Motion Planning for Autonomous Vehicles

Frenet frame trajectory planning

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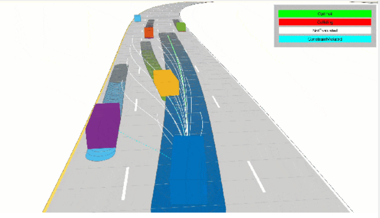
March 31, 2023

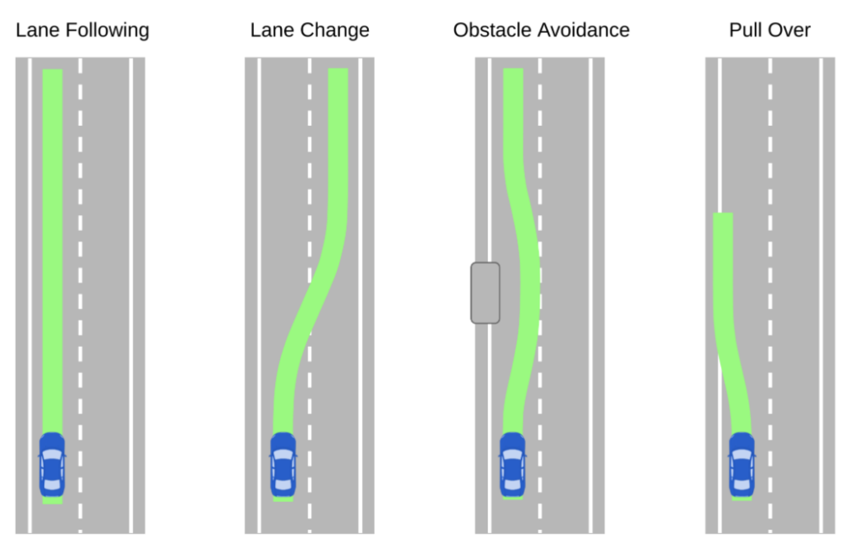


https://www.mathworks.com/help/nav/ug/highway-trajectory-planning-using-frenet.html

Frenet frame trajectory planning

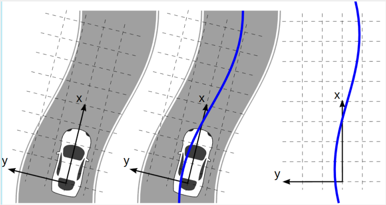
Contents

Different scenarios

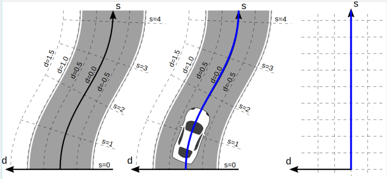


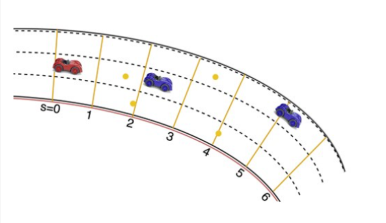
https://autowarefoundation.github.io/autoware.universe/main/planning/behavior_path_planner/

World frame W



Frenent frame F

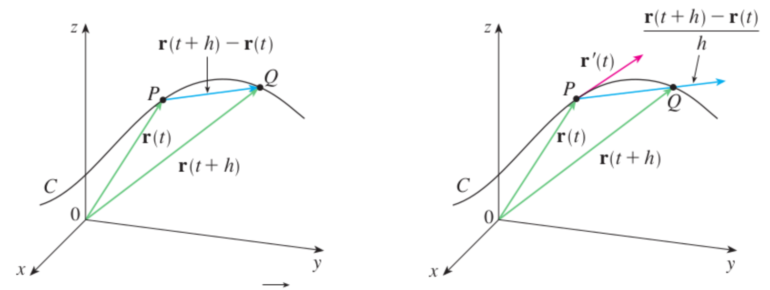




https://raw.githubusercontent.com/fjp/frenet/master/docs/images/cart_refpath.svg?sanitize=true, https://caseypen.github.io/posts/2021/01/FrenetFrame/

