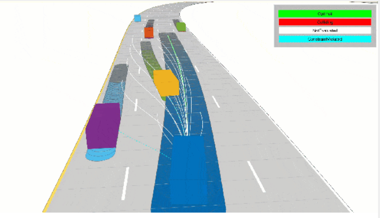
Motion Planning for Autonomous Vehicles

Frenet frame trajectory planning

Geesara Kulathunga



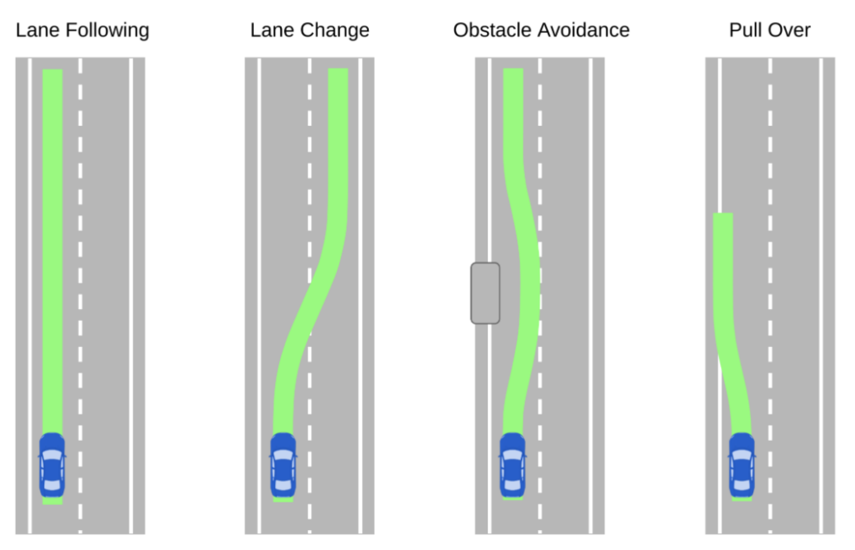
https://www.mathworks.com/help/nav/ug/highway-trajectory-planning-using-frenet.html

April 28, 2023

Frenet frame trajectory planning

Contents

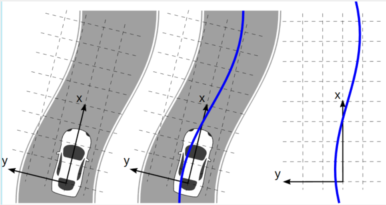
Different scenarios



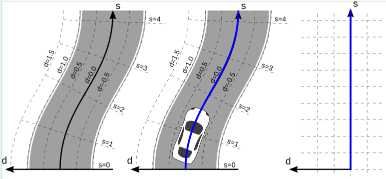
https://autowarefoundation.github.io/autoware.universe/main/planning/behavior_path_planner/

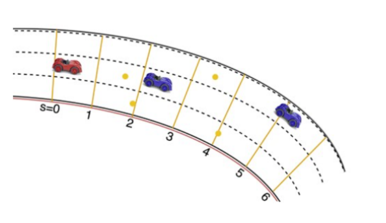
Frenet frame

World frame W



Frenent frame F

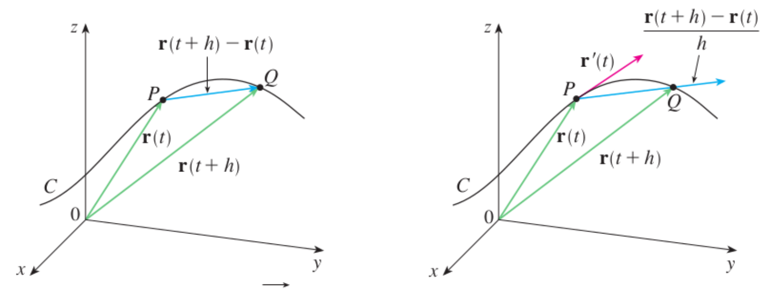




https://raw.githubusercontent.com/fjp/frenet/master/docs/images/cart_refpath.svg?sanitize=true, https://caseypen.github.io/posts/2021/01/FrenetFrame/

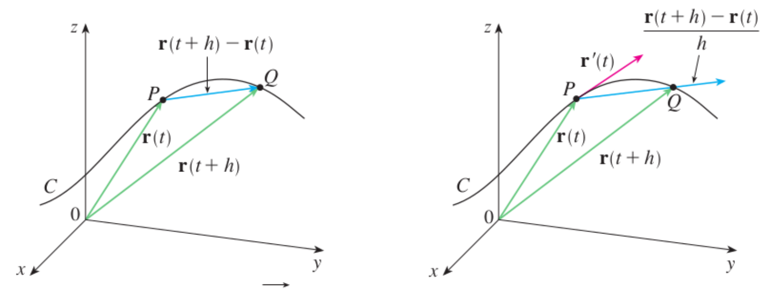
Let $\mathbf{r}(t) = x(t)i + y(t)j + z(t)k$ be a vector-valued function. That is, for every t, there is unique vector in $\mathbf{V}_3$ denoted by $\mathbf{r}(t)$ whose components are $x(t), y(t)$, and $z(t)$.
The derivative $\dot{\mathbf{r}}(t)$

$$\frac{d\mathbf{r}}{dt} = \frac{lim}{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$



Multivariable Calculus: Stewart, James

The vector $\dot{\mathbf{r}}(t)$ is called tangent line to the defined curve $\mathbf{r}$ at point P, provided that $\dot{\mathbf{r}}(t)$ exists and $\dot{\mathbf{r}}(t) \neq 0$



Unit tangent vector

$$\mathbf{t} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|}$$

Example 01

Show that if $|\mathbf{r}(t)| = c$ (a constant), then $\dot{\mathbf{r}}(t)$ is orthogonal to $\mathbf{r}$ for all t.

