Motion Planning for Autonomous Vehicles

Linear Quadratic Regulator (LQR)

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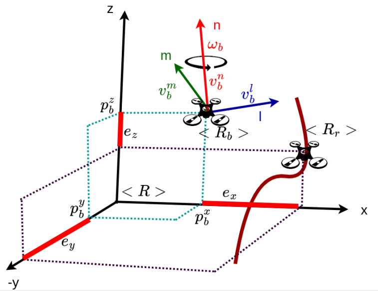

Linear Quadratic Regulator

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LQR Formulation

In general, discrete linear system, which can be either LTI or LTV, dynamics is described by:

$$\mathbf{x}_{k+1} = \mathbf{f}_d(\mathbf{x}_k, \mathbf{u}_k) = A_k \mathbf{x}_k + B_k \mathbf{u}_k, \tag{1}$$

where $k = 0, \ldots, n$, $\mathbf{x}_k \in \mathbb{R}^n$, and $\mathbf{u}_k \in \mathbb{R}^m$. For the continuous time system

$$\dot{\mathbf{x}} = \mathbf{f}_c(t) = A(t)\mathbf{x}(\mathbf{t}) + B(t)\mathbf{u}(\mathbf{t}) \tag{2}$$

If the system dynamics is non-linear, A_k and B_k are recalculated by linearizing the $\mathbf{f}_c$ at each time index.

Since linearization has to be carried out in each iteration, **ILQR and ELQR** are such variants, consider nominal trajectory, $\mathbf{x_0(t)}, \mathbf{u_0(t)} \quad \forall\, t[t_1, t_2]$.
Using first-order Taylor series approximation, the increment $\Delta\dot{\mathbf{x}} = \dot{\mathbf{x}} - \dot{\mathbf{x}}_0 = \mathbf{f_c}(\mathbf{x}, \mathbf{u}) - \mathbf{f_c}(\mathbf{x_0}, \mathbf{u_0})$ can be expressed by

$$\begin{aligned} \Delta\dot{\mathbf{x}} &\approx \mathbf{f_c}(\mathbf{x_0}, \mathbf{u_0}) + \frac{\partial \mathbf{f_c}(\mathbf{x_0}, \mathbf{u_0})}{\partial \mathbf{x}}(\mathbf{x} - \mathbf{x_0}) + \frac{\partial \mathbf{f_c}(\mathbf{x_0}, \mathbf{u_0})}{\partial \mathbf{u}}(\mathbf{u} - \mathbf{u_0}) - \mathbf{f_c}(\mathbf{x_0}, \mathbf{u_0}) \\ &= A(t)\Delta\mathbf{x(t)} + B(t)\Delta\mathbf{u(t)} \end{aligned} \tag{3}$$

where $\Delta\mathbf{x(t)} = \mathbf{x(t)} - \mathbf{x(t_0)}$ and $\Delta\mathbf{u(t)} = \mathbf{u(t)} - \mathbf{u(t_0)}$ and $A(t) = \frac{\partial \mathbf{f}_c}{\partial \mathbf{x}}(\mathbf{x}_0, \mathbf{u}_0), \quad B(t) = \frac{\partial \mathbf{f}_c}{\partial \mathbf{u}}(\mathbf{x}_0, \mathbf{u}_0)$.

Consider **initial state** x_0 at each time instance t_0 is given, the objective is to **find the optimal control input sequence** $\mathbf{u}$ for a given initial condition x_0, to reach the final state x_T, i.e., **estimate the optimal state prediction**, an optimal control sequence (or control policy) has to be calculated.

LQR Formulation

Such a control policy can be estimated by minimizing the following quadratic cost:

$$J(\mathbf{x},\mathbf{u}) = \underbrace{\|x_n\|^2_{Q_n}}_{\text{terminal cost}} + \underbrace{\sum_{k=0}^{n-1} \|x_k\|^2_Q + \|u_k\|^2_R}_{\text{running cost}} \tag{4}$$

$$J(\mathbf{x},\mathbf{u}) = \int_0^{\infty} \left(\|x(t)\|^2_Q + \|u(t)\|^2_R \right) dt,$$

where $k \in \{0, 1, ..., n-1\}$, $Q, Q_n \in \mathbb{R}^{n_x \times n_x}, R \in \mathbb{R}^{n_u \times n_u}, P \in \mathbb{R}^{n_x \times n_x}$ are predefined in which $\mathbf{Q} = \mathbf{Q}^\top \geq \mathbf{0}$ is a **positive definite** and $\mathbf{R} = \mathbf{R}^\top \succ 0$ is a **positive semi-definite**. However, if the **system is nonlinear**, need to estimate the second-order approximation of the non-linear cost functions to **define Q(t) and R(t)**.

LQR via least squares

- For a linear system

$$
\begin{aligned}
\min_{\mathbf{u}} \quad & \sum_{k=0}^{n-1} \mathbf{x}_k^\top \mathbf{Q_k} \mathbf{x}_k + \mathbf{u}_k^\top \mathbf{R_k} \mathbf{u}_k + \mathbf{x}_n^\top \mathbf{Q_n} \mathbf{x}_n, \mathbf{Q_k} = \mathbf{Q_k}^\top \succeq 0, \mathbf{R_k} = \mathbf{R_k}^\top \succ 0 \\
\text{s.t.} \quad & \mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \\
& \mathbf{x}_0
\end{aligned}
\tag{5}
$$

https://markcannon.github.io/assets/downloads/teaching/C21_Model_Predictive_Control/mpc_notes.pdf

LQR via least squares

- For a linear system

$$\begin{aligned} \min_{\mathbf{u}} \quad & \sum_{k=0}^{n-1} \mathbf{x}_k^\top \mathbf{Q_k} \mathbf{x}_k + \mathbf{u}_k^\top \mathbf{R_k} \mathbf{u}_k + \mathbf{x}_n^\top \mathbf{Q_n} \mathbf{x}_n, \mathbf{Q_k} = \mathbf{Q_k}^\top \succeq 0, \mathbf{R_k} = \mathbf{R_k}^\top \succ 0 \\ \text{s.t.} \quad & \mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \\ & \mathbf{x}_0 \end{aligned} \tag{5}$$

- The state prediction sequence can be written in a compact sequence as follows:

$$\mathbf{x} = M x_0 + C\mathbf{u}, \quad M = \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & \dots & 0 \\ B & & & \\ AB & B & & \\ \vdots & \vdots & \ddots & \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix}$$

https://markcannon.github.io/assets/downloads/teaching/C21_Model_Predictive_Control/mpc_notes.pdf

LQR via least squares

- The defined quadratic cost (5) can be written in terms of $\mathbf{x}$ and $\mathbf{u}$ as

$$J = \mathbf{x}^\top \tilde{Q} \mathbf{x} + \mathbf{u}^\top \tilde{R} \mathbf{u} = \mathbf{u}^\top H \mathbf{u} + 2 x_0^\top F^\top \mathbf{u} + x_0^\top G x_0 \tag{6}$$

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- Can you define the $\tilde{Q}$ and $\tilde{R}$? as well as prove that H, F, and G are given by $C^\top \tilde{Q} C + \tilde{R}$, $C^\top \tilde{Q} M$, and $M^\top \tilde{Q} M$, respectively.

