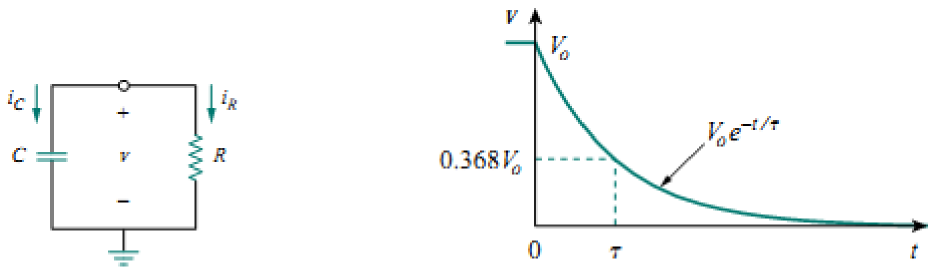
In this lecture, we shall examine two types of simple circuits: a circuit comprising a resistor and capacitor and a circuit comprising a resistor and an inductor. These are called RC and RL circuits, respectively. As simple as these circuits are, they find continual applications in electronics, communications, and control systems, as we shall see.

We carry out the analysis of RC and RL circuits by applying Kirchhoff's laws, as we did for resistive circuits. The only difference is that applying Kirchhoff's laws to purely resistive circuits results in algebraic equations, while applying the laws to RC and RL circuits produces differential equations, which are more difficult to solve than algebraic equations. The differential equations resulting from analyzing RC and RL circuits are of the first order. Hence, the circuits are collectively known as first-order circuits.

In addition to there being two types of first-order circuits (RC and RL), there are two ways to excite the circuits. The first way is by initial conditions of the storage elements in the circuits. In these so-called source-free circuits, we assume that energy is initially stored in the capacitive or inductive element. The energy causes current to flow in the circuit and is gradually dissipated in the resistors. Although source-free circuits are by definition free of independent sources, they may have dependent sources. The second way of exciting first-order circuits is by independent sources. In this lecture, the independent sources we will consider are dc sources. (In later chapters, we shall consider sinusoidal and exponential sources.)

Source free RC circuit [Natural Response]

A source-free RC circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors.



(a) (b)

Figure 1 (a) A source free RC circuit (b) its Voltage response

Consider a series combination of a resistor and an initially charged capacitor, as shown in Fig. 1. (The resistor and capacitor may be the equivalent resistance and equivalent capacitance of combinations of resistors and capacitors.) Our objective is to determine the circuit response, which, for pedagogic reasons, we assume to be the voltage v(t) across the capacitor. Since the capacitor is initially charged, we can assume that at time t = 0, the initial voltage is

$$v(0) = V_0 \qquad (6.1)$$

with the corresponding value of the energy stored as

$$w(0) = \frac{1}{2}CV_0^2 \qquad (6.2)$$

Applying KCL at the top node of figure 1

$$i_C + i_R = 0 \qquad (6.3)$$

By definition, $i_C = C\,dv/dt$ and $i_R = v/R$. Thus,

$$C\frac{dv}{dt} + \frac{v}{R} = 0 \qquad (6.4a)$$

or

$$\frac{dv}{dt} + \frac{v}{RC} = 0 \qquad (6.4b)$$

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This is a first-order differential equation, since only the first derivative of v is involved. To solve it, we rearrange the terms as

$$\frac{dv}{v} = -\frac{1}{RC}dt \qquad (6.5)$$

Integrating both sides, we get

$$\ln v = -\frac{t}{RC} + \ln A$$

Where $\ln A$ is the integration constant. Thus,

$$\ln\frac{v}{A} = -\frac{t}{RC} \qquad (6.6)$$

Taking powers of e produces

$$v(t) = Ae^{-\frac{t}{RC}}$$

But from the initial conditions, $v(0) = A = V_0$. Hence,

$$\boldsymbol{v(t) = V_0 e^{-\frac{t}{RC}}} \qquad (6.7)$$

This shows that the voltage response of the RC circuit is an exponential decay of the initial voltage. Since the response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called the ***natural response*** of the circuit.

The natural response of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.

The natural response is illustrated graphically in figure 1(b). Note that at t=0, we have correct initial condition as in equation 6.1. As t increases, the voltage decreases toward zero. The rapidity with which the voltage decreases is expressed in terms of the time constant, denoted by the lower case Greek letter tau, τ.

The time constant of a circuit is the time required for the response to decay by a factor of 1/e or 36.8 percent of its initial value.

This implies that at time $t = \tau$ equation 6.7 becomes

$$V_0 e^{-\frac{\tau}{RC}} = V_0 e^{-1} = 0.368V_0$$

Or

$$\boldsymbol{\tau = RC} \qquad (6.8)$$

In terms of time constant the equation 6.7 can be written as

$$v(t) = V_0 e^{-\frac{t}{\tau}} \qquad (6.9)$$

With a calculator it is easy to show that the value of $v(t)/V_0$ is as shown in Table 1. It is evident from Table 1 that the voltage v(t)is less than 1 percent of V_0 after 5τ (five time constants). Thus, **it is customary to assume that the capacitor is fully discharged (or charged) after five time constants**. In other words, it takes 5τ for the circuit to reach its final state or steady state when no changes take place with time. Notice that for every time interval of τ, the voltage is reduced by 36.8 percent of its previous value, $v(t + \tau) = v(t)/e = 0.368v(t)$, regardless of the value of t.

