Appendix B

Calculus Review

B.1 Limits and Continuity

B.1.1 Limits

The expression

$$L = \lim_{x \to a} f(x)$$

is read as: L is the *limit* of a function f as x approaches a given value a. Informally, this represents that as x gets closer to a, $f(x)$ will get closer to L. We can more formally represent the notion of "closeness" to L and a by using the following definition:

> A function $f(x)$ has a limit L at a, if given any $\epsilon > 0$ there exists
> $\delta > 0$ such that $|f(x) - L| < \epsilon$ when $0 < |x - a| < \delta$.

In other words, for each value of ϵ larger than zero, $f(x)$ is less than ϵ away from L for all x sufficiently close to a. The value of δ provides a measure of what "sufficiently close" means.

In many cases, the limit is just the value of the function at a. For example, if we have the function

$$f(x) = x^2 \tag{B.1}$$

then

$$\lim_{x \to a} f(x) = \lim_{x \to a} x^2 = a^2 = f(a)$$

for all values of a. However, consider

$$g(x) = \frac{x^2 - 1}{x - 1} \tag{B.2}$$

At $x = 1$, the value of $g(x)$ is undefined since the resulting denominator is 0. But if we graph g, as in Figure B.1, it appears that as we get close to 1 the function value gets close to 2. So in this case, we can say that while at $x = 1$, the function value is undefined, but:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

or 2 is the limit of $g(x)$ as x approaches 1.

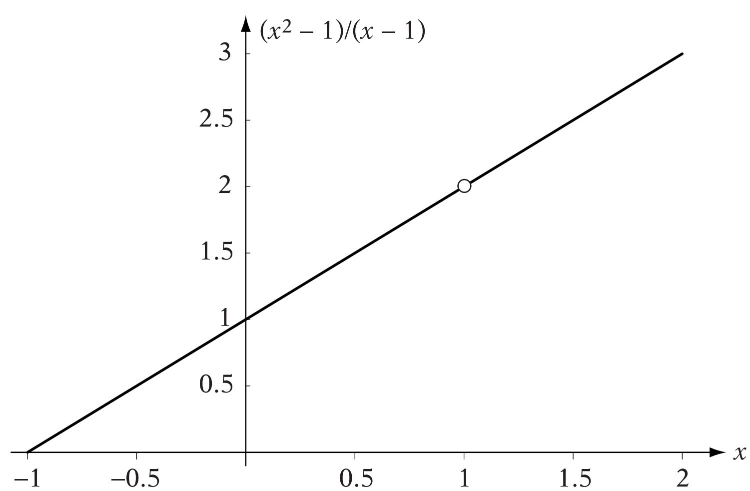

Figure B.1: Part of function with discontinuity but valid limit at $x = 1$.

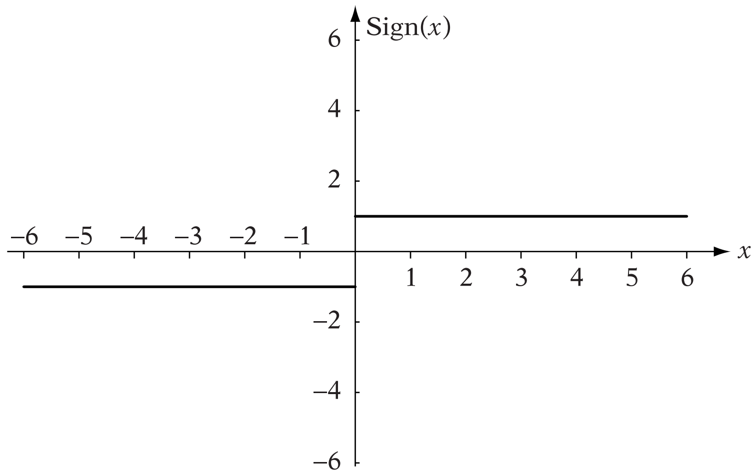

Figure B.2: Function with discontinuity and no two-sided limit at $x = 0$.

