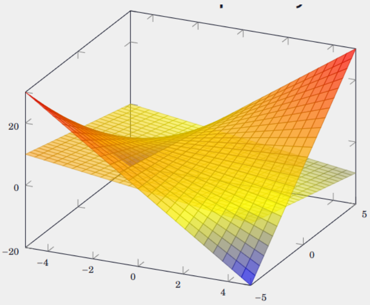
Motion Planning for Autonomous Vehicles

Pontryagin's Minimum Principle

Geesara Kulathunga



March 10, 2023

Pontryagin's Minimum Principle (Optimal control theory)

Contents

Consider that the system

$$\dot{x}(t) = f(x(t), u(t), t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ to follow an admissible trajectory x^* that minimizes the following objective function

$$J(u) = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), u(t), t)dt \tag{2}$$

The initial condition $x(t_0) = x_0$ is given.

Hamiltonian

$$H(x(t), u(t), P(t), t) := g(x(t), u(t), t) + P^\top(t) f(x(t), u(t), t) \tag{3}$$

Necessary conditions

$$\begin{aligned}
\dot{x}^*(t) &= \frac{H(\cdot)}{\partial P} \\
\dot{P}^*(t) = -\frac{H(\cdot)}{\partial x} &= -\left(\frac{\partial f(\cdot)}{\partial x}\right)^\top P^*(t) - \frac{\partial g(\cdot)}{\partial x} \\
0 = \frac{H(\cdot)}{\partial u} &= \left(\frac{\partial g(\cdot)}{\partial u}\right)^\top P^*(t) + \frac{\partial f(\cdot)}{\partial u} \\
\left(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\right)^\top \delta x_f + \left(H(\cdot) + \frac{\partial h(\cdot)}{\partial t}\right)\delta t_f &= 0
\end{aligned} \tag{4}$$

where $H(\cdot) = H(x^*(t), u^*(t), P^*(t), t)$ and $\forall\, t \in [t_0, t_f]$. State $x(t)$ and inputs $u(t)$ are **unconstrained**.

The control $\mathbf{u}^*$ causes the functional "J" to have a relative minima if

$$J(u) - J(u^*) = \Delta J \geq 0$$

for all **admissible controls** sufficiently close to u^*, i.e., u^* is the relative minima

- Consider such control $u = u^* + \delta u$, the increment in 'J' can be expressed as

$$\Delta J(u^*, \delta u) = \delta J(u^*, \delta u) + H.O.T \tag{5}$$

where, the first variation $\delta J = \frac{\partial J}{\partial u} \delta u(t)$.

- Consider such control $u = u^* + \delta u$, the increment in 'J' can be expressed as

$$\Delta J(u^*, \delta u) = \delta J(u^*, \delta u) + H.O.T \tag{5}$$

  where, the first variation $\delta J = \frac{\partial J}{\partial u} \delta u(t)$.
- When δu is arbitrary to obtain an extremal solution $\delta J = 0$.

- Consider such control $u = u^* + \delta u$, the increment in 'J' can be expressed as

$$\Delta J(u^*, \delta u) = \delta J(u^*, \delta u) + H.O.T \tag{5}$$

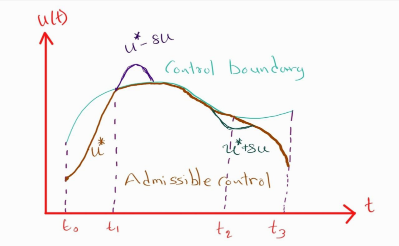
  where, the first variation $\delta J = \frac{\partial J}{\partial u}\delta u(t)$.
- When δu is arbitrary to obtain an extremal solution $\delta J = 0$.
- However, control is bounded if the optimal control exceeds the control boundary in the sub-interval.

- Consider such control $u = u^* + \delta u$, the increment in 'J' can be expressed as

$$\Delta J(u^*, \delta u) = \delta J(u^*, \delta u) + H.O.T \tag{5}$$

where, the first variation $\delta J = \frac{\partial J}{\partial u}\delta u(t)$.