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- If no additional constraints are given, eq.6 has a **closed-form solution** that is derived by minimizing the J with respect to $\mathbf{u}$. Show that $\mathbf{u}^* = -H^{-1} F x_0$.

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- Can you define the $\tilde{Q}$ and $\tilde{R}$? as well as prove that H, F, and G are given by $C^\top \tilde{Q} C + \tilde{R}$, $C^\top \tilde{Q} M$, and $M^\top \tilde{Q} M$, respectively.
- If no additional constraints are given, eq.6 has a **closed-form solution** that is derived by minimizing the J with respect to $\mathbf{u}$. Show that $\mathbf{u}^* = -H^{-1} F x_0$.
- What can you say about when H is **singular whose determinant is 0 (the rank is given by non-zero eigenvalues)** (i.e., positive semi-definite rather than positive definite); this implies **multiple optimal solutions** can exist.

LQR via least squares

Since H and F are constant matrices, which can be calculated offline, at every sampling time, the first element of the optimal control can be applied to the system. This is called **time-invariant feedback controller**.

$$\mathbf{u} = Lx$$

where $L = -[I_{n_u}\ 0\ 0\ , ..., \ 0]H^{-1}F$.

Example 01

Estimate feedback control law, considering the following system with

$$A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & -1 \end{bmatrix} \tag{7}$$

for horizon N = 4, you may assume $Q = D^\top D$, $R = 0.01$.

The continuous time system or the plant is expressed as

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(\mathbf{t}) + B(t)\mathbf{u}(\mathbf{t}) = f_c(\mathbf{x}(t), \mathbf{u}(t), t) \tag{8}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, and $\mathbf{u}(t) \in \mathbb{R}^m$. And performance index is defined as:

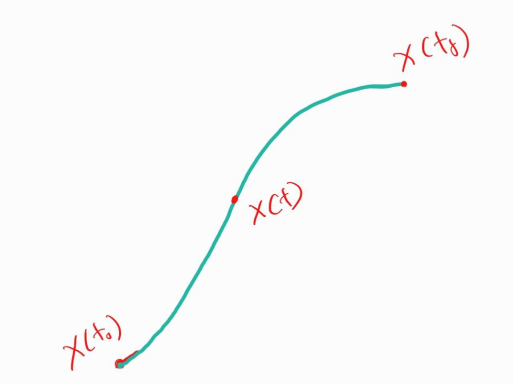
$$J(\mathbf{x}(t), \mathbf{u}(t), t_0, t_f) = Q(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t) dt \tag{9}$$

The objective is to find the **optimal feedback control minimizing the above cost function**, i.e., the optimal solution from any time instance t to the final time t_f as:

$$\begin{aligned} J^*(\mathbf{x}(t_0), t_0, t_f) &= \int_{t}^{t_f} g(\mathbf{x}^*(\tau), \mathbf{u}^*(\tau), \tau) d\tau, \\ \Rightarrow V(\mathbf{x}(t_0), t_0, t_f) &= \min_{\mathbf{u}(t)} \Big(J(\mathbf{x}(t), \mathbf{u}(t), t_0, t_f) \Big) \end{aligned} \tag{10}$$

Hence, $V(\mathbf{x}(t_0), t_0, t_f)$ **does not depend of $\mathbf{u}$**

Bellman Optimality



$$V(\mathbf{x}(t_0), t_0, t_f) = V(\mathbf{x}(t_0), t_0, t) + V(\mathbf{x}(t), t, t_f) \tag{11}$$

Hamilton Jacobi Bellman Approach

Taking time derivative

$$
\begin{aligned}
V(\mathbf{x}(t_0), t_0, t_f) &= \min_{\mathbf{u}(t)} \Big(J(\mathbf{x}(t), \mathbf{u}(t), t_0, t_f) \Big) \\
\frac{dV(\mathbf{x}(t), t, t_f)}{dt} &= \left[\frac{\partial V(\mathbf{x}(t), t, t_f)}{\partial x} \right]^\top \dot{x}(t) + \frac{\partial V(\mathbf{x}(t), t, t_f)}{\partial t} \\
&= \min_{\mathbf{u}(t)} \frac{d}{dt} \Big(Q(\mathbf{x}(t_f), t_f) + \int_t^{t_f} g(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau) d\tau \Big) \\
&= \min_{\mathbf{u}(t)} \left(\frac{d}{dt} \int_t^{t_f} g(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau) d\tau \right), \quad \frac{d}{dt} \Big(Q(\mathbf{x}(t_f), t_f) \Big) = 0 \\
&= \min_{\mathbf{u}(t)} -g(\mathbf{x}(t), \mathbf{u}(t), t) \quad \text{where } g(\mathbf{x}(t_f), t_f) \text{ is a constant} \\
\Rightarrow -\frac{\partial V(\mathbf{x}(t), t, t_f)}{\partial t} &= \min_{\mathbf{u}(t)} \left(\left(\frac{\partial V(\mathbf{x}(t), t, t_f)}{\partial x} \right)^\top \dot{x}(t) + g(\mathbf{x}(t), \mathbf{u}(t), t) \right)
\end{aligned}
\tag{12}
$$

Hamilton Jacobi Bellman Approach

- Given system dynamics and the performance index, the Hamiltonian can be determined as

$$H = g(\mathbf{x}(t), \mathbf{u}(t), t) + \underbrace{\left[\frac{\partial J(\mathbf{x}(t), t)}{\partial x}\right]^{\top}}_{\lambda^{\top}} \dot{x}(t) = 0 \tag{13}$$

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- After considering the boundary conditions: $J^*(\mathbf{x}^*(t_f), t_f) = \frac{1}{2}\mathbf{x}(t_f)^\top Q(t_f)\mathbf{x}(t_f)$,

$$\underbrace{\min_{\mathbf{u}(t)} \left(\left[\frac{\partial V(\mathbf{x}(t), t, t_f)}{\partial x}\right]^\top \dot{x}(t) + g(\mathbf{x}(t), \mathbf{u}(t), t) \right)}_{H^*} + \frac{\partial V(\mathbf{x}(t), t, t_f)}{\partial t} = 0$$

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- This equation is called the Hamilton-Jacobi equation. Since it is used in Bellman's dynamic programming, it is also known as Hamilton-Jacobi-Bellman (HJB) equation.