Note that there may not necessarily be a limit at a given a. For example, as graphed in Figure B.2, the step function,

$$h(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases} \tag{B.3}$$

has no limit at 0. In this case, we can talk about a right-hand limit (approaching only from the positive direction) or left-hand limit (approaching from the negative direction) or, respectively,

$$\lim_{x \to 0^+} h(x) = 1$$
$$\lim_{x \to 0^-} h(x) = -1$$

B.1.2 Continuity

There are three possibilities with regard to the limit of a function $f(x)$ as x approaches a:

1. $\lim_{x \to a} f(x)$ exists and equals $f(a)$ (e.g., equation B.1).

2. $\lim_{x \to a} f(x)$ exists and does not equal $f(a)$ (e.g., equation B.2).

3. $\lim_{x \to a} f(x)$ does not exist (e.g., equation B.3).

In the first case, we say that f is *continuous* at a. Otherwise, it is *discontinuous* at a.

We also say that a function $f(x)$ is continuous over an interval (a, b) (or $[a, b]$) if it is continuous for every value x in the interval. Informally, we can think of a continuous function as one that we can draw without ever lifting the pen from the page.

B.2 Derivatives

B.2.1 Definition

Suppose we have a function $f(x)$. If we take two points on the curve at time x and time $x + h$, then we can compute the slope of the secant that passes through the points by the function

$$\frac{f(x+h) - f(x)}{h} \tag{B.4}$$

As the value of h approaches 0, the limit (if it exists) approaches the slope of a line tangent to the function at the point x. We can use this to create a new function of x, which computes slopes of $f(x)$ for every value of x where the limit exists:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{B.5}$$

This function is called the *first derivative*, or simply the *derivative*, which we have represented as $f'(x)$. Other common representations are df/dx (also known as *Leibnitz notation*), or when taken with respect to time we place a dot over the function, as $\dot{f}(t)$.

The derivative $f'(x)$ describes the instantaneous rate of change of $f(x)$ at the value x. If $f'(x)$ is positive, $f(x)$ is said to be *increasing* at that point. Correspondingly, if $f'(x)$ is negative, $f(x)$ is said to be *decreasing*. The magnitude of $f'(x)$ describes how great the rate of change is.

A derivative may not necessarily exist for every value in the domain of a function. If this is the case for a particular value x, we say the function is not differentiable at x. If a function is discontinuous, it is not differentiable at the discontinuity. However, even if it is continuous, it may

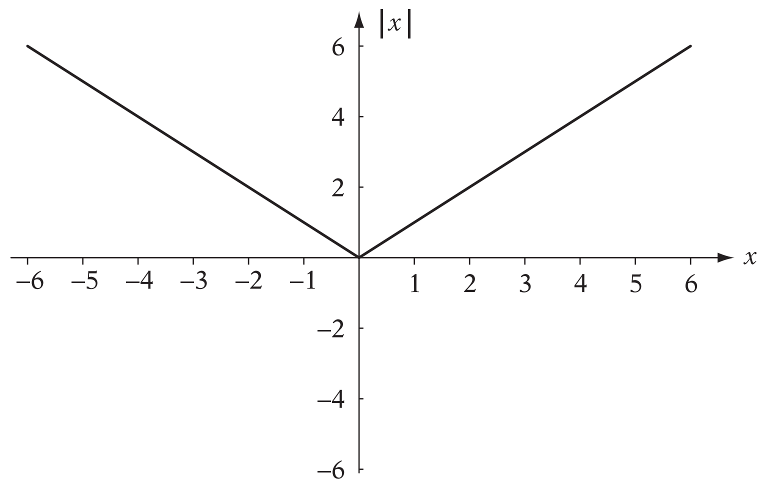

Figure B.3: Function that is continuous but has discontinuity in its first derivative at $x = 0$.

not be possible. For example, in Figure B.3, the absolute value function,

$$|x| = \begin{cases} x; & x \geq 0 \\ -x; & x < 0 \end{cases}$$

has no derivative at $x = 0$. This discontinuity represents a sudden change in slope, or if our function represents a path in space, a sudden change in direction.

