Motion Planning for Autonomous Vehicles

Variation of Calculus

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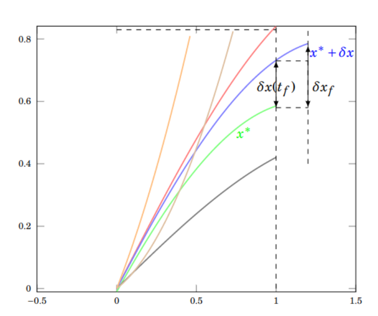


Variation of Calculus

Contents

Optimal control problem

To find a control $u^* \in U$ which causes the system $\dot{x}(t) = f(t, x(t), u(t))$ to follow a trajectory $x^* \in X$ that minimize the given objective function

$$J := h(t, x(t_f)) + \int_{t_0}^{t_f} g(u(t), x(t), t)\,dt \tag{1}$$

Different types of optimal control problems.

- Minimum-time problem
  from a given arbitrary initial state to a specified target set in a minimum time

$$\underset{t}{\text{minimize}} \quad \int_{t_0}^{t_f} dt = t_f - t_0 = t^*,$$

where $x(t_0), t_0$ is the initial state at time t_0, and $x(t_f), t_f$ is the final state at time t_f.

- Terminal control problem
  minimize the residual between the system's final state and its desired state

$$\underset{\mathbf{x}}{\text{minimize}} \quad J = \Sigma_{i=0}^{n}\left(x_i(t_f) - x_{d_i}(t_f)\right)^2,$$

where J can be formulated in the following ways as well:

$$\begin{aligned} J &= (\mathbf{x}(t_f) - \mathbf{x}_d(t_f))^\top (\mathbf{x}(t_f) - \mathbf{x}_d(t_f)) \\ &= \left\|\mathbf{x}(t_f) - \mathbf{x}_d(t_f)\right\|_2 = (\mathbf{x}(t_f) - \mathbf{x}_d(t_f))^\top H(\mathbf{x}(t_f) - \mathbf{x}_d(t_f)), \end{aligned}$$

where $H \geq 0$ is a real positive semi-definite matrix. For a given matrix is positive semi-definite if for all vectors z, $z^\top H z \geq 0$

- Minimum-control effect problems
  from a given arbitrary initial state to a specified target set in a minimum control effect

$$\underset{\mathbf{u}}{\text{minimize}} \quad J = \int_{t_0}^{t_n} |\mathbf{u}(t)| dt = \int_{t_0}^{t_n} \left(\Sigma_{i=0}^{m} \beta_i |u(t)| \right) dt,$$

  , where each β_i, denoted weighting factor of the corresponding control.

- Tracking problem
  minimize the residual between the system's current state and its desired state

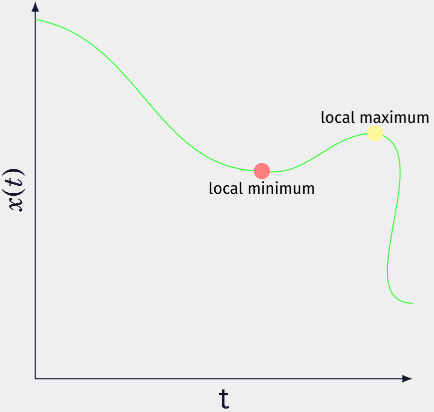
$$\underset{\mathbf{x}}{\text{minimize}} \quad \left\|\mathbf{x}(t_f) - \mathbf{x}_d(t_f)\right\|_H^2 + \int_{t_0}^{t_f} \left(\|\mathbf{x}(t) - \mathbf{x}_d(t)\|_{Q(t)}^2 + \|\mathbf{u}(t)\|_{R(t)} \right) dt$$

  where $Q(t), H$ are positive semi-definite matrices $\forall\, t \in [t_0, t_f]$
  and $R(t)$ is a positive definite matrix $\forall\, t \in [t_0, t_f]$

- Regulator problem
  minimize the residual between the system's current state and the final desired state

  $$\underset{\mathbf{x}}{\text{minimize}} \quad J = \left\|\mathbf{x}(t_f) - \mathbf{x}_d\right\|_H^2 + \int_{t_0}^{t_f} \left(\|\mathbf{x}(t) - \mathbf{x}_d\|_{Q(t)}^2 + \|\mathbf{u}(t)\|_{R(t)} \right) dt$$

  where $Q(t), H$ are positive semi-definite matrices $\forall\, t \in [t_0, t_f]$ and $R(t)$ is a positive definite matrix $\forall\, t \in [t_0, t_f]$



Local minimum

$$x(t^*) \leq x(t^* + \delta t), |\delta t| < \epsilon, \exists \epsilon > 0$$

with δt perturbation.
Global minimum

$$x(t^*) \leq x(t^* + \delta t), |\delta t| < \epsilon, \exists \epsilon > 0$$

where $x(\cdot)$ should be **smooth** (**exits 1st and 2nd derivatives**) and **convex**

If there is a **discontinuity** in the **first derivative** of a function, it means that it has a **sharp corner**, i.e., a place where there is an abrupt change in direction, and if not **function** is **continuous**. If there is a **discontinuity** in **second derivative**, it means there is an **abrupt change** in **curvature** (or radius of curvature)

A set $\Omega \subseteq \mathbb{R}^n$ is convex if and only if the line segment between any two points in Ω lies in Ω, i.e., $\forall x_1, x_2 \Omega$ and $0 \le \lambda \le 1$

$$\lambda x_1 + (1-\lambda)x_2 \in \Omega \tag{2}$$

$\lambda x_1 + (1-\lambda)x_2$, $\lambda \in [0,1]$ is called convex combination of x_1 and x_2. This can be generalized up to n points
$\lambda_1 x_1 + \ldots + \lambda_n x_n$, $\lambda_1 + \ldots + \lambda_n = 1$

