TD Algebre : Application linéaire

* $f : E \to F$

- $\forall\, x, y \in E \quad f(x+y) = f(x) + f(y)$
- $\forall\, \alpha \in \mathbb{R}, X \in E \Rightarrow f(\alpha x) = \alpha f(x)$

Ex :

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$(x,y,z) \longmapsto (x+y, x+z)$$

soit $X = (x,y,z) \in \mathbb{R}^3$

$Y = (a,b,c) \in \mathbb{R}^3$

$$f(X+Y) = f((x,y,z)+(a,b,c))$$
$$= f(x+a, y+b, z+c)$$
$$= (x+a+y+b, x+a+z+c)$$
$$= (x+y, x+z) + (a+b, a+c)$$
$$= f(x,y,z) + f(a,b,c) = f(x) + f(y)$$

soit $\alpha \in \mathbb{R}$

$$f(\alpha X) = f(\alpha(x,y,z))$$
$$= f(\alpha x, \alpha y, \alpha z)$$
$$= (\alpha x + \alpha y, \alpha x + \alpha z)$$
$$= \alpha(x+y, x+z) = \alpha f(x)$$

* $\mathrm{Ker}(f) = \{X \in E \text{ tq } f(x) = 0_F\}$

soi $\forall X \in \mathrm{Ker}(f) \Leftrightarrow f(x) = 0_F$

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$(x,y,z) \longmapsto (x+y, x+z)$$

soit $X = (x,y,z) \in \mathrm{Ker}(f)$

soi $f(x) = 0_{\mathbb{R}^2}$

$\Leftrightarrow f(x,y,z) = (0,0)$

$\Leftrightarrow (x+y, x+z) = (0,0)$

$$\Leftrightarrow \begin{cases} x+y=0 \\ x+z=0 \end{cases} \Leftrightarrow \begin{cases} x=-y \\ x=-z \end{cases} \Leftrightarrow \begin{cases} x=-y \\ z=y \end{cases}$$

$\Leftrightarrow X = (-y, y, y) = y(-1,1,1)$

$\mathrm{Ker}(f) = \mathrm{Vect}\{(-1,1,1)\}$

* $f : \mathbb{R}^3 \to \mathbb{R}^3$

$$(x,y,z) \longmapsto (x+y, x, z)$$

soit $X \in (x,y,z) \in \mathbb{R}^3 \quad f(X) = 0_{\mathbb{R}^3}$

$\Leftrightarrow f(x,y,z) = (0,0,0)$

$\Leftrightarrow (x+y, x, z) = (0,0,0)$

$$\Leftrightarrow \begin{cases} x+y=0 \\ x=0 \\ z=0 \end{cases} \Leftrightarrow \begin{cases} y=0 \\ x=0 \\ z=0 \end{cases}$$

$\mathrm{Ker}(f) = \{\mathbb{R}^3\}$

$f(X) = f(Y) \Leftrightarrow Y = X$

$f(X) - f(Y) = 0_E \Leftrightarrow -Y + X = 0_F$

$f(X) - f(-Y) = 0_E \Leftrightarrow X - Y = 0_F$

$f(X - Y) = 0_E$

* $\mathrm{Im}(f) = \{Y \in F, f(X) = Y, X \in E\}$

$= f(E) = \{f(X), X \in E\}$

$\dim(\ker f) + \dim(\operatorname{In} f) = \dim(E)$

$\{V_1, \dots, V_n\}$ base de E

$= \{f(V_1) \dots f(V_n)\}$ generatrice de $\operatorname{In}(f)$

$f(x,y,z) = (x+y, x+z)$

$\{e_1, e_2, e_3\}$ $e_1 = (1,0,0), e_2(0,1,0)$
$e_3 = (0,0,1)$

base de $\mathbb{R}^3$

$\Rightarrow \{f(e_1), f(e_2), f(e_3)\}$ generatrice de $\operatorname{In}(f)$

$f(e_1) = (1,1) = V_1 \quad f(1,0,0) = (\ , \ )$

$f(e_2) = (1,0) = V_2 \quad f(0,1,0) = (\ , \ )$

$f(e_3) = (0,1) = V_3 \quad f(0,0,1) = (\ , \ )$

$\{V_1, V_2, V_3\}$ generatrice de $\operatorname{In}(f)$

$V_2 + 1.V_3 + V_1 = (0,0)$

$\{V_1, V_2\}$ libre

$\Rightarrow \{V_1, V_2\}$ base de $\operatorname{In}(f)$

$\{V_1, V_2, V_3\}$

$V_1 = 2V_2$

$X = \alpha V_1 + \beta V_2 + \gamma V_3$

$= \alpha V_1 + \frac{1}{2}\beta V_2 + \gamma V_3$

$= (\alpha + \frac{1}{2}\beta) V_1 + \gamma V_3$

$\{V_1, V_2\}$

Examen finale Exo 1 :