A function f is differentiable on an open interval (a, b) if it is differentiable at each point in (a, b). It is differentiable on a closed interval $[a, b]$ if it is differentiable on (a, b) and the limits

$$\lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h}$$

and

$$\lim_{h \to 0^-} \frac{f(b + h) - f(b)}{h}$$

exist. If either limit exists at a point x, then we say that f has a one-sided derivative at x. For example, the absolute value function is differentiable on the intervals $[c, 0)$ and $(0, d]$, where $c < 0$ and $d > 0$, despite not being differentiable at 0.

Since the derivative is itself a function, assuming it is differentiable we can take its derivative to get the *second derivative*, represented by $f''(x)$. If the second derivative is positive, it represents a part of the function that is concave-up (the cross section of a bowl). If it is negative, that part of the function is concave-down (an arch). If the first derivative is continuous but there is a discontinuity in the second derivative, then this represents a sudden change in concavity.

So long as a function and its subsequent derivatives are differentiable, we can continue this process of taking the derivative of derivatives. In general, the nth derivative of a function f at x is represented as $f^{(n)}(x)$, and if such a derivative exists, we say that f is differentiable to order n. If we can keep differentiating in perpetuity, we say that f is infinitely differentiable.

B.2.2 Basic Derivatives

Power of a Variable

The derivative for the power of a variable x, or $f(x) = x^k$, is

$$f'(x) = kx^{k-1} \tag{B.6}$$

By this, the derivative for a linear function $g(x) = x$ is just

$$g'(x) = 1 \cdot x^0 = 1$$

The derivative of a constant term $f(x) = a$ is

$$f'(x) = 0$$

Arithmetic Operations on Functions

The derivative of the sum of two functions is the sum of the derivatives:

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x) \tag{B.7}$$

The derivative of the difference of two functions is the difference of the derivatives:

$$\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x) \tag{B.8}$$

The derivative of the product of two functions is

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + g'(x)f(x) \tag{B.9}$$

The derivative of the quotient of two functions is

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} \tag{B.10}$$

Composite Functions

If we have the composite of two functions

$$h(x) = f(g(x)) = (f \circ g)(x)$$

then the derivative is found by using the *chain rule*. We take the derivative of f with respect to the function g, and multiply that by the derivative of g with respect to the variable x, or

$$h'(x) = f'(g(x))g'(x) \tag{B.11}$$

For example, suppose we have

$$h(x) = (2x^2 + 1)^5$$

We change variables to set $f(u) = u^5$ and $g(x) = 2x^2 + 1$, so that $h(x) = f(g(x))$. Then,

$$\begin{aligned} h'(x) = f'(g(x))g'(x) &= 5(2x^2 + 1)^4 \cdot 4x \\ &= 20x(2x^2 + 1)^4 \end{aligned}$$

General Polynomials

If we have a general polynomial

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

we can combine equations A.6, A.7, and A.9 to find its resulting derivative:

$$f'(x) = \sum_{i=0}^{n} a_i i x^{i-1}$$

B.2.3 Derivatives of Transcendental Functions

Trigonometric Functions

The derivatives of the standard trigonometric functions are as follows:

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\tan x = \mathtt{sec}^2 x = 1 + \tan^2 x$$
$$\frac{d}{dx}\cot x = -\mathtt{csc}^2 x = -(1 + \cot^2 x)$$
$$\frac{d}{dx}\mathtt{sec} x = \mathtt{sec} x \tan x$$
$$\frac{d}{dx}\mathtt{csc} x = -\mathtt{csc} x \cot x$$

Trigonometric Inverses

The derivatives of the trigonometric inverses are as follows:

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}; \qquad |x| < 1$$
$$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}; \quad |x| < 1$$
$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$
$$\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$$
$$\frac{d}{dx}\texttt{sec}^{-1}x = \frac{1}{x\sqrt{x^2-1}}; \qquad |x| > 1$$
$$\frac{d}{dx}\texttt{csc}^{-1}x = -\frac{1}{x\sqrt{x^2-1}}; \quad |x| > 1$$

Exponentials and Logarithms

The derivative of the natural exponential function $f(x) = e^x$ is

$$\frac{d}{dx}e^x = e^x$$

That is, the exponential is its own derivative.