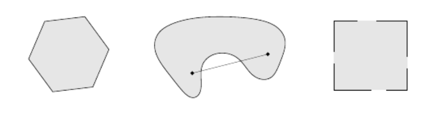
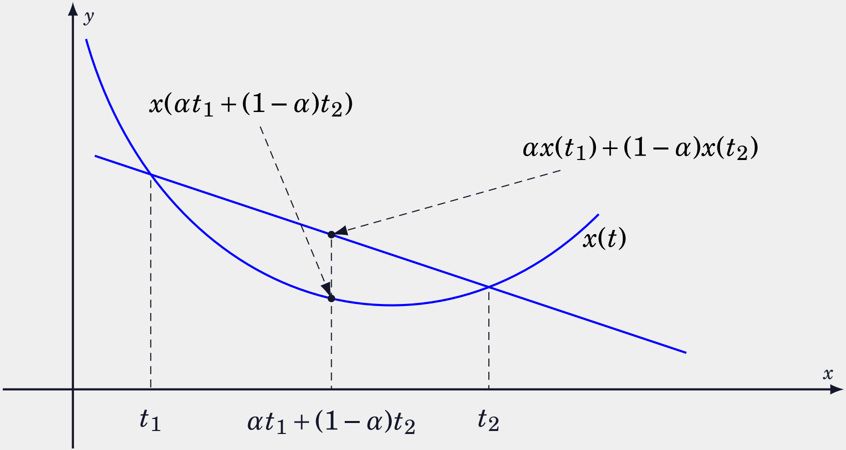


Figure: Some convex and nonconvex sets [1], convex set is much more general than convex function

[1]. Boyd, S., Boyd, S. P., Vandenberghe, L. (2004). Convex optimization. Cambridge university press.

Convexity



$$x(\alpha t_1 + (1-\alpha)t_2) \leq \alpha x(t_1) + (1-\alpha)x(t_2), \alpha \in [0,1] \tag{3}$$

Check the **Hessian matrix** of the function. If the matrix is **Positive-definite** then the function is **strictly convex**, Positive **semi-definite** then the function is **convex**.

$$\operatorname{Hess} f_p(\mathbf{v}) = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

For the determinant, $|\operatorname{Hess} f_p|$.

Example 01

Calculate the Hessian matrix at the point (4,2) of the following multivariable function and decide it is a convex function or not

$$f(x,y) = y^4 + x^4 + 3x^2 + 4y^2 - 4xy - 5y + 8$$

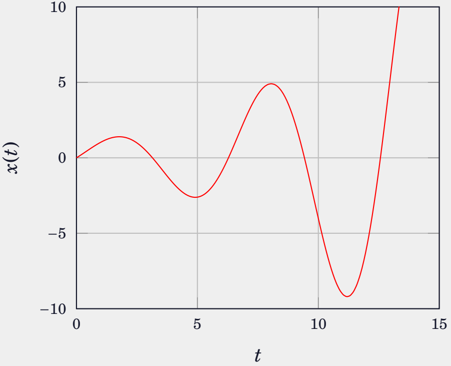
A **strictly convex function** will always take a **unique minimum**. For a **convex function** which is not strictly convex the **minimum needs not to be unique**
Consider the following function

$$f(x) = \begin{cases} -x-4, \text{ if } x \leq -4 \\ 0, \text{ if } -4 < x < 4 \\ x-4, \text{ if } x \geq 4 \end{cases}$$

f is convex because the first inequality above holds. However it is not strictly convex because for $x = -2$ and $y = 2$ the inequality does not hold strictly.

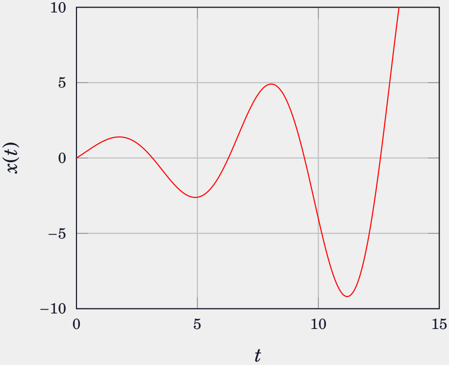
Local minimum of a function

Consider a local minimum of $x(t) = e^{0.2 \cdot t} \cdot sin(t)$



Use CasADi `https://web.casadi.org/` toolbox to solve this

Consider a local minimum of $x(t) = e^{0.2\cdot t} \cdot sin(t)$ s.t. $t \geq 0, t \leq 4\pi$

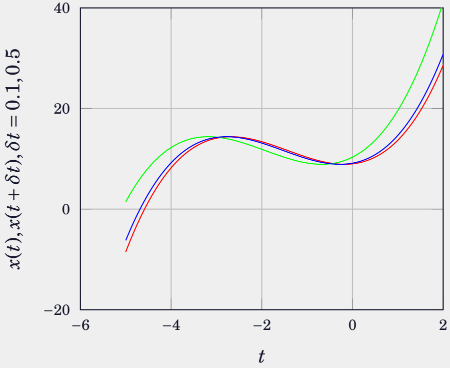


Use CasADi `https://web.casadi.org/` toolbox to solve this

Linearization of function up to the second variation

Consider

$x(t) = f(t) = 0.7t^3 + 3t^2 + t + 9$



Taylor series expansion around local minimum or maximum, e.g., $t = t^*$, for a function

$$f(t^* + \delta t) = f(t^*) + \frac{\partial f(t)}{\partial t}\bigg|_{t=t^*} \delta t + \frac{1}{2}\frac{\partial^2 f(t)}{\partial t^2}\bigg|_{t=t^*} (\delta t)^2 + H.O.C \tag{4}$$

- Incremental of a function

$$\Delta f = \delta f(t, \delta t) = f(t + \delta t) - f(t) \tag{5}$$

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- Incremental of a function around an extremum, e.g., $t = t^*$,

$$\begin{aligned} \Delta f &= \delta f(t^*, \delta t) = f(t^* + \delta t) - f(t^*) \\ &= f(t^*) + \left.\frac{\partial f(t)}{\partial t}\right|_{t=t^*} \delta t + \underbrace{\frac{1}{2}\left.\frac{\partial^2 f(t)}{\partial t^2}\right|_{t=t^*} (\delta t)^2 + \ldots}_{H.O.T} - f(t^*) = \frac{\partial f(t^*)}{\partial t} \delta t, \end{aligned} \tag{6}$$

where Δf is the differential of a function at t^*, $\dot{f}(t^*)$ is the derivative of f at t^*.