- When δu is arbitrary to obtain an extremal solution $\delta J = 0$.
- However, control is bounded if the optimal control exceeds the control boundary in the sub-interval.
- Therefore, δu can not be arbitrary in the interval t_0, t_f.



- Hence, the **necessary condition** for u^* to minimize J is that $\delta J(u^*, \delta u) = \Delta J \geq 0$. On the other hand, if the u^* **lies within** the acceptable **boundary** then $\delta J(u^*, \delta u) = 0$. Thus, the necessary condition

$$\delta J(u^*(t), \delta u(t)) = \int_{t_0}^{t_f} \left(\frac{\partial H(\cdot)}{\partial u}\right)^\top \delta u(t) dt \tag{6}$$

- Hence, the **necessary condition** for u^* to minimize J is that $\delta J(u^*, \delta u) = \Delta J \geq 0$. On the other hand, if the u^* **lies within** the acceptable **boundary** then $\delta J(u^*, \delta u) = 0$. Thus, the necessary condition

$$\delta J(u^*(t), \delta u(t)) = \int_{t_0}^{t_f} \left(\frac{\partial H(\cdot)}{\partial u}\right)^\top \delta u(t) dt \tag{6}$$

- By taking the first-order approximation of H,

$$\begin{aligned}\left(\frac{\partial H(x(t), u(t), P(t), t)}{\partial u(t)}\right)^\top \delta u(t) = H(x^*(t), u^*(t) + \delta u(t), P^*(t), t) \\ -H(x^*(t), u^*(t), P^*(t), t)\end{aligned} \tag{7}$$

- Hence, the **necessary condition** for u^* to minimize J is that $\delta J(u^*, \delta u) = \Delta J \geq 0$. On the other hand, if the u^* **lies within** the acceptable **boundary** then $\delta J(u^*, \delta u) = 0$. Thus, the necessary condition

$$\delta J(u^*(t), \delta u(t)) = \int_{t_0}^{t_f} \left(\frac{\partial H(\cdot)}{\partial u}\right)^\top \delta u(t) dt \tag{6}$$

- By taking the first-order approximation of H,

$$\begin{aligned}\left(\frac{\partial H(x(t), u(t), P(t), t)}{\partial u(t)}\right)^\top \delta u(t) = H(x^*(t), u^*(t) + \delta u(t), P^*(t), t) \\ - H(x^*(t), u^*(t), P^*(t), t)\end{aligned} \tag{7}$$

- Therefore, the necessary condition becomes

$$\begin{aligned}\delta J(u^*(t), \delta u(t)) = \int_{t_0}^{t_f} \Big(H(x^*(t), u^*(t) + \delta u(t), P^*(t), t) \\ - H(x^*(t), u^*(t), P^*(t), t)\Big) dt \geq 0\end{aligned} \tag{8}$$

Hence, the following inequality must be satisfied.

$$
\begin{aligned}
H(x^*(t), u^*(t) + \delta u(t), P^*(t), t) \geq H(x^*(t), u^*(t), P^*(t), t) \\
\Rightarrow H(x^*(t), {\color{red}u^*(t)}, P^*(t), t) \leq H(x^*(t), {\color{red}u(t)}, P^*(t), t)
\end{aligned}
\tag{9}
$$

where $u(t) = u^*(t) + \delta u(t)$. In other words, any $\delta u(t)$ is added to $u^*(t)$, which holds this inequality.