The inverse of an exponential function is a logarithmic function. For example, the inverse of the natural exponential function e^x is $\log_e x$, usually written as $\ln x$ and called the natural logarithm. The derivative of the natural logarithm is

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

A general exponential function a^x can be represented in terms of the natural exponential as $a^x = e^{x \ln a}$. So, by the Chain rule:

$$\frac{d}{dx}a^x = \ln a \cdot a^x$$

A logarithm with an arbitrary base a can be represented in terms of the natural logarithm as

$$\log_a x = \frac{\ln x}{\ln a}$$

Using this, the derivative is

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$$

B.2.4 Taylor's Series

A *power series centered on* h is an infinite summation of the form

$$\sum_{k=0}^{\infty} a_k (x-h)^k$$

Suppose it is possible to represent a function f as a power series centered on h. Expanding terms, we can then write $f(x)$ as

$$f(x) = a_0 + a_1(x-h) + a_2(x-h)^2 + \cdots$$

To solve for $a_0, a_1, \ldots$, we begin by finding the value at $f(h)$:

$$f(h) = a_0 + a_1(h-h) + a_2(h-h)^2 + \cdots$$

All terms but the first cancel, and so $a_0 = f(h)$. Assuming that f is differentiable at h, we can differentiate both sides and again evaluate at h to get

$$f'(h) = a_1 + 2a_2(h-h) + 3a_3(h-h)^2 \cdots$$

So, $a_1 = f'(h)$. Differentiating one more time (again, assuming that it is possible) gives us

$$f''(h) = 2a_2 + 6a_3(h-h) + 12a_4(h-h)^2 \cdots$$

giving $a_2 = f''(h)/2$. Continuing this process gives us a general formula for a_k of

$$a_k = \frac{f^{(k)}(h)}{k!}$$

Assuming that f is infinitely differentiable, the *Taylor series expansion* for f is

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(h)}{k!}(x-h)^k \tag{B.12}$$

The first few terms of this look like:

$$f(x) = f(h) + f'(h)(x-h) + \frac{f''(h)}{2}(x-h)^2 + \frac{f'''(h)}{6}(x-h)^3 + \cdots$$

In general, a function f may not be infinitely differentiable, so another form is used. Suppose f is differentiable to degree $n+1$ within an interval I, and h lies within I. Then we can approximate f with p_n, the nth Taylor polynomial:

$$f(x) \approx p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(h)}{k!}(x-h)^k$$

The error of the approximation is given by r_n, the nth Taylor remainder, where

$$r_n(x) = f(x) - p_n(x)$$

It can be proved that for every x in I, there is a value $\xi(x)$ between x and h that allows us to represent $r_n(x)$ as

$$r_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-h)^{n+1}$$

This is also known as the *Lagrange remainder formula.*

B.3 Integrals

B.3.1 Definition

Given a function $f(x)$, the *indefinite integral* (also known as the *antiderivative*) of $f(x)$ is represented as

$$\int f(x)\ dx$$

The term dx, or differential, represents the fact that we are integrating with respect to the variable x; any other variables will be considered constant. The result of the indefinite integral for $f(x)$ is a function $F(x)+C$, where $F'(x)=f(x)$.

The arbitrary constant C is appended to indicate a possible constant term, the value of which will differentiate to 0. For example, differentiating the functions $f(x) = x^2 + x + 1$ and $g(x) = x^2 + x - 12$ produces $f'(x) = g'(x) = 2x + 1$. Integrating $2x + 1$ with respect to x gives the result $x^2 + x + C$.