Slope, the average rate of change, $\frac{\partial f}{\partial x}$, is generally applicable when only 2 variables are in consideration. The slope is the tangent or the derivative to the function's curve that connects the 2 variables, i.e., a measure of the rate of change of a function f(x) with respect to the x.
A tangent line is a straight line that touches a function at only one point. The **tangent line** represents the instantaneous rate of change of the function (level set) at that one point. The **slope** of the tangent line at a point on the function is equal to the **derivative** of the function at the same point.

https://clas.sa.ucsb.edu/staff/lee/secant,%20tangent,%20and%20derivatives.ht

THE FIRST ORDER APPROXIMATION Δf TO INCREMENT δt

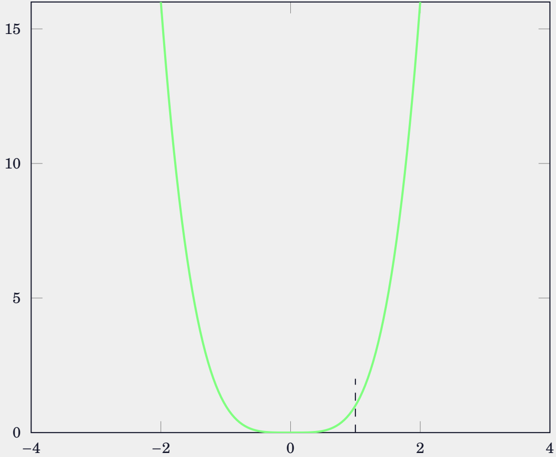
The $f(t)$ is said to have a local optimal at point t^*, if there is a positive parameter ϵ that satisfy $|t - t^*| < \epsilon$, also increment of $f(t)$ has the same sign (positive or negative)

- $\Delta f = f(t) - f(t^*) \geq 0$ then, $f(t^*)$ is a local minimum
- $\Delta f = f(t) - f(t^*) \leq 0$ then, $f(t^*)$ is a local maximum

Hence, **necessary condition** for optimal of a function $\left.\frac{\partial f(t)}{\partial t}\right|_{t=t^*} = 0$, which $f(t^*)$ is called critical point or stationary point, and the **sufficient condition**

- $\left.\frac{\partial^2 f(t)}{\partial t^2}\right|_{t=t^*} > 0$ then, $f(t^*)$ is a local minimum
- $\left.\frac{\partial^2 f(t)}{\partial t^2}\right|_{t=t^*} < 0$ then, $f(t^*)$ is a local maximum

Consider $x(t) = t^4$, Only the **fourth derivative** is **non-zero**.



In the case of higher order $t \in \mathbb{R}^n$, **Approach 01**;

$$\frac{\partial f(t^* + d \cdot \delta t)}{\partial \delta t} = \Sigma_{i=1}^{n} \frac{\partial f(t^* + d \cdot \delta t)}{\partial t_i} \cdot d_i = \Big(\Delta f(t^* + d \cdot \delta t)\Big)^\top \cdot d$$

$d \in \mathbb{R}^n$, where **d is arbitrary direction but fixed**. Hence, the **first-order necessary condition**: $\Big(\Delta f(t^*)\Big) = 0, \delta t = 0$, where the gradient as a column vector $\Delta f = < \frac{\partial f}{\partial t_1}, \ldots, \frac{\partial f}{\partial t_n}^\top >$. The **gradient** is the **transpose derivatives**, i.e., the gradient is just the vector of partial derivatives.

$$\frac{\partial^2 f(t^* + d \cdot \delta t)}{\partial \delta t^2} = \Sigma_{i=1}^{n} \Sigma_{j=1}^{n} \frac{\partial^2 f(t^* + d \cdot \delta t)}{\partial t_i \partial t_j} \cdot d_i d_j = d^\top \cdot \underbrace{\Big(\Delta^2 f(t^* + d \cdot \delta t)\Big)^\top}_{\text{Hessian}} \cdot d$$

the **second-order necessary condition** $d^\top \Big(\Delta^2 f(t^*)\Big) d \geq 0, \delta t = 0$, or $\Delta^2 f(t^*) > 0$ or **eigen values** λ must be higher than zero, namely **positive definite matrix** $\lambda : \Big(\Delta^2 f(t^*)\Big) > 0$

Approach 02; $t^* \Rightarrow t^* + d, d \in \mathbb{R}^n$, where d for all the directions
Consider f(x,y) = $x^2 + y^2$ calculate its Hessian and check it has a local minimum.

Incremental of a functional (gradient-based first order conditions)

A functional is simply a **function** that **maps** to $\mathbb{R}$. A function y(t) takes as input a number t and returns a number. A functional F(y) takes as input a function y(t) and returns a number.

$$
\begin{aligned}
\Delta J = \Delta J(x(t), \delta x(t)) = J(x(t) + \delta x(t)) - J(x(t)) \\
= J(x(t)) + \frac{\partial J}{\partial x}\delta x(t) + \frac{1}{2!}\frac{\partial^2 J}{\partial \delta x^2}(\delta x(t))^2 + \underbrace{H.O.C}_{=0} - J(x(t)) \\
= \delta J + \delta^2 J,
\end{aligned}
\tag{7}
$$

where δJ (**first variation**) is **not zero** then the **sign** will be governed by the **first variation,** likewise, if δJ (**first variation**) is **zero** then the sign will be governed by the **second variation**.

Consider $x(t) = t^2 + 4$. The sign gives whether it is a minimum or maximum.
Now consider the functional at a optimal value $x(t) = x^*(t)$ and obtain expression for the first variation δJ

$$\begin{aligned}
\Delta J(x^*(t), \delta x(t)) &= J(x^*(t) + \delta x(t), \dot{x}^*(t) + \delta \dot{x}(t), t) - J(x^*(t), \dot{x}^*(t), t) \\
&= \int_{t_0}^{t_f} g(x^*(t) + \delta x(t), \dot{x}^*(t) + \delta \dot{x}(t), t) dt - \int_{t_0}^{t_f} g(x^*(t), \dot{x}^*(t), t) dt \\
&= \int_{t_0}^{t_f} g(x^*(t) + \delta x(t), \dot{x}^*(t) + \delta \dot{x}(t), t) - g(x^*(t), \dot{x}^*(t), t) dt
\end{aligned} \tag{8}$$

- By considering the **functional incremental** (eq.7), For the simplicity, let $g(x^*(t)+\delta x(t),\dot{x}^*(t)+\delta\dot{x}(t),t)$ be $g(\cdot)$. Then eq.(8) can be written as

$$\Delta J(x^*(t),\delta x(t)) = \int_{t_0}^{t_f} \Big[\frac{\partial g(\cdot)}{\partial x}\delta x(t) + \frac{\partial g(\cdot)}{\partial \dot{x}}\delta\dot{x}(t)\Big]dt \qquad (9)$$

