Motion Planning for Autonomous Vehicles

Linear Quadratic Regulator (LQR)

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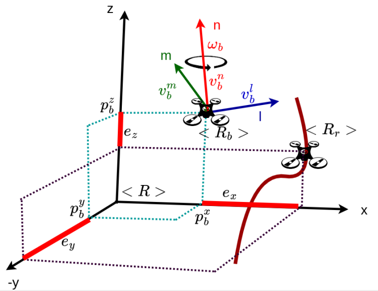

Linear Quadratic Regulator

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In general, discrete linear system, which can be either LTI or LTV, dynamics is described by:

$$\mathbf{x}_{k+1} = \mathbf{f}_d(\mathbf{x}_k, \mathbf{u}_k) = A_k \mathbf{x}_k + B_k \mathbf{u}_k, \tag{1}$$

where $k = 0, \dots, n$, $\mathbf{x}_k \in \mathbb{R}^n$, and $\mathbf{u}_k \in \mathbb{R}^m$. For the continuous time system

$$\dot{\mathbf{x}} = \mathbf{f}_c(t) = A(t)\mathbf{x}(\mathbf{t}) + B(t)\mathbf{u}(\mathbf{t}) \tag{2}$$

If the system dynamics is non-linear, A_k and B_k are recalculated by linearizing the $\mathbf{f}_c$ at each time index.

Since linearization has to be carried out in each iteration, **ILQR and ELQR** are such variants, consider nominal trajectory, $\mathbf{x_0(t)}, \mathbf{u_0(t)} \quad \forall\, t[t_1, t_2]$.
Using first-order Taylor series approximation, the increment $\Delta\dot{\mathbf{x}} = \dot{\mathbf{x}} - \dot{\mathbf{x}}_0 = \mathbf{f_c}(\mathbf{x}, \mathbf{u}) - \mathbf{f_c}(\mathbf{x_0}, \mathbf{u_0})$ can be expressed by

$$\begin{aligned} \Delta\dot{\mathbf{x}} &\approx \mathbf{f_c}(\mathbf{x_0}, \mathbf{u_0}) + \frac{\partial \mathbf{f_c}(\mathbf{x_0}, \mathbf{u_0})}{\partial \mathbf{x}}(\mathbf{x} - \mathbf{x_0}) + \frac{\partial \mathbf{f_c}(\mathbf{x_0}, \mathbf{u_0})}{\partial \mathbf{u}}(\mathbf{u} - \mathbf{u_0}) - \mathbf{f_c}(\mathbf{x_0}, \mathbf{u_0}) \\ &= A(t)\Delta\mathbf{x(t)} + B(t)\Delta\mathbf{u(t)} \end{aligned} \tag{3}$$

where $\Delta\mathbf{x(t)} = \mathbf{x(t)} - \mathbf{x(t_0)}$ and $\Delta\mathbf{u(t)} = \mathbf{u(t)} - \mathbf{u(t_0)}$ and $A(t) = \frac{\partial \mathbf{f}_c}{\partial \mathbf{x}}(\mathbf{x}_0, \mathbf{u}_0), \quad B(t) = \frac{\partial \mathbf{f}_c}{\partial \mathbf{u}}(\mathbf{x}_0, \mathbf{u}_0)$.

LQR Formulation

Consider **initial state** x_0 at each time instance t_0 is given, the objective is to **find the optimal control input sequence $\mathbf{u}$** for a given initial condition x_0, to reach the final state x_T, i.e., **estimate the optimal state prediction**, an optimal control sequence (or control policy) has to be calculated.

LQR Formulation

Such a control policy can be estimated by minimizing the following quadratic cost:

$$J(\mathbf{x}, \mathbf{u}) = \underbrace{\|x_n\|^2_{Q_n}}_{\text{terminal cost}} + \underbrace{\sum_{k=0}^{n-1} \|x_k\|^2_Q + \|u_k\|^2_R}_{\text{running cost}}$$

$$J(\mathbf{x}, \mathbf{u}) = \int_0^{\infty} \left(\|x(t)\|^2_Q + \|u(t)\|^2_R \right) dt, \tag{4}$$

where $k \in \{0, 1, \ldots, n-1\}$, $Q, Q_n \in \mathbb{R}^{n_x \times n_x}, R \in \mathbb{R}^{n_u \times n_u}, P \in \mathbb{R}^{n_x \times n_x}$ are predefined in which $\mathbf{Q} = \mathbf{Q}^\top \succeq \mathbf{0}$ is a **positive definite** and $\mathbf{R} = \mathbf{R}^\top \succ 0$ is a **positive semi-definite**. However, if the **system is nonlinear**, need to estimate the second-order approximation of the non-linear cost functions to **define Q(t) and R(t)**.

- For a linear system

$$\begin{aligned} \min_{\mathbf{u}} \quad & \sum_{k=0}^{n-1} \mathbf{x}_k^\top \mathbf{Q_k} \mathbf{x}_k + \mathbf{u}_k^\top \mathbf{R_k} \mathbf{u}_k + \mathbf{x}_n^\top \mathbf{Q_n} \mathbf{x}_n, \mathbf{Q_k} = \mathbf{Q_k}^\top \succeq 0, \mathbf{R_k} = \mathbf{R_k}^\top \succ 0 \\ \text{s.t.} \quad & \mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \\ & \mathbf{x}_0 \end{aligned} \tag{5}$$

https://markcannon.github.io/assets/downloads/teaching/C21_Model_Predictive_Control/mpc_notes.pdf

LQR VIA LEAST SQUARES

- For a linear system

$$\begin{aligned}\min_{\mathbf{u}} \quad & \sum_{k=0}^{n-1} \mathbf{x}_k^\top \mathbf{Q_k} \mathbf{x}_k + \mathbf{u}_k^\top \mathbf{R_k} \mathbf{u}_k + \mathbf{x}_n^\top \mathbf{Q_n} \mathbf{x}_n, \mathbf{Q_k} = \mathbf{Q_k}^\top \succeq 0, \mathbf{R_k} = \mathbf{R_k}^\top \succ 0 \\ \text{s.t.} \quad & \mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \\ & \mathbf{x}_0 \end{aligned} \tag{5}$$

- The state prediction sequence can be written in a compact sequence as follows:

$$\mathbf{x} = Mx_0 + C\mathbf{u}, \quad M = \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & \dots & 0 \\ B & & & \\ AB & B & & \\ \vdots & \vdots & \ddots & \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix}$$

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