$$H(x^*(t), u^*(t), P^*(t), t) \leq H(x^*(t), u(t), P^*(t), t) \tag{10}$$

where $u(t) = u^*(t) + \delta u(t)$. However, this does not guarantee to be ensured $x^*(t), P^*(t)$

$$\begin{aligned}
\dot{x}^*(t) &= \frac{H(\cdot)}{\partial P} \\
\dot{P}^*(t) = -\frac{H(\cdot)}{\partial x} &= -\left(\frac{\partial f(\cdot)}{\partial x}\right)^\top P^*(t) - \frac{\partial g(\cdot)}{\partial x} \\
H(\cdot) \leq H(x^*(t), u(t), P^*(t), t), &\quad \forall\, u(t) \in U \\
\left(\frac{\partial h(\cdot)}{\partial x} - P^*(t_f)\right)^\top \delta x_f + \left(H(\cdot) + \frac{\partial h(\cdot)}{\partial t}\right)\delta t_f &= 0
\end{aligned} \tag{11}$$

where $H(\cdot) = H(x^*(t), u^*(t), P^*(t), t)$ and $\forall\, t \in [t_0, t_f]$. State $x(t)$ and inputs $u(t)$ are unconstrained.
$u^* = \text{argmax}\, H(x^*(t), u(t), P^*(t), t) \quad \forall\, u(t) \in U$

- Consider the system having the state equations

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_2(t) + u(t) \end{aligned} \tag{12}$$

with initial condition $x(t) = x_0$.

- Consider the system having the state equations

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_2(t) + u(t) \end{aligned} \tag{12}$$

  with initial condition $x(t) = x_0$.
- The objective function is given as $J(u) = \int_{t_0}^{t_f} \frac{1}{2}(x_1^2(t) + u^2(t))dt$, where t_f is specified and final state $x(t_f)$ is free.

- Consider the system having the state equations

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_2(t) + u(t) \end{aligned} \tag{12}$$

  with initial condition $x(t) = x_0$.
- The objective function is given as $J(u) = \int_{t_0}^{t_f} \frac{1}{2}(x_1^2(t) + u^2(t))dt$, where t_f is specified and final state $x(t_f)$ is free.
- The Hamiltonian is

$$\begin{aligned} H(x(t), u(t), P(t), t) &:= g(x(t), u(t), t) + P^\top(t) f(x(t), u(t), t) \\ &\Rightarrow \frac{1}{2}(x_1^2(t) + u^2(t)) + p_1(t)x_2(t) - p_2(t)x_2(t) + p_2(t)u(t) \end{aligned} \tag{13}$$

- Costate equations are

$$\begin{aligned} \dot{p}_1^*(t) &= -\frac{\partial H(\cdot)}{\partial x_1} = -x_1^*(t) \\ \dot{p}_2^*(t) &= -\frac{\partial H(\cdot)}{\partial x_2} = -p_1^*(t) + -p_2^*(t) \end{aligned} \tag{14}$$

- Costate equations are

$$\begin{aligned} \dot{p}_1^*(t) &= -\frac{\partial H(\cdot)}{\partial x_1} = -x_1^*(t) \\ \dot{p}_2^*(t) &= -\frac{\partial H(\cdot)}{\partial x_2} = -p_1^*(t) + -p_2^*(t) \end{aligned} \tag{14}$$

- If controls are not-bounded

$$\begin{aligned} 0 &= \frac{H(\cdot)}{\partial u} \\ \Rightarrow u^*(t) + p_2^*(t) &= 0 \\ \Rightarrow u^*(t) &= -p_2^*(t) \end{aligned} \tag{15}$$

- Costate equations are

$$\begin{aligned} \dot{p}_1^*(t) &= -\frac{\partial H(\cdot)}{\partial x_1} = -x_1^*(t) \\ \dot{p}_2^*(t) &= -\frac{\partial H(\cdot)}{\partial x_2} = -p_1^*(t) + -p_2^*(t) \end{aligned} \tag{14}$$

- If controls are not-bounded

$$\begin{aligned} 0 &= \frac{H(\cdot)}{\partial u} \\ \Rightarrow u^*(t) + p_2^*(t) &= 0 \\ \Rightarrow u^*(t) &= -p_2^*(t) \end{aligned} \tag{15}$$

- The boundary conditions are $p^*(t_f) = 0$,

Pontryagin's Minimum Principle

- Consider the control is bounded $-1 \leq u(t) \leq 1\,; \forall\, t \in [t_0, t_f]$. The state and costate equations remain the same. However, 'u' must be selected to minimize.

