Appendix A

Trigonometry Review

A.1 Basic Definitions

A.1.1 Ratios on the Right Triangle

The trigonometric functions sine, cosine, and tangent are based on ratios of the sides of a right triangle, relative to one acute angle θ (Figure A.1):

$$\sin\theta = opp/hyp$$

$$\cos\theta = adj/hyp$$

$$\tan\theta = opp/adj = \sin\theta/\cos\theta$$

We also define the reciprocal functions secant, cosecant, and cotangent as follows:

$$\texttt{sec}\,\theta = hyp/adj = 1/\cos\theta$$

$$\texttt{csc}\,\theta = hyp/opp = 1/\sin\theta$$

$$\texttt{cot}\,\theta = adj/opp = 1/\tan\theta$$

$$= \cos\theta/\sin\theta = \texttt{sec}\,\theta/\texttt{csc}\,\theta$$

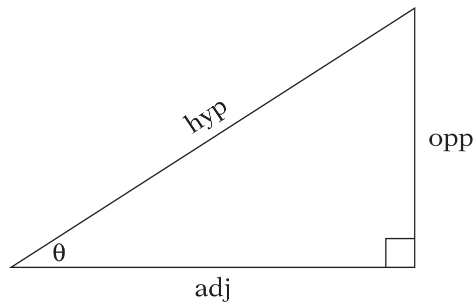

Figure A.1: Computing trigonometric functions on the right triangle.

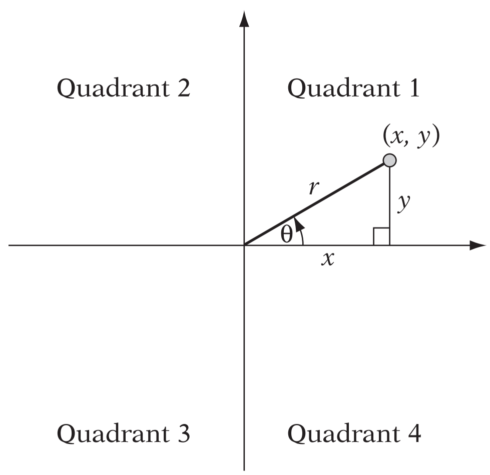

Figure A.2: Computing trigonometric functions on the standard Cartesian frame, showing the four ordered quadrants.

A.1.2 Extending to General Angles

Consider a standard Cartesian frame for $\mathbb{R}^2$. We place a line segment, or radius, with length r and one endpoint fixed at the origin. The other endpoint is located at a point (x, y). We define θ as the angle between the radius and the positive x-axis. The angle is positive if the direction of rotation from the x-axis to the radius is counterclockwise, and negative if clockwise. A full rotation is broken into 2π radians, or 360 degrees. The coordinate axes divide the plane into four quadrants: they are numbered in the order of rotation. Within this we can inscribe a right triangle, with the radius as hypotenuse and one side incident with the x-axis (Figure A.2).

We can represent the sine and cosine based on the length r of the radius and the location (x, y)

of the free endpoint:

$$\sin\theta = y/r$$
$$\cos\theta = x/r$$

In this case, the tangent becomes the slope of the radius:

$$\tan\theta = y/x$$

For angles greater than $\pi/2$, the magnitude of the result is the same, but the sign may be negative depending on which quadrant the angle is in:

Functions	*Quadrant*	*Sign*
sin, csc	1,2	$+$
	3,4	$-$
cos, sec	1,4	$+$
	2,3	$-$
tan, cot	1,3	$+$
	2,4	$-$

The tangent, cotangent, secant, and cosecant all involve divisions by x or y, which may be 0. This leads to singularities at those locations, which can be seen in the function graphs in Figures

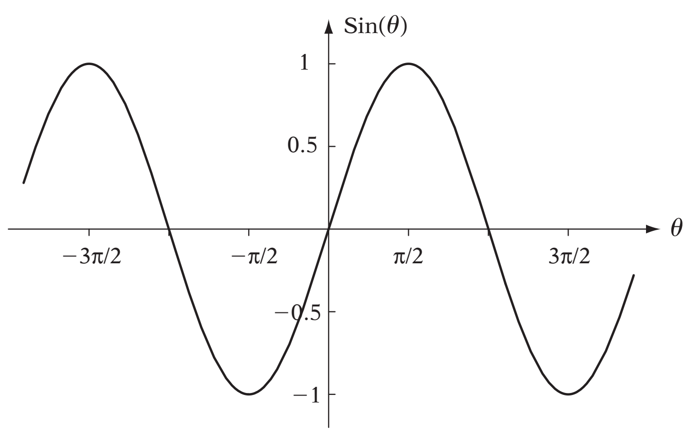

Figure A.3: Graph of $\sin\theta$.