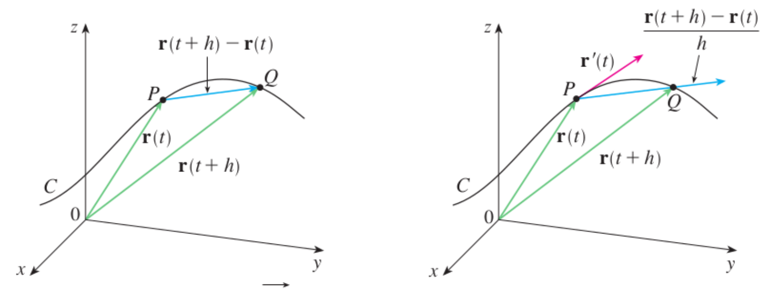
Let $\mathbf{r}(t) = x(t)i + y(t)j + z(t)k$ be a vector-valued function. That is, for every t, there is unique vector in $\mathbf{V}_3$ denoted by $\mathbf{r}(t)$ whose components are $x(t), y(t)$, and $z(t)$.
The derivative $\dot{\mathbf{r}}(t)$

$$\frac{d\mathbf{r}}{dt} = \underset{h\to 0}{\overrightarrow{lim}} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$



Multivariable Calculus: Stewart, James

The vector $\dot{\mathbf{r}}(t)$ is called tangent line to the defined curve $\mathbf{r}$ at point P, provided that $\dot{\mathbf{r}}(t)$ exists and $\dot{\mathbf{r}}(t) \neq 0$



Unit tangent vector

$$\mathbf{t} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|}$$

Example 01

Show that if $|\mathbf{r}(t)| = c$ (a constant), then $\dot{\mathbf{r}}(t)$ is orthogonal to $\mathbf{r}$ for all t.

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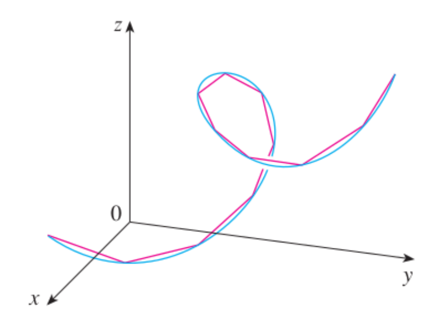
$$\mathbf{r}(t) \cdot \mathbf{r}(t) = |\mathbf{r}(t)|^2 = c^2$$

$$0 = \frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] = \dot{\mathbf{r}}(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \dot{\mathbf{r}}(t) = 2\dot{\mathbf{r}}(t)\mathbf{r}(t)$$

Length of a curve

For a considered range, e.g., a and b,

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt = \int_a^b |\dot{\mathbf{r}}(t)|dt$$



The Fundamental Theorem of Calculus, 1

If r is continuous on [a,b], then the function defined by

$$s(t) = \int_a^b r(u)du, \quad a \leq t \leq b$$

is continuous on [a,b] and differentiable on (a,b), and $s'(t) = r(t)$.

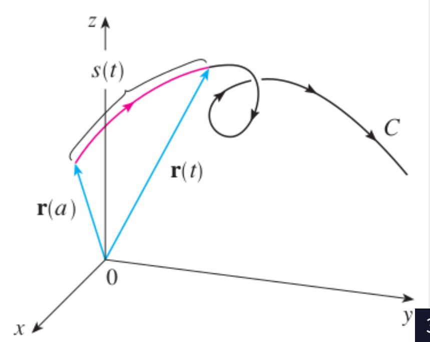
The Arc length function

For a function $\mathbf{r}(t) = x(t)i + y(t)j + z(t)k, \quad a \le t \le b$, arc length function s by

$$s(t) = \int_a^t |\dot{\mathbf{r}}(u)| du = \int_a^t \sqrt{\left\{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2\right\}} du$$

Thus, $s(t)$ is **the length of the path** between $\mathbf{r}(a)$ and $\mathbf{r}(t)$. When differentiating both sides,

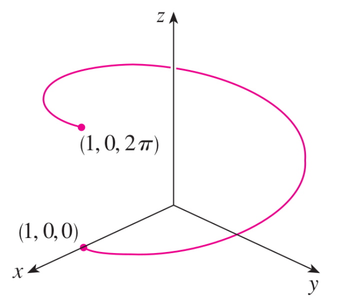
$$\frac{ds}{dt} = |\dot{\mathbf{r}}(t)|$$



Parameterise a curve with respect to **arc length** is quite useful since the **shape of the curve** does not depend on a **particular coordinate system.**, i.e., the arc length is invariant to reparameterization of a curve.