- Consider the control is bounded $-1 \leq u(t) \leq 1\,; \forall\, t \in [t_0, t_f]$. The state and costate equations remain the same. However, 'u' must be selected to minimize.
- Considering terms that depend on u(t)

$$\begin{aligned} H(x(t),u(t),P(t),t) &:= g(x(t),u(t),t) + P^\top(t) f(x(t),u(t),t) \\ &\Rightarrow \frac{1}{2}u^2(t) + p_2^*(t)u(t) \end{aligned} \tag{16}$$

- Consider the control is bounded $-1 \le u(t) \le 1\,; \forall\, t \in [t_0, t_f]$. The state and costate equations remain the same. However, 'u' must be selected to minimize.
- Considering terms that depend on u(t)

$$\begin{aligned} H(x(t),u(t),P(t),t) &:= g(x(t),u(t),t) + P^\top(t) f(x(t),u(t),t) \\ &\Rightarrow \frac{1}{2}u^2(t) + p_2^*(t)u(t) \end{aligned} \tag{16}$$

- Within the control boundary $u^*(t) = -p_2^*(t)$ is valid.

- Consider the control is bounded $-1 \leq u(t) \leq 1\,; \forall\, t \in [t_0, t_f]$. The state and costate equations remain the same. However, 'u' must be selected to minimize.
- Considering terms that depend on u(t)

$$\begin{aligned} H(x(t),u(t),P(t),t) &:= g(x(t),u(t),t) + P^{\top}(t) f(x(t),u(t),t) \\ &\Rightarrow \frac{1}{2}u^2(t) + p_2^*(t)u(t) \end{aligned} \tag{16}$$

- Within the control boundary $u^*(t) = -p_2^*(t)$ is valid.
- However, $|p_2^*(t) > 1|$

$$u^*(t) = \begin{cases} -1 & \text{for} \quad p_2^*(t) > 1 \\ -p_2^*(t) & \text{for} \quad -1 \leq \quad p_2^*(t) \leq 1 \\ 1 & \text{for} \quad p_2^*(t) < -1 \end{cases} \tag{17}$$

Optimal boundary value problem

Let $\sigma(t)$ be the translational variable of the quadrocopter, consisting of its position, velocity, and acceleration, such that

$$\sigma(t) = (\mathbf{x}(t), \dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t)) \in \mathbb{R}^9 \tag{18}$$

If T be the motion duration, and $\bar{\sigma}_i$, $i \in I \subseteq \{1, 2, \ldots, 9\}$ be the components of desired translational variables at the end of the motion, then to achieve the target goal

$$\sigma_i(T) = \bar{\sigma}_i \ \forall i \in I \tag{19}$$

Mueller, M. W., Hehn, M., D'Andrea, R. (2015). A computationally efficient motion primitive for quadrocopter trajectory generation. IEEE transactions on robotics, 31(6), 1294-1310.

- In general, design a trajectory x(t) such that x_0 = a, and x_T = b whose order of **degree five**, which is called a **quintic polynomial**:

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 \tag{20}$$

Minimizing the square of the jerk

- In general, design a trajectory x(t) such that x_0 = a, and x_T = b whose order of **degree five**, which is called a **quintic polynomial**:

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 \tag{20}$$

- Consider the initial condition $x_0 = a, \dot{x}_0 = 0, \ddot{x}_0 = 0$ at $t = 0$ and final condition $x_T = b, \dot{x}_T = 0, \ddot{x}_T = 0$ at $t = T$ are given.

