for optimal control: with free final time

Consider the final state of a provided system that is required to lie on the circle $h(x(t)) = (x_1(t) - 3)^2 + (x_2(t) - 4)^2 - 4$.

- Admissible changes in $x(t_f)$ are tangent to the $h(x(t))$ at point $x^*(t_f), t_f$.

Boundary conditions for optimal control: with free final time

Consider the final state of a provided system that is required to lie on the circle $h(x(t)) = (x_1(t)-3)^2 + (x_2(t)-4)^2 - 4$.

- Admissible changes in $x(t_f)$ are tangent to the $h(x(t))$ at point $x^*(t_f), t_f$.
- Moreover, the change in $x(t_f)$ or (δx_f) is independent of δt_f.

Boundary conditions for optimal control: with free final time

Consider the final state of a provided system that is required to lie on the circle $h(x(t)) = (x_1(t)-3)^2 + (x_2(t)-4)^2 - 4$.

- Admissible changes in $x(t_f)$ are tangent to the $h(x(t))$ at point $x^*(t_f), t_f$.
- Moreover, the change in $x(t_f)$ or (δx_f) is independent of δt_f.
- Hence, the coefficients of δt_f must be zero.

Boundary conditions for optimal control: with free final time

Consider the final state of a provided system that is required to lie on the circle $h(x(t)) = (x_1(t) - 3)^2 + (x_2(t) - 4)^2 - 4$.

- Admissible changes in $x(t_f)$ are tangent to the $h(x(t))$ at point $x^*(t_f), t_f$.
- Moreover, the change in $x(t_f)$ or (δx_f) is independent of δt_f.
- Hence, the coefficients of δt_f must be zero.
- Therefore,

$$\begin{aligned} \left(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\right)^\top \delta x_f + \left(H(\cdot) + \frac{\partial h(\cdot)}{\partial t}\right)\delta t_f &= 0 \\ H(\cdot) + \frac{\partial h(\cdot)}{\partial t} &= 0 \end{aligned} \tag{54}$$

Boundary conditions for optimal control: with free final time

- The tangent lies normal to the gradient vector

$$\frac{h(x)(t)}{\partial x}|_{x^*(t_f)} = \begin{bmatrix} 2(x_1^*(t_f)-3) \\ 2(x_2^*(t_f)-4) \end{bmatrix} \tag{55}$$

at point $(x^*(t_f), t_f)$.

Boundary conditions for optimal control: with free final time

- The tangent lies normal to the gradient vector

$$\frac{h(x)(t)}{\partial x}|_{x^*(t_f)} = \begin{bmatrix} 2(x_1^*(t_f) - 3) \\ 2(x_2^*(t_f) - 4) \end{bmatrix} \tag{55}$$

  at point $(x^*(t_f), t_f)$.
- Term $\delta x(t_f)$ must be normal to the gradient, so that

$$\begin{aligned} \left(\frac{h(x)(t)}{\partial x}|_{x^*(t_f)}\right)^\top \delta x(t_f) = \begin{bmatrix} 2(x_1^*(t_f) - 3) \\ 2(x_2^*(t_f) - 4) \end{bmatrix}^\top \delta x(t_f) = 0 \\ \Rightarrow \delta x_2(t_f) = \frac{-(x_1^*(t_f) - 3)}{(x_2^*(t_f) - 4)} \delta x_1(t_f) \end{aligned} \tag{56}$$

Boundary conditions for optimal control: with free final time

- Therefore, eq.56 becomes

$$\begin{aligned}
\Big(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\Big)^\top \delta x_f = 0 \\
\Big(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\Big)^\top \begin{bmatrix} 1 \\ \frac{-(x_1^*(t_f)-3)}{(x_2^*(t_f)-4)} \end{bmatrix} = 0
\end{aligned} \tag{57}$$

Boundary conditions for optimal control: with free final time

- Therefore, eq.56 becomes

$$\begin{aligned}\Big(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\Big)^\top \delta x_f = 0 \\ \Big(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\Big)^\top \begin{bmatrix} 1 \\ \frac{-(x_1^*(t_f)-3)}{(x_2^*(t_f)-4)} \end{bmatrix} = 0\end{aligned} \tag{57}$$

- Since the final state $h(x(t_f))$ at t_f must satisfy the $m(x^*(t_f)) = (x_1^*(t_f)-3)^2 + (x_2^*(t_f)-4)^2 - 4$

Boundary conditions for optimal control: with free final time

Consider the final state of a provided system that is required to lie on the circle $h(x(t),t) = (x_1(t)-3)^2 + (x_2(t)-4-t)^2 - 4$. Admissible changes in $x(t_f)$ are tangent to the $m(x(t))$ at point $x^*(t_f), t_f$. Moreover, the change in $x(t_f)$ or (δx_f) is dependent on δt_f.