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$$\Delta J(x^*(t), \delta x(t)) = \int_{t_0}^{t_f} \Big[\frac{\partial g(\cdot)}{\partial x}\delta x(t) + \frac{\partial g(\cdot)}{\partial \dot{x}}\dot{\delta x}(t)\Big] dt \tag{9}$$

- And using **integration by parts**: $\int u dv = uv - \int v du$, $\frac{\partial g(\cdot)}{\partial \dot{x}}\dot{\delta x}(t)$ can be expanded as

$$\int_{t_0}^{t_f} \underbrace{\frac{\partial g(\cdot)}{\partial \dot{x}}}_{u} \underbrace{\dot{\delta x}(t) dt}_{dv} = \int_{t_0}^{t_f} \frac{\partial g(\cdot)}{\partial \dot{x}} \frac{d}{dt}(\delta x(t)) dt = \int_{t_0}^{t_f} \frac{\partial g(\cdot)}{\partial \dot{x}} d(\delta x(t))$$

$$= \Big[\frac{\partial g(\cdot)}{\partial \dot{x}}\delta x(t)\Big]_{t_0}^{t_f} - \int_{t_0}^{t_f} \delta x(t) \frac{d}{dt}\Big(\frac{\partial g(\cdot)}{\partial \dot{x}}\Big) dt \tag{10}$$

- Now eq.(9) can be rewritten after incorporating eq.(10)

$$\begin{aligned}\Delta J(x^*(t),\delta x(t)) &= \int_{t_0}^{t_f}\Big[\Big(\frac{\partial g(\cdot)}{\partial x(t)}\Big)\delta x(t)dt + \Big[\Big(\frac{\partial g(\cdot)}{\partial \dot{x}(t)}\Big)\delta x(t)\Big]_{t_0}^{t_f} \\ &\quad - \int_{t_0}^{t_f}\frac{d}{dt}\Big(\frac{\partial g(\cdot)}{\partial \dot{x}(t)}\Big)\Big]\delta x(t)dt \\ &= \int_{t_0}^{t_f}\Big[\Big(\frac{\partial g(\cdot)}{\partial x(t)}\Big) - \frac{d}{dt}\Big(\frac{\partial g(\cdot)}{\partial \dot{x}(t)}\Big)\Big]\delta x(t)dt + \Big[\Big(\frac{\partial g(\cdot)}{\partial \dot{x}(t)}\Big)\delta x(t)\Big]_{t_0}^{t_f}\end{aligned} \tag{11}$$

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$$\Delta J(x^*(t),\delta x(t)) = \int_{t_0}^{t_f}\Big[\Big(\frac{\partial g(\cdot)}{\partial x(t)}\Big)\delta x(t)dt + \Big[\Big(\frac{\partial g(\cdot)}{\partial \dot{x}(t)}\Big)\delta x(t)\Big]_{t_0}^{t_f} - \int_{t_0}^{t_f}\frac{d}{dt}\Big(\frac{\partial g(\cdot)}{\partial \dot{x}(t)}\Big)\Big]\delta x(t)dt \tag{11}$$

$$= \int_{t_0}^{t_f}\Big[\Big(\frac{\partial g(\cdot)}{\partial x(t)}\Big) - \frac{d}{dt}\Big(\frac{\partial g(\cdot)}{\partial \dot{x}(t)}\Big)\Big]\delta x(t)dt + \Big[\Big(\frac{\partial g(\cdot)}{\partial \dot{x}(t)}\Big)\delta x(t)\Big]_{t_0}^{t_f}$$

- To find the **second variation** (δJ^2), again consider eq.(8)

$$\Delta J(x^*(t),\delta x(t)) = \int_{t_0}^{t_f}\Big[\frac{\partial g(\cdot)}{\partial x}\delta x(t) + \frac{\partial g(\cdot)}{\partial \dot{x}}\dot{\delta x}(t) + \frac{1}{2!}\Big[\frac{\partial^2 g(\cdot)}{\partial x^2}(\delta x(t))^2 + \frac{\partial^2 g(\cdot)}{\partial \dot{x}^2}(\dot{\delta x}(t))^2 + 2\frac{\partial^2 g(\cdot)}{\partial \dot{x}\partial x}(\dot{\delta x}(t))\cdot\delta x(t) + \ldots\Big]\Big]dt \tag{12}$$

- When considering only the second variation

$$\delta J^2 = \int_{t_0}^{t_f} \frac{1}{2!}\bigg[\Big(\frac{\partial^2 g(\cdot)}{\partial x^2}\Big)(\delta x(t))^2 + \Big(\frac{\partial^2 g(\cdot)}{\partial (\dot{x})^2}\Big)(\dot{\delta x}(t))^2 + \Big(2\frac{\partial^2 g(\cdot)}{\partial \dot{x}\cdot\partial x}\delta\dot{x}(t)\delta x(t)\Big)\bigg] d \tag{13}$$

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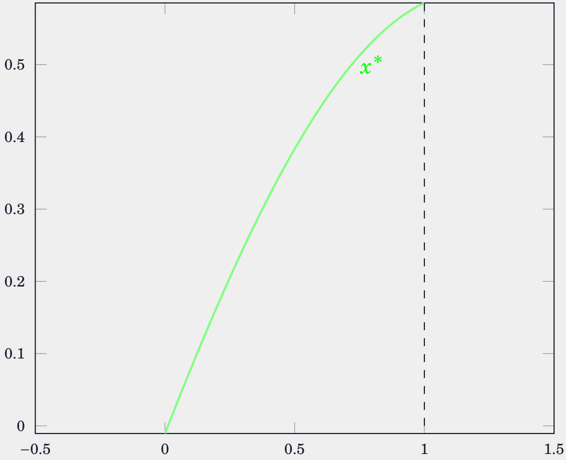
- Expanding the last term using integration by parts, where $u = \frac{\partial^2 g(\cdot)}{\partial \dot{x}\cdot\partial x}\delta x(t)$ and $dv = \delta\dot{x}(t)dt$

$$\delta J^2 = \frac{1}{2}\int_{t_0}^{t_f} \Big[\Big[\Big(\frac{\partial^2 g}{\partial x^2}\Big) - \frac{d}{dt}\big(\frac{\partial^2 g}{\partial \dot{x}\cdot\partial x}\big)\Big](\delta x(t))^2 + \Big(\frac{\partial^2 g}{\partial (\dot{x})^2}\Big)(\dot{\delta x}(t))^2\Big]dt + \Big[\frac{\partial^2 g(\cdot)}{\partial \dot{x}\cdot\partial x}\delta x(t)\delta x(t)\Big]_{t_0}^{t_f} \tag{14}$$