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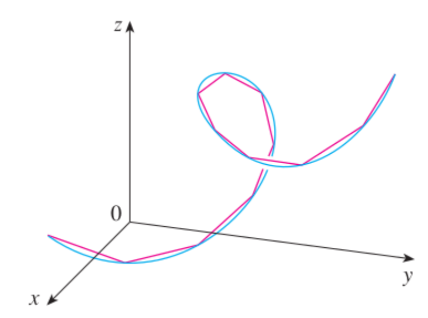
$$\mathbf{r}(t) \cdot \mathbf{r}(t) = |\mathbf{r}(t)|^2 = c^2$$

$$0 = \frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] = \dot{\mathbf{r}}(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \dot{\mathbf{r}}(t) = 2\dot{\mathbf{r}}(t)\mathbf{r}(t)$$

Length of a curve

For a considered range, e.g., a and b,

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt = \int_a^b |\dot{\mathbf{r}}(t)| dt$$



The Fundamental Theorem of Calculus, 1

If r is continuous on [a,b], then the function defined by

$$s(t) = \int_a^b r(u)du, \quad a \leq t \leq b$$

is continuous on [a,b] and differentiable on (a,b), and $s'(t) = r(t)$.

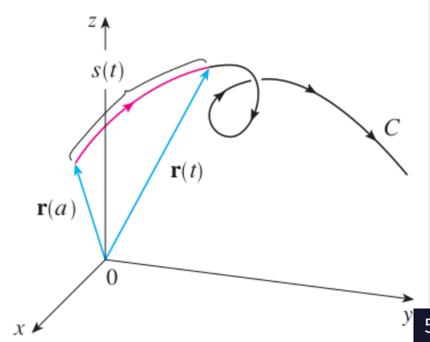
The Arc length function

For a function $\mathbf{r}(t) = x(t)i + y(t)j + z(t)k, \quad a \le t \le b$, arc length function s by

$$s(t) = \int_a^t |\dot{\mathbf{r}}(u)| du = \int_a^t \sqrt{\left\{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2\right\}} du$$

Thus, $s(t)$ is **the length of the path** between $\mathbf{r}(a)$ and $\mathbf{r}(t)$.
When differentiating both sides,

$$\frac{ds}{dt} = |\dot{\mathbf{r}}(t)|$$

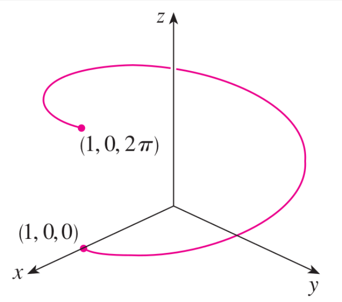


Parameterise a curve with respect to arc length

Parameterise a curve with respect to **arc length** is quite useful since the **shape of the curve** does not depend on a **particular coordinate system.**, i.e., the arc length is invariant to reparameterization of a curve.

Example 02

Reparametrize the $\mathbf{r}(t) = cost\, i + sint\, j + tk$ with respect to arc length measured from (1,0,0) in the direction of increasing t.



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$$\frac{ds}{dt} = |\dot{\mathbf{r}}(t)| = \sqrt{(-sin(t))^2 + cos(t)^2 + 1} = \sqrt{2}$$

- Hence,

$$s = s(t) = \int_0^t |\dot{\mathbf{r}}(u)| du = \int_0^t \sqrt{2} du = \sqrt{2} t$$

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- Therefore, $t = s/\sqrt{2}$.

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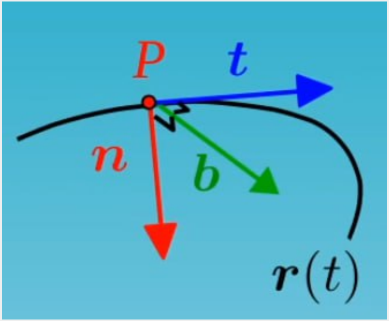
- Therefore, $t = s/\sqrt{2}$.
- Thus,

$$\mathbf{r}(t(s)) = cos(s/\sqrt{2})i + sin(s/\sqrt{2})j + s/\sqrt{2}k$$

- Time derivative of curve

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{r}'\dot{s} = \mathbf{r}'|\dot{\mathbf{r}}|, \quad \mathbf{r}' = \frac{d\mathbf{r}}{ds} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} \rightarrow |\mathbf{r}'| = 1$$

$$\ddot{\mathbf{r}} = \frac{d\dot{\mathbf{r}}}{dt} = \frac{d}{dt}(\mathbf{r}'\dot{s}) = \frac{d\mathbf{r}'}{dt}\dot{s} + \mathbf{r}'\ddot{s} = \dot{s}^2\mathbf{r}'' + \ddot{s}\mathbf{r}'$$



https://www.youtube.com/watch?v=aFCMIt63pgc

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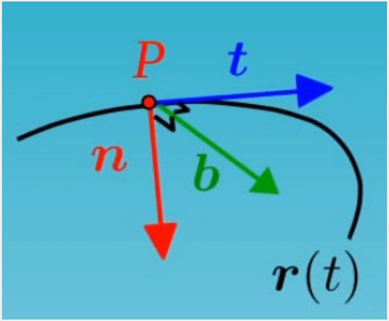
- tangent vector

$$\mathbf{t} = \mathbf{r}' = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|}$$



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Frenent-Serret Frame

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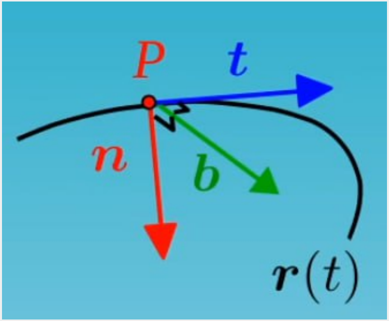
$$\mathbf{n} = \frac{\mathbf{t}'}{|\mathbf{t}'|}$$



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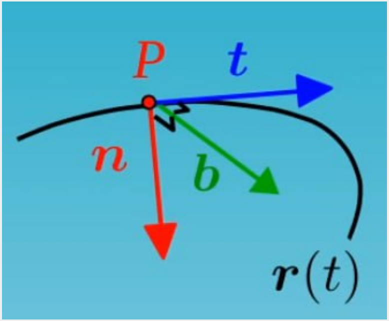
- binormal vector $\mathbf{b} = \mathbf{t} \times \mathbf{n}$