The *definite integral* of a function $f(x)$ across an interval $[a, b]$ is represented as

$$\int_a^b f(x)\ dx$$

We say in this case that we are integrating from a to b. The result of the definite integral is a quantity. In particular, when $f(x) \geq 0$ it equals the area between the curve and the axis represented by the differential—in this case, the x-axis. For example, the definite integral,

$$\int_0^1 x^2\ dx$$

computes the area (also known as the *area under the curve*) shown in Figure B.4. The result is 1/3.

If any of the curve being evaluated is negative along the interval, the area computed by the definite integral between that section of curve and the axis in question is also negative. For example,

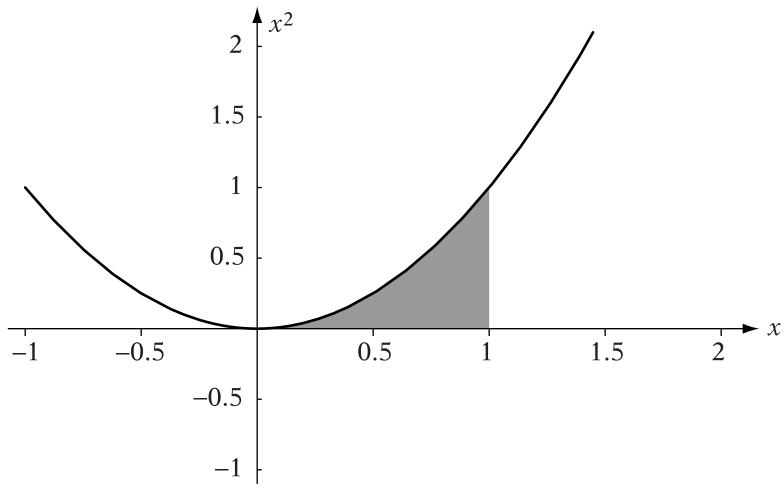

Figure B.4: Definite integral returns area between curves and x-axis. The result in this case is $1/3$.

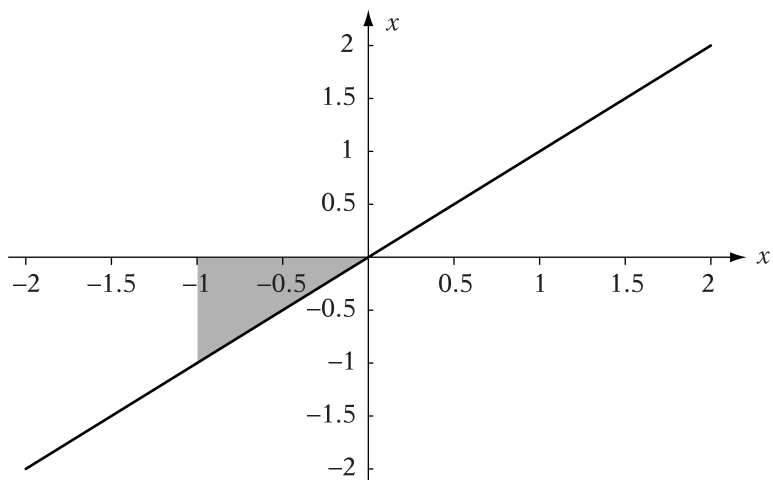

Figure B.5: Definite integral of areas of curve below axis produces negative results. The result in this case is $-1/2$.

the definite integral,

$$\int_{-1}^{0} x \, dx$$

computes the area shown in Figure B.5. The result of the definite integral is $-1/2$.

The *fundamental theorem of calculus* states that a definite integral can be computed from two evaluations of the indefinite integral. More specifically, if $f(x)$ is a continuous function on a closed interval $[a, b]$, and an antiderivative $F(x)$ can be found such that $F'(x) = f(x)$ for all x in $[a, b]$, then

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$$

B.3.2 Evaluating Integrals

Computing an integral for a general function is often not easy, if it can be done at all. Most of the time in games numerical methods are used for evaluation of definite integrals. However, knowing some simple integrals can be useful. For more complex forms, the reader is directed to a more detailed calculus reference.

