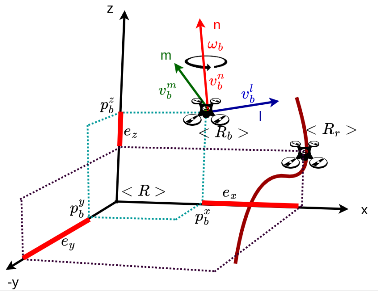
Motion Planning for Autonomous Vehicles

Linear Quadratic Regulator (LQR)

Geesara Kulathunga

April 1, 2023



Linear Quadratic Regulator

Contents

In general, discrete linear system, which can be either LTI or LTV, dynamics is described by:

$$\mathbf{x}_{k+1} = \mathbf{f}_d(\mathbf{x}_k, \mathbf{u}_k) = A_k \mathbf{x}_k + B_k \mathbf{u}_k, \tag{1}$$

where $k = 0, \ldots, n$, $\mathbf{x}_k \in \mathbb{R}^n$, and $\mathbf{u}_k \in \mathbb{R}^m$. For the continuous time system

$$\dot{\mathbf{x}} = \mathbf{f}_c(t) = A(t)\mathbf{x}(\mathbf{t}) + B(t)\mathbf{u}(\mathbf{t}) \tag{2}$$

If the system dynamics is non-linear, A_k and B_k are recalculated by linearizing the $\mathbf{f}_c$ at each time index.

Since linearization has to be carried out in each iteration, **ILQR and ELQR** are such variants, consider nominal trajectory, $\mathbf{x_0(t)}, \mathbf{u_0(t)} \quad \forall\, t[t_1, t_2]$.
Using first-order Taylor series approximation, the increment $\Delta\dot{\mathbf{x}} = \dot{\mathbf{x}} - \dot{\mathbf{x}}_0 = \mathbf{f_c}(\mathbf{x}, \mathbf{u}) - \mathbf{f_c}(\mathbf{x_0}, \mathbf{u_0})$ can be expressed by

$$\begin{aligned}\Delta\dot{\mathbf{x}} &\approx \mathbf{f_c}(\mathbf{x_0}, \mathbf{u_0}) + \frac{\partial \mathbf{f_c}(\mathbf{x_0}, \mathbf{u_0})}{\partial \mathbf{x}}(\mathbf{x} - \mathbf{x_0}) + \frac{\partial \mathbf{f_c}(\mathbf{x_0}, \mathbf{u_0})}{\partial \mathbf{u}}(\mathbf{u} - \mathbf{u_0}) - \mathbf{f_c}(\mathbf{x_0}, \mathbf{u_0}) \\ &= A(t)\Delta\mathbf{x(t)} + B(t)\Delta\mathbf{u(t)}\end{aligned} \tag{3}$$

where $\Delta\mathbf{x(t)} = \mathbf{x(t)} - \mathbf{x(t_0)}$ and $\Delta\mathbf{u(t)} = \mathbf{u(t)} - \mathbf{u(t_0)}$ and $A(t) = \frac{\partial \mathbf{f}_c}{\partial \mathbf{x}}(\mathbf{x}_0, \mathbf{u}_0), \quad B(t) = \frac{\partial \mathbf{f}_c}{\partial \mathbf{u}}(\mathbf{x}_0, \mathbf{u}_0)$.

LQR Formulation

Consider **initial state** x_0 at each time instance t_0 is given, the objective is to **find the optimal control input sequence** $\mathbf{u}$ for a given initial condition x_0, to reach the final state x_T, i.e., **estimate the optimal state prediction**, an optimal control sequence (or control policy) has to be calculated.

Such a control policy can be estimated by minimizing the following quadratic cost:

$$J(\mathbf{x},\mathbf{u}) = \underbrace{\|x_n\|^2_{Q_n}}_{\text{terminal cost}} + \underbrace{\sum_{k=0}^{n-1} \|x_k\|^2_Q + \|u_k\|^2_R}_{\text{running cost}} \tag{4}$$

$$J(\mathbf{x},\mathbf{u}) = \int_0^{\infty} \left(\|x(t)\|^2_Q + \|u(t)\|^2_R \right) dt,$$

where $k \in \{0,1,\dots,n-1\}$, $Q, Q_n \in \mathbb{R}^{n_x \times n_x}, R \in \mathbb{R}^{n_u \times n_u}, P \in \mathbb{R}^{n_x \times n_x}$ are predefined in which $\mathbf{Q} = \mathbf{Q}^\top \succeq \mathbf{0}$ is a **positive definite** and $\mathbf{R} = \mathbf{R}^\top \succ 0$ is a **positive semi-definite**. However, if the **system is nonlinear**, need to estimate the second-order approximation of the non-linear cost functions to **define Q(t) and R(t)**.