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- **t,n, b**: an orthogonal triplet of vectors

$$|\mathbf{t}| = |\mathbf{n}| = 1$$
$$0 = (\mathbf{t} \cdot \mathbf{t})' = 2\mathbf{t} \cdot \mathbf{t}'$$
$$\mathbf{t} \perp \mathbf{t}' \rightarrow \mathbf{t} \perp \mathbf{n}$$
$$|\mathbf{b}|^2 = |\mathbf{t} \times \mathbf{n}|^2 = |\mathbf{t}|^2 |\mathbf{n}|^2$$

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$$
\begin{aligned}
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\end{aligned}
$$

- In Frenent-Serret Frame, **t,n, b** are selected as an orthonormal basis along the curve, i.e., $\{\mathbf{e}_i\}_{i=1}^3 = ($**t,n, b**$)$. Hence, expansion of vector function $\mathbf{r}$ in the basis

$$\mathbf{r} = \Sigma_{i=1}^3(\mathbf{r}\cdot\mathbf{e}_i)\cdot\mathbf{e}_i$$

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$$\mathbf{r} = \Sigma_{i=1}^3(\mathbf{r}\cdot\mathbf{e}_i)\cdot\mathbf{e}_i$$

- Expansion of the derivative of the basis $\mathbf{e}'$

$$\mathbf{e}_i' = \Sigma_{j=1}^3(\mathbf{e}_i'\cdot\mathbf{e}_j)\cdot\mathbf{e}_j$$

- Also,

$$0 = (\mathbf{e}_i \cdot \mathbf{e}_j)' = \mathbf{e}_i' \cdot \mathbf{e}_j + \mathbf{e}_i \cdot \mathbf{e}_j' \rightarrow \mathbf{e}_i' \cdot \mathbf{e}_j = -\mathbf{e}_i \cdot \mathbf{e}_j', \quad \mathbf{e}_i \cdot \mathbf{e}_i' = 0$$

$$a_{ij} = \mathbf{e}_i' \cdot \mathbf{e}_j, \quad a_{ji} = a_{ij}, \quad a_{ii} = 0$$

$$a_{ij} = \begin{pmatrix} 0 & k & \alpha \\ -k & 0 & \tau \\ -\alpha & -\tau & 0 \end{pmatrix}$$

Frenent-Serret Frame

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- Expansion of the derivative of the basis $\mathbf{e}'$

$$\mathbf{e}_i' = \Sigma_{j=1}^3 (\mathbf{e}_i' \cdot \mathbf{e}_j) \cdot \mathbf{e}_j = \Sigma_{j=1}^3 a_{ij} \cdot \mathbf{e}_j$$

$$\Rightarrow \mathbf{e}_1' = k\mathbf{e}_2 + \alpha\mathbf{e}_3,\ \mathbf{e}_2' = -k\mathbf{e}_1 + \tau\mathbf{e}_3,\ \mathbf{e}_3' = -\alpha\mathbf{e}_1 - \tau\mathbf{e}_2,$$

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- Expansion of the derivative of the Frenet-Serret frame $\mathbf{e}_1 = \mathbf{t}, \mathbf{e}_2 = \mathbf{n}, \mathbf{e}_3 = \mathbf{b}$

$$\Rightarrow \mathbf{t}' = k\mathbf{n} + \alpha\mathbf{b},\ \mathbf{n}' = -k\mathbf{t} + \tau\mathbf{b},\ \mathbf{b}' = -\alpha\mathbf{t} - \tau\mathbf{n},$$

- Also,

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-

- Since $\mathbf{t}' = \left\|\mathbf{t}'\right\| \mathbf{n}$

$$\begin{aligned}
\mathbf{t}' &= k\mathbf{n} + \alpha\mathbf{b} \rightarrow \alpha = 0, \quad k = \left\|\mathbf{t}'\right\| \\
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\end{aligned}$$

Frenent-Serret Frame

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- Also,

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Frenent-Serret formular

$$\frac{d}{ds}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix} = \begin{pmatrix}0 & k & 0\\ -k & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix},$$

the curvature $k(s) = -\mathbf{t}\cdot\mathbf{n}' = \mathbf{n}\cdot\mathbf{t}' = \frac{\mathbf{t}'}{\|\mathbf{t}'\|}\cdot\mathbf{t}' = \|\mathbf{t}'\|$

A parametrization $\mathbf{r}(t)$ is smooth on an interval I, if $\dot{\mathbf{r}}(t)$ is continuous and $\dot{\mathbf{r}}(t) \neq 0$ on I, the smooth curve has no **corners** or **cusps**.

- The curvature of a curve is

$$k = \|\mathbf{t}'\| = \left\| \frac{d\mathbf{t}}{ds} \right\|$$

  where $\mathbf{t}$ unit tangent vector.

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$$\frac{d\mathbf{t}}{dt} = \frac{d\mathbf{t}}{ds} \frac{ds}{dt}$$

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$$\frac{d\mathbf{t}}{dt} = \frac{d\mathbf{t}}{ds}\frac{ds}{dt}$$
- Hence,
$$k = \left\| \frac{d\mathbf{t}}{ds} \right\| = \left\| \frac{d\mathbf{t}/dt}{ds/dt} \right\|$$

A parametrization $\mathbf{r}(t)$ is smooth on an interval I, if $\dot{\mathbf{r}}(t)$ is continuous and $\dot{\mathbf{r}}(t) \neq 0$ on I, the smooth curve has no **corners** or **cusps**.

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$$k = \|\mathbf{t}'\| = \left\| \frac{d\mathbf{t}}{ds} \right\|$$
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- The curvature is easier to compute if it is expressed in terms of t,
$$\frac{d\mathbf{t}}{dt} = \frac{d\mathbf{t}}{ds}\frac{ds}{dt}$$
- Hence,
$$k = \left\| \frac{d\mathbf{t}}{ds} \right\| = \left\| \frac{d\mathbf{t}/dt}{ds/dt} \right\|$$
- But $ds/dt = \|\dot{\mathbf{r}}(t)\|$,
$$k(t) = \left\| \frac{\dot{\mathbf{t}}(t)}{\dot{\mathbf{r}}(t)} \right\|$$

Example 03

Show that the curvature of a circle of radius a is 1/a, assume that center of the circle is at the origin.