The integral of the sum of two functions is the sum of the integrals of the functions:

$$\int f(x) + g(x)\ dx = \int f(x)\ dx + \int g(x)\ dx$$

If a function is multiplied by a constant, we can pull the constant out of the integral:

$$\int a \cdot f(x)\ dx = a \int f(x)\ dx$$

If the limits of integration are reversed, then the result is negated:

$$\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$$

The integral of a polynomial term x^k, where $k \neq -1$, is

$$\int x^k\ dx = \frac{x^{k+1}}{k+1} + C$$

If $k = -1$, then we note that

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

so

$$\int \frac{1}{x}\ dx = \ln x + C$$

Tables of integrals can be found in most standard calculus references. A few selected examples are as follows:

$$\int \cos x\ dx = \sin x + C$$

$$\int \sin x\ dx = -\cos x + C$$

$$\int \tan x\ dx = -\ln|\cos x| + C$$

$$\int \cot x\ dx = \ln|\sin x| + C$$

$$\int \mathtt{sec} x\ dx = \ln|\mathtt{sec} x + \tan x| + C$$

$$\int \mathtt{csc} x\ dx = -\ln|\mathtt{csc} x + \cot x| + C$$

$$\int e^x\ dx = e^x + C$$

$$\int a^x\ dx = \frac{a^x}{\ln a} + C$$

$$\int \ln x\ dx = x\ln x - x + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\ dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2 + x^2}\ dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{x\sqrt{x^2 - a^2}}\ dx = \frac{1}{a}\mathtt{sec}^{-1}\left|\frac{x}{a}\right| + C$$

B.3.3 Trapezoidal Rule

In many cases it is either inconvenient or impossible to compute the integral directly. For example, the sinc function $f(x) = \sin x / x$ cannot be integrated analytically. In these cases, numerical meth-

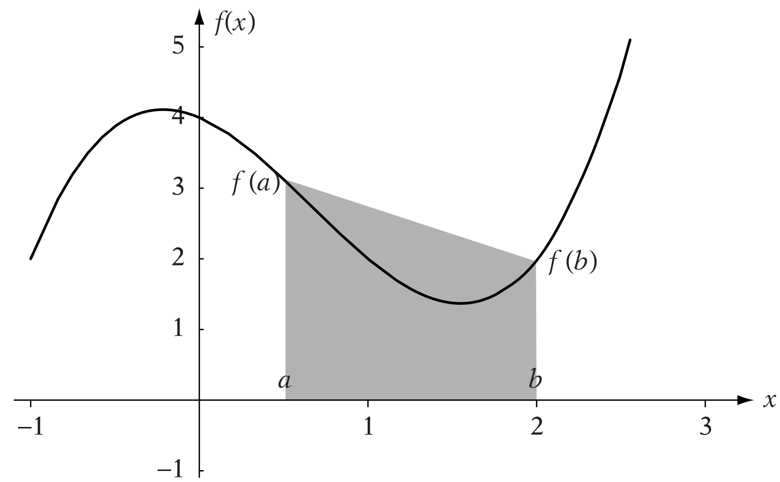

Figure B.6: Approximating the definite integral using a single trapezoid.

ods are used to approximate the value of a definite integral. One of the simplest such methods is the *trapezoidal rule*.

Figure B.6 shows a function that we want to integrate. We can approximate the curve between a and b by using a line segment, and the area under the curve is approximated by the area of a trapezoid:

$$\int_a^b f(x)\ dx \approx \frac{1}{2}(b-a)[f(b)+f(a)]$$

We can get a better approximation by slicing the interval into n equally spaced subintervals, computing the areas of the resulting trapezoids and adding them together (Figure B.7). This is equal to

$$\begin{aligned}\int_a^b f(x)\ dx &\approx \frac{b-a}{2n}\sum_{i=0}^{n-1}[f(x_{i+1})+f(x_i)] \\ &= \frac{b-a}{2n}[f(b)+f(a)] + \frac{b-a}{n}\sum_{i=1}^{n-1} f(x_i)\end{aligned}$$

where each $x_i = a + (b-a)i/n$.

