Serie numerique

Soit une suite (U_n)

$S_n = U_0 + U_1 + \dots + U_n = \sum_{k=0}^{n} U_k$

↳ somme partielle

une serie : $\sum U_n$ converge ssi $\lim S_n \in \mathbb{R}$ existe

$* \; S_n = ???$

① suite geometrique

② suite arithmetique

③ suite Telescopique

① $U_0 \xrightarrow{\times q} U_1 \xrightarrow{\times q} U_2 + \dots + U_n$, soit $U_0 + U_1 + U_2 + \dots + U_n$

② $U_0 \xrightarrow{+r} U_1 \xrightarrow{+r} U_2 + \dots + U_n$, soit $U_0 + U_1 + U_2 + \dots + U_n$

exemple suite geometrique

$U_n = \frac{1}{3^n}$

$U_n = \left(\frac{1}{3}\right)^n = 1 \times \left(\frac{1}{3}\right)^n = U_0 \times q^n$

$S_n = U_0 \times \frac{1 - q^{n-0+1}}{1-q} = 1 \times \frac{1 - \left(\frac{1}{3}\right)^{n-0+1}}{1 - \left(\frac{1}{3}\right)} = \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{n+1}\right)$

$$\lim_{n \to +\infty} q^n = \begin{cases} +\infty & \text{si } q > 1 \\ 0 & \text{si } -1 < q < 1 \\ \text{n'existe pas} & q \leq -1 \end{cases}$$

$\lim_{n \to +\infty} S_n = \frac{3}{2} \in \mathbb{R}$

Donc $\sum U_n$ converge

Une serie $\sum U_n$ $\overset{\text{convergente}}{C.V} \Rightarrow \lim_{n \to +\infty} U_n = 0$

preuve

Une serie $\sum U_n$ convergente càd $\lim_{n \to +\infty} S_n = \ell \in \mathbb{R}$

$$S_n = \underbrace{U_0 + U_1 + \dots + U_{n-1}}_{S_{n-1}} + U_n$$

$$U_n = S_n - S_{n-1}$$

or $\begin{cases} \lim_{n \to +\infty} S_n = \ell \\ \lim_{n \to +\infty} S_{n-1} = \ell \end{cases} \Rightarrow \lim_{n \to +\infty} U_n = \ell - \ell = 0$

$\lim_{n \to +\infty} U_n \neq 0 \Rightarrow \sum U_n$ divergente

exemple : $U_n = n \ln\left(1 + \frac{1}{n}\right)$

$$\lim_{n \to +\infty} U_n = \lim_{n \to +\infty} n \ln\left(1 + \frac{1}{n}\right)$$

soit $x = 1/n$

$n \to +\infty$
$x \to 0^+$

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$$

$$= \lim_{n \to +\infty} \frac{\ln(1 + \overset{\to 0}{1/n})}{\underset{\to 0}{1/n}} = 1$$

$\lim_{n \to +\infty} U_n \neq 0 \Rightarrow \sum U_n$ diverge

Serie de Rieman

si $U_n = \frac{cte}{n^\alpha}$ alors $\sum U_n$ converge ssi $\alpha > 1$

Regle de comparaison

$0 \leq U_n \leq V_n$

- $\sum V_n$ converge $\Rightarrow \sum U_n$ converge
- $\sum U_n$ diverge $\Rightarrow \sum V_n$ diverge

exemple 1: $U_n = \dfrac{1+\sin(n)}{n^2}$

$-1 \leq \sin(n) \leq 1$

$0 \leq \sin(n)+1 \leq 2$

$0 \leq \dfrac{\sin(n)+1}{n^2} \leq \dfrac{2}{n^2}$

$0 \leq U_n \leq V_n$

or $\sum \frac{2}{n^2}$ converge $(\alpha = 2 > 1)$ (serie de Rieman)

donc $\sum U_n$ converge.

exemple 2 : $U_n = \dfrac{\cos(n)}{n^2}$

$-1 \leq \cos(n) \leq 1$

$-\dfrac{1}{n^2} \leq \dfrac{\cos(n)}{n^2} \leq \dfrac{1}{n^2}$

$\left| \dfrac{\cos(n)}{n^2} \right| \leq 1/n^2$

$\sum \frac{1}{n^2}$ converge $(\alpha = 2 > 1)$

$\sum |U_n|$ converge ($\sum U_n$ absoolument converge

$\sum |U_n|$ converge $\Rightarrow \sum U_n$ converge

* $\sum |U_n|$ diverge $\not\Rightarrow \sum U_n$ diverge !!!

Regle d'equivalence :

- Si $U_n \geq 0$ et $U_n \underset{n \to +\infty}{\sim} V_n$

alors $\sum U_n$ et $\sum V_n$ sont de meme nature

exemple

$$U_n = \frac{\sin\left(\frac{1}{n(n+1)}\right)}{\cos\left(\frac{1}{n}\right) \cdot \cos\left(\frac{1}{n+1}\right)}$$

- $\sin\left(\frac{1}{n(n+1)}\right) \underset{\substack{n \to +\infty \\ x \to 0^+}}{\sim} \frac{1}{n(n+1)} \underset{+\infty}{\sim} \frac{1}{n^2}$

- $\cos\left(\overset{x}{\frac{1}{n}}\right) \underset{\substack{n \to +\infty \\ x \to 0}}{\sim} 1$

- $\cos\left(\overset{x}{\frac{1}{n+1}}\right) \underset{\substack{n \to +\infty \\ x \to 0^+}}{\sim} 1$

$\Rightarrow U_n \underset{+\infty}{\sim} \frac{\frac{1}{n^2}}{1 \times 1} = \frac{1}{n^2}$

or $\sum \frac{1}{n^2}$ converge $(\alpha = 2 > 1)$

donc $\sum U_n$ converge

- Regle de cauchy

- Si $\lim\limits_{n\to+\infty} \left|\sqrt[n]{U_n}\right| = \ell < 1 \Rightarrow \sum U_n$ converge

- Si $\lim\limits_{n\to+\infty} \left|\sqrt[n]{U_n}\right| = \ell > 1 \Rightarrow \sum U_n$ diverge

exemple :

$$U_n = \frac{1}{2^n}\left(\frac{n}{n+1}\right)^{n^2}$$

$$\left|\sqrt[n]{U_n}\right| = \left|(U_n)^{1/n}\right| = \frac{1}{2}\cdot\left(\frac{n}{n+1}\right)^n = \frac{1}{2}e^{\ln\left[\left(\frac{n}{n+1}\right)^n\right]}$$

$$= \frac{1}{2}e^{n\ln\left(\frac{n}{n+1}\right)} = \frac{1}{2}e^{-n\ln\left(\frac{n+1}{n}\right)} = \frac{1}{2}e^{-n\ln\left(1+\frac{1}{n}\right)}$$

$$= \frac{1}{2}e^{-\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}}$$

on $\lim\limits_{x\to 0} \frac{\ln(1+x)}{x} = 1$

$$\lim\limits_{n\to+\infty} \sqrt[n]{U_n} = \frac{1}{2}e^{-1} = \frac{1}{2e} < 1$$

$\Rightarrow \sum U_n$ converge