Example 02

Reparametrize the $\mathbf{r}(t) = costi + sintj + tk$ with respect to arc length measured from (1,0,0) in the direction of increasing t.



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Reparametrize the $\mathbf{r}(t) = costi + sintj + tk$ with respect to arc length measured from (1,0,0) in the direction of increasing t.

$$\frac{ds}{dt} = |\dot{\mathbf{r}}(t)| = \sqrt{(-sin(t))^2 + cos(t)^2 + 1} = \sqrt{2}$$

- Hence,

$$s = s(t) = \int_0^t |\dot{\mathbf{r}}(u)| du = \int_0^t \sqrt{2} du = \sqrt{2}t$$

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- Therefore, $t = s/\sqrt{2}$.

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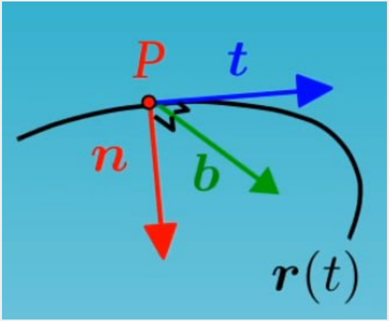
- Therefore, $t = s/\sqrt{2}$.
- Thus,

$$\mathbf{r}(t(s)) = cos(s/\sqrt{2})i + sin(s/\sqrt{2})j + s/\sqrt{2}k$$

- Time derivative of curve

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{r}'\dot{s} = \mathbf{r}'|\dot{\mathbf{r}}|, \quad \mathbf{r}' = \frac{d\mathbf{r}}{ds} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} \rightarrow |\mathbf{r}'| = 1$$

$$\ddot{\mathbf{r}} = \frac{d\dot{\mathbf{r}}}{dt} = \frac{d}{dt}(\mathbf{r}'\dot{s}) = \frac{d\mathbf{r}'}{dt}\dot{s} + \mathbf{r}'\ddot{s} = \dot{s}^2\mathbf{r}'' + \ddot{s}\mathbf{r}'$$



https://www.youtube.com/watch?v=aFCMIt63pgc

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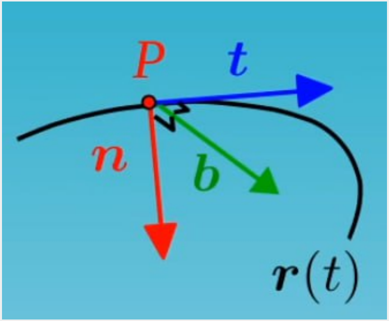
- tangent vector

$$\mathbf{t} = \mathbf{r}' = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|}$$



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Frenent-Serret Frame

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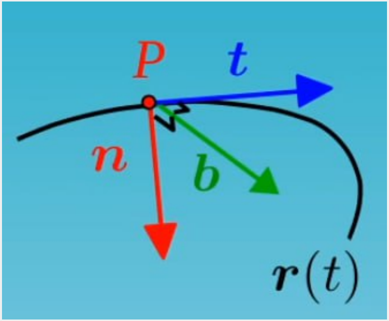
$$\mathbf{n} = \frac{\mathbf{t}'}{|\mathbf{t}'|}$$



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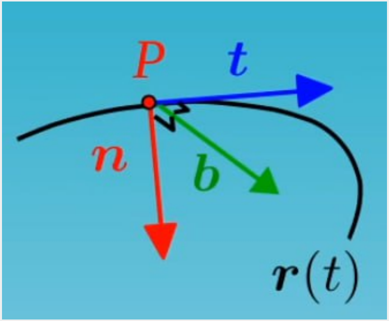
- binormal vector $\mathbf{b} = \mathbf{t} \times \mathbf{n}$