- In general, design a trajectory x(t) such that x_0 = a, and x_T = b whose order of **degree five**, which is called a **quintic polynomial**:

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 \tag{20}$$

- Consider the initial condition $x_0 = a, \dot{x}_0 = 0, \ddot{x}_0 = 0$ at $t = 0$ and final condition $x_T = b, \dot{x}_T = 0, \ddot{x}_T = 0$ at $t = T$ are given.
- Hence, the objective is to find the optimal value of T

$$a_0 = x_0, \quad a_1 = \dot{x}_0, \quad a_2 = \ddot{x}_0/2$$

$$A = \begin{bmatrix} T^3 & T^4 & T^5 \\ 3T^2 & 4T^3 & 5T^4 \\ 6T & 12T^2 & 20T^3 \end{bmatrix}, \quad b = \begin{bmatrix} x_f - a_0 - a_1 - a_2 T^2 \\ \dot{x}_f - a_1 - 2a_2 T \\ \ddot{x}_f - 2a_2 \end{bmatrix} \Rightarrow \begin{bmatrix} b - a \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_3 \\ a_4 \\ a_5 \end{bmatrix} = A^{-1} b \tag{21}$$

The higher-order derivatives can be estimated for a given time index t:

$$
\begin{aligned}
x(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 \\
\dot{x}(t) &= a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 \\
\ddot{x}(t) &= 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3
\end{aligned}
\tag{22}
$$

Consider a UAV moves between the two positions within the time interval T.

$$
J = \int_0^T L dt \tag{23}
$$

where T is the motion duration and the performance index, in general, is L, the minimum jerk trajectory J_Σ whose cost is decoupled into a per-axis cost J_k.

Minimizing the square of the jerk

Assume UAV trajectory is represented in $\mathbb{R}^3$, hence minimizing the integral of the squared jerk

$$J_\Sigma = \Sigma_{k=1}^{3} J_k, \quad J_k = \frac{1}{T}\int_0^T j_k(t)^2 dt, \tag{24}$$

where for each axis k, system state $s_k = (p_k, v_k, a_k)$, system input $u_k = j_k$, and system motion model $\dot{s}_k = f_s(s_k, u_k) = (v_k, a_k, j_k)$. Pontryagin's minimum principle can be used to solve this problem: $\dot{s}^*(t) = f(s^*(t), u^*(t))$, $s^*(0) = s(0)$

- Define the Hamiltonian

$$
\begin{aligned}
H(s,u,\lambda) &:= g(s,u) + \lambda^\top f(s,u), \quad \lambda = (\lambda_1, \lambda_2, \lambda_3) \\
\Rightarrow &= \frac{1}{T} j^2 + \lambda^\top f(s,u) = \frac{1}{T} j^2 + \lambda_1 v + \lambda_2 a + \lambda_3 j
\end{aligned}
\tag{25}
$$

- Define the Hamiltonian

$$
\begin{aligned}
H(s,u,\lambda) &:= g(s,u) + \lambda^\top f(s,u), \quad \lambda = (\lambda_1, \lambda_2, \lambda_3) \\
\Rightarrow &= \frac{1}{T} j^2 + \lambda^\top f(s,u) = \frac{1}{T} j^2 + \lambda_1 v + \lambda_2 a + \lambda_3 j
\end{aligned} \tag{25}
$$

- Since s is a function of p,v, and a, (for the sake of readability, the axis subscript k will be discarded)

$$
\begin{aligned}
\dot{\lambda}^*(t) = -\frac{H(\cdot)}{\partial s} = -\left(\frac{\partial f(\cdot)}{\partial s}\right)^\top \lambda^*(t) - \frac{\partial g(\cdot)}{\partial s} \\
\dot{\lambda} = 0, -\lambda_1, -\lambda_2
\end{aligned} \tag{26}
$$

- Define the Hamiltonian

$$
\begin{aligned}
H(s,u,\lambda) &:= g(s,u) + \lambda^\top f(s,u), \quad \lambda = (\lambda_1,\lambda_2,\lambda_3) \\
\Rightarrow &= \frac{1}{T} j^2 + \lambda^\top f(s,u) = \frac{1}{T} j^2 + \lambda_1 v + \lambda_2 a + \lambda_3 j
\end{aligned}
\tag{25}
$$