Hamilton Jacobi Bellman Approach

Hence, the procedure for the HJB approach is as follows:

1. Define the Hamiltonian

$$H = g(\mathbf{x}(t), \mathbf{u}(t), t) + \left[\frac{\partial J(\mathbf{x}(t), t)}{\partial x} \right]^\top \dot{x}(t) = 0 \tag{14}$$

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2. Minimize the H with respect to u(t), i.e., $\frac{\partial H}{\partial u} = 0$, for solving $\mathbf{u}^*(t)$
3. Rewrite $H \rightarrow H^*$ substituting the optimal $u^*(t)$
4. Solve for HJB

$$H^* + \frac{\partial J^*(\mathbf{x}^*(t), t)}{\partial t^*} = 0 \tag{15}$$

considering the boundary conditions: $J^*(\mathbf{x}^*(t_f), t_f) = 0$
whose solution provides an expression for $\mathbf{u}^*$

Consider a linear time-varying system

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(\mathbf{t}) + B(t)\mathbf{u}(\mathbf{t}) = f_c(x(t), u(t), t) \tag{16}$$

that should minimize the following cost function

$$J(\mathbf{x}, \mathbf{u}) = \int_{t_0}^{t_f} \frac{1}{2}\Big(\|x(t)\|_Q^2 + \|u(t)\|_R^2 \Big) dt, \tag{17}$$

with these assumptions: the **control** inputs are unconstrained and the **system** must be **controllable**. The objective is to find the optimal **cost-to-go** function J^* that satisfies the (Hamilton-Jacobi-Bellman Equation) for a finite time horizon

$$\forall \mathbf{x}, \quad 0 = \min_{\mathbf{u}} \left[\frac{1}{2}\Big(\|\mathbf{x}\|_Q^2 + \|\mathbf{u}\|_R^2 \Big) + \frac{\partial J^*}{\partial \mathbf{x}} (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) + \frac{\partial J^*}{\partial t} \right]. \tag{18}$$

LQR with HJB

- Define the Hamiltonian

$$
\begin{aligned}
H &= g(\mathbf{x}(t), \mathbf{u}(t), t) + \left[\frac{\partial J(\mathbf{x}(t), t)}{\partial x}\right]^\top f_c(\mathbf{x}(t), \mathbf{u}(t), t) = 0 \\
&= \frac{1}{2}\mathbf{x}^\top Q \mathbf{x} + \frac{1}{2}\mathbf{u}^\top R \mathbf{u} + \left[\frac{\partial J(\mathbf{x}(t), t)}{\partial x}\right]^\top \Big(A\mathbf{x} + B\mathbf{u}\Big)
\end{aligned}
\tag{19}
$$

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$$
\begin{aligned}
H &= g(\mathbf{x}(t), \mathbf{u}(t), t) + \left[\frac{\partial J(\mathbf{x}(t), t)}{\partial x}\right]^\top f_c(\mathbf{x}(t), \mathbf{u}(t), t) = 0 \\
&= \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} + \frac{1}{2}\mathbf{u}^\top R\mathbf{u} + \left[\frac{\partial J(\mathbf{x}(t), t)}{\partial x}\right]^\top \left(A\mathbf{x} + B\mathbf{u}\right)
\end{aligned}
\tag{19}
$$

- Minimize the H with respect to u(t), i.e., $\frac{\partial H^*}{\partial u} = 0$, for solving $\mathbf{u}^*(t)$

$$
R\mathbf{u} + B^\top \frac{\partial J(\mathbf{x}(t), t)}{\partial x} = 0 \quad \Rightarrow \mathbf{u} = -R^{-1}B^\top \underbrace{\frac{\partial J(\mathbf{x}(t), t)}{\partial x}}_{\lambda}
\tag{20}
$$

LQR with HJB

- Rewrite H substituting the optimal $u^*(t)$

$$\begin{aligned}
&= \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} + \frac{1}{2}\left[R^{-1}B^\top \frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right]^\top R\left[R^{-1}B^\top \frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right] \\
&\quad + \left[\frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right]^\top \left(A\mathbf{x} - B\left[R^{-1}B^\top \frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right]\right) \\
&= \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} + \frac{1}{2}\left[\frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right]^\top \left[BR^{-1}B^\top\right]\left[\frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right] \\
&\quad + \left[\frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right]^\top \left(A\mathbf{x} - B\left[R^{-1}B^\top \frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right]\right) \\
&= \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} - \frac{1}{2}\left[\frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right]^\top \left[BR^{-1}B^\top\right]\left[\frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right] \\
&\quad + \left[\frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right]^\top A\mathbf{x}
\end{aligned} \tag{21}$$

- Solve for HJB

$$
\begin{aligned}
H^* + \frac{\partial J(\mathbf{x}(t),t)}{\partial t} &= 0 \\
\frac{\partial J(\mathbf{x}(t),t)}{\partial t} + \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} - \frac{1}{2}\left[\frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right]^\top \left[BR^{-1}B^\top\right]\left[\frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right] \\
+ \left[\frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right]^\top A\mathbf{x}
\end{aligned} \tag{22}
$$

- Considering **terminal cost**

$$J(\mathbf{x}(t_f), t_f) = h(t_f) = \frac{1}{2}\mathbf{x}^\top(t_f)Q(t_f)\mathbf{x}(t_f)$$

whose solution provides an expression for $\mathbf{u}^*$. Since the **cost function** is **quadratic**, the control input $\mathbf{u}^*$ is in terms of J^*. To seek **feedback control**, i.e., $\mathbf{u}^*$ in terms of $\mathbf{x}(t)$, it is **reasonable to consider** $J^*(\mathbf{x}^*(t), t) = \frac{1}{2}\mathbf{x}^\top(t)P(t)\mathbf{x}(t)$

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- Therefore,

$$J^*(\mathbf{x}^*(t), t) = \frac{1}{2}\mathbf{x}^\top(t)P(t)\mathbf{x}(t)$$
$$\frac{\partial J^*(\mathbf{x}^*(t), t)}{\partial t} = \frac{1}{2}\mathbf{x}^\top(t)\dot{P}(t)\mathbf{x}(t), \quad \frac{\partial J^*(\mathbf{x}^*(t), t)}{\partial x} = P(t)\mathbf{x}(t) = \lambda(t_f) \tag{23}$$

- Hence, rewriting the eq.22,

$$
\begin{aligned}
H^* + \frac{\partial J^*(\mathbf{x}^*(t), t)}{\partial t^*} &= 0 \\
\frac{1}{2}\mathbf{x}^\top \dot{P}\mathbf{x} + \frac{1}{2}\left(\mathbf{x}^\top Q\mathbf{x} - \mathbf{x}^\top PBR^{-1}B^\top P\mathbf{x}\right) + \mathbf{x}^\top PA\mathbf{x} &= 0
\end{aligned}
\tag{24}
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\end{aligned}
\tag{24}
$$

- However, $\mathbf{x}^\top PA\mathbf{x}$ is a scalar term, this can be rewritten as $2\mathbf{x}^\top PA\mathbf{x} = \mathbf{x}^\top PA\mathbf{x} + \mathbf{x}^\top A^\top P\mathbf{x}$.

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\begin{aligned}
H^* + \frac{\partial J^*(\mathbf{x}^*(t),t)}{\partial t^*} &= 0 \\
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- Therefore,

$$
\dot{P} + PA + A^\top P - PBR^{-1}B^\top P + Q = 0 \tag{25}
$$

This is called Differential Riccati Equation. And the optimal control becomes $\mathbf{u} = -R^{-1}B^\top P\mathbf{x} = -K\mathbf{x}$, with $P(t_f) = Q(t_f)$

Example 01

Consider $\lambda(t) = P(t)\mathbf{x}(t)$. Using the Hamilton operator try to derive the **Differential Riccati Equation**.