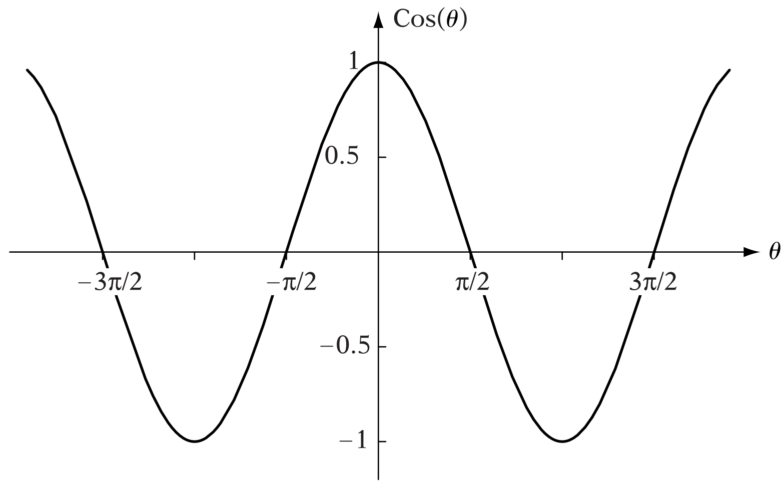

Figure A.4: Graph of $\cos\theta$.

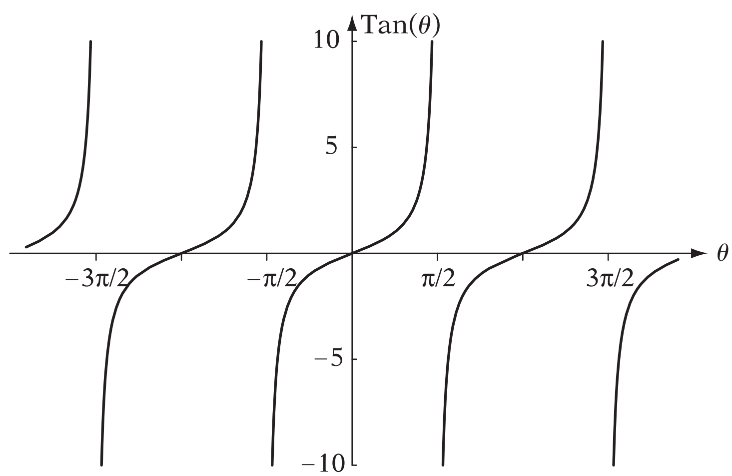

Figure A.5: Graph of $\tan\theta$.

A.3 through A.8. This sequence of figures shows the six trigonometric functions graphed against θ (in radians).

Also note that these functions are periodic. For example, $\sin(0) = \sin(2\pi) = \sin(-4\pi)$. In general, $\sin(x) = \sin(n \cdot 2\pi + x)$, for any integer n. The same is true for cosine, secant, and cosecant. Tangent and cotangent are periodic with period π: $\tan(x) = \tan(n \cdot \pi + x)$.

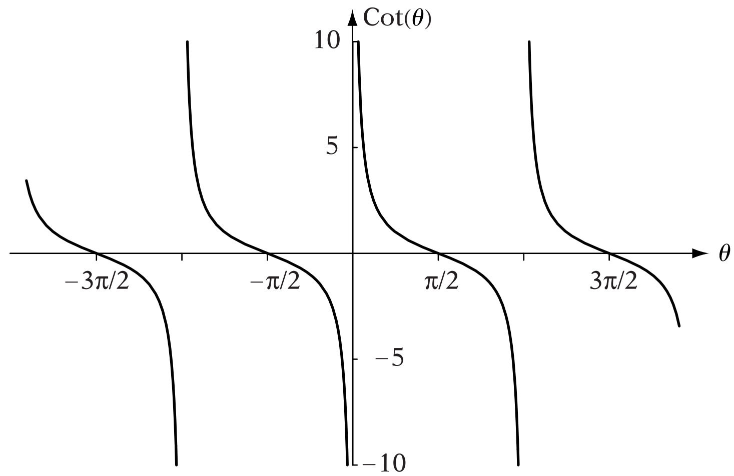

Figure A.6: Graph of $\cot \theta$.

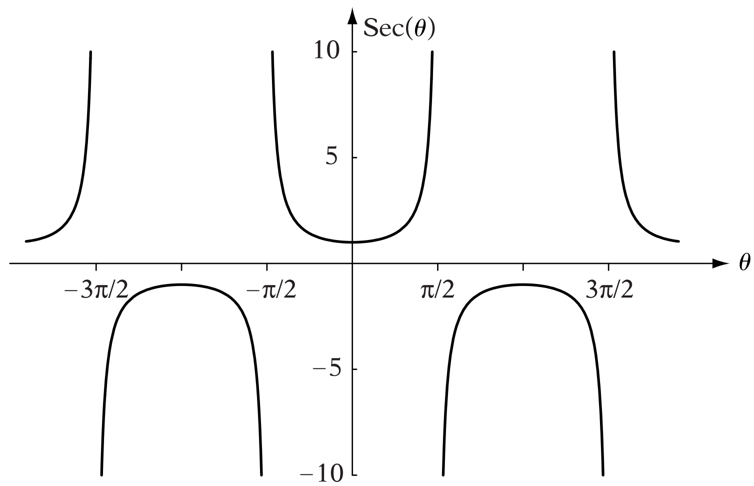

Figure A.7: Graph of $\sec \theta$.