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Frenent-Serret Frame

- **t,n, b**: an orthogonal triplet of vectors

$$|\mathbf{t}| = |\mathbf{n}| = 1$$

$$0 = (\mathbf{t} \cdot \mathbf{t})' = 2\mathbf{t} \cdot \mathbf{t}'$$

$$\mathbf{t} \perp \mathbf{t}' \rightarrow \mathbf{t} \perp \mathbf{n}$$

$$|\mathbf{b}|^2 = |\mathbf{t} \times \mathbf{n}|^2 = |\mathbf{t}|^2 |\mathbf{n}|^2$$

- **t,n, b**: an orthogonal triplet of vectors

$$
\begin{aligned}
|\mathbf{t}| = |\mathbf{n}| &= 1 \\
0 = (\mathbf{t}\cdot\mathbf{t})' &= 2\mathbf{t}\cdot\mathbf{t}' \\
\mathbf{t}\perp\mathbf{t}' &\to \mathbf{t}\perp\mathbf{n} \\
|\mathbf{b}|^2 = |\mathbf{t}\times\mathbf{n}|^2 &= |\mathbf{t}|^2|\mathbf{n}|^2
\end{aligned}
$$

- In Frenent-Serret Frame, **t,n, b** are selected as an orthonormal basis along the curve, i.e., $\{\mathbf{e}_i\}_{i=1}^3 =$ (**t,n, b**). Hence, expansion of vector function $\mathbf{r}$ in the basis

$$\mathbf{r} = \Sigma_{i=1}^3(\mathbf{r}\cdot\mathbf{e}_i)\cdot\mathbf{e}_i$$

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$$\mathbf{r} = \Sigma_{i=1}^3(\mathbf{r}\cdot\mathbf{e}_i)\cdot\mathbf{e}_i$$

- Expansion of the derivative of the basis $\mathbf{e}'$

$$\mathbf{e}_i' = \Sigma_{j=1}^3(\mathbf{e}_i'\cdot\mathbf{e}_j)\cdot\mathbf{e}_j$$

- Also,

$$0 = (\mathbf{e}_i \cdot \mathbf{e}_j)' = \mathbf{e}_i' \cdot \mathbf{e}_j + \mathbf{e}_i \cdot \mathbf{e}_j' \rightarrow \mathbf{e}_i' \cdot \mathbf{e}_j = -\mathbf{e}_i \cdot \mathbf{e}_j', \quad \mathbf{e}_i \cdot \mathbf{e}_i' = 0$$

$$a_{ij} = \mathbf{e}_i' \cdot \mathbf{e}_j, \quad a_{ji} = a_{ij}, \quad a_{ii} = 0$$

$$a_{ij} = \begin{pmatrix} 0 & k & \alpha \\ -k & 0 & \tau \\ -\alpha & -\tau & 0 \end{pmatrix}$$

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$$\mathbf{e}_i' = \Sigma_{j=1}^{3}(\mathbf{e}_i' \cdot \mathbf{e}_j) \cdot \mathbf{e}_j = \Sigma_{j=1}^{3} a_{ij} \cdot \mathbf{e}_j$$

$$\Rightarrow \mathbf{e}_1' = k\mathbf{e}_2 + \alpha\mathbf{e}_3,\ \mathbf{e}_2' = -k\mathbf{e}_1 + \tau\mathbf{e}_3,\ \mathbf{e}_3' = -\alpha\mathbf{e}_1 - \tau\mathbf{e}_2,$$

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- Expansion of the derivative of the Frenet-Serret frame $\mathbf{e}_1 = \mathbf{t}, \mathbf{e}_2 = \mathbf{n}, \mathbf{e}_3 = \mathbf{b}$

$$\Rightarrow \mathbf{t}' = k\mathbf{n} + \alpha\mathbf{b},\ \mathbf{n}' = -k\mathbf{t} + \tau\mathbf{b},\ \mathbf{b}' = -\alpha\mathbf{t} - \tau\mathbf{n},$$

- Also,

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-

- Since $\mathbf{t}' = \|\mathbf{t}'\| \mathbf{n}$

$$\begin{aligned}
\mathbf{t}' &= k\mathbf{n} + \alpha\mathbf{b} \rightarrow \alpha = 0, \quad k = \|\mathbf{t}'\| \\
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\end{aligned}$$