LQR via least squares

- For a linear system

$$
\begin{aligned}
\min_{\mathbf{u}} \quad & \sum_{k=0}^{n-1} \mathbf{x}_k^\top \mathbf{Q_k} \mathbf{x}_k + \mathbf{u}_k^\top \mathbf{R_k} \mathbf{u}_k + \mathbf{x}_n^\top \mathbf{Q_n} \mathbf{x}_n, \mathbf{Q_k} = \mathbf{Q_k}^\top \succeq 0, \mathbf{R_k} = \mathbf{R_k}^\top \succ 0 \\
\text{s.t.} \quad & \mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \\
& \mathbf{x}_0
\end{aligned} \tag{5}
$$

https://markcannon.github.io/assets/downloads/teaching/C21_Model_Predictive_Control/mpc_notes.pdf

- For a linear system

$$\begin{aligned}
\min_{\mathbf{u}} \quad & \sum_{k=0}^{n-1} \mathbf{x}_k^\top \mathbf{Q_k} \mathbf{x}_k + \mathbf{u}_k^\top \mathbf{R_k} \mathbf{u}_k + \mathbf{x}_n^\top \mathbf{Q_n} \mathbf{x}_n, \mathbf{Q_k} = \mathbf{Q_k}^\top \succeq 0, \mathbf{R_k} = \mathbf{R_k}^\top \succ 0 \\
\text{s.t.} \quad & \mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \\
& \mathbf{x}_0
\end{aligned} \tag{5}$$

- The state prediction sequence can be written in a compact sequence as follows:

$$\mathbf{x} = Mx_0 + C\mathbf{u}, \quad M = \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & \dots & 0 \\ B & & & \\ AB & B & & \\ \vdots & \vdots & \ddots & \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix}$$

https://markcannon.github.io/assets/downloads/teaching/C21_Model_Predictive_Control/mpc_notes.pdf

LQR via least squares

- The defined quadratic cost (5) can be written in terms of $\mathbf{x}$ and $\mathbf{u}$ as

$$J = \mathbf{x}^\top \tilde{Q}\mathbf{x} + \mathbf{u}^\top \tilde{R}\mathbf{u} = \mathbf{u}^\top H\mathbf{u} + 2x_0^\top F^\top \mathbf{u} + x_0^\top G x_0 \tag{6}$$

- The defined quadratic cost (5) can be written in terms of $\mathbf{x}$ and $\mathbf{u}$ as

$$J = \mathbf{x}^\top \tilde{Q} \mathbf{x} + \mathbf{u}^\top \tilde{R} \mathbf{u} = \mathbf{u}^\top H \mathbf{u} + 2 x_0^\top F^\top \mathbf{u} + x_0^\top G x_0 \tag{6}$$

- Can you define the $\tilde{Q}$ and $\tilde{R}$? as well as prove that H, F, and G are given by $C^\top \tilde{Q} C + \tilde{R}$, $C^\top \tilde{Q} M$, and $M^\top \tilde{Q} M$, respectively.

- The defined quadratic cost (5) can be written in terms of $\mathbf{x}$ and $\mathbf{u}$ as

$$J = \mathbf{x}^\top \tilde{Q} \mathbf{x} + \mathbf{u}^\top \tilde{R} \mathbf{u} = \mathbf{u}^\top H \mathbf{u} + 2 x_0^\top F^\top \mathbf{u} + x_0^\top G x_0 \tag{6}$$

- Can you define the $\tilde{Q}$ and $\tilde{R}$? as well as prove that H, F, and G are given by $C^\top \tilde{Q} C + \tilde{R}$, $C^\top \tilde{Q} M$, and $M^\top \tilde{Q} M$, respectively.