1) soit $X = (x, y, z) \in \mathbb{R}^3$

$Y = (a, b, c) \in \mathbb{R}^3$

on a $f(X, Y) = f(a+x, y+b, z+c)$

$= (-2(a+x) + y + b + z + c, x + a - 2(y+b) + z + c, a + x + b + y - 2(z+c))$

$= (-2x + y + z, x - 2y + z, x + y - 2z)$
$+ (-2a + b + c, a - 2b + c, a + b - 2c)$
$= f(X) + f(Y)$

soit $\alpha \in \mathbb{R}$

$f(\alpha X) = f(\alpha(x, y, z)) = f(\alpha x, \alpha y, \alpha z)$
$= (-2\alpha x + \alpha y + \alpha z, \alpha x - 2\alpha y + \alpha z, \alpha x + \alpha y - 2\alpha z)$

$= \alpha(-2x + y + z, x - 2y + z, x + y - 2z)$
$= \alpha f(x)$

$\Rightarrow f$ est lineaire de $\mathbb{R}^3$ sur $\mathbb{R}^3$

2) soit $X = (x, y, z) \in (\ker(f))$

$\Leftrightarrow f(x, y, z) = (0,0,0)$

$\Leftrightarrow (-2x + y + z, x - 2y + z, x + y - 2z) = (0,0,0)$

$\Leftrightarrow \begin{cases} -2x + y + z = 0 \\ x - 2y + z = 0 \\ x + y - 2z = 0 \end{cases} \Leftrightarrow \begin{cases} y = 2x - z \\ x - 2(2x - z) + z = 0 \\ x + y - 2z = 0 \end{cases}$

$\Rightarrow \begin{cases} y = 2x - z \\ x = z \\ y = x \end{cases} \Leftrightarrow \begin{cases} x = z \\ y = z \end{cases}$

$\Leftrightarrow x = y = z$

$\Leftrightarrow X = (x, x, x)$

$= x(1,1,1)$

$\ker f = \text{Vect}\{(1,1,1)\}$

or $(1,1,1) \neq (0,0,0)$

donc est libre

$\Rightarrow$ donc [illegible] est une bose

de In f

$\Rightarrow \dim(\ker f) = 1$

3) on a $\{e_1, e_2, e_3\}$ base de $\mathbb{R}^3$

$\Rightarrow \{f(e_1), f(e_2), f(e_3)\}$

generatrice de $In(f)$

tq :

$V_1 = f(e_1) = (-2,1,1)$

$V_2 = f(e_2) = (1,-2,1)$

$V_3 = f(e_3) = (1,1,-2)$

on a $V_1 + V_2 + V_3 = (0,0,0)$

$\{V_1 + V_2 + V_3\}$ n'est pas libre

on a $V_1 + V_2 = -V_3$

et on a $\forall X \in Im(f)$

$\Leftrightarrow X = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3$

$= \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3(-V_2 - V_1)$

$= (\alpha_1 - \alpha_3) V_1 + (\alpha_1 - \alpha_3) V_2$

$\Leftrightarrow \{V_1, V_2\}$ generatrice de $In(f)$

$\{V_1, V_2\}$ libre

$\Rightarrow \{V_1, V_2\}$ base de $In(f)$

$\dim(In\, f) = 2$

2ème methode

$In(f) = f(\mathbb{R}^3)$

soit $y = (a,b,c) \in In(f)$

$\Rightarrow \exists X = (x,y,z) \in \mathbb{R}^3$

tq : $(a,b,c) = f(x,y,z)$

$= (-2x+y+z, x-2y+z, x+y-2z)$

$$\Leftrightarrow \begin{cases} a = -2x+y+z \\ b = x-2y+z \\ c = x+y-2z \end{cases} \Rightarrow a = -b-c$$

$y = (-b-c, b, c)$

$= (-b, b, 0) + (-c, 0, c$

$y = b\underbrace{(-1,1,0)}_{V_1} + c\underbrace{(-1,0,1)}_{V_2}$

$\Rightarrow Im\, f = \text{vect}\{V_1, V_2\}$

$\{V_1, V_2\}$ libre

$\Rightarrow \{V_1, V_2\}$ base de $In(f)$