Boundary conditions for optimal control: with free final time

- Hence, the coefficients of δt_f must be zero. Therefore, The tangent lies normal to the gradient vector

$$\begin{bmatrix} \frac{h(x)(t)}{\partial x_1}(x^*(t_f),t_f) \\ \frac{h(x)(t)}{\partial x_2}(x^*(t_f),t_f) \\ \frac{h(x)(t)}{\partial t}(x^*(t_f),t_f) \end{bmatrix} = \begin{bmatrix} 2(x_1^*(t_f)-3) \\ 2(x_2^*(t_f)-4) \\ -2(x_2^*(t_f)-4-t_f) \end{bmatrix} \tag{58}$$

at point $(x^*(t_f),t_f)$.

Boundary conditions for optimal control: with free final time

- Hence, the coefficients of δt_f must be zero. Therefore, The tangent lies normal to the gradient vector

$$\begin{bmatrix} \frac{h(x)(t)}{\partial x_1}(x^*(t_f),t_f) \\ \frac{h(x)(t)}{\partial x_2}(x^*(t_f),t_f) \\ \frac{h(x)(t)}{\partial t}(x^*(t_f),t_f) \end{bmatrix} = \begin{bmatrix} 2(x_1^*(t_f)-3) \\ 2(x_2^*(t_f)-4) \\ -2(x_2^*(t_f)-4-t_f) \end{bmatrix} \tag{58}$$

at point $(x^*(t_f),t_f)$.

- Term $\delta x(t_f)$ must be normal to the gradient, so that

$$\left(\begin{bmatrix} \frac{h(x)(t)}{\partial x_1}(x^*(t_f),t_f) \\ \frac{h(x)(t)}{\partial x_2}(x^*(t_f),t_f) \\ \frac{h(x)(t)}{\partial t}(x^*(t_f),t_f) \end{bmatrix} \right)^\top \begin{bmatrix} \delta x_{1f} \\ \delta x_{2f} \\ \delta t_f \end{bmatrix} = \begin{bmatrix} 2(x_1^*(t_f)-3) \\ 2(x_2^*(t_f)-4) \\ -2(x_2^*(t_f)-4-t_f) \end{bmatrix}^\top \begin{bmatrix} \delta x_{1f} \\ \delta x_{2f} \\ \delta t_f \end{bmatrix} = 0$$

$$\Rightarrow \delta t_f = \frac{-(x_1^*(t_f)-3)}{(x_2^*(t_f)-4-t_f)} \delta x_{1f}(t_f) + \delta x_{2f} \tag{59}$$

With free final time

Therefore, boundary conditions become

$$\Big(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\Big)^\top \delta x_f + \Big(H(\cdot) + \frac{\partial h(\cdot)}{\partial t}\Big)\delta t_f = 0$$

$$\begin{aligned} \bigg(\Big(\frac{\partial h(\cdot)}{\partial x_1} - P^*(t_f)\Big) + \Big(H(\cdot) + \frac{\partial h(\cdot)}{\partial t}\Big)\Big(\frac{-(x_1^*(t_f)-3)}{(x_2^*(t_f)-4-t_f)}\Big)\bigg)\delta x_{1f} \\ + \bigg(\Big(\frac{\partial h(\cdot)}{\partial x_2} - P^*(t_f)\Big) + H(\cdot) + \frac{\partial h(\cdot)}{\partial t}\bigg)\delta x_{2f} = 0 \end{aligned} \tag{60}$$

Since coefficients of δx_{2f} and δx_{1f} must be zero,

$$\begin{aligned} \Big(\frac{\partial h(\cdot)}{\partial x_1} - P^*(t_f)\Big) + \Big(H(\cdot) + \frac{\partial h(\cdot)}{\partial t}\Big)\Big(\frac{-(x_1^*(t_f)-3)}{(x_2^*(t_f)-4-t_f)}\Big) = 0 \\ \Big(\frac{\partial h(\cdot)}{\partial x_2} - P^*(t_f)\Big) + H(\cdot) + \frac{\partial h(\cdot)}{\partial t} = 0 \end{aligned} \tag{61}$$

Since the final state $m(x(t_f))$ at t_f must satisfy the
$m(x^*(t_f)) = (x_1^*(t_f)-3)^2 + (x_2^*(t_f)-4)^2 - 4 = 0$