- Since $\mathbf{t}' = \|\mathbf{t}'\|\,\mathbf{n}$

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- Also,

$$\begin{aligned}\mathbf{t}\cdot\mathbf{n}' = -k\cdot\mathbf{t}\cdot\mathbf{t} + \tau\cdot\mathbf{t}\cdot\mathbf{b} \rightarrow k &= -\mathbf{t}\cdot\mathbf{n}' \\ 0 = (\mathbf{n}\cdot\mathbf{t})' &= \mathbf{n}'\cdot\mathbf{t} + \mathbf{n}\cdot\mathbf{t}'\end{aligned}$$

Frenent-Serret formular

$$\frac{d}{ds}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix} = \begin{pmatrix}0 & k & 0\\ -k & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix},$$

the curvature $k(s) = -\mathbf{t}\cdot\mathbf{n}' = \mathbf{n}\cdot\mathbf{t}' = \frac{\mathbf{t}'}{\|\mathbf{t}'\|}\cdot\mathbf{t}' = \left\|\mathbf{t}'\right\|$

A parametrization $\mathbf{r}(t)$ is smooth on an interval I, if $\dot{\mathbf{r}}(t)$ is continuous and $\dot{\mathbf{r}}(t) \neq 0$ on I, the smooth curve has no **corners** or **cusps**.

- The curvature of a curve is

$$k = \|\mathbf{t}'\| = \left\| \frac{d\mathbf{t}}{ds} \right\|$$

where $\mathbf{t}$ unit tangent vector.

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- Hence,
$$k = \left\| \frac{d\mathbf{t}}{ds} \right\| = \left\| \frac{d\mathbf{t}/dt}{ds/dt} \right\|$$

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$$k = \|\mathbf{t}'\| = \left\| \frac{d\mathbf{t}}{ds} \right\|$$
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$$\frac{d\mathbf{t}}{dt} = \frac{d\mathbf{t}}{ds}\frac{ds}{dt}$$
- Hence,
$$k = \left\| \frac{d\mathbf{t}}{ds} \right\| = \left\| \frac{d\mathbf{t}/dt}{ds/dt} \right\|$$
- But $ds/dt = \|\dot{\mathbf{r}}(t)\|$,
$$k(t) = \left\| \frac{\dot{\mathbf{t}}(t)}{\dot{\mathbf{r}}(t)} \right\|$$

Example 03

Show that the curvature of a circle of radius a is 1/a, assume that center of the circle is at the origin.

Example 03

- Let $\mathbf{r}(t) = acos(t)i + asin(t)j$

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- Hence, $\dot{\mathbf{r}}(t) = -asin(t)i + acos(t)j$ and $\|\dot{\mathbf{r}}(t)\| = a$
- That is, $\mathbf{t}(t) = \frac{\dot{\mathbf{r}}(t)}{\|\dot{\mathbf{r}}(t)\|} = -sin(t)i + cos(t)j$ and $\dot{\mathbf{t}}(t) = -cos(t)i - sin(t)j$

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- That is, $\mathbf{t}(t) = \frac{\dot{\mathbf{r}}(t)}{\|\dot{\mathbf{r}}(t)\|} = -sin(t)i + cos(t)j$ and $\dot{\mathbf{t}}(t) = -cos(t)i - sin(t)j$
- Therefore, $\left\|\dot{\mathbf{t}}(t)\right\| = 1$. Thus,

$$k(t) = \frac{\left\|\dot{\mathbf{t}}(t)\right\|}{\|\dot{\mathbf{r}}(t)\|} = \frac{1}{a}$$

Curvature

The curvature can be formed using the vector function of a curve $\mathbf{r}$

$$k(t) = \frac{|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)|}{|\dot{\mathbf{r}}(t)|^3}$$

- Since $\mathbf{t}(t) = \frac{\dot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t)|}$ and $|\dot{\mathbf{r}}(t)| = \frac{ds}{dt}$,

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- That is,

$$k = \frac{|\dot{\mathbf{t}}(t)|}{|\dot{\mathbf{r}}(t)|} = \frac{|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)|}{|\dot{\mathbf{r}}(t)|^3}$$

Example 04

Find the curvature of the trajectory $\mathbf{r}(t) = ti + t^2 j + t^3 k$ when t = 0.