Example 01

Consider $\lambda(t) = P(t)\mathbf{x}(t)$. Using the Hamilton operator try to derive the Differential Riccati Equation.

$$
\begin{aligned}
\lambda(t) &= P(t)\mathbf{x}(t) \\
\dot{\lambda}(t) &= \dot{P}(t)\mathbf{x}(t) + P(t)\dot{\mathbf{x}}(t) \\
&= \dot{P}(t)\mathbf{x}(t) + P(t)(A\mathbf{x}(t) - BR^{-1}B^{\top}\lambda(t)) \\
&= \dot{P}(t)\mathbf{x}(t) + P(t)(A\mathbf{x}(t) - BR^{-1}B^{\top}P(t)\mathbf{x}(t))
\end{aligned}
$$

Using costate equation, $\dot{\lambda}(t) = -\frac{\partial H}{\partial \mathbf{x}} = -Q\mathbf{x}(t) + A^{\top}\lambda(t)$

$$
\begin{aligned}
\dot{\lambda}(t) &= \dot{P}(t)\mathbf{x}(t) + P(t)(A\mathbf{x}(t) - BR^{-1}B^{\top}P(t)\mathbf{x}(t)) \\
-Q\mathbf{x}(t) + A^{\top}P\mathbf{x}(t) &= \dot{P}(t)\mathbf{x}(t) + P(t)(A\mathbf{x}(t) - BR^{-1}B^{\top}P(t)\mathbf{x}(t)) \\
(\dot{P} + PA + A^{\top}P - PBR^{-1}B^{\top}P + Q)\mathbf{x}(t) &= 0
\end{aligned}
$$

LQR with HJB

- If the system dynamics is nonlinear (eq.3), the control law becomes $\mathbf{u}^* = \mathbf{u}_0(t) - \mathbf{K}(t)(\mathbf{x} - \mathbf{x}_0(t))$.

- If the system dynamics is nonlinear (eq.3), the control law becomes $\mathbf{u}^* = \mathbf{u}_0(t) - \mathbf{K}(t)(\mathbf{x} - \mathbf{x}_0(t))$.
- In the case of infinite horizon problem formulation, the objective is to find the optimal cost-to-go function $J^*(\mathbf{x})$ that satisfies the (Hamilton-Jacobi-Bellman Equation) with $\frac{\partial J^*}{\partial t} = 0$

$$\forall \mathbf{x}, \quad 0 = \min_{\mathbf{u}} \left[\frac{1}{2} \left(\|\mathbf{x}\|_Q^2 + \|\mathbf{u}\|_R^2 \right) + \frac{\partial J^*}{\partial \mathbf{x}} (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) \right]. \tag{26}$$

where it gives the Algebraic Riccati Equation which is similar to the differential Riccati equation.

$$PA + A^\top P - PBR^{-1}B^\top P + Q = 0 \tag{27}$$

- Discrete-time linear quadratic control problem to minimize

$$\Sigma_{t=1}^{T} \mathbf{x}(t)^\top Q \mathbf{x}(t) + \mathbf{u}(t)^\top R \mathbf{u}(t)$$

subject to $\mathbf{x}(t) = A\mathbf{x}(t-1) + B\mathbf{u}(t-1)$, where $\mathbf{x}(t)$ is an $n \times 1$ vector of state variables, $\mathbf{u}(t)$ is a $m \times 1$ vector of control variables, A is the $n \times n$ state transition matrix, B is the $n \times m$ matrix of control multipliers,$Q(n \times n)$ is a **symmetric positive semi-definite state cost matrix**, and $R(m \times m)$ is a **symmetric positive definite control cost matrix**.

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- Optimal cost

$$\mathbf{u}^*(t)=K\mathbf{x}(t-1)=-(B^\top P_tB+R)^{-1}(B^\top P_tA)\mathbf{x}(t-1)$$

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$$\mathbf{u}^*(t)=K\mathbf{x}(t-1)=-(B^\top P_tB+R)^{-1}(B^\top P_tA)\mathbf{x}(t-1)$$

- Discrete-time algebraic Riccati equation (DARE):

$$P_{t-1}=Q+A^\top P_tA-A^\top P_tB(B^\top P_tB+R)^{-1}B^\top P_tA \tag{28}$$

  with $P_T=Q$

Hamiltonian formulation to find the optimal control policy

$$H(x(t),u(t),P(t),t) := g(\mathbf{x}(t),\mathbf{u}(t),t) + P^\top(t) f(\mathbf{x}(t),\mathbf{u}(t),t)$$

$$H(x(t),u(t),P(t),t) := \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} + \frac{1}{2}\mathbf{u}^\top R\mathbf{u} + \lambda^\top(t)\Big(A\mathbf{x} + B\mathbf{u}\Big), \quad \lambda \in \mathbb{R}^n \tag{29}$$

Necessary conditions

$$\dot{x}^*(t) = \frac{H(\cdot)}{\partial P} \Rightarrow \dot{x} = \frac{\partial H}{\partial \lambda} = A\mathbf{x} + B\mathbf{u}$$

$$\dot{P}^*(t) = -\frac{H(\cdot)}{\partial x} = -\Big(\frac{\partial f(\cdot)}{\partial x}\Big)^\top P^*(t) - \frac{\partial g(\cdot)}{\partial x} \Rightarrow -\dot{\lambda} = \frac{\partial H}{\partial x} = Q\mathbf{x} + A^\top \lambda$$

$$0 = \frac{H(\cdot)}{\partial u} = \Big(\frac{\partial g(\cdot)}{\partial u}\Big)^\top P^*(t) + \frac{\partial f(\cdot)}{\partial u} \Rightarrow 0 = \frac{\partial H}{\partial u} = R\mathbf{u} + B^\top \lambda \Rightarrow \mathbf{u}^* = -R^{-1}B^\top \lambda \tag{30}$$

where $H(\cdot) = H(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda^*(t), t)$ and $\forall\, t \in [t_0 = 0, t_f = T]$

Hamiltonian formulation to find the optimal control policy

This kind of problem is considered as a two-point boundary value problem.

- $\mathbf{x}_0$ is given
- **fixed** final state $\mathbf{x}(t_f)$
- **free** final state $h(t_f) = \frac{1}{2}\mathbf{x}(t_f)^\top P(t_f)\mathbf{x}(t_f)$

$$\begin{aligned} \left(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\right)^\top \delta x_f + \left(H(\cdot) + \frac{\partial h(\cdot)}{\partial t}\right)\delta t_f = 0 \\ \lambda_{t_f} = \frac{\partial h(t_f)}{\partial x}\bigg|_{t_f} \Rightarrow P(t_f)\mathbf{x}(t_f) \end{aligned} \tag{31}$$