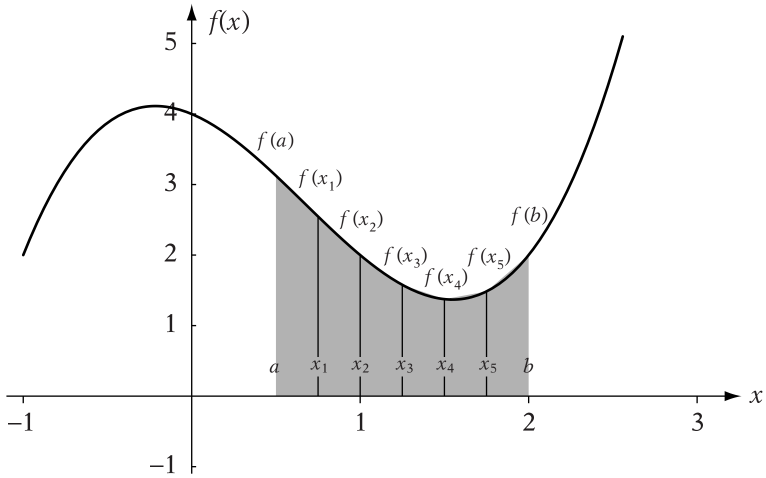

Figure B.7: Approximating the definite integral using multiple trapezoids.

B.3.4 Gaussian Quadrature

While the trapezoid rule provides reasonable approximation of a definite integral for little cost, we can get a better approximation using a method called *Gaussian quadrature.*

The trapezoid rule can be rewritten as a summation of the form

$$\int_a^b f(x)\ dx \approx \sum_{i=0}^{n} c_i f(x_i)$$

where our c_i and x_i are as follows:

$$c_i = \begin{cases} (b-a)/2n; & i = 0, i = n \\ (b-a)/n; & 0 < i < n \end{cases}$$

$$x_i = a + (b-a)i/n$$

Gaussian quadrature uses a similar form, except that it uses nonuniform samples and calculates weights to minimize error and get a better approximation. The error is measured relative to a polynomial; using Gaussian quadrature with n samples, we want the exact result when integrating a polynomial P of degree $2n-1$ or less.

It can be shown that for a given value of n and limits of integration of $[-1,1]$, the values of x_i needed to meet this criteria are the roots of the nth member of a set of polynomials called the *Legendre polynomials*. The corresponding values of c_i are given by

$$c_i = \int_{-1}^{1} \prod_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}\ dx$$

The roots x_i and the associated constants c_i are easily precomputed for a given n. The first few are as follows:

n	x_i	c_i
2	$\pm\sqrt{1/3}$	1
3	0	8/9
	$\pm\sqrt{3/5}$	5/9
4	$\pm\,0.3399810436$	0.6521451549
	$\pm\,0.8611363116$	0.3478548451
5	0.0000000000	0.5688888889
	$\pm\,0.5384693101$	0.4786286705
	$\pm\,0.9061798459$	0.2369268850

Note that using these values is valid only when integrating from -1 to 1. If our integral has a general interval of $[a,b]$, we can use the following to transform it so it can be used with Gaussian quadrature:

$$\int_a^b f(x)\ dx = \int_{-1}^{1} f\left(\frac{(b-a)t + b + a}{2}\right) \frac{b-a}{2}\ dt$$

B.4 Space Curves

A parametric curve is a function $Q(t)$ that maps a set of real values (represented by the parameter t) to a set of points. When mapping to $\mathbb{R}^3$, we commonly use a parametric curve broken into three separate functions, one for each coordinate: $Q(t) = (x(t), y(t), z(t))$. This is also known as a *space curve*.

