# Motion Planning for Autonomous Vehicles

## Pontryagin's Minimum Principle

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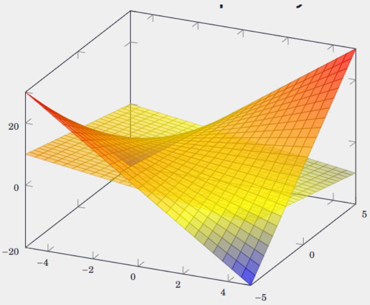

# Pontryagin's Minimum Principle (Optimal control theory)

# Contents

Consider that the system

$$\dot{x}(t) = f(x(t), u(t), t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ to follow an admissible trajectory $x^*$ that minimizes the following objective function

$$J(u) = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), u(t), t) dt \tag{2}$$

The initial condition $x(t_0) = x_0$ is given.

# Hamiltonian

$$H(x(t),u(t),P(t),t) := g(x(t),u(t),t) + P^\top(t) f(x(t),u(t),t) \tag{3}$$

Necessary conditions

$$\begin{aligned}
\dot{x}^*(t) &= \frac{H(\cdot)}{\partial P} \\
\dot{P}^*(t) = -\frac{H(\cdot)}{\partial x} &= -\left(\frac{\partial f(\cdot)}{\partial x}\right)^\top P^*(t) - \frac{\partial g(\cdot)}{\partial x} \\
0 = \frac{H(\cdot)}{\partial u} &= \left(\frac{\partial g(\cdot)}{\partial u}\right)^\top P^*(t) + \frac{\partial f(\cdot)}{\partial u} \\
\left(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\right)^\top \delta x_f + \left(H(\cdot) + \frac{\partial h(\cdot)}{\partial t}\right)\delta t_f &= 0
\end{aligned} \tag{4}$$

where $H(\cdot) = H(x^*(t), u^*(t), P^*(t), t)$ and $\forall\, t \in [t_0, t_f]$. State $x(t)$ and inputs $u(t)$ are **unconstrained**.

# Pontryagin's Minimum Principle

The control $\mathbf{u}^*$ causes the functional "J" to have a relative minima if

$$J(u) - J(u^*) = \Delta J \geq 0$$

for all **admissible controls** sufficiently close to $u^*$, i.e., $u^*$ is the relative minima

- Consider such control $u = u^* + \delta u$, the increment in 'J' can be expressed as

$$\Delta J(u^*, \delta u) = \delta J(u^*, \delta u) + H.O.T \tag{5}$$

where, the first variation $\delta J = \frac{\partial J}{\partial u} \delta u(t)$.

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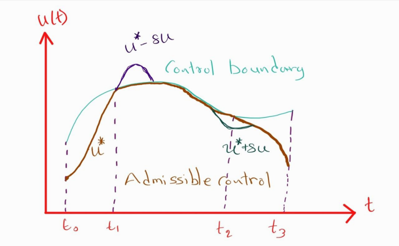
- When $\delta u$ is arbitrary to obtain an extremal solution $\delta J = 0$.
- However, control is bounded if the optimal control exceeds the control boundary in the sub-interval.
- Therefore, $\delta u$ can not be arbitrary in the interval $t_0, t_f$.



- Hence, the **necessary condition** for $u^*$ to minimize J is that $\delta J(u^*, \delta u) = \Delta J \geq 0$. On the other hand, if the $u^*$ **lies within** the acceptable **boundary** then $\delta J(u^*, \delta u) = 0$. Thus, the necessary condition

$$\delta J(u^*(t), \delta u(t)) = \int_{t_0}^{t_f} \left(\frac{\partial H(\cdot)}{\partial u}\right)^\top \delta u(t) dt \tag{6}$$

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- By taking the first-order approximation of H,

$$\begin{aligned} \left(\frac{\partial H(x(t), u(t), P(t), t)}{\partial u(t)}\right)^\top \delta u(t) = H(x^*(t), u^*(t) + \delta u(t), P^*(t), t) \\ - H(x^*(t), u^*(t), P^*(t), t) \end{aligned} \tag{7}$$

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- Therefore, the necessary condition becomes

$$\begin{aligned}\delta J(u^*(t), \delta u(t)) = \int_{t_0}^{t_f} \Big(H(x^*(t), u^*(t) + \delta u(t), P^*(t), t)& \\ - H(x^*(t), u^*(t), P^*(t), t)\Big) dt \geq 0&\end{aligned} \tag{8}$$

Hence, the following inequality must be satisfied.