Hamiltonian formulation to find the optimal control policy

Let's assume $P(t)\mathbf{x}(t) = \lambda(t)$, for any t, i.e., since the performance index is quadratic, $\mathbf{u}^*$ is in terms of J^*, and we seek a feedback control in term of $\mathbf{x}(t)$. Hence, it is reasonable to consider $J^*(\mathbf{x}^*(t), t) = \frac{1}{2}\mathbf{x}^\top(t)P(t)\mathbf{x}(t) \Rightarrow \frac{\partial J^*(\mathbf{x}^*(t),t)}{\partial x} = \lambda(t) = P(t)\mathbf{x}(t)$

$$
\begin{aligned}
\dot{\lambda} = \frac{d}{dt}P\mathbf{x} = \dot{P}\mathbf{x} + P\dot{\mathbf{x}} = \dot{P}\mathbf{x} + P(A\mathbf{x} + B\mathbf{u}) \\
= \dot{P}\mathbf{x} + PA\mathbf{x} + PB(-R^{-1}B^\top\lambda) = \dot{P}\mathbf{x} + PA\mathbf{x} - PBR^{-1}B^\top P\mathbf{x} \\
= \left[\dot{P} + PA - PBR^{-1}B^\top P\right]\mathbf{x} \\
\Rightarrow -\left[Q + A^\top P\right]\mathbf{x} = \left[\dot{P} + PA - PBR^{-1}B^\top P\right]\mathbf{x} \quad \forall\, x \in \mathbb{R}^n \\
\Rightarrow \underbrace{-\dot{P} = PA + A^\top P - PBR^{-1}B^\top P + Q}_{\text{Differential Riccati Equation}}
\end{aligned}
\tag{32}
$$

Hamiltonian formulation to find the optimal control policy

- What are the boundary conditions since every differential equation must have **boundary conditions**? $\lambda_{t_f} = P(t_f)\mathbf{x}(t_f)$

Hamiltonian formulation to find the optimal control policy

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- For the infinite-horizon LQR, $\dot{P}(t) = 0$ that results in a **steady-state solution**. However, for **controllable systems**, the **finite-horizon solution** is also **stable enough**, but the finite-horizon solution converges on the infinite-horizon solution as the horizon **time limit goes to infinity**.

- Consider a time-varying continuous-time dynamical system

$$\dot{\mathbf{x}} = f_c(\mathbf{x}(t), \mathbf{u}(t), t) = A\mathbf{x} + B\mathbf{u} \tag{33}$$

  or, in general, consider a time-varying affine continuous-time dynamical system form:

$$\dot{\mathbf{x}} = f_c(\mathbf{x}(t), \mathbf{u}(t), t) = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u} + \mathbf{c}(t), \tag{34}$$

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- With the terminal cost

$$h_f = \Big(\mathbf{x}(t_f) - \mathbf{x}_d(t_f)\Big)^T \mathbf{Q}_f \Big(\mathbf{x}(t_f) - \mathbf{x}_d(t_f)\Big), \quad \mathbf{Q}_f = \mathbf{Q}_f^T \geq 0 \tag{35}$$

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- The running cost can be formulated as

$$\begin{aligned} g(\mathbf{x}, \mathbf{x_d}, \mathbf{u}, t) = \\ \left(\mathbf{x}(t) - \mathbf{x}_d(t)\right)^T \mathbf{Q}\left(\mathbf{x} - \mathbf{x}_d(t)\right) + \left(\mathbf{u}(t) - \mathbf{u}_d(t)\right)^T \mathbf{R}\left(\mathbf{u}(t) - \mathbf{u}_d(t)\right), \\ \mathbf{Q} = \mathbf{Q}^T \geq 0, \mathbf{R} = \mathbf{R}^T \succ 0 \end{aligned} \tag{36}$$

In compact matrix form the running cost for a time-varying affine continuous-time dynamical system

$$g(\mathbf{x},\mathbf{u},t) = \begin{bmatrix}\mathbf{x}\\1\end{bmatrix}^T \mathbf{Q}(t)\begin{bmatrix}\mathbf{x}\\1\end{bmatrix} + \begin{bmatrix}\mathbf{u}\\1\end{bmatrix}^T \mathbf{R}(t)\begin{bmatrix}\mathbf{u}\\1\end{bmatrix} + 2\mathbf{x}^T\mathbf{N}(t)\mathbf{u},$$

$$\forall t \in [t_0, t_f], \quad \mathbf{Q}(t) = \begin{bmatrix}\mathbf{Q}_{xx}(t) & \mathbf{Q}_x(t)\\ \mathbf{Q}_x^T(t) & q_0(t)\end{bmatrix}, \mathbf{Q}_{xx}(t) \succeq 0, \tag{37}$$

$$\mathbf{R}(t) = \begin{bmatrix}\mathbf{R}_{uu}(t) & \mathbf{R}_u(t)\\ \mathbf{R}_u^T(t) & r_0(t)\end{bmatrix}, \mathbf{R}_{uu}(t) \succ 0.$$

$$g(\mathbf{x},\mathbf{u},t) = \begin{bmatrix}\mathbf{x}\\1\end{bmatrix}^T \mathbf{Q}(t)\begin{bmatrix}\mathbf{x}\\1\end{bmatrix} + \begin{bmatrix}\mathbf{u}\\1\end{bmatrix}^T \mathbf{R}(t)\begin{bmatrix}\mathbf{u}\\1\end{bmatrix} + 2\mathbf{x}^T\mathbf{N}(t)\mathbf{u}, \tag{38}$$

In **case of tracking**,

$$\begin{aligned} g(\mathbf{x},\mathbf{x_d},\mathbf{u},t) = (\mathbf{x(t)} - \mathbf{x}_d(t))^T\mathbf{Q}_t(\mathbf{x}(t)-\mathbf{x}_d(t)) \\ +(\mathbf{u}(t)-\mathbf{u}_d(t))^T\mathbf{R}_t(\mathbf{u}(t)-\mathbf{u}_d(t)) + 2(\mathbf{x}(t)-\mathbf{x}_d(t))^T\mathbf{N}_t(\mathbf{u}(t)-\mathbf{u}_d(t)), \end{aligned} \tag{39}$$

where

$$\begin{aligned} &\mathbf{Q}_{xx} = \mathbf{Q}_t, \quad \mathbf{Q}_x = -\mathbf{Q}_t\mathbf{x}_d - \mathbf{N}_t\mathbf{u}_d, \quad q_0 = \mathbf{x}_d^T\mathbf{Q}_t\mathbf{x}_d + 2\mathbf{x}_d^T\mathbf{N}_t\mathbf{u}_d, \\ &\mathbf{R}_{uu} = \mathbf{R}_t, \quad \mathbf{R}_u = -\mathbf{R}_t\mathbf{u}_d - \mathbf{N}_t^T\mathbf{x}_d, \quad r_0 = \mathbf{u}_d^T\mathbf{R}_t\mathbf{u}_d, \quad \mathbf{N} = \mathbf{N}_t. \end{aligned}$$

The optimal **state feedback control problem** can be formulated as

$$\forall \mathbf{x}, \quad 0 = \min_{\mathbf{u}} \left[g(\mathbf{x}, \mathbf{u}, t) + \frac{\partial J^*}{\partial \mathbf{x}} (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c}) + \frac{\partial J^*}{\partial t} \right]. \tag{40}$$

The optimal **linear tracking problem** can be formulated as

$$\forall \mathbf{x}, \quad 0 = \min_{\mathbf{u}} \left[g(\mathbf{x}, \mathbf{x_d}, \mathbf{u}, t) + \frac{\partial J^*}{\partial \mathbf{x}} (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c}) + \frac{\partial J^*}{\partial t} \right]. \tag{41}$$

The **Hamiltonian-Jacobi-Belman approach** can be employed to solve this.