- If no additional constraints are given, eq.6 has a **closed-form solution** that is derived by minimizing the J with respect to $\mathbf{u}$. Show that $\mathbf{u}^* = -H^{-1} F x_0$.

- The defined quadratic cost (5) can be written in terms of $\mathbf{x}$ and $\mathbf{u}$ as

$$J = \mathbf{x}^\top \tilde{Q}\mathbf{x} + \mathbf{u}^\top \tilde{R}\mathbf{u} = \mathbf{u}^\top H\mathbf{u} + 2x_0^\top F^\top \mathbf{u} + x_0^\top G x_0 \qquad (6)$$

- Can you define the $\tilde{Q}$ and $\tilde{R}$? as well as prove that H, F, and G are given by $C^\top \tilde{Q} C + \tilde{R}$, $C^\top \tilde{Q} M$, and $M^\top \tilde{Q} M$, respectively.

- If no additional constraints are given, eq.6 has a **closed-form solution** that is derived by minimizing the J with respect to $\mathbf{u}$. Show that $\mathbf{u}^* = -H^{-1} F x_0$.

- What can you say about when H is **singular whose determinant is 0 (the rank is given by non-zero eigenvalues)** (i.e., positive semi-definite rather than positive definite); this implies **multiple optimal solutions** can exist.

Since H and F are constant matrices, which can be calculated offline, at every sampling time, the first element of the optimal control can be applied to the system. This is called **time-invariant feedback controller**.

$$\mathbf{u} = Lx$$

where $L = -[I_{n_u}\ 0\ 0\ , ..., \ 0]H^{-1}F$.

Example 01

Estimate feedback control law, considering the following system with

$$A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & -1 \end{bmatrix} \tag{7}$$

for horizon N = 4, you may assume $Q = D^\top D$, $R = 0.01$.

The continuous time system or the plant is expressed as

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(\mathbf{t}) + B(t)\mathbf{u}(\mathbf{t}) = f_c(\mathbf{x}(t), \mathbf{u}(t), t) \tag{8}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, and $\mathbf{u}(t) \in \mathbb{R}^m$. And performance index is defined as:

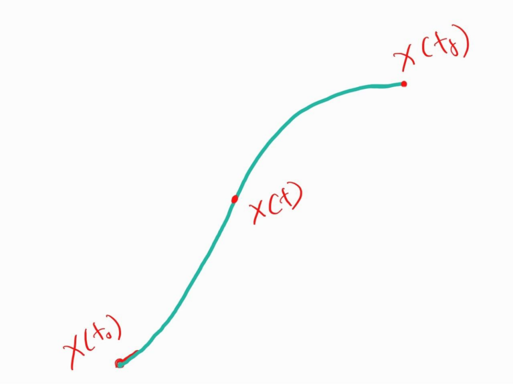
$$J(\mathbf{x}(t), \mathbf{u}(t), t_0, t_f) = Q(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)dt \tag{9}$$

The objective is to find the **optimal feedback control minimizing the above cost function**, i.e., the optimal solution from any time instance t to the final time t_f as:

$$\begin{aligned} J^*(\mathbf{x}(t_0), t_0, t_f) &= \int_t^{t_f} g(\mathbf{x}^*(\tau), \mathbf{u}^*(\tau), \tau)d\tau, \\ \Rightarrow V(\mathbf{x}(t_0), t_0, t_f) &= \min_{\mathbf{u}(t)} \Big(J(\mathbf{x}(t), \mathbf{u}(t), t_0, t_f) \Big) \end{aligned} \tag{10}$$

Hence, $V(\mathbf{x}(t_0), t_0, t_f)$ **does not depend of** $\mathbf{u}$

Bellman Optimality



$$V(\mathbf{x}(t_0), t_0, t_f) = V(\mathbf{x}(t_0), t_0, t) + V(\mathbf{x}(t), t, t_f) \tag{11}$$