The first derivative of a space curve is found by computing the derivatives of the functions $x(t)$, $y(t)$, and $z(t)$, so $\mathbf{Q}'(t) = (x'(t), y'(t), z'(t))$. The result of $\mathbf{Q}'$ at parameter t is a vector tangent to the curve at location $Q(t)$, instead of a single slope value. The magnitude of the vector represents the speed at which $Q(t)$ changes relative to time; the larger the vector, the faster the position changes. $\mathbf{Q}'$ is also known as the velocity $\mathbf{v}(t)$.

Computing the second derivative of $Q(t)$ is done similarly, by computing the second derivatives of the individual functions x, y, and z: $\mathbf{Q}''(t) = (x'(t), y'(t), z'(t))$. This represents the change in velocity and is also known as acceleration, or $\mathbf{a}(t)$.

If we normalize $\mathbf{Q}'(t)$ at each parameter t, we get the tangent $\mathbf{T}(t)$:

$$\mathbf{T}(t) = \frac{\mathbf{Q}'(t)}{\|\mathbf{Q}'(t)\|}$$

We can also compute the derivative of $\mathbf{T}(t)$ and normalize it to get the normal $\mathbf{N}(t)$:

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

Note that this is not the same as the acceleration. While the acceleration's direction may vary relative to the velocity, the result of $\mathbf{N}(t)$ is always perpendicular to $\mathbf{T}(t)$. By taking the cross product of $\mathbf{T}$ and $\mathbf{N}$, we get the binormal $\mathbf{B}(t)$:

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

Using $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ as an orthonormal basis and $Q(t)$ as the origin, this gives us a coordinate frame for every parameter t, known as the Frenet frame.

As mentioned, $\mathbf{N}(t)$ is not the same as acceleration. The acceleration vector lies in the subspace formed by using $\mathbf{T}$ and $\mathbf{N}$ as basis vectors, or

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$$

where

$$\begin{aligned} a_T &= \frac{d\|\mathbf{v}\|}{dt} \\ a_N &= \|\mathbf{v}\| \left\| \frac{d\mathbf{T}}{dt} \right\| \end{aligned}$$

A parametric curve $Q(t)$ is *smooth* on an interval $[a, b]$ if it has a continuous derivative on $[a, b]$ and $Q'(t) \neq 0$ for all t in (a, b). A parametric curve $Q(t)$ is *piecewise smooth* on an interval $[a, b]$ if it can be broken into a finite number of subintervals, where it is smooth on each subinterval and Q has one-sided derivatives on (a, b).

For a given point P on a smooth curve $Q(t)$, we define a circle with radius ρ and first and second derivative vectors equal to those at P as the osculating circle. The *curvature* κ at P is $1/\rho$. We can also define the curvature of Q as

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{Q}'(t)\|} \tag{B.13}$$

The curvature at any point is always nonnegative. The higher the curvature, the more the curve bends at that point; the curvature of a straight line is 0.

We can compute the length $\mathcal{L}$ of a piecewise smooth-space curve Q on an interval $[a, b]$ by

$$\mathcal{L} = \int_a^b \|\mathbf{Q}'(t)\| \; dt \tag{B.14}$$

If $Q(t)$ is smooth, we can also define the *arc length* function $s(t)$ as

$$s(t) = \int_a^t \|\mathbf{Q}'(u)\| \, du$$

If $t \geq a$, this measures the length of the curve from a given point $Q(a)$ to a variable point $Q(t)$. If we differentiate both sides with respect to t, we get

$$s'(t) = \|\mathbf{Q}'(t)\| = \|\mathbf{v}(t)\|$$

Since vector length is nonnegative, and we also know that $\mathbf{Q}'(t) \neq 0$ (since Q is smooth), we know that $s(t)$ is strictly increasing and thus invertible to a function $t(s)$. Based on this, we can reparameterize a curve represented by $Q(t)$ by s, by using $Q(t(s))$. This is known as reparameterization by arc length. Rather than mapping a time t to a position on the curve, we can map a length $\mathcal{L}$ to a position on the curve.

It is usually impossible to evaluate the integral in equation B.14, and hence the arc length, directly. Instead the length is approximated by using numerical methods, such as the trapezoid rule or Gaussian quadrature.