Define the **Hamiltonian for optimal feedback control**

$$H = g(\mathbf{x}, \mathbf{u}, t) + \left[\frac{\partial J(\mathbf{x}(t), t)}{\partial x}\right]^\top f_c(\mathbf{x}(t), \mathbf{u}(t), t) = 0 \tag{42}$$

Define the **Hamiltonian for optimal linear tracking** problem

$$H = g(\mathbf{x}, \mathbf{x_d}, \mathbf{u}, t) + \left[\frac{\partial J(\mathbf{x}^*(t), t)}{\partial x}\right]^\top f_c(\mathbf{x}(t), \mathbf{u}(t), t) = 0 \tag{43}$$

- Minimize the H with respect to $\mathbf{u}(t)$, i.e., $\frac{\partial H}{\partial u} = 0$, for solving $\mathbf{u}^*(t)$

- Minimize the H with respect to $\mathbf{u}(t)$, i.e., $\frac{\partial H}{\partial u} = 0$, for solving $\mathbf{u}^*(t)$
- When considering the **terminal condition**: $J(\mathbf{x}(t_f), t_f) = h(t_f) = \left(\mathbf{x}(t_f) - \mathbf{x}_d(t_f)\right)^T \mathbf{Q}_f \left(\mathbf{x}(t_f) - \mathbf{x}_d(t_f)\right)$ whose solution provides an expression for $\mathbf{u}^*$.

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- Since the cost function is quadratic, the control input $\mathbf{u}^*$ is in terms of J, to seek feedback control, i.e., $\mathbf{u}^*$, in terms of $\mathbf{x}(t)$, it is reasonable to consider [next slide]

General case

$$J^*(\mathbf{x}(t),t) = \begin{bmatrix}\mathbf{x}\\1\end{bmatrix}^T \mathbf{P}(t)\begin{bmatrix}\mathbf{x}\\1\end{bmatrix}, \quad \mathbf{P}(t) = \begin{bmatrix}\mathbf{P}_{xx}(t) & \mathbf{p}_x(t)\\ \mathbf{p}_x^T(t) & \mathbf{p}_0(t)\end{bmatrix}, \mathbf{P}_{xx}(t) = \mathbf{P}_{xx}^\top \succeq 0$$
$$\frac{\partial J(\mathbf{x}(t),t)}{\partial t} = \begin{bmatrix}\mathbf{x}\\1\end{bmatrix}^T \dot{\mathbf{P}}(t)\begin{bmatrix}\mathbf{x}\\1\end{bmatrix}.$$
$$\frac{\partial J(\mathbf{x}(t),t)}{\partial x} = 2\mathbf{x}^\top \mathbf{P}_{xx}(t) + 2\mathbf{p}_x^T(t) \tag{44}$$

Tracking

$$J^*(\mathbf{x}(t),t) = \mathbf{x}^\top(t)\mathbf{P}_{xx}(t)\mathbf{x}(t) + 2\mathbf{x}^\top \mathbf{p}_x(t) + \mathbf{p}_0(t)$$
$$\frac{\partial J(\mathbf{x}(t),t)}{\partial t} = \mathbf{x}^\top \dot{\mathbf{P}}_{xx}(t)\mathbf{x} + \mathbf{x}^\top \dot{\mathbf{p}}_x + \dot{\mathbf{p}}_0(t) \tag{45}$$
$$\frac{\partial J(\mathbf{x}(t),t)}{\partial x} = 2\mathbf{x}^\top \mathbf{P}_{xx}(t) + 2\mathbf{p}_x^T(t)$$

General case

$$\begin{aligned}\frac{\partial H}{\partial \mathbf{u}} &= 2\mathbf{u}^T\mathbf{R}_{uu} + 2\mathbf{R}_u^T + 2\mathbf{x}^T\mathbf{N} + (2\mathbf{x}^T\mathbf{P}_{xx} + 2\mathbf{p}_x^T)\mathbf{B} = 0 \\ \mathbf{u}^* &= -\mathbf{R}_{uu}^{-1}\begin{bmatrix}\mathbf{N}+\mathbf{P}_{xx}\mathbf{B} \\ \mathbf{R}_u^T + \mathbf{p}_x^T\mathbf{B}\end{bmatrix}^T \begin{bmatrix}\mathbf{x} \\ 1\end{bmatrix} = -\mathbf{K}(t)\begin{bmatrix}\mathbf{x} \\ 1\end{bmatrix} = -\mathbf{K}_x(t)\mathbf{x} - \mathbf{k}_0(t).\end{aligned} \tag{46}$$

Tracking

$$\begin{aligned}\frac{\partial H}{\partial \mathbf{u}} &= 2\Big(\mathbf{u}(t) - \mathbf{u}_d(t)\Big)^T\mathbf{R} + \left[\frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right]^\top B = 2\Big(\mathbf{u}(t) - \mathbf{u}_d(t)\Big)^T\mathbf{R} \\ &\quad + \Big(2\mathbf{x}^\top\mathbf{P}_{xx}(t) + 2\mathbf{p}_x^T(t)\Big)B = 0 \\ \mathbf{u}^* &= \mathbf{u}_d(t) - \mathbf{R}^{-1}\mathbf{B}^\top\Big(\mathbf{P}_{xx}(t)\mathbf{x} + \mathbf{p}_x(t)\Big)\end{aligned} \tag{47}$$

H substituting with the optimal $u^*(t)$ and solve for HJB

$$H^* + \frac{\partial J^*(\mathbf{x}(t), t)}{\partial t} = 0 \tag{48}$$

After solving the updated **Differential Riccati Equation** by setting each individually equal to zero, yielding:

General case

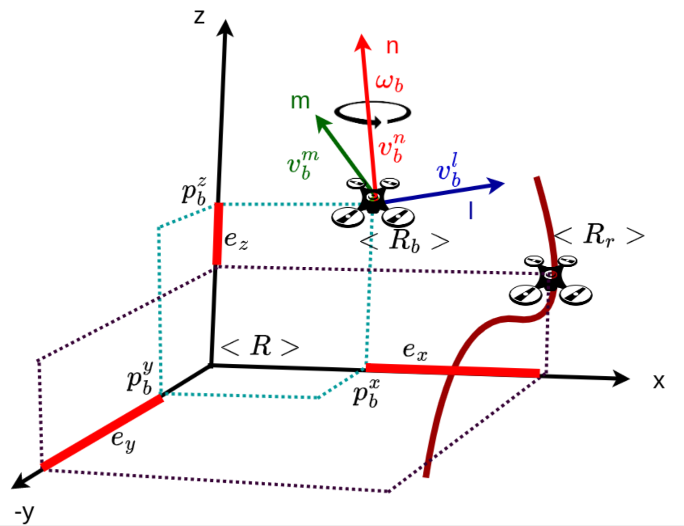
$$
\begin{aligned}
-\dot{\mathbf{P}}_{xx} =& \mathbf{Q}_{xx} - (\mathbf{N} + \mathbf{P}_{xx}\mathbf{B})\mathbf{R}_{uu}^{-1}(\mathbf{N} + \mathbf{P}_{xx}\mathbf{B})^T + \mathbf{P}_{xx}\mathbf{A} + \mathbf{A}^T\mathbf{P}_{xx}, \\
-\dot{\mathbf{p}}_x =& \mathbf{Q}_x - (\mathbf{N} + \mathbf{P}_{xx}\mathbf{B})\mathbf{R}_{uu}^{-1}(\mathbf{R}_u + \mathbf{B}^T\mathbf{p}_x) + \mathbf{A}^T\mathbf{p}_x + \mathbf{P}_{xx}\mathbf{c}, \\
-\dot{p}_0 =& q_0 + r_0 - (\mathbf{R}_u + \mathbf{B}^T\mathbf{p}_x)^T\mathbf{R}_{uu}^{-1}(\mathbf{R}_u + \mathbf{B}^T\mathbf{p}_x) + 2\mathbf{p}_x^T\mathbf{c},
\end{aligned}
\tag{49}
$$