Example 03

- Let $\mathbf{r}(t) = acos(t)i + asin(t)j$

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- Hence, $\dot{\mathbf{r}}(t) = -asin(t)i + acos(t)j$ and $\|\dot{\mathbf{r}}(t)\| = a$
- That is, $\mathbf{t}(t) = \frac{\dot{\mathbf{r}}(t)}{\|\dot{\mathbf{r}}(t)\|} = -sin(t)i + cos(t)j$ and $\dot{\mathbf{t}}(t) = -cos(t)i - sin(t)j$

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- Hence, $\dot{\mathbf{r}}(t) = -asin(t)i + acos(t)j$ and $\|\dot{\mathbf{r}}(t)\| = a$
- That is, $\mathbf{t}(t) = \frac{\dot{\mathbf{r}}(t)}{\|\dot{\mathbf{r}}(t)\|} = -sin(t)i + cos(t)j$ and $\dot{\mathbf{t}}(t) = -cos(t)i - sin(t)j$
- Therefore, $\left\|\dot{\mathbf{t}}(t)\right\| = 1$. Thus,

$$k(t) = \frac{\left\|\dot{\mathbf{t}}(t)\right\|}{\|\dot{\mathbf{r}}(t)\|} = \frac{1}{a}$$

Curvature

The curvature can be formed using the vector function of a curve $\mathbf{r}$

$$k(t) = \frac{|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)|}{|\dot{\mathbf{r}}(t)|^3}$$

- Since $\mathbf{t}(t) = \frac{\dot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t)|}$ and $|\dot{\mathbf{r}}(t)| = \frac{ds}{dt}$,

$$\dot{\mathbf{r}}(t) = |\dot{\mathbf{r}}(t)|\mathbf{t} = \frac{ds}{dt}\mathbf{t}$$

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- Since $\mathbf{t} \times \mathbf{t} = 0$

$$\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t) = \left(\frac{ds}{dt}\right)^2 (\mathbf{t} \times \dot{\mathbf{t}})$$

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- Thus,

$$|\dot{\mathbf{t}}| = \frac{|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)|}{\left(\frac{ds}{dt}\right)^2} = \frac{|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)|}{|\dot{\mathbf{r}}(t)|^2}$$

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- That is,

$$k = \frac{|\dot{\mathbf{t}}(t)|}{|\dot{\mathbf{r}}(t)|} = \frac{|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)|}{|\dot{\mathbf{r}}(t)|^3}$$

Example 04

Find the curvature of the trajectory $\mathbf{r}(t) = ti + t^2 j + t^3 k$ when t = 0.

Example 04

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$$\mathbf{r}(t) = ti + t^2 j + t^3 k, \quad ,\dot{\mathbf{r}}(t) = 1i + 2tj + 3t^2 k, \quad \ddot{\mathbf{r}}(t) = 0i + 2j + 6tk$$

$$\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t) = \begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2 i - 6tj + 2k$$

$$|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)| = 2\sqrt{9t^4 + 9t^2 + 1}$$

$$k(t) = \frac{|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)|}{|\dot{\mathbf{r}}(t)|^3} = \frac{2\sqrt{9t^4 + 9t^2 + 1}}{(1 + 4t^2 + 9t^4)^{3/2}}$$

Special form of k(t) for a plane curve with equation $y = f(x)$

- Let x be the parameter, that is

$$\mathbf{r}(x) = xi + f(x)j$$

- Also,

$$\dot{\mathbf{r}}(x) = i + \dot{f}(x)j, \quad \ddot{\mathbf{r}}(x) = \ddot{f}(x)j$$

- Since $i \times j = k$ and $j \times j = 0$

$$\dot{\mathbf{r}}(x) \times \ddot{\mathbf{r}}(x) = \ddot{f}(x)k, \quad |\dot{\mathbf{r}}(x)| = \sqrt{1 + (\dot{f}(x))^2}$$

- Hence,

$$k(x) = \frac{|\ddot{f}(x)|}{[1 + (\dot{f}(x))^2]^{3/2}}$$

Example 05

If a curve is defined in parametric form by the equations $x = x(t)$ and $y = y(t)$, i.e., $\mathbf{r}(t) = x(t)i + y(t)j$, derive a general expression for curvature.

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$$k(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$$

Example 06

Find the curvature of the parabola $y = x^2$ at points (0,0), (1,1), and (2,4).

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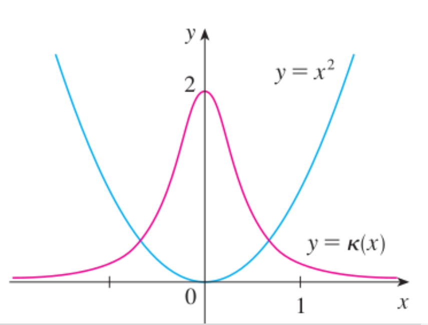


Since $\dot{\mathbf{y}} = 2x$ and $\ddot{\mathbf{y}} = 2$

$$k(x) = \frac{|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)|}{|\dot{\mathbf{r}}(t)|^3}$$

$$= \frac{|\ddot{\mathbf{y}}|}{\left[1 + (\dot{\mathbf{y}})^2\right]^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}$$

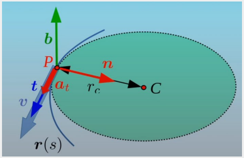
Velocity and acceleration in Frenent-Serret frame

- Derivative of $\mathbf{r}(s)$ with respect to time:

$$\dot{\mathbf{r}} = \dot{s}\mathbf{r}', \quad v = \dot{s}, \quad \mathbf{t} = \mathbf{r}' \rightarrow \mathbf{v} = v\mathbf{t}$$

$$\ddot{\mathbf{r}} = \dot{s}^2\mathbf{r}'' + \ddot{s}\mathbf{r}', \quad a = \ddot{s} = \dot{v}, \quad \mathbf{t}' = \mathbf{r}''$$

$$\rightarrow \mathbf{a} = a\mathbf{t} + v^2\mathbf{t}'$$



https://www.youtube.com/watch?v=aFCMIt63pgc

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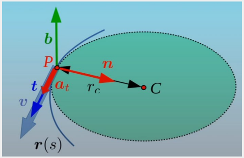
$$\rightarrow \mathbf{a} = a\mathbf{t} + v^2\mathbf{t}'$$