Hamilton Jacobi Bellman Approach

Taking time derivative

$$
\begin{aligned}
V(\mathbf{x}(t_0), t_0, t_f) &= \min_{\mathbf{u}(t)} \Big(J(\mathbf{x}(t), \mathbf{u}(t), t_0, t_f) \Big) \\
\frac{dV(\mathbf{x}(t), t, t_f)}{dt} &= \left[\frac{\partial V(\mathbf{x}(t), t, t_f)}{\partial x} \right]^\top \dot{x}(t) + \frac{\partial V(\mathbf{x}(t), t, t_f)}{\partial t} \\
&= \min_{\mathbf{u}(t)} \frac{d}{dt} \Big(Q(\mathbf{x}(t_f), t_f) + \int_t^{t_f} g(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau) d\tau \Big) \\
&= \min_{\mathbf{u}(t)} \Big(\frac{d}{dt} \int_t^{t_f} g(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau) d\tau \Big), \quad \frac{d}{dt} \Big(Q(\mathbf{x}(t_f), t_f) \Big) = 0 \\
&= \min_{\mathbf{u}(t)} - g(\mathbf{x}(t), \mathbf{u}(t), t) \quad \text{where } g(\mathbf{x}(t_f), t_f) \text{ is a constant}
\end{aligned} \tag{12}
$$

$$
\Rightarrow -\frac{\partial V(\mathbf{x}(t), t, t_f)}{\partial t} = \min_{\mathbf{u}(t)} \left(\Big(\frac{\partial V(\mathbf{x}(t), t, t_f)}{\partial x} \Big)^\top \dot{x}(t) + g(\mathbf{x}(t), \mathbf{u}(t), t) \right)
$$

- Given system dynamics and the performance index, the Hamiltonian can be determined as

$$H = g(\mathbf{x}(t), \mathbf{u}(t), t) + \underbrace{\left[\frac{\partial J(\mathbf{x}(t), t)}{\partial x}\right]^\top}_{\lambda^\top} \dot{x}(t) = 0 \tag{13}$$

- Given system dynamics and the performance index, the Hamiltonian can be determined as

$$H = g(\mathbf{x}(t), \mathbf{u}(t), t) + \underbrace{\left[\frac{\partial J(\mathbf{x}(t), t)}{\partial x}\right]^\top}_{\lambda^\top} \dot{x}(t) = 0 \tag{13}$$

- After considering the boundary conditions: $J^*(\mathbf{x}^*(t_f), t_f) = \frac{1}{2}\mathbf{x}(t_f)^\top Q(t_f)\mathbf{x}(t_f)$,

$$\underbrace{\min_{\mathbf{u}(t)}\left(\left[\frac{\partial V(\mathbf{x}(t), t, t_f)}{\partial x}\right]^\top \dot{x}(t) + g(\mathbf{x}(t), \mathbf{u}(t), t)\right)}_{H^*} + \frac{\partial V(\mathbf{x}(t), t, t_f)}{\partial t} = 0$$

- Given system dynamics and the performance index, the Hamiltonian can be determined as

$$H = g(\mathbf{x}(t), \mathbf{u}(t), t) + \underbrace{\left[\frac{\partial J(\mathbf{x}(t), t)}{\partial x}\right]^{\top}}_{\lambda^{\top}} \dot{x}(t) = 0 \tag{13}$$

- After considering the boundary conditions: $J^*(\mathbf{x}^*(t_f), t_f) = \frac{1}{2}\mathbf{x}(t_f)^{\top} Q(t_f)\mathbf{x}(t_f)$,

$$\underbrace{\min_{\mathbf{u}(t)} \left(\left[\frac{\partial V(\mathbf{x}(t), t, t_f)}{\partial x}\right]^{\top} \dot{x}(t) + g(\mathbf{x}(t), \mathbf{u}(t), t) \right)}_{H^*} + \frac{\partial V(\mathbf{x}(t), t, t_f)}{\partial t} = 0$$

- This equation is called the Hamilton-Jacobi equation. Since it is used in Bellman's dynamic programming, it is also known as Hamilton-Jacobi-Bellman (HJB) equation.