Tracking

$$
\begin{aligned}
-\dot{\mathbf{P}}_{xx} =& \mathbf{Q} - \mathbf{P}_{xx}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top\mathbf{P}_{xx} + \mathbf{P}_{xx}\mathbf{A} + \mathbf{A}^T\mathbf{P}_{xx}, \\
-\dot{\mathbf{p}}_x =& -\mathbf{Q}\mathbf{x}_d + \left(\mathbf{A}^T - \mathbf{P}_{xx}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\right)\mathbf{p}_x + \mathbf{P}_{xx}\mathbf{B}\mathbf{u}_d \\
-\dot{p}_0 =& \mathbf{x}_d^\top\mathbf{Q}\mathbf{x}_d - \mathbf{p}_x^\top\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{p}_x + 2\mathbf{p}_x^\top\mathbf{B}\mathbf{u}_d
\end{aligned}
\tag{50}
$$

with terminal conditions $\mathbf{P}_{xx}(t_f) = \mathbf{Q}_f$, $\mathbf{p}_x(t_f) = -\mathbf{Q}_f\mathbf{x}_d(t_f)$, and $p_0(t_f) = \mathbf{x}_d^T(t_f)\mathbf{Q}_f\mathbf{x}_d(t_f)$ https://underactuated.csail.mit.edu/lqr.html

Reference trajectory tracking scheme



The simplified motion model is expressed by $\dot{\mathbf{x}}_b = \mathbf{f}_c(\mathbf{x}_b, \mathbf{u}_b)$ over the fixed frame of reference $< R >$ is defined as:

$$\dot{\mathbf{x}}_b = \begin{bmatrix} \dot{p}_b^x \\ \dot{p}_b^y \\ \dot{p}_b^z \\ \dot{\alpha}_b^z \end{bmatrix} = \begin{bmatrix} v_b^l cos(\alpha_b) - v_b^m sin(\alpha_b) \\ v_b^l sin(\alpha_b) + v_b^m cos(\alpha_b) \\ v_b^n \\ \omega_b \end{bmatrix} = \underbrace{\begin{bmatrix} cos(\alpha_a) & -sin(\alpha_a) & 0 & 0 \\ sin(\alpha_a) & cos(\alpha_a) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\varrho_{\mathbf{b}}} \underbrace{\begin{bmatrix} v_b^l \\ v_b^m \\ v_b^n \\ \omega_b \end{bmatrix}}_{\mathbf{u}_b} \tag{51}$$

where $\mathbf{f}_c(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ and $n_x = n_u = 4$. The system states $\mathbf{x}_b = [p_b^l, p_b^m, p_b^n, \alpha_b]^T \in \mathbb{R}^{n_x}$ and control inputs $\mathbf{u}_b = [v_b^l, v_b^m, v_b^n, \omega_b]^T \in \mathbb{R}^{n_u}$. p_b^i and $v_b^i, i \in \{l, m, n\}$ denote the quadrotor center position(m) and velocity (m/s) in each direction, i.e., l (front), m (lateral), and n (altitude), in the local coordinate frame; α_b and ω_b denote the yaw angle (rad) and yaw rate (rad/s) around n direction, respectively.

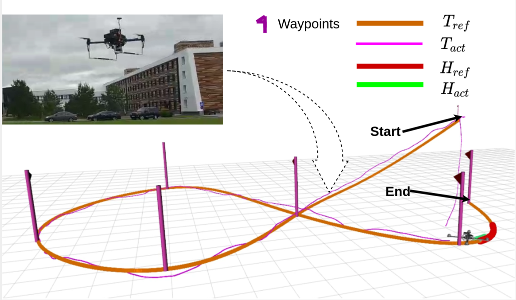
The tracking problem can be formulated as a discrete dynamical model. Forward Euler discretization, $\mathbf{x}_{k+1} = \mathbf{f}_d(\mathbf{x}_k, \mathbf{u}_k)$ is introduced for a given sampling period in seconds, $\delta \in \mathbb{R} > 0$, e.g., $\delta = 0.05s$

$$\mathbf{x}_b^{k+1} = \mathbf{x}_b^k + \delta \cdot \varrho_b^k \cdot \mathbf{u}_b^k, \tag{52}$$

where $\mathbf{f}_d(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$.

Formulation of uniform cubic B-spline

Knot sequence $p^{knot} = \{t_0, t_1, ..., t_{n_k}\}$ and control points $p^{ref} = \{p_0, p_1, ..., p_{n_p}\}$, where $t_* \in \mathbb{R}$, $p_* \in \mathbb{R}^d$ and $n_k = n_p + d + 1$; $*$ denotes the indexing of p^{ref} and p^{knot} and $p_i^{ref} = < x_i, y_i, z_i >$ in $\mathbb{R}^3$ where $i = 0, ..., n_p$



Reference position $c^{ref}(k)$ or velocity $\dot{c}^{ref}(k)$ estimation for a time t

1. Time index t, corresponding position, $c^{ref}(t) = DeBoorCox(t, p^{ref})$, $c^{ref}(t) \in \mathbf{R}^3$
2. Reference velocity and acceleration are estimated by taking first and second derivative of p^{ref} and $c^{ref^{(*)}}(t) = DeBoorCox(t, p^{ref^{(*)}})$
3. $c^{ref}(k)$ is continuous and it may or may not pass through control points due to B-spline interpolation

Virtual quadrotor that moves on a generated reference trajectory can be formulated using the same **simplified motion model** defined in eq.(51):

$$\dot{p}^{ref} = \dot{\mathbf{x}}_r = \begin{bmatrix} \dot{p}_r^x \\ \dot{p}_r^y \\ \dot{p}_r^z \\ \dot{\alpha}_r^z \end{bmatrix} = \underbrace{\begin{bmatrix} cos(\alpha_r) & -sin(\alpha_r) & 0 & 0 \\ sin(\alpha_r) & cos(\alpha_r) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\varrho_{\mathbf{r}}} \underbrace{\begin{bmatrix} v_r^l \\ v_r^m \\ v_r^n \\ \omega_r \end{bmatrix}}_{\mathbf{u}_r}, \qquad (53)$$

where control inputs (velocities) $\mathbf{u}_r = [v_r^l, v_r^m, v_r^n, \omega_r]$, in which linear velocities of **front, lateral, and altitude** are defined as v_r^l, v_r^m, and v_r^n, respectively. **Angular velocity** around **altitude** is given by ω_r. The **transformation of reference control velocities** into **reference frame**, denoted ϱ_r.