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$$\mathbf{r}(t) = ti + t^2 j + t^3 k, \quad ,\dot{\mathbf{r}}(t) = 1i + 2tj + 3t^2 k, \quad \ddot{\mathbf{r}}(t) = 0i + 2j + 6tk$$

$$\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t) = \begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2 i - 6tj + 2k$$

$$|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)| = 2\sqrt{9t^4 + 9t^2 + 1}$$

$$k(t) = \frac{|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)|}{|\dot{\mathbf{r}}(t)|^3} = \frac{2\sqrt{9t^4 + 9t^2 + 1}}{(1 + 4t^2 + 9t^4)^{3/2}}$$

Special form of k(t) for a plane curve with equation $y = f(x)$

- Let x be the parameter, that is

$$\mathbf{r}(x) = xi + f(x)j$$

- Also,

$$\dot{\mathbf{r}}(x) = i + \dot{f}(x)j, \quad \ddot{\mathbf{r}}(x) = \ddot{f}(x)j$$

- Since $i \times j = k$ and $j \times j = 0$

$$\dot{\mathbf{r}}(x) \times \ddot{\mathbf{r}}(x) = \ddot{f}(x)k, \quad |\dot{\mathbf{r}}(x)| = \sqrt{1 + (\dot{f}(x))^2}$$

- Hence,

$$k(x) = \frac{|\ddot{f}(x)|}{[1 + (\dot{f}(x))^2]^{3/2}}$$

Example 05

If a curve is defined in parametric form by the equations $x = x(t)$ and $y = y(t)$, i.e., $\mathbf{r}(t) = x(t)i + y(t)j$, derive a general expression for curvature.

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$$k(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$$

Example 06

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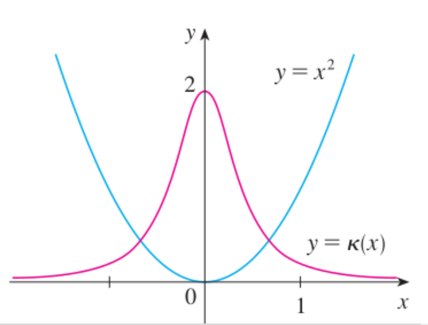


Since $\dot{\mathbf{y}} = 2x$ and $\ddot{\mathbf{y}} = 2$

$$k(x) = \frac{|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)|}{|\dot{\mathbf{r}}(t)|^3}$$

$$= \frac{|\ddot{\mathbf{y}}|}{\left[1 + (\dot{\mathbf{y}})^2\right]^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}$$

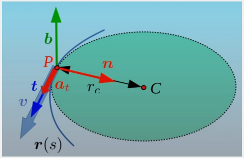
Velocity and acceleration in Frenent-Serret frame

- Derivative of $\mathbf{r}(s)$ with respect to time:

$$\dot{\mathbf{r}} = \dot{s}\mathbf{r}', \quad v = \dot{s}, \quad \mathbf{t} = \mathbf{r}' \to \mathbf{v} = v\mathbf{t}$$

$$\ddot{\mathbf{r}} = \dot{s}^2\mathbf{r}'' + \ddot{s}\mathbf{r}', \quad a = \ddot{s} = \dot{v}, \quad \mathbf{t}' = \mathbf{r}''$$

$$\to \mathbf{a} = a\mathbf{t} + v^2\mathbf{t}'$$



https://www.youtube.com/watch?v=aFCMIt63pgc

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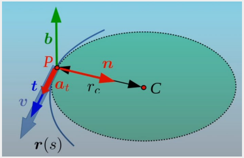

- Applying the first Frenent-Serret formula: $\mathbf{t}' = k\mathbf{n}$

$$\mathbf{a} = a\mathbf{t} + v^2 k\mathbf{n} = a\mathbf{t} + (v^2/r_c)\mathbf{n}$$

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