Hence, the procedure for the HJB approach is as follows:

1. Define the Hamiltonian

$$H = g(\mathbf{x}(t), \mathbf{u}(t), t) + \left[\frac{\partial J(\mathbf{x}(t), t)}{\partial x}\right]^\top \dot{x}(t) = 0 \tag{14}$$

Hence, the procedure for the HJB approach is as follows:

1. Define the Hamiltonian

$$H = g(\mathbf{x}(t), \mathbf{u}(t), t) + \left[\frac{\partial J(\mathbf{x}(t), t)}{\partial x} \right]^{\top} \dot{x}(t) = 0 \tag{14}$$

2. Minimize the H with respect to u(t), i.e., $\frac{\partial H}{\partial u} = 0$, for solving $\mathbf{u}^*(t)$

Hence, the procedure for the HJB approach is as follows:

1. Define the Hamiltonian

$$H = g(\mathbf{x}(t), \mathbf{u}(t), t) + \left[\frac{\partial J(\mathbf{x}(t), t)}{\partial x}\right]^{\top} \dot{x}(t) = 0 \tag{14}$$

2. Minimize the H with respect to u(t), i.e., $\frac{\partial H}{\partial u} = 0$, for solving $\mathbf{u}^*(t)$
3. Rewrite $H \rightarrow H^*$ substituting the optimal $u^*(t)$

Hence, the procedure for the HJB approach is as follows:

1. Define the Hamiltonian

$$H = g(\mathbf{x}(t), \mathbf{u}(t), t) + \left[\frac{\partial J(\mathbf{x}(t), t)}{\partial x}\right]^{\top} \dot{x}(t) = 0 \tag{14}$$

2. Minimize the H with respect to u(t), i.e., $\frac{\partial H}{\partial u} = 0$, for solving $\mathbf{u}^*(t)$
3. Rewrite $H \rightarrow H^*$ substituting the optimal $u^*(t)$
4. Solve for HJB

$$H^* + \frac{\partial J^*(\mathbf{x}^*(t), t)}{\partial t^*} = 0 \tag{15}$$

considering the boundary conditions: $J^*(\mathbf{x}^*(t_f), t_f) = 0$
whose solution provides an expression for $\mathbf{u}^*$

LQR with HJB

Consider a linear time-varying system

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(\mathbf{t}) + B(t)\mathbf{u}(\mathbf{t}) = f_c(x(t), u(t), t) \tag{16}$$

that should minimize the following cost function

$$J(\mathbf{x}, \mathbf{u}) = \int_{t_0}^{t_f} \frac{1}{2}\left(\|x(t)\|_Q^2 + \|u(t)\|_R^2 \right) dt, \tag{17}$$

with these assumptions: the **control** inputs are unconstrained and the **system** must be **controllable**. The objective is to find the optimal **cost-to-go** function J^* that satisfies the (Hamilton-Jacobi-Bellman Equation) for a finite time horizon

$$\forall \mathbf{x}, \quad 0 = \min_{\mathbf{u}} \left[\frac{1}{2}\left(\|\mathbf{x}\|_Q^2 + \|\mathbf{u}\|_R^2 \right) + \frac{\partial J^*}{\partial \mathbf{x}} (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) + \frac{\partial J^*}{\partial t} \right]. \tag{18}$$

- Define the Hamiltonian

$$
\begin{aligned}
H &= g(\mathbf{x}(t), \mathbf{u}(t), t) + \left[\frac{\partial J(\mathbf{x}(t), t)}{\partial x}\right]^\top f_c(\mathbf{x}(t), \mathbf{u}(t), t) = 0 \\
&= \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} + \frac{1}{2}\mathbf{u}^\top R\mathbf{u} + \left[\frac{\partial J(\mathbf{x}(t), t)}{\partial x}\right]^\top \left(A\mathbf{x} + B\mathbf{u}\right)
\end{aligned}
\tag{19}
$$

- Define the Hamiltonian

$$
\begin{aligned}
H &= g(\mathbf{x}(t), \mathbf{u}(t), t) + \left[\frac{\partial J(\mathbf{x}(t), t)}{\partial x}\right]^\top f_c(\mathbf{x}(t), \mathbf{u}(t), t) = 0 \\
&= \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} + \frac{1}{2}\mathbf{u}^\top R\mathbf{u} + \left[\frac{\partial J(\mathbf{x}(t), t)}{\partial x}\right]^\top \Big(A\mathbf{x} + B\mathbf{u}\Big)
\end{aligned}
\tag{19}
$$