Hence, the first order approximation δJ to increment δt. J is said to have a **local extremum**, if there is a positive parameter ϵ that satisfy for all functions $|x - x^*| < \epsilon$, also increment of J has the same sign (positive or negative).
If the mentioned condition is valid for **large** ϵ, then $J(x^*)$ value gives the **global extremum**. Hence, **necessary condition** for optimal of a functional $\delta J = 0$ for all admissible value of $\delta x(t)$, and the **sufficient condition**

- $\delta J^2 > 0$ then, $J(x^*)$ is a local minimum
- $\delta J^2 < 0$ then, $J(x^*)$ is a local maximum

Consider the **initial** and the **final** values are fixed. In other words, boundary conditions are specified $x(t_0), t_0$ and $x(t_f), t_f$ are given.



- When initial and final values are fixed, to obtain the optimal $x^*(t)$, $\delta J(x^*(t), \delta x(t)) = 0$ has to be **zero**. Let's consider the **first variation**.

$$\Delta J(x^*(t), \delta x(t)) = \int_{t_0}^{t_f} \left[\left(\frac{\partial g(\cdot)}{\partial x(t)} \right) - \frac{d}{dt} \left(\frac{\partial g(\cdot)}{\partial \dot{x}(t)} \right) \right] \delta x(t) dt + \left[\left(\frac{\partial g(\cdot)}{\partial \dot{x}(t)} \right) \delta x(t) \right]_{t_0}^{t_f} \tag{15}$$

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- Since the **initial** and **final** values are **fixed**, which is no variation at the start and final point $(\delta x(t_0) = 0, \quad \delta x(t_f) = 0)$, $\left[\left(\frac{\partial g(\cdot)}{\partial \dot{x}(t)}\right)\delta x(t)\right]_{t_0}^{t_f} = 0$, then eq.(15) becomes

$$\Delta J(x^*(t), \delta x(t)) = \int_{t_0}^{t_f} \underbrace{\left[\left(\frac{\partial g(\cdot)}{\partial x(t)}\right) - \frac{d}{dt}\left(\frac{\partial g(\cdot)}{\partial \dot{x}(t)}\right)\right]}_{g(t)} \delta x(t)dt \tag{16}$$

Fixed value problem

Between the **start** and the **final** point, $\delta x(t)$ usually can **not be zero** since it is **arbitrary**. Hence, **g(t) must be zero**.

Lemma

If a continuous function g(t) on an open interval (t_0, t_f) satisfies the equality

$$\int_{t_0}^{t_f} g(t)\delta x(t)dt = 0, \tag{17}$$

where the function $\delta x(t)$ is continuous in the interval $[t_0, t_f]$, then g(t) is identically zero

https://en.wikipedia.org/wiki/Fundamental_lemma_of_calculus_of_variations

- After considering the Lemma 1, from eq.(16) following condition, i.e., Euler–Lagrange equation, can be derived

$$\left(\frac{\partial g(\cdot)}{\partial x(t)}\right) - \frac{d}{dt}\left(\frac{\partial g(\cdot)}{\partial \dot{x}(t)}\right) = 0 \tag{18}$$

- After considering the Lemma 1, from eq.(16) following condition, i.e., Euler–Lagrange equation, can be derived

$$\left(\frac{\partial g(\cdot)}{\partial x(t)}\right) - \frac{d}{dt}\left(\frac{\partial g(\cdot)}{\partial \dot{x}(t)}\right) = 0 \tag{18}$$

- The **sufficient condition for a minimum** is $\delta^2 J > 0$

$$\delta J^2 = \frac{1}{2}\int_{t_0}^{t_f} \left[\left[\left(\frac{\partial^2 g(\cdot)}{\partial x^2}\right) - \frac{d}{dt}\left(\frac{\partial^2 g(\cdot)}{\partial \dot{x}\cdot\partial x}\right)\right](\delta x(t))^2 + \left(\frac{\partial^2 g(\cdot)}{\partial (\dot{x})^2}\right)(\delta \dot{x}(t))^2\right] dt + \underbrace{\left[\frac{\partial^2 g(\cdot)}{\partial \dot{x}\cdot\partial x}\delta x(t)\delta x(t)\right]_{t_0}^{t_f}}_{=0} \tag{19}$$

- Since $(\dot{\delta x}(t))^2 > 0$ and $(\delta x(t))^2 > 0$, the following two conditions must be satisfied

$$\begin{aligned} \Big(\frac{\partial^2 g}{\partial x^2}\Big) - \frac{d}{dt}\big(\frac{\partial^2 g}{\partial \dot{x}\cdot\partial x}\big) > 0 \\ \frac{\partial^2 g}{\partial (\dot{x})^2}\Big) > 0 \end{aligned} \tag{20}$$

- Since $(\dot{\delta x}(t))^2 > 0$ and $(\delta x(t))^2 > 0$, the following two conditions must be satisfied

$$
\begin{aligned}
\Big(\frac{\partial^2 g}{\partial x^2}\Big) - \frac{d}{dt}\Big(\frac{\partial^2 g}{\partial \dot{x}\cdot\partial x}\Big) > 0 \\
\frac{\partial^2 g}{\partial (\dot{x})^2}\Big) > 0
\end{aligned}
\tag{20}
$$