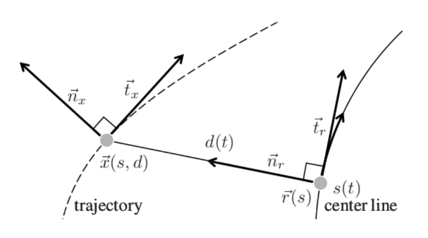
- Applying the first Frenent-Serret formula: $\mathbf{t}' = k\mathbf{n}$

$$\mathbf{a} = a\mathbf{t} + v^2 k\mathbf{n} = a\mathbf{t} + (v^2/r_c)\mathbf{n}$$



https://www.youtube.com/watch?v=aFCMIt63pgc

Frenet frame F coordinate built on a curve, which is composed of **five components**: location of the curve $\vec{r}(s)$, corresponding tangential vector $\vec{t}_r$, the perpendicular distance to the reference location $d(t)$, and corresponding tangential and normal vectors $\vec{t}_x$ $\vec{n}_x$



where the moving reference frame is given by tangential vector $\vec{t}_r = [cos\psi_r(s)\, sin_r\psi(s)]$ and normal vector $\vec{n}_r = [-sin\psi_r(s)\, cos_r\psi(s)]$ around a curve, namely, center line.

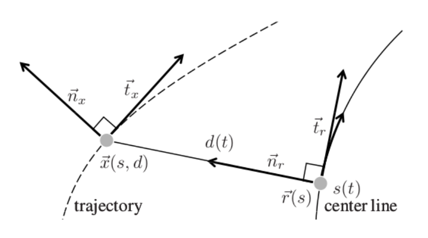
[1]. Werling, M., Ziegler, J., Kammel, S., Thrun, S. (2010, May). Optimal trajectory generation for dynamic street scenarios in a frenet frame. In 2010 IEEE International Conference on Robotics and Automation (pp. 987-993). IEEE.

Frenet frame

Car location in the Cartesian coordinates $\vec{x}$ and moving point in the moving reference curve, depicted $\vec{r}$, and the offset between $\vec{x}$ and $\vec{r}$ is given by d(t) at time t. Therefore, the following relationship can be built.

$$\vec{x}(s(t), d(t)) = \vec{r}(s(t)) + d(t)\vec{n}_r(s(t)), \tag{1}$$

where s denotes the arc length of the center line, and $\vec{t}_x$ and $\vec{n}_x$ are the tangential and normal vector of the resulting trajectory.



Curve parameterization of the reference trajectory

As the first step, it is required to calculate the initial reference trajectory

Algorithm 1 Generate reference trajectory

procedure GETTARGETTRAJ(x, t)

$\quad spline \leftarrow getCubicSpline < 2D > (x, t)$

$\quad s \leftarrow getS(spline, \delta s)$ ▷ get arc length along the reference trajectory

$\quad r_x, r_y, r_{yaw}, r_k$ ▷ reference trajectory pose and curvature

$\quad$ **for** $s_i \leftarrow s_0$ to s_n **do**

$\qquad r_x, r_y << \leftarrow getPositionVector(spline, s_i)$

$\qquad r_{yaw} << \leftarrow getYaw(spline, s_i)$

$\qquad r_k << \leftarrow getCurvature(spline, s_i)$

$\quad$ **end for**

end procedure

where yaw angle is determined by $atan2(\dot{y}, \dot{x})$ and the curvature is determined by $\frac{\ddot{y} \cdot \dot{x} - \ddot{x} \cdot \dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$.

Estimate the position of a given Spline

For a given $slipne$ and time index t, the position is determined as follows:

Algorithm 2 Get position given time index

procedure GETPOSITION($t, slipne$)
    a_i, b_i, c, d_i ▷ depicted the spline coefficients
    t_i ▷ knok vector of the spline
    if $t_d < t_0$ or $(t_d > t_n)$ **then**
        none ▷ when given time is invalid
    end if
    $i \leftarrow getIndex(t_d)$ ▷ get the closet index, namely knok, of the polynomial
    $dt \leftarrow t_d - t_i$
    position $\leftarrow a_i + b_i dt + c_i dt^2 + d_i dt^3$
end procedure

Transformations from Frenent coordinates to global coordinates

We seek transformations

$$[s, \dot{s}, \ddot{s}, d, \dot{d}, \ddot{d}, d', d''] \rightarrow [\vec{x}, \psi, k, v_x, a_x] \tag{2}$$

- s longitudinal displacement
- $\dot{s} = \frac{ds}{dt}$ longitudinal velocity
- $\ddot{s} = \frac{d^2s}{dt^2}$ longitudinal acceleration
- d lateral displacement
- $\dot{d} = \frac{dd}{dt}$ lateral velocity
- $\ddot{d} = \frac{d\dot{d}}{dt}$ lateral acceleration
- d’ = $\frac{dd}{ds}$ the first derivative of the lateral displacement with respect to the longitudinal coordinate
- d” = $\frac{dd'}{ds}$ the second derivative of the lateral displacement with respect to the longitudinal coordinate

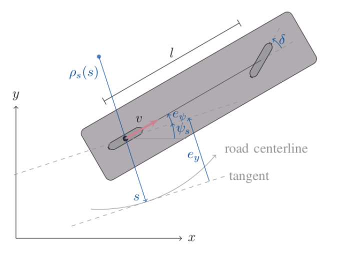
Transformations from Frenent coordinates to global coordinates

We seek transformations

$$[s, \dot{s}, \ddot{s}, d, \dot{d}, \ddot{d}, d', d''] \rightarrow [\vec{x}, \psi, k, v_x, a_x] \tag{3}$$