- Since s is a function of p,v, and a, (for the sake of readability, the axis subscript k will be discarded)

$$
\begin{aligned}
\dot{\lambda}^*(t) = -\frac{H(\cdot)}{\partial s} = -\left(\frac{\partial f(\cdot)}{\partial s}\right)^\top \lambda^*(t) - \frac{\partial g(\cdot)}{\partial s} \\
\dot{\lambda} = 0, -\lambda_1, -\lambda_2
\end{aligned}
\tag{26}
$$

- The costate equation is determined easily, for the latter convenience the solution is written in constants α, β, γ

$$
\lambda(t) = \frac{1}{T}\begin{bmatrix} -\alpha \\ \alpha t + \beta \\ -\frac{1}{2}\alpha t^2 - \beta t - \gamma \end{bmatrix} \Rightarrow \quad or \quad \lambda(t) = \frac{1}{T}\begin{bmatrix} -2\alpha \\ 2\alpha t + 2\beta \\ -\alpha t^2 - 2\beta t - 2\gamma \end{bmatrix}
$$

Minimizing the square of the jerk

- The optimal input is solved as:

$$
\begin{aligned}
0 = \frac{H(\cdot)}{\partial u} &= \left(\frac{\partial g(\cdot)}{\partial u}\right)^{\top} \lambda^*(t) + \frac{\partial f(\cdot)}{\partial u} \\
j^*(t) &= \frac{1}{2}\alpha t^2 + \beta t + \gamma
\end{aligned} \tag{27}
$$

- The optimal input is solved as:

$$0 = \frac{H(\cdot)}{\partial u} = \left(\frac{\partial g(\cdot)}{\partial u}\right)^{\top} \lambda^*(t) + \frac{\partial f(\cdot)}{\partial u}$$
$$j^*(t) = \frac{1}{2}\alpha t^2 + \beta t + \gamma \tag{27}$$

- From which, i.e., $j^*(t)$, the optimal state trajectory is solved by integration, i.e., an integral on u to get a, get v in the integral of a, and get p in the integral v:

$$s^*(t) = \begin{bmatrix} \frac{\alpha}{120}t^5 + \frac{\beta}{24}t^4 + \frac{\gamma}{6}t^3 + \frac{a_0}{2}t^2 + v_0 t + p_0 \\ \frac{\alpha}{24}t^4 + \frac{\beta}{6}t^3 + \frac{\gamma}{2}t^2 + a_0 t + v_0 \\ \frac{\alpha}{6}t^3 + \frac{\beta}{2}t^3 + \gamma t + a_0 \end{bmatrix}, \tag{28}$$

where initial state $s(0) = (p_0, v_0, a_0)$.

The remaining unknowns α, β, γ are solved for as a function of the desired end transnational variable components as defined in eq. 19

$$\begin{bmatrix} \frac{1}{120}T^5 & \frac{1}{24}T^4 & \frac{1}{6}T^3 \\ \frac{1}{24}T^4 & \frac{1}{6}T^3 & \frac{1}{2}T^2 \\ \frac{1}{6}T^3 & \frac{1}{2}T^2 & T \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \underbrace{\begin{bmatrix} \Delta p \\ \Delta v \\ \Delta a \end{bmatrix}}_{s(T)-s(0)} = \begin{bmatrix} p_f - \frac{a_0}{2}T^2 - v_0 T - p_0 \\ v_f - a_0 T - v_0 \\ a_f - a_0 \end{bmatrix}, \tag{29}$$

where final state $s(T) = (p_f, v_f, a_f)$.