The quadrotor **trajectory tracking error**

$$\hat{\mathbf{x}} = \varrho \bar{\mathbf{x}}$$

where **the proportional error**

$$\bar{\mathbf{x}} = \mathbf{x}_r - \mathbf{x}_b = [p_r^x - p_b^x,\, p_r^y - p_b^y,\, p_r^z - p_b^z,\, \alpha_r - \alpha_b]$$

where the proportional error with respect to reference frame $< R >$ is given by $\hat{\mathbf{x}} = [e_x,\, e_y,\, e_z\, , e_\alpha]$, and the transformation to the quadrotor frame

$$\varrho = {\varrho_{\mathbf{b}}}^{-1}$$

Therefore,

$$\hat{\mathbf{x}} = \begin{bmatrix} e_x \\ e_y \\ e_z \\ e_\alpha \end{bmatrix} = \begin{bmatrix} cos(\alpha_b) & sin(\alpha_b) & 0 & 0 \\ -sin(\alpha_b) & cos(\alpha_b) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_r^x - p_b^x \\ p_r^y - p_b^y \\ p_r^z - p_b^z \\ \alpha_r - \alpha_b \end{bmatrix} \tag{54}$$

The **residual dynamics** or error dynamics can be obtained by differentiating the error model eq.(54) with respect to time:

$$\dot{\hat{\mathbf{x}}} = \dot{\varrho}\bar{\mathbf{x}} + \varrho\dot{\bar{\mathbf{x}}}$$

$$\dot{\hat{\mathbf{x}}} = -\dot{\alpha}_b \begin{bmatrix} sin(\alpha_b) & -cos(\alpha_b) & 0 & 0 \\ cos(\alpha_b) & sin(\alpha_b) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \bar{\mathbf{x}} + \varrho \begin{bmatrix} \dot{p}_r^x \\ \dot{p}_r^y \\ \dot{p}_r^z \\ \dot{\alpha}_r \end{bmatrix} - \varrho \begin{bmatrix} \dot{p}_b^x \\ \dot{p}_b^y \\ \dot{p}_b^z \\ \dot{\alpha}_b \end{bmatrix} \tag{55}$$

Considering the simplified kinematic model eq.(51) and $\varrho = {\varrho_{\mathbf{b}}}^{-1}$, the eq.(55) can be rewritten as

$$\dot{\hat{\mathbf{x}}} = -\dot{\alpha}_b \begin{bmatrix} sin(\alpha_b) & -cos(\alpha_b) & 0 & 0 \\ cos(\alpha_b) & sin(\alpha_b) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \bar{\mathbf{x}} + \varrho \begin{bmatrix} \dot{p}_r^x \\ \dot{p}_r^y \\ \dot{p}_r^z \\ \dot{\alpha}_r \end{bmatrix} - \begin{bmatrix} v_b^l \\ v_b^m \\ v_b^n \\ \omega_b \end{bmatrix} \tag{56}$$

Reference trajectory tracking

The reference velocity $\dot{\mathbf{x}}_r$ in the fixed reference frame

$$\varrho_r \mathbf{u}_r$$

Further, by considering the simplified reference motion model eq.(53) and the proportional error estimation eq.(54), the eq.(56) result in

$$\dot{\hat{\mathbf{x}}} = -\dot{\alpha}_b \begin{bmatrix} -e_y \\ e_x \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} cos(e_\alpha) & -sin(e_\alpha) & 0 & 0 \\ sin(e_\alpha) & cos(e_\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_r^l \\ v_r^m \\ v_r^n \\ \omega_r \end{bmatrix} - \begin{bmatrix} v_b^l \\ v_b^m \\ v_b^n \\ \omega_b \end{bmatrix} \quad (57)$$

Eq.57 can be rearranged considering $\dot{\alpha}_b = \omega_b$,

$$\dot{\hat{\mathbf{x}}} = \begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \\ \dot{e}_\alpha \end{bmatrix} = \begin{bmatrix} \omega_b e_y - v_b^l + v_r^l cos(e_\alpha) - v_r^m sin(e_\alpha) \\ -\omega_b e_x - v_b^m + v_r^l sin(e_\alpha) + v_r^m cos(e_\alpha) \\ v_r^n - v_b^n \\ \omega_r - \omega_b \end{bmatrix} \tag{58}$$

In order to design a **linear feedback control law, error dynamics** are linearized around the **desired at the operating point**. In such an operational point, the following assumptions are made:

$$cos(e_\alpha) \approx 1, \quad sin(e_\alpha) \approx e_\alpha, e_\alpha \approx 0$$

Thus, linearized error model for the nonlinear error model eq.(58) is expressed as follows:

$$\dot{\hat{\mathbf{x}}} = \begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \\ \dot{e}_\alpha \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \omega_b & 0 & 0 \\ -\omega_b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} e_x \\ e_y \\ e_z \\ e_\alpha \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} v_r^l - v_b^l \\ v_r^m - v_b^m \\ v_r^n - v_b^n \\ \omega_r - \omega_b \end{bmatrix}}_{\Delta\mathbf{U}} \quad (59)$$

The **trajectory tracking error**, i.e., position error, converges to zero when applying the optimal $\Delta\mathbf{U}^*$ to the system. Since the approximated motion model is linearized, **Linear Quadratic Regulator (LQR)** can be employed to obtain $\Delta\mathbf{U}^*$.

$$J = \frac{1}{2}\int_0^\infty (\hat{\mathbf{x}}^\top Q\hat{\mathbf{x}} + \Delta\mathbf{U}^\top R\Delta\mathbf{U})dt, \tag{60}$$

where $Q \preceq 0 \in R^{4\times4}$ and $R \prec 0 \in R^{4\times4}$ are positive semi-definite and positive definite matrices, respectively, which contains the weights for positional error correction.

After minimizing the cost J, a control law $\Delta \mathbf{U}^* = -K\mathbf{x}$, where K is the optimal control gain and given state of the quadrotor $\mathbf{x}$, can be obtained. Hence, the optimal control commands are calculated as follows:

$$\bar{\mathbf{U}} = -K\mathbf{x} + \mathbf{U}_{ref}, \tag{61}$$

where reference control command, denoted $\mathbf{U}_{ref}$. Eq.(61) can be expressed in the following way:

$$\begin{bmatrix} \hat{v}_b^l \\ \hat{v}_b^m \\ \hat{v}_b^n \\ \hat{\omega}_b \end{bmatrix} = \underbrace{\begin{bmatrix} \Delta v_*^l \\ \Delta v_*^m \\ \Delta v_*^n \\ \Delta \omega_* \end{bmatrix}}_{\Delta \mathbf{U}^*} + \begin{bmatrix} v_r cos(e_\alpha) \\ v_r sin(e_\alpha) \\ v_r^n \\ \omega_r \end{bmatrix} \tag{62}$$

Reference trajectory tracking problem formulation with LQR (Linear Quadratic Regulator)