- $\vec{x}$ the current position of the vehicle
- ψ the orientation in the global coordinate system
- k the curvature
- v_x linear velocity in the Cartesian coordinate system
- $a_x = \frac{dv_x}{dt}$ acceleration

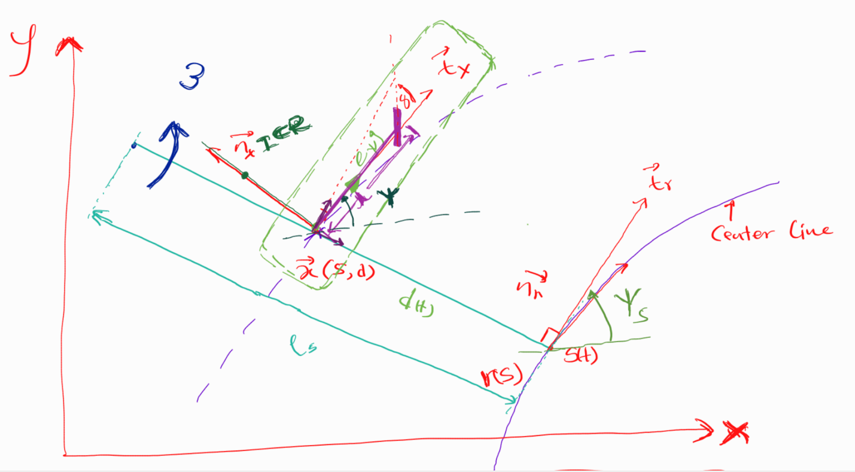
The road-aligned coordinate system with a non-linear dynamic bicycle model



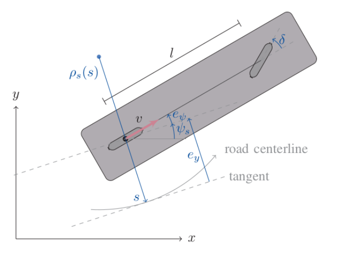
where x and y are the vehicle coordinates in the global coordinate system, yaw angle, wheelbase, longitudinal velocity in the vehicle coordinate system, and steering angle of the front wheel are given by ψ, l,v, and δ, respectively. Vehicle curvature is given by $k = \frac{tan(\delta)}{l}$ whilst rode curvature is given by $k_r = \frac{1}{\rho_s}$

Lima, P. F., Mårtensson, J., Wahlberg, B. (2017, December). Stability conditions for linear time-varying model predictive control in autonomous driving. In 2017 IEEE 56th Annual Conference on Decision and Control (CDC) (pp. 2775-2782). IEEE.

Frenet frame trajectory tracking using a non-linear bicycle model



The vehicle model in the time domain



$$\dot{x} = \frac{dx}{dt} = v_x cos(\psi),$$
$$\dot{y} = \frac{dy}{dt} = v_x sin(\psi), \qquad (4)$$
$$\dot{\psi} = \frac{d\psi}{dt} = \frac{v}{l} tan(\delta)$$

Transformations from Frenent coordinates to global coordinates

By considering the mentioned vehicle model, the following set of expressions can be derived.

$$\dot{e}_y = \dot{d}(t) = v_x sin(e_\psi),$$
$$\dot{e}_\psi = \dot{\psi} - \dot{\psi}_s,$$
$$\dot{s} = \omega \cdot \frac{1}{k_r} = \frac{v_x cos(e_\psi)}{k_r(1/k_r - d(t))} = \frac{v_x cos(e_\psi)}{1 - k_r d(t)} = \frac{\rho_s v cos(e_\psi)}{\rho_s - e_y}, \quad \omega = \frac{v_x cos(e\psi)}{(\frac{1}{k_r} - d(t))} \tag{5}$$

where ρ_s is the radius of curvature of the road ψ_s is the road heading angle.

Transformations from Frenent coordinates to global coordinates

The next step is to derive with respect to s, i.e., $\frac{d(\cdot)}{ds} = \frac{d(\cdot)}{dt}\frac{dt}{ds} = \frac{d(\cdot)}{dt}\frac{1}{\dot{s}}$.
Using eq 1 and $\dot{\vec{n}}_r(s) = -\begin{pmatrix} cos_r\psi_r(s) & sin_r\psi(s)\end{pmatrix}\dot{\psi}_r = -k_r\dot{s}\vec{t}_r$ where $\vec{n}_r = -sin(\psi)i + cos(\psi)j$ and $\dot{\psi} = \dot{s}k_r$

$$\begin{aligned} d &= [\vec{x} - \vec{r}(s)]^\top \vec{n}_r \\ \dot{d} &= [\dot{\vec{x}} - \dot{\vec{r}}(s)]^\top \vec{n}_r + [\vec{x} - \vec{r}(s)]^\top \dot{\vec{n}}_r \\ = v_x \vec{t}_x^\top \vec{n}_r - \dot{s}\underbrace{\vec{t}_r^\top \vec{n}_r}_{=0} &- k_r\dot{s}\underbrace{[\vec{x} - \vec{r(s)}]^\top \vec{t}_r}_{=0} = v_x sin(e_\psi) \end{aligned} \tag{6}$$

note: $sin(a-b) = sin(a)cos(b) - cos(a)sin(b)$

Transformations from Frenent coordinates to global coordinates

The velocity of the robot v_x can be expressed as

$$\begin{aligned} v_x = \|\dot{x}\|_2 = \left\|(1-k_r d(t))\dot{s}\vec{t}_r + \dot{d}\vec{n}_r\right\|_2 &= \left\|\begin{bmatrix}\vec{t}_r & \vec{n}_r\end{bmatrix}\begin{bmatrix}1-k_r d & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}\dot{s} \\ \dot{d}\end{bmatrix}\right\|_2 \\ &= \left\|\begin{bmatrix}1-k_r d & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}\dot{s} \\ \dot{d}\end{bmatrix}\right\|_2 = \sqrt{[1-k_r d(t)]^2\dot{s}^2 + \dot{d}^2} \end{aligned} \tag{7}$$