$$
\begin{aligned}
H(x^*(t), u^*(t)+\delta u(t), P^*(t), t) &\geq H(x^*(t), u^*(t), P^*(t), t) \\
\Rightarrow H(x^*(t), {\color{red}u^*(t)}, P^*(t), t) &\leq H(x^*(t), {\color{red}u(t)}, P^*(t), t)
\end{aligned} \tag{9}
$$

where $u(t) = u^*(t) + \delta u(t)$. In other words, any $\delta u(t)$ is added to $u^*(t)$, which holds this inequality.

$$H(x^*(t), u^*(t), P^*(t), t) \leq H(x^*(t), u(t), P^*(t), t) \tag{10}$$

where $u(t) = u^*(t) + \delta u(t)$. However, this does not guarantee to be ensured $x^*(t), P^*(t)$

$$\begin{aligned} \dot{x}^*(t) &= \frac{H(\cdot)}{\partial P} \\ \dot{P}^*(t) = -\frac{H(\cdot)}{\partial x} = -\left(\frac{\partial f(\cdot)}{\partial x}\right)^\top P^*(t) &- \frac{\partial g(\cdot)}{\partial x} \\ H(\cdot) \leq H(x^*(t), u(t), P^*(t), t), \quad &\forall\, u(t) \in U \\ \left(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\right)^\top \delta x_f + \left(H(\cdot) + \frac{\partial h(\cdot)}{\partial t}\right) &\delta t_f = 0 \end{aligned} \tag{11}$$

where $H(\cdot) = H(x^*(t), u^*(t), P^*(t), t)$ and $\forall\, t \in [t_0, t_f]$. State $x(t)$ and inputs $u(t)$ are unconstrained.
$u^* = \text{argmax}\, H(x^*(t), u(t), P^*(t), t) \quad \forall\, u(t) \in U$

- Consider the system having the state equations

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -x_2(t) + u(t)
\end{aligned}
\tag{12}
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  with initial condition $x(t) = x_0$.
- The objective function is given as $J(u) = \int_{t_0}^{t_f} \frac{1}{2}(x_1^2(t) + u^2(t))dt$, where $t_f$ is specified and final state $x(t_f)$ is free.

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- The Hamiltonian is

$$\begin{aligned} H(x(t), u(t), P(t), t) &:= g(x(t), u(t), t) + P^\top(t) f(x(t), u(t), t) \\ \Rightarrow &\frac{1}{2}(x_1^2(t) + u^2(t)) + p_1(t)x_2(t) - p_2(t)x_2(t) + p_2(t)u(t) \end{aligned} \tag{13}$$

- Costate equations are

$$
\begin{aligned}
\dot{p}_1^*(t) &= -\frac{\partial H(\cdot)}{\partial x_1} = -x_1^*(t) \\
\dot{p}_2^*(t) &= -\frac{\partial H(\cdot)}{\partial x_2} = -p_1^*(t) + -p_2^*(t)
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- If controls are not-bounded

$$\begin{aligned} 0 &= \frac{H(\cdot)}{\partial u} \\ \Rightarrow u^*(t) + p_2^*(t) &= 0 \\ \Rightarrow u^*(t) &= -p_2^*(t) \end{aligned} \tag{15}$$

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- The boundary conditions are $p^*(t_f) = 0$,

- Consider the control is bounded $-1 \leq u(t) \leq 1 \,; \forall\, t \in [t_0, t_f]$. The state and costate equations remain the same. However, 'u' must be selected to minimize.

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- Considering terms that depend on u(t)

$$\begin{aligned} H(x(t), u(t), P(t), t) &:= g(x(t), u(t), t) + P^\top(t) f(x(t), u(t), t) \\ &\Rightarrow \frac{1}{2} u^2(t) + p_2^*(t) u(t) \end{aligned} \tag{16}$$

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- Within the control boundary $u^*(t) = -p_2^*(t)$ is valid.
- However, $|p_2^*(t) > 1|$

$$u^*(t) = \begin{cases} -1 & \text{for} \quad p_2^*(t) > 1 \\ -p_2^*(t) & \text{for} \quad -1 \leq \quad p_2^*(t) \leq 1 \\ 1 & \text{for} \quad p_2^*(t) < -1 \end{cases} \tag{17}$$

# Optimal boundary value problem

Let $\sigma(t)$ be the translational variable of the quadrocopter, consisting of its position, velocity, and acceleration, such that

$$\sigma(t) = (\mathbf{x}(t), \dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t)) \in \mathbb{R}^9 \tag{18}$$

If T be the motion duration, and $\bar{\sigma}_i$, $i \in I \subseteq \{1, 2, \ldots, 9\}$ be the components of desired translational variables at the end of the motion, then to achieve the target goal

$$\sigma_i(T) = \bar{\sigma}_i \; \forall i \in I \tag{19}$$

Mueller, M. W., Hehn, M., D'Andrea, R. (2015). A computationally efficient motion primitive for quadrocopter trajectory generation. IEEE transactions on robotics, 31(6), 1294-1310.