- Minimize the H with respect to u(t), i.e., $\frac{\partial H^*}{\partial u} = 0$, for solving $\mathbf{u}^*(t)$

$$
R\mathbf{u} + B^\top \frac{\partial J(\mathbf{x}(t), t)}{\partial x} = 0 \quad \Rightarrow \mathbf{u} = -R^{-1}B^\top \underbrace{\frac{\partial J(\mathbf{x}(t), t)}{\partial x}}_{\lambda} \tag{20}
$$

- Rewrite H substituting the optimal $u^*(t)$

$$
\begin{aligned}
&= \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} + \frac{1}{2}\left[R^{-1}B^\top \frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right]^\top R\left[R^{-1}B^\top \frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right] \\
&\qquad + \left[\frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right]^\top \left(A\mathbf{x} - B\left[R^{-1}B^\top \frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right]\right) \\
&= \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} + \frac{1}{2}\left[\frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right]^\top \left[BR^{-1}B^\top\right]\left[\frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right] \\
&\qquad + \left[\frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right]^\top \left(A\mathbf{x} - B\left[R^{-1}B^\top \frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right]\right) \\
&= \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} - \frac{1}{2}\left[\frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right]^\top \left[BR^{-1}B^\top\right]\left[\frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right] \\
&\qquad + \left[\frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right]^\top A\mathbf{x}
\end{aligned} \tag{21}
$$

- Solve for HJB

$$
\begin{aligned}
H^* + \frac{\partial J(\mathbf{x}(t),t)}{\partial t} &= 0 \\
\frac{\partial J(\mathbf{x}(t),t)}{\partial t} + \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} - \frac{1}{2}\left[\frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right]^\top \left[BR^{-1}B^\top\right]\left[\frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right] & \\
+\left[\frac{\partial J(\mathbf{x}(t),t)}{\partial x}\right]^\top A\mathbf{x} &
\end{aligned} \tag{22}
$$

- Considering **terminal cost**

$$J(\mathbf{x}(t_f), t_f) = h(t_f) = \frac{1}{2}\mathbf{x}^\top(t_f)Q(t_f)\mathbf{x}(t_f)$$

whose solution provides an expression for $\mathbf{u}^*$. Since the **cost function** is **quadratic**, the control input $\mathbf{u}^*$ is in terms of J^*. To seek **feedback control**, i.e., $\mathbf{u}^*$ in terms of $\mathbf{x}(t)$, it is **reasonable to consider** $J^*(\mathbf{x}^*(t), t) = \frac{1}{2}\mathbf{x}^\top(t)P(t)\mathbf{x}(t)$

- Considering **terminal cost**

$$J(\mathbf{x}(t_f), t_f) = h(t_f) = \frac{1}{2}\mathbf{x}^\top(t_f)Q(t_f)\mathbf{x}(t_f)$$

  whose solution provides an expression for $\mathbf{u}^*$. Since the **cost function** is **quadratic**, the control input $\mathbf{u}^*$ is in terms of J^*. To seek **feedback control**, i.e., $\mathbf{u}^*$ in terms of $\mathbf{x}(t)$, it is **reasonable to consider** $J^*(\mathbf{x}^*(t), t) = \frac{1}{2}\mathbf{x}^\top(t)P(t)\mathbf{x}(t)$

- Therefore,

$$\begin{aligned} J^*(\mathbf{x}^*(t), t) &= \frac{1}{2}\mathbf{x}^\top(t)P(t)\mathbf{x}(t) \\ \frac{\partial J^*(\mathbf{x}^*(t), t)}{\partial t} &= \frac{1}{2}\mathbf{x}^\top(t)\dot{P}(t)\mathbf{x}(t), \quad \frac{\partial J^*(\mathbf{x}^*(t), t)}{\partial x} = P(t)\mathbf{x}(t) = \lambda(t_f) \end{aligned} \tag{23}$$