Transformations from Frenent coordinates to global coordinates

$$e'_y = \frac{\dot{e}_y}{\dot{s}} = \frac{v_x sin(e_\psi)}{v_x cos(e_\psi)}(1 - d(t)k_r) = \Big(1 - d(t)k_r\Big) tan(e_\psi)$$

$$e'_\psi = \frac{\frac{v_x}{k} - \frac{v_x cos(e_\psi)}{1/k_r - d(t)}}{\frac{v_x cos(e_\psi)}{1 - k_r d(t)}} = \frac{k(\rho_s - d(t))}{\rho_s cos(e_\psi)} - \frac{1}{\rho_s} = \frac{k(\rho_s - d(t))}{\rho_s cos(e_\psi)} - \psi'_s, \qquad (8)$$

$$\psi'_s = \frac{\dot{\psi}_s}{\dot{s}} = \frac{\frac{v_x cos(e_\psi)k_r}{1 - d(t)k_r}}{\frac{v_x cos(e_\psi)k_r}{1 - d(t)}} = \frac{1}{\rho_s}$$

Transformations from Frenent coordinates to global coordinates

$$
\begin{aligned}
e'_y = d' = \frac{d}{ds}d = \frac{dt}{ds}\frac{d}{dt}d = \frac{1}{\dot{s}}\dot{d} = \frac{v_x}{\dot{s}}sin(e_\psi) = sin(e_\psi)\sqrt{[1-k_r d(t)]^2 + d'^2} \\
d'^2 = \left([1-k_r d(t)]^2 + d'^2\right)sin^2(e_\psi) \\
\Rightarrow d'^2[1-sin^2(e_\psi)] = [1-k_r d(t)]^2 sin^2(e_\psi) \\
d' = \left(1-d(t)k_r\right)tan(e_\psi)
\end{aligned} \tag{9}
$$

Transformations from Frenent coordinates to global coordinates

In additionally, $[\vec{x} - \vec{r(s)}]^\top \vec{t}_r = 0$ at all times, so that differentiation with respect to time gives

$$
\begin{aligned}
[\dot{\vec{x}} - \dot{\vec{r(s)}}]^\top \vec{t}_r + [\vec{x} - \vec{r(s)}]^\top \dot{\vec{t}}_r = 0, \quad \dot{\vec{t}}_r = \vec{n}_r \dot{s} k_r \\
v_x \vec{t}_x^\top \vec{t}_r - \dot{s} \vec{t}_r^\top \vec{t}_r + d(t) \dot{s} k_r = 0 \\
\frac{v_x cos(e_\psi)}{\dot{s}} - 1 + k_r d = 0 \Rightarrow v_x = \dot{s} \frac{1 - k_r d}{cos(e_\psi)}
\end{aligned}
\tag{10}
$$

If s_x is the arc length of the the trajectory $\vec{x}$

$$
\frac{d}{ds} = \frac{ds_x}{ds} \frac{d}{ds_x} = \frac{ds_x}{dt} \frac{dt}{ds} \frac{d}{ds_x} = \frac{v_x}{\dot{s}} \frac{d}{ds_x} = \frac{1 - k_r d}{cos(e_\psi)} \frac{d}{ds_x}
\tag{11}
$$

Transformations from Frenent coordinates to global coordinates

Hence, the second derivative of d can be calculated as

$$\begin{aligned} d'' &= -[k_r d(t)]' tan(e_\psi) + \frac{1-k_r d}{cos^2(e_\psi)}\left[\frac{de_\psi}{ds}\right] \\ &= -[k_r d(t)]' tan(e_\psi) + \frac{1-k_r d}{cos^2(e_\psi)}\left[e'_\psi\right] \\ &= -[k'_r d(t) + k_r d'] tan(e_\psi) + \frac{1-k_r d}{cos^2(e_\psi)}\left[\frac{k(\rho_s - d(t))}{\rho_s cos(e_\psi)} - \frac{1}{\rho_s}\right] \\ &= -[k'_r d(t) + k_r d'] tan(e_\psi) + \frac{1-k_r d}{cos^2(e_\psi)}\left[k\frac{(1-k_r d)}{cos(e_\psi)} - k_r\right] \end{aligned} \tag{12}$$

Transformations from Frenent coordinates to global coordinates

Time differentiating velocity more time yields the last unknown $a_x := \dot{v}_x$

$$a_x = \dot{v}_x = \ddot{s}\frac{1-k_r d}{cos(e_\psi)} + \dot{s}\frac{d}{ds}\frac{1-k_r d}{cos(e_\psi)}\dot{s} = \ddot{s}\frac{1-k_r d}{cos(e_\psi)} + \frac{\dot{s}^2}{cos(e_\psi)}\left[[1-k_r d]tan(e_\psi)e'_\psi - [k'_r d + k_r d']\right] \quad (13)$$

For high-speed driving $\dot{d} = \frac{dd}{dt} = \frac{ds}{dt}\frac{dd}{ds} = \dot{s}d$ and $\ddot{d} = d''\dot{s}^2 + d'\ddot{s}$

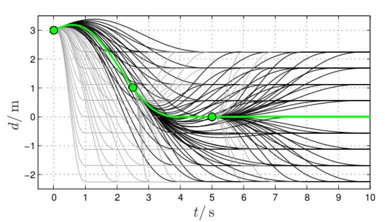
The Quntic polynomial is generated to represent the motion of the vehicle

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$
$$y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 \tag{14}$$

The objective is to minimize the square of the jerk,

$$C = \frac{1}{2} \int_0^T (\frac{d^3 x}{dt^3})^2 + (\frac{d^3 y}{dt^3})^2 dt \tag{15}$$

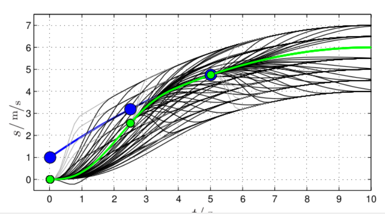
Optimal lateral trajectory generation



where green being the optimal trajectory, black the valid, and gray the invalid alternatives

[1]. Werling, M., Ziegler, J., Kammel, S., Thrun, S. (2010, May). Optimal trajectory generation for dynamic street scenarios in a frenet frame. In 2010 IEEE International Conference on Robotics and Automation (pp. 987-993). IEEE.