A.2 Properties of Triangles

There are three laws that relate angles in a triangle to sides of a triangle, using trigonometric functions. Figure A.9 shows a general triangle with sides of length a, b, and c, and corresponding opposite angles α, β, and γ.

The *law of sines* relates angles to their opposing sides as a constant ratio for each pair:

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} \tag{A.1}$$

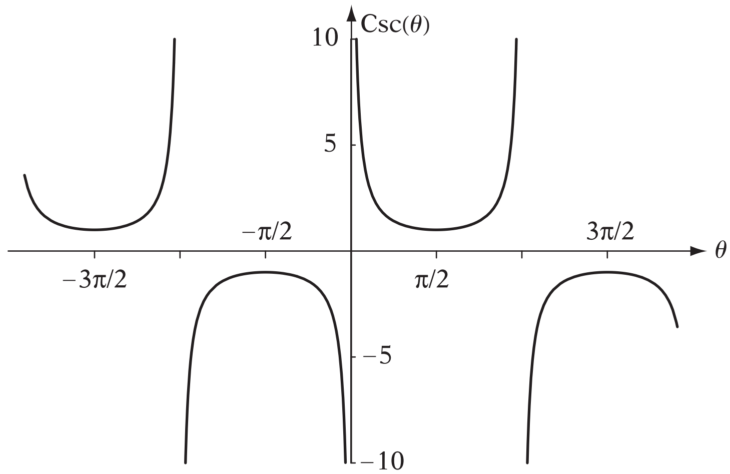

Figure A.8: Graph of $\csc\theta$.

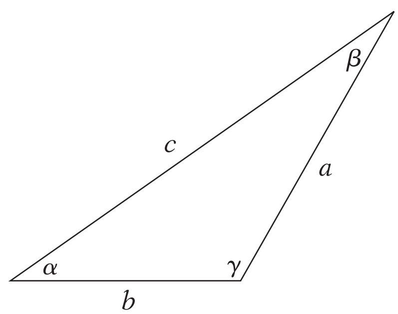

Figure A.9: General triangle, with sides and angles labeled.

Recall the Pythagorean theorem,

$$c^2 = a^2 + b^2$$

which relates two sides of a right triangle to the hypotenuse. The *law of cosines* is an extension to this, which can be used to compute the length of a side from the length of two other sides and the angle between them:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \tag{A.2}$$

Substituting $\pi/2$ for γ produces the specific case of the Pythagorean theorem.

The *law of tangents* relates two angles and their corresponding opposite sides:

$$\frac{a-b}{a+b} = \frac{\tan(\frac{1}{2}(\alpha-\beta))}{\tan(\frac{1}{2}(\alpha+\beta))} \tag{A.3}$$

All of these can be used to construct information about a triangle from partial data.

While not specifically one of the laws, a related set of formulas computes the area of a triangle:

$$\frac{ab\sin\gamma}{2} = \frac{bc\sin\alpha}{2} = \frac{ac\sin\beta}{2} \tag{A.4}$$

A.3 Trigonometric Identities

A.3.1 Pythagorean Identities

Again, from the Pythagorean theorem we know that

$$a^2 + b^2 = c^2$$

where c is the length of the hypotenuse and a and b are the lengths of the other two sides. In the case where the length of the hypotenuse is 1, the length of the other two sides are $\cos\theta$ and $\sin\theta$, so

$$\sin^2\theta + \cos^2\theta = 1 \tag{A.5}$$

where $\sin^2\theta = (\sin\theta)(\sin\theta)$, and similarly for $\cos^2\theta$.