- Hence, solving for the unknown coefficients yields

$$\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix} = \frac{1}{T^5}\begin{bmatrix}720 & -360T & 60T^2\\ -360T & 168T^2 & -24T^3\\ 60T^2 & -24T^3 & 3T^4\end{bmatrix}\begin{bmatrix}\Delta p\\ \Delta v\\ \Delta a\end{bmatrix} \tag{30}$$

- Hence, solving for the unknown coefficients yields

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \frac{1}{T^5} \begin{bmatrix} 720 & -360T & 60T^2 \\ -360T & 168T^2 & -24T^3 \\ 60T^2 & -24T^3 & 3T^4 \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta v \\ \Delta a \end{bmatrix} \tag{30}$$

- After finding these, $s^*(t)$ and cost function J_k can be determined.

$$J_k = \frac{1}{T}\int_0^T j_k(t)^2 dt = \gamma^2 + \beta\gamma T + \frac{1}{3}\beta^2 T^2 + \frac{1}{3}\alpha\gamma T^2 + \frac{1}{4}\alpha\beta T^3 + \frac{1}{20}\alpha^2 T^4 \tag{31}$$

- Hence, solving for the unknown coefficients yields

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \frac{1}{T^5} \begin{bmatrix} 720 & -360T & 60T^2 \\ -360T & 168T^2 & -24T^3 \\ 60T^2 & -24T^3 & 3T^4 \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta v \\ \Delta a \end{bmatrix} \tag{30}$$

- After finding these, $s^*(t)$ and cost function J_k can be determined.

$$J_k = \frac{1}{T}\int_0^T j_k(t)^2 dt = \gamma^2 + \beta\gamma T + \frac{1}{3}\beta^2 T^2 + \frac{1}{3}\alpha\gamma T^2 + \frac{1}{4}\alpha\beta T^3 + \frac{1}{20}\alpha^2 T^4 \tag{31}$$

- The cost function is only a function of time, hence this kind of problem is called a minimum time problem. Afterwards, the **extremum value of T** is calculated by **the root-finding of the polynomial**. After finding the optimal T, the **rest of the parameters can be calculated**.

Assume that

$$\gamma^2+\beta\gamma T+\frac{1}{3}\beta^2T^2+\frac{1}{3}\alpha\gamma T^2+\frac{1}{4}\alpha\beta T^3+\frac{1}{20}\alpha^2T^4 := c_0+c_1x+c_2x^2+c_3x^3+c_4x^4$$

- To find roots of a such polynomial, **matrix eigenvalue method** can be employed. The matrix eigenvalues are defined as

  $$A\mathbf{y} = \lambda\mathbf{y}$$

  If the roots of the polynomial are the eigenvalues of the matrix

  $$A\mathbf{y} = x\mathbf{y}$$

- That is to say if $\mathbf{y} = [x^3\ x^2\ x\ 1]^\top$, matrix A is constructed as

$$\underbrace{\begin{bmatrix} -\frac{c_3}{c_4} & -\frac{c_2}{c_4} & -\frac{c_1}{c_4} & -\frac{c_0}{c_4} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x^3 \\ x^2 \\ x \\ 1 \end{bmatrix} = x \begin{bmatrix} x^3 \\ x^2 \\ x \\ 1 \end{bmatrix}$$

- That is to say if $\mathbf{y} = [x^3\ x^2\ x\ 1]^\top$, matrix A is constructed as

$$\underbrace{\begin{bmatrix} -\frac{c_3}{c_4} & -\frac{c_2}{c_4} & -\frac{c_1}{c_4} & -\frac{c_0}{c_4} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x^3 \\ x^2 \\ x \\ 1 \end{bmatrix} = x \begin{bmatrix} x^3 \\ x^2 \\ x \\ 1 \end{bmatrix}$$

- To obtain matrix A
  `https://colab.research.google.com/drive/1R6lIak_5zqUq40a8FDcHh3wzykpIIkQT?usp=sharing`

After finding eigenvalues of the matrix

$$A\mathbf{y} = x\mathbf{y}$$

, **highest magnitude eigenvalue** is considered as T