Optimal longitudinal tracking of a target position in blue



where green being the optimal trajectory, black the valid, and gray the invalid alternatives in each replanning step

[1]. Werling, M., Ziegler, J., Kammel, S., Thrun, S. (2010, May). Optimal trajectory generation for dynamic street scenarios in a frenet frame. In 2010 IEEE International Conference on Robotics and Automation (pp. 987-993). IEEE.

Frenet frame trajectory generation algorithm

Algorithm 3 Frenet frame trajectory generation algorithm

procedure FRENETTRAJGEN($v_x, \dot{v}_x, \dot{d}_0, \ddot{d}_0, s_0$) ▷
 $frenet\_paths \leftarrow []$
 for $d_i \leftarrow D_{min}$ to D_{max}, D_{width} **do**
  $fp \leftarrow FrenetPath()$ // placeholder for lat and log trajectories
  for $t_i \leftarrow T_{min}$ to $T_{max}, \Delta T$ **do**
   $lat \leftarrow generateQuinticPolynomial(d_0, \dot{d}_0, \ddot{d}_0, d_i, 0.0, 0.0, t_i)$
   for $t_j \leftarrow 0$ to $t_i, \Delta T$ **do**
    $fp.t << \leftarrow t_j$
   end for
   for $t \leftarrow 0$ to $fp.t$ **do**
    $fp.d << \leftarrow lat.get\_d(t)$
    $fp.\dot{d} << \leftarrow lat.get\_ddot(t)$
    $fp.\ddot{d} << \leftarrow lat.get\_dddot(t)$
   end for
   for $v_t \leftarrow v_T - \Delta V$ to $v_T + \Delta V, D_S$ **do**
    fps = copy(fp) // copy the lat trajectory and adding lon trajectory into fps
    $lon \leftarrow generateQuarticPolynomial(s_0, v_x, \dot{v}_x, v_t, 0.0, t_i)$
    for $t \leftarrow 0$ to $fp.t$ **do**
     $fps.s << \leftarrow lon.get\_s(t)$
     $fps.\dot{s} << \leftarrow lon.get\_sdot(t)$
     $fps.\ddot{s} << \leftarrow lon.get\_sddot(t)$
    end for
    $J_d \leftarrow \Sigma_{i=0}^{fp.t} fps.\dddot{d}^2(t_i)$
    $J_s \leftarrow \Sigma_{i=0}^{fp.t} fps.\ddot{s}^2(t_i)$
    $fps.c_d \leftarrow k_j \cdot J_d + k_t \cdot t_i + k_d \Sigma_{i=0}^{fp.t} \cdot fps.d(t_i)^2$
    $fps.c_v \leftarrow k_j \cdot J_s + k_t \cdot t_i + k_d \Sigma_{i=0}^{fp.t} (v_T - fps.s(t_i))^2$
    $fps.c_f \leftarrow k_{lat} fps.c_d + k_{lon} fps.c_v$
    $frenet\_paths << fps$
   end for
  end for
 end for
end procedure

Calculate global trajectories

After Frenet trajectories are estimated, the following algorithm can be utilized to transform the franent frame to the cartesian coordinate system.

- The position

$$x, y \rightarrow s, d$$

$$\vec{x}(s(t), d(t)) = \vec{r}(s(t)) + d(t)\vec{n}_r(s(t)),$$

- The velocity

$$\dot{s}, \dot{d} \rightarrow v_x, \psi$$

$$\dot{d} = v_x sin(\psi - \psi_r)$$

$$\dot{s} = \frac{v cos(\psi - \psi_r)}{1 - \kappa_r d}$$

$$v = \sqrt{\dot{s}^2(1 - \kappa_r d)^2 + \dot{d}^2}$$

$$\psi = arccos(\frac{\dot{s}(1 - \kappa_r d)}{v} + \psi_r)$$

- The acceleration

$$\ddot{s} = 0 \text{ for small } \Delta v$$

$$\ddot{d} = d''\dot{s}^2 + d'\ddot{s}$$

$$d'' = -(\kappa_r' d + \kappa_r d')tan(\theta - \theta_r) + \frac{1 - \kappa_r d}{cos^2 \Delta\theta}(\kappa_x \frac{1 - \kappa_r d}{cos\Delta\theta} - \kappa_r)$$

$$d' = (1 - \kappa_r d)tan\Delta\theta$$

$$\kappa_r' = \frac{(x'^2 + y'^2)(x'y''' + y'x''') - 3(x'y' - y'x'')(x'x'' + y'y'')}{(x'^2 + y'^2)^3}$$

where x' and y' are the parameterization of curve to the arc segment s

Calculate global trajectories

Algorithm 4 Calculate global trajectories

procedure CALGLOBALTRAJ($frenet\_paths, spline$)
    for $fp \leftarrow frenet\_paths_0$ to $frenet\_paths_m$ **do**
        for $i \leftarrow 0$ to $len(fp.s)$ **do**
            $x_i, y_i << \leftarrow getPositionVector(spline, s_i)$
            $yaw_i << \leftarrow getYaw(spline, s_i)$
            $d_i \leftarrow fp.d_i$
            $fp.x << \leftarrow x_i + d_i \cdot cos(yaw_i + \pi/2)$
            $fp.y << \leftarrow y_i + d_i \cdot sin(yaw_i + \pi/2)$
        end for
        for $i \leftarrow 0$ to $len(fp.x) - 1$ **do**
            $\delta x \leftarrow fp.x_{i+1} - fp.x_i$
            $\delta y \leftarrow fp.y_{i+1} - fp.y_i$
            $fp.yaw << \leftarrow atan2(\delta y, \delta x)$
            $fp.ds << \leftarrow hypotenuse(\delta x, \delta y)$
        end for
        $\triangleright$ set the last fp.yaw and fp.ds as before the last values that calculated
        for $i \leftarrow 0$ to $len(fp.yaw) - 1$ **do**
            $fp.k << \leftarrow \frac{fp.yaw_{i+1} - fp.yaw_i}{fp.ds_i}$
        end for
    end for
    $return frenet\_paths$
end procedure