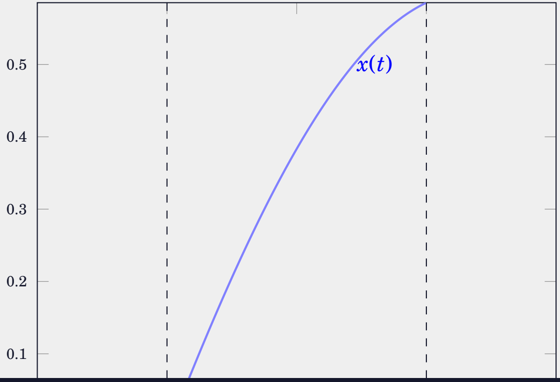
- The eq.(19) can be rearrange into the following form:

$$
\begin{aligned}
\delta J^2 = \frac{1}{2}\int_{t_0}^{t_f} \begin{bmatrix}\delta x(t) & \dot{\delta x}(t)\end{bmatrix} \begin{bmatrix} \frac{\partial^2 g(\cdot)}{\partial x^2} & \frac{\partial^2 g(\cdot)}{\partial \dot{x}\cdot\partial x} \\ \frac{\partial^2 g(\cdot)}{\partial \dot{x}\cdot\partial x} & \frac{\partial^2 g(\cdot)}{\partial (\dot{x})^2} \end{bmatrix} \begin{bmatrix}\delta x(t) \\ \dot{\delta x}(t)\end{bmatrix} dt \\
\frac{1}{2}\int_{t_0}^{t_f} \begin{bmatrix}\delta x(t) & \dot{\delta x}(t)\end{bmatrix} \Xi \begin{bmatrix}\delta x(t) \\ \dot{\delta x}(t)\end{bmatrix} dt,
\end{aligned}
\tag{21}
$$

where if $\Xi = \begin{bmatrix} \frac{\partial^2 g}{\partial x^2} & \frac{\partial^2 g}{\partial \dot{x}\cdot\partial x} \\ \frac{\partial^2 g}{\partial \dot{x}\cdot\partial x} & \frac{\partial^2 g}{\partial (\dot{x})^2} \end{bmatrix}_*$ is **positive definite**, the result will be **minimum** otherwise, i.e., **negative definite**, **maximum**. This way we can define the **objective function** that gives **minimum** or **maximum** optimal

Example 02

Consider the initial and final conditions given as $t_0 = e, x(t_0) = f$ and $t_f = g, x(t_f) = h$, respectively. Find the shortest path possible between the interval [e,g]. A small distance along the curve $x(t)$ can be defined as $ds = \sqrt{dx^2 + dt^2}$.



Fixed value problem

$$s = \int_e^g \sqrt{dx^2 + dt^2} = \int_e^g \sqrt{1 + d\dot{x}^2} dt \tag{22}$$

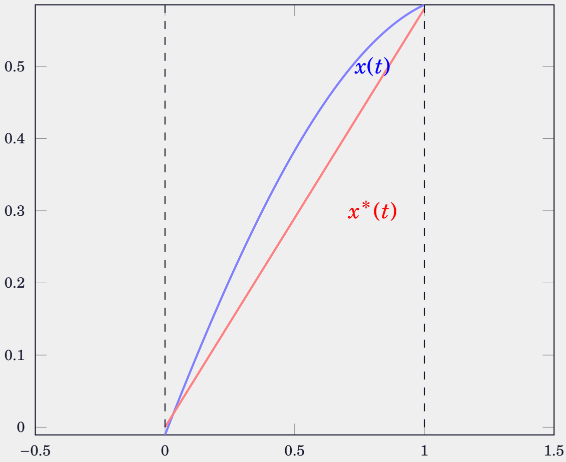
Term $\sqrt{1 + d\dot{x}^2}$ can be considered as $g(\cdot)$ as given in eq.18. To obtain the optimal curve, the following condition must be satisfied.

$$\left(\frac{\partial g(\cdot)}{\partial x(t)}\right) - \frac{d}{dt}\left(\frac{\partial g(\cdot)}{\partial \dot{x}(t)}\right) = 0 \tag{23}$$

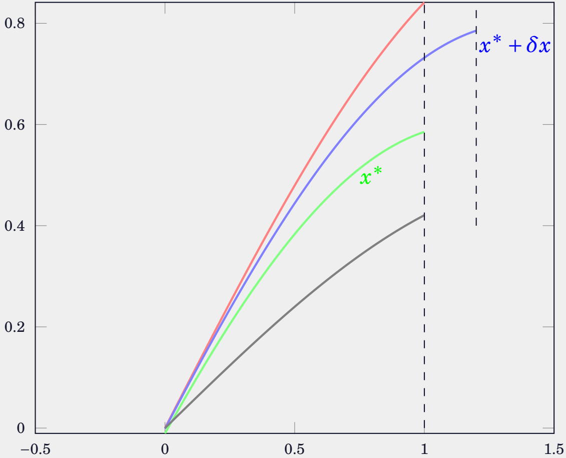
The partial derivatives of $g(\cdot)$ are $\frac{\partial g(\cdot)}{\partial \dot{x}(t)} = \frac{\dot{x}}{\sqrt{1+\dot{x}^2}}$ and $\frac{\partial g(\cdot)}{\partial x(t)} = 0$, where $g(\cdot) = g(t, x, \dot{x})$. Therefore, by substituting these into eq.23

$$\begin{aligned}
\frac{d}{dt}\Big(\frac{\partial g(\cdot)}{\partial \dot{x}(t)}\Big) &= 0 \\
\frac{d}{dt}\Big(\frac{\dot{x}(t)}{\sqrt{1+(\dot{x}(t))^2}}\Big) &= 0 \\
\frac{\dot{x}(t)}{\sqrt{1+(\dot{x}(t))^2}} = c, \quad c \in \mathbb{R} \quad or \quad \ddot{x}(t) &= 0 \\
\Rightarrow \dot{x}(t) = \frac{c}{\sqrt{1-c^2}} &=: a \\
\Rightarrow x(t) &= at + b
\end{aligned} \tag{24}$$

Fixed value problem



If $x(t_0) = x_0, t_0$ is fixed, and $x(t_f)$, t_f is free.



Consider $g(\cdot) = g(x^*(t) + \delta x(t), \dot{x^*}(t) + \dot{\delta x}(t), t)$

$$\begin{aligned}
\Delta J(x^*(t), \delta x(t)) &= J(x^*(t) + \delta x(t), \dot{x^*}(t) + \dot{\delta x}(t), t) - J(x^*(t), \dot{x^*}(t), t) \\
&= \int_{t_0}^{t_f + \delta t_f} g(\cdot) dt - \int_{t_0}^{t_f} g(x^*(t), \dot{x^*}(t), t) dt \\
&= \int_{t_0}^{t_f} \Big(g(\cdot) - g(x^*(t), \dot{x^*}(t), t) \Big) dt + \int_{t_f}^{t_f + \delta t_f} g(\cdot) dt \\
&= \int_{t_0}^{t_f} \Big[\frac{\partial g(\cdot)}{\partial x} \delta x(t) + \frac{\partial g(\cdot)}{\partial \dot{x}} \delta \dot{x}(t) + H.O.C \Big] dt + \int_{t_f}^{t_f + \delta t_f} g(\cdot) dt
\end{aligned} \tag{25}$$

expanding $\int_{t_0}^{t_f} \frac{\partial g(\cdot)}{\partial \dot{x}} \delta \dot{x}(t) dt$ by integration by parts