LQR with HJB

- Hence, rewriting the eq.22,

$$\begin{aligned} H^* + \frac{\partial J^*(\mathbf{x}^*(t), t)}{\partial t^*} &= 0 \\ \frac{1}{2}\mathbf{x}^\top \dot{P}\mathbf{x} + \frac{1}{2}\Big(\mathbf{x}^\top Q\mathbf{x} - \mathbf{x}^\top PBR^{-1}B^\top P\mathbf{x}\Big) + \mathbf{x}^\top PA\mathbf{x} &= 0 \end{aligned} \tag{24}$$

LQR with HJB

- Hence, rewriting the eq.22,

$$\begin{aligned} H^* + \frac{\partial J^*(\mathbf{x}^*(t), t)}{\partial t^*} &= 0 \\ \frac{1}{2}\mathbf{x}^\top \dot{P}\mathbf{x} + \frac{1}{2}\left(\mathbf{x}^\top Q\mathbf{x} - \mathbf{x}^\top PBR^{-1}B^\top P\mathbf{x}\right) + \mathbf{x}^\top PA\mathbf{x} &= 0 \end{aligned} \tag{24}$$

- However, $\mathbf{x}^\top PA\mathbf{x}$ is a scalar term, this can be rewritten as $2\mathbf{x}^\top PA\mathbf{x} = \mathbf{x}^\top PA\mathbf{x} + \mathbf{x}^\top A^\top P\mathbf{x}$.

- Hence, rewriting the eq.22,

$$\begin{aligned} H^* + \frac{\partial J^*(\mathbf{x}^*(t), t)}{\partial t^*} &= 0 \\ \frac{1}{2}\mathbf{x}^\top \dot{P}\mathbf{x} + \frac{1}{2}\left(\mathbf{x}^\top Q\mathbf{x} - \mathbf{x}^\top PBR^{-1}B^\top P\mathbf{x}\right) + \mathbf{x}^\top PA\mathbf{x} &= 0 \end{aligned} \tag{24}$$

- However, $\mathbf{x}^\top PA\mathbf{x}$ is a scalar term, this can be rewritten as $2\mathbf{x}^\top PA\mathbf{x} = \mathbf{x}^\top PA\mathbf{x} + \mathbf{x}^\top A^\top P\mathbf{x}$.
- Therefore,

$$\dot{P} + PA + A^\top P - PBR^{-1}B^\top P + Q = 0 \tag{25}$$

This is called Differential Riccati Equation. And the optimal control becomes $\mathbf{u} = -R^{-1}B^\top P\mathbf{x} = -K\mathbf{x}$, with $P(t_f) = Q(t_f)$

Example 01

Consider $\lambda(t) = P(t)\mathbf{x}(t)$. Using the Hamilton operator try to derive the **Differential Riccati Equation**.

Example 01

Consider $\lambda(t) = P(t)\mathbf{x}(t)$. Using the Hamilton operator try to derive the Differential Riccati Equation.

$$
\begin{aligned}
\lambda(t) &= P(t)\mathbf{x}(t) \\
\dot{\lambda}(t) &= \dot{P}(t)\mathbf{x}(t) + P(t)\dot{\mathbf{x}}(t) \\
&= \dot{P}(t)\mathbf{x}(t) + P(t)(A\mathbf{x}(t) - BR^{-1}B^\top\lambda(t)) \\
&= \dot{P}(t)\mathbf{x}(t) + P(t)(A\mathbf{x}(t) - BR^{-1}B^\top P(t)\mathbf{x}(t))
\end{aligned}
$$

Using costate equation, $\dot{\lambda}(t) = -\frac{\partial H}{\partial \mathbf{x}} = -Q\mathbf{x}(t) + A^\top\lambda(t)$

$$
\begin{aligned}
\dot{\lambda}(t) &= \dot{P}(t)\mathbf{x}(t) + P(t)(A\mathbf{x}(t) - BR^{-1}B^\top P(t)\mathbf{x}(t)) \\
-Q\mathbf{x}(t) + A^\top P\mathbf{x}(t) &= \dot{P}(t)\mathbf{x}(t) + P(t)(A\mathbf{x}(t) - BR^{-1}B^\top P(t)\mathbf{x}(t)) \\
(\dot{P} + PA + A^\top P - PBR^{-1}B^\top P + Q)\mathbf{x}(t) &= 0
\end{aligned}
$$

LQR with HJB

- If the system dynamics is nonlinear (eq.3), the control law becomes $\mathbf{u}^* = \mathbf{u}_0(t) - \mathbf{K}(t)(\mathbf{x} - \mathbf{x}_0(t))$.