Dividing equation A.5 through by $\cos^2\theta$, we get

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta \tag{A.6}$$

If we instead divide equation A.5 by $\sin^2\theta$, we get

$$\frac{\sin^2\theta}{\sin^2\theta}+\frac{\cos^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}$$

$$\texttt{cot}^2\theta+1=\texttt{csc}^2\theta$$

A.3.2 Complementary Angle

If we consider one acute angle θ in a right triangle, the other acute angle is its complement $\frac{\pi}{2}-\theta$. We can compute trigonometric functions for the complementary angle by changing the sides we use when computing the ratios; for example,

$$\sin\left(\frac{\pi}{2}-\theta\right)=\text{adj}/\text{hyp}=\cos(\theta)$$

The complementary angle identities are as follows:

$$\cos\theta=\sin\left(\frac{\pi}{2}-\theta\right) \tag{A.7}$$

$$\sin\theta=\cos\left(\frac{\pi}{2}-\theta\right) \tag{A.8}$$

$$\cot\theta=\tan\left(\frac{\pi}{2}-\theta\right) \tag{A.9}$$

$$\tan\theta=\cot\left(\frac{\pi}{2}-\theta\right) \tag{A.10}$$

$$\texttt{csc}\,\theta=\texttt{sec}\left(\frac{\pi}{2}-\theta\right) \tag{A.11}$$

$$\texttt{sec}\,\theta=\texttt{csc}\left(\frac{\pi}{2}-\theta\right) \tag{A.12}$$

A.3.3 Even–Odd

Two of the trigonometric functions, cosine and secant, are symmetric across $\theta = 0$ and are called *even* functions:

$$\cos(-\theta) = \cos\theta \tag{A.13}$$

$$\mathtt{sec}\,(-\theta) = \mathtt{sec}\,\theta \tag{A.14}$$

The remainder are antisymmetric across $\theta = 0$ and are called *odd* functions:

$$\sin(-\theta) = -\sin\theta \tag{A.15}$$

$$\mathtt{csc}(-\theta) = -\mathtt{csc}\,\theta \tag{A.16}$$

$$\tan(-\theta) = -\tan\theta \tag{A.17}$$

$$\cot(-\theta) = -\cot\theta \tag{A.18}$$

A.3.4 Compound Angle

For two angles α and β, the sines of the sum and difference of the angles are, respectively,

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{A.19}$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \tag{A.20}$$

Similarly, the cosines of the sum and difference of the angles are

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{A.21}$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \tag{A.22}$$

These can be combined to create the compound angle formulas for the tangent:

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \tag{A.23}$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} \tag{A.24}$$

A.3.5 Double Angle

If we substitute the same angle θ for both α and β into the compound angle identities, we get the double angle identities:

$$\sin 2\theta = 2 \sin \theta \cos \theta \tag{A.25}$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta \tag{A.26}$$

The latter can be rewritten using the Pythagorean identity as

$$\cos 2\theta = 1 - 2\sin^2 \theta \tag{A.27}$$

$$= 2\cos^2 \theta - 1 \tag{A.28}$$

The double angle identity for tangent is

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} \tag{A.29}$$

A.3.6 Half-Angle

Equations A.27 and A.28 can be rewritten as as follows:

$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2} \tag{A.30}$$

$$\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2} \tag{A.31}$$

Substituting $\theta/2$ for α and taking the square roots gives

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}} \tag{A.32}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}} \tag{A.33}$$

Note that due to the square root, there are two choices for each identity, positive and negative—the one chosen depends on what quadrant $\theta/2$ is in.

A.4 Inverses

The trigonometric functions invert to multivalued functions because they are periodic. For example, the graph of the inverse $\sin^{-1}\theta$, or *arcsine*, can be seen in Figure A.10. Its domain is the interval $[-1, 1]$ and its range is $\mathbb{R}$.

Because of this, it is common to restrict the range of an inverse trignometric function so that

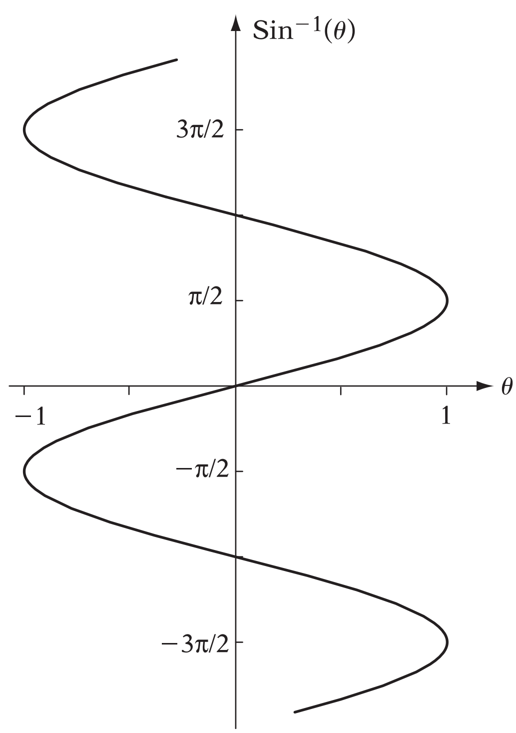

Figure A.10: Graph of $\arcsin\theta$.

it maps only to one value, given a value in the domain. Standard choices for these restrictions are as follows:

Function	*Domain*	*Range*
$\sin^{-1}$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}$	$\mathbb{R}$	$[-\pi/2, \pi/2]$