- $$\int_{t_0}^{t_f} \frac{\partial g(\cdot)}{\partial \dot{x}} \delta \dot{x}(t) dt = \left(\frac{\partial g(\cdot)}{\partial \dot{x}} \right) \delta x(t) \Big|_{t_0}^{t_f} - \int_{t_0}^{t_f} \frac{d}{dt} \left(\frac{\partial g}{\partial \dot{x}} \right) \delta x(t) dt \tag{26}$$

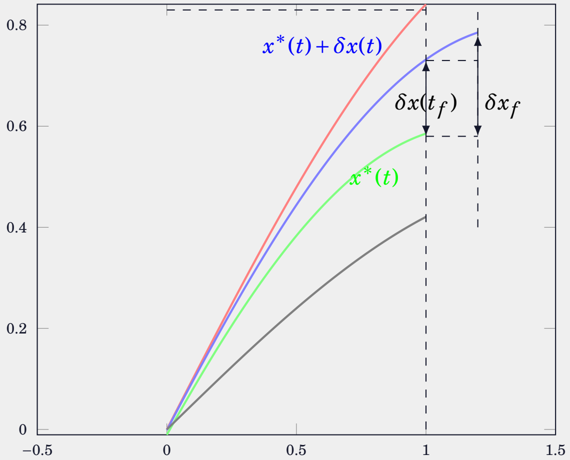
- $$\int_{t_0}^{t_f} \frac{\partial g(\cdot)}{\partial \dot{x}} \delta \dot{x}(t) dt = \left(\frac{\partial g(\cdot)}{\partial \dot{x}}\right) \delta x(t) \Big|_{t_0}^{t_f} - \int_{t_0}^{t_f} \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}\right) \delta x(t) dt \tag{26}$$
- However, t_0 is fixed, eq.26 can be written as

$$\int_{t_0}^{t_f} \frac{\partial g(\cdot)}{\partial \dot{x}} \delta \dot{x}(t) dt = \left(\frac{\partial g(\cdot)}{\dot{x}}\right) \Big|_{t_f} \delta x(t_f) - \int_{t_0}^{t_f} \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}\right) \delta x(t) dt \tag{27}$$

Free terminal point problem

- $$\int_{t_0}^{t_f} \frac{\partial g(\cdot)}{\partial \dot{x}} \delta \dot{x}(t) dt = \left(\frac{\partial g(\cdot)}{\partial \dot{x}}\right) \delta x(t) \Big|_{t_0}^{t_f} - \int_{t_0}^{t_f} \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}\right) \delta x(t) dt \qquad (26)$$
- However, t_0 is fixed, eq.26 can be written as
$$\int_{t_0}^{t_f} \frac{\partial g(\cdot)}{\partial \dot{x}} \delta \dot{x}(t) dt = \left(\frac{\partial g(\cdot)}{\dot{x}}\right)\Big|_{t_f} \delta x(t_f) - \int_{t_0}^{t_f} \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}\right) \delta x(t) dt \qquad (27)$$
- Hence, eq.(25) can be reformulated as
$$\Delta J(x^*(t), \delta x(t)) = \left(\frac{\partial g(\cdot)}{\dot{x}}\right)\Big|_{t_f} \delta x(t_f) + \int_{t_0}^{t_f} \left[\frac{\partial g(\cdot)}{\partial x} - \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}\right)\right] \delta x(t) + H.O.C\Big] dt + \int_{t_f}^{t_f+\delta t_f} g(\cdot) dt \qquad (28)$$
The term $\int_{t_f}^{t_f+\delta t_f} g(\cdot) dt$ can **approximated** by **taking area under curve from** t_f to $t_f+\delta t_f$

Free terminal point problem



- $$\begin{aligned}\int_{t_f}^{t_f+\delta t_f} g(\cdot)dt &= g(x(t_f),\dot{x}(t_f),t_f)\delta t_f + H.O.C \\ &= g(x^*(t_f)+\delta x(t_f),\dot{x}^*(t_f)+\delta\dot{x}(t_f),t_f)\delta t_f + H.O.C\end{aligned} \tag{29}$$

- $$\int_{t_f}^{t_f+\delta t_f} g(\cdot)dt = g(x(t_f),\dot{x}(t_f),t_f)\delta t_f + H.O.C$$
  $$= g(x^*(t_f)+\delta x(t_f),\dot{x}^*(t_f)+\delta\dot{x}(t_f),t_f)\delta t_f + H.O.C \tag{29}$$
- Expanding eq.29 with Taylor series,

$$\int_{t_f}^{t_f+\delta t_f} g(\cdot)dt = g(x^*(t_f),\dot{x}^*(t_f),t_f)\delta t_f + \frac{\partial g(\cdot)}{\delta x}\bigg|_{t_f}\delta x(t_f)\delta t_f + \left(\frac{\partial g(\cdot)}{\partial \dot{x}}\right)\bigg|_{t=t_f}\delta\dot{x}(t_f)\delta t_f + H.O.C\delta t_f \tag{30}$$

- After eliminating higher order terms eq.30 becomes

$$\int_{t_f}^{t_f+\delta t_f} g(\cdot)dt = g(x^*(t_f), \dot{x}^*(t_f), t_f)\delta t_f = g(\cdot)\Big|_{t_f} \delta t_f \tag{31}$$

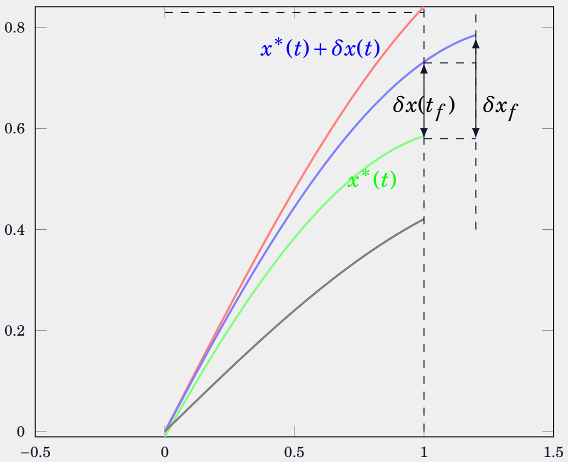
- After eliminating higher order terms eq.30 becomes

$$\int_{t_f}^{t_f+\delta t_f} g(\cdot)dt = g(x^*(t_f), \dot{x}^*(t_f), t_f)\delta t_f = g(\cdot)\Big|_{t_f}\delta t_f \tag{31}$$