LQR with HJB

- If the system dynamics is nonlinear (eq.3), the control law becomes $\mathbf{u}^* = \mathbf{u}_0(t) - \mathbf{K}(t)(\mathbf{x} - \mathbf{x}_0(t))$.
- In the case of infinite horizon problem formulation, the objective is to find the optimal cost-to-go function $J^*(\mathbf{x})$ that satisfies the (Hamilton-Jacobi-Bellman Equation) with $\frac{\partial J^*}{\partial t} = 0$

$$\forall \mathbf{x}, \quad 0 = \min_{\mathbf{u}} \left[\frac{1}{2} \left(\|\mathbf{x}\|_Q^2 + \|\mathbf{u}\|_R^2 \right) + \frac{\partial J^*}{\partial \mathbf{x}} (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) \right]. \tag{26}$$

where it gives the Algebraic Riccati Equation which is similar to the differential Riccati equation.

$$PA + A^\top P - PBR^{-1}B^\top P + Q = 0 \tag{27}$$

LQR with HJB

- Discrete-time linear quadratic control problem to minimize

$$\Sigma_{t=1}^{T}\mathbf{x}(t)^\top Q\mathbf{x}(t) + \mathbf{u}(t)^\top R\mathbf{u}(t)$$

  subject to $\mathbf{x}(t) = A\mathbf{x}(t-1) + B\mathbf{u}(t-1)$, where $\mathbf{x}(t)$ is an $n \times 1$ vector of state variables, $\mathbf{u}(t)$ is a $m \times 1$ vector of control variables, A is the $n \times n$ state transition matrix, B is the $n \times m$ matrix of control multipliers,$Q(n \times n)$ is a **symmetric positive semi-definite state cost matrix**, and $R(m \times m)$ is a **symmetric positive definite control cost matrix**.

- Discrete-time linear quadratic control problem to minimize

$$\Sigma_{t=1}^{T}\mathbf{x}(t)^\top Q\mathbf{x}(t)+\mathbf{u}(t)^\top R\mathbf{u}(t)$$

  subject to $\mathbf{x}(t)=A\mathbf{x}(t-1)+B\mathbf{u}(t-1)$, where $\mathbf{x}(t)$ is an $n\times 1$ vector of state variables, $\mathbf{u}(t)$ is a $m\times 1$ vector of control variables, A is the $n\times n$ state transition matrix, B is the $n\times m$ matrix of control multipliers,$Q(n\times n)$ is a **symmetric positive semi-definite state cost matrix**, and $R(m\times m)$ is a **symmetric positive definite control cost matrix**.
- Optimal cost

$$\mathbf{u}^*(t)=K\mathbf{x}(t-1)=-(B^\top P_tB+R)^{-1}(B^\top P_tA)\mathbf{x}(t-1)$$

- Discrete-time linear quadratic control problem to minimize

$$\Sigma_{t=1}^{T} \mathbf{x}(t)^\top Q \mathbf{x}(t) + \mathbf{u}(t)^\top R \mathbf{u}(t)$$

subject to $\mathbf{x}(t) = A\mathbf{x}(t-1) + B\mathbf{u}(t-1)$, where $\mathbf{x}(t)$ is an $n \times 1$ vector of state variables, $\mathbf{u}(t)$ is a $m \times 1$ vector of control variables, A is the $n \times n$ state transition matrix, B is the $n \times m$ matrix of control multipliers,$Q(n \times n)$ is a **symmetric positive semi-definite state cost matrix**, and $R(m \times m)$ is a **symmetric positive definite control cost matrix**.

- Optimal cost

$$\mathbf{u}^*(t) = K\mathbf{x}(t-1) = -(B^\top P_t B + R)^{-1}(B^\top P_t A)\mathbf{x}(t-1)$$

- Discrete-time algebraic Riccati equation (DARE):

$$P_{t-1} = Q + A^\top P_t A - A^\top P_t B(B^\top P_t B + R)^{-1} B^\top P_t A \tag{28}$$

with $P_T = Q$