- After substituting the result from eq.31 to eq.28

$$\Delta J(x^*(t), \delta x(t)) = \left(\frac{\partial g(\cdot)}{\partial \dot{x}}\right)\Big|_{t_f}\delta x(t_f) + \int_{t_0}^{t_f}\left[\frac{\partial g(\cdot)}{\partial x} - \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}\right)\right]\delta x(t) + H.O.C \\ + g(\cdot)\Big|_{t_f}\delta t_f \tag{32}$$

Free terminal point problem



- Term δt_f **depends** on $\delta x(t_f)$, the relationship between these can be obtained by considering the following **linear approximation**, i.e., derivative of a function at a point is the slope of the tangent to the curve at that point.

$$\begin{aligned} \dot{x}(t_f) + \delta\dot{x}(t_f) &\approx \frac{\delta x_f - \delta x(t_f)}{\delta t_f} \\ \dot{x}(t_f)\cdot\delta t_f + \underbrace{\delta\dot{x}(t_f)\cdot\delta t_f}_{higher\,order} &\approx \delta x_f - \delta x(t_f), \\ \delta x(t_f) &= \delta x_f - \dot{x}(t_f)\delta t_f, \end{aligned} \tag{33}$$

where term $\delta\dot{x}(t_f)\cdot\delta t_f = 0$ due to higher order term.

- The overall expression for eq.32 can be represented as

$$\Delta J(x^*(t),\delta x(t)) = \int_{t_0}^{t_f} \Big[\frac{\partial g(\cdot)}{\partial x} - \frac{d}{dt}\Big(\frac{\partial g}{\partial \dot{x}}\Big)\Big]\delta x(t) + H.O.D\Big]dt$$
$$+\Big(\frac{\partial g(\cdot)}{\partial \dot{x}}\Big)\Big|_{t_f}\delta x_f + \Big[g(\cdot)\Big|_{t_f} - \frac{\partial g(\cdot)}{\partial \dot{x}}\Big|_{t_f}\dot{x}(t_f)\Big]\delta t_f \tag{34}$$

- The overall expression for eq.32 can be represented as

$$\begin{aligned}\Delta J(x^*(t),\delta x(t)) = \int_{t_0}^{t_f} \Big[\frac{\partial g(\cdot)}{\partial x} - \frac{d}{dt}\Big(\frac{\partial g}{\partial \dot{x}}\Big)\Big]\delta x(t) + H.O.D\Big]dt \\ +\Big(\frac{\partial g(\cdot)}{\partial \dot{x}}\Big)\Big|_{t_f}\delta x_f + \Big[g(\cdot)\Big|_{t_f} - \frac{\partial g(\cdot)}{\partial \dot{x}}\Big|_{t_f}\dot{x}(t_f)\Big]\delta t_f\end{aligned} \tag{34}$$

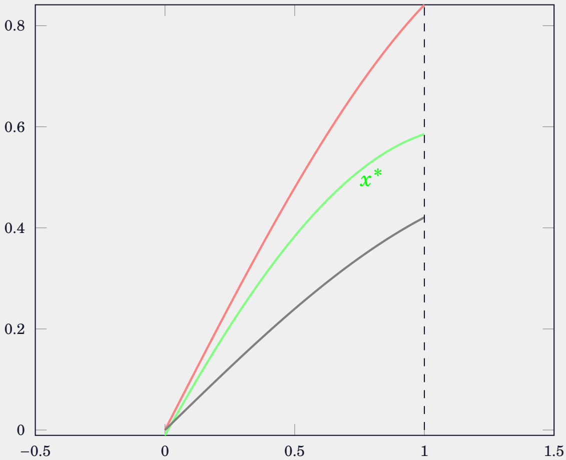
- According the lemma.1, to have a minimum or maximum value $\delta J = 0$

$$\begin{aligned}\Big[\frac{\partial g(\cdot)}{\partial x} - \frac{d}{dt}\Big(\frac{\partial g}{\partial \dot{x}}\Big)\Big]\delta x(t) = 0 \\ \frac{\partial g(\cdot)}{\partial x} - \frac{d}{dt}\Big(\frac{\partial g}{\partial \dot{x}}\Big) = 0,\end{aligned} \tag{35}$$

where $\delta x(t)$ is **arbitrary**. The **boundary conditions** can be obtained as

$$\Big(\frac{\partial g(\cdot)}{\partial \dot{x}}\Big)\Big|_{t_f}\delta x_f + \Big[g(\cdot)\Big|_{t_f} - \frac{\partial g(\cdot)}{\partial \dot{x}}\Big|_{t_f}\dot{x}(t_f)\Big]\delta t_f = 0 \tag{36}$$

Fix point problem (t_f is fixed and $x(t_f)$ is free)



Fix point problem (t_f is fixed and $x(t_f)$ is free)

Term t_f is fixed, and $x(t_f)$ is free, hence, $\delta t_f = 0$. Therefore, the boundary value constraints after considering eq.36,

$$\begin{aligned} \Big(\frac{\partial g(\cdot)}{\partial \dot{x}}\Big)\Big|_{t_f}\delta x_f + \Big[g(\cdot)\Big|_{t_f} - \frac{\partial g(\cdot)}{\partial \dot{x}}\Big|_{t_f}\dot{x}(t_f)\Big]\delta t_f = 0 \\ \Rightarrow \Big[(\frac{\partial g}{\partial \dot{x}})\Big|_{t_f}\Big]\delta x_f = 0 \end{aligned} \tag{37}$$

Term δx_f is arbitrary, then the value constraints, i.e., the final point condition,

$$\frac{\partial g(x(t_f), \dot{x}(t_f), t_f)}{\partial \dot{x}(t_f)} = 0 \tag{38}$$

Example 03

Consider the initial position given as $t_0 = e, x(t_0) = f$. Find the shortest path between on the interval [e,h], where $h = t_f$ and $x(t_f)$ is free. A small distance along the curve $x(t)$ can be defined as $ds = \sqrt{dx^2 + dt^2}$.