Table 1 Values of $\frac{v(t)}{V_0} = e^{-\frac{t}{\tau}}$

t	$v(t)/V_0$
τ	0.36788
2τ	0.13534
3τ	0.04979
4τ	0.01832
5τ	0.00674

Observe from Eq. (6.8) that the smaller the time constant, the more rapidly the voltage decreases, that is, the faster the response. This is illustrated in Fig. 2. A circuit with a small time constant gives a fast response in that it reaches the steady state (or final state) quickly due to quick dissipation of energy stored, whereas a circuit with a large time constant gives a slow response because it takes longer to reach steady state. At any rate, whether the time constant is small or large, the circuit reaches steady state in five time constants.

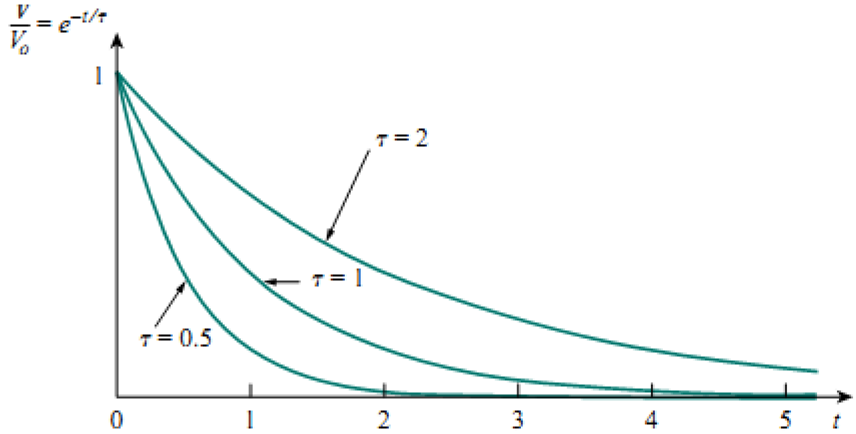


Figure 2 Plot of $v/V_0 = e^{-\frac{t}{\tau}}$ for various values of the time constant.

With the voltage v(t) in Eq. (6.9), we can find the current $i_R(t)$,

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-\frac{t}{\tau}} \qquad (6.10)$$

The power dissipated in the resistor is

$$p(t) = v i_R = \frac{V_0^2}{R} e^{-\frac{2t}{\tau}} \qquad (6.11)$$

The energy absorbed by the resistor up to time t is

$$w_R(t) = \int_0^t p\,dt = \int_0^t \frac{V_0^2}{R} e^{-\frac{2t}{\tau}} dt = -\frac{\tau V_0^2}{2R} e^{-2t/\tau}\Big|_0^t = \frac{1}{2} C V_0^2 (1 - e^{-2t/\tau}), \qquad \tau = \mathrm{RC} \qquad (6.12)$$

Notice that as $t \to \infty$, $w_R(\infty) \to \frac{1}{2} C V_0^2$, which is the same as $w_C(0)$, the energy initially stored in the capacitor. The energy that was initially stored in the capacitor is eventually dissipated in the resistor.

In summary, The Key to Working with a Source-free RC Circuit is finding:

1. The initial voltage v(0) = V0 across the capacitor.
2. The time constant τ.

With these two items, we obtain the response as the capacitor voltage $v_C(t) = v(t) = v(0)e^{-\frac{t}{\tau}}$. Once the capacitor voltage is first obtained, other variables (capacitor current i_C, resistor voltage v_R, and resistor current i_R) can be determined. In finding the time constant τ = RC, R is often the Thevenin equivalent resistance at the terminals of the capacitor; that is, we take out the capacitor C and find R = R_{Th} at its terminals.

Example 7.1

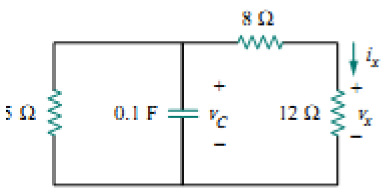


Figure 7.5 For Example 7.1.

In Fig. 7.5, let $v_C(0) = 15$ V. Find v_C, v_x, and i_x for $t > 0$.

Solution:

We first need to make the circuit in Fig. 7.5 conform with the standard *RC* circuit in Fig. 7.1. We find the equivalent resistance or the Thevenin resistance at the capacitor terminals. Our objective is always to first obtain capacitor voltage v_C. From this, we can determine v_x and i_x.

The 8-Ω and 12-Ω resistors in series can be combined to give a 20-Ω resistor. This 20-Ω resistor in parallel with the 5-Ω resistor can be combined so that the equivalent resistance is

$$R_{eq} = \frac{20 \times 5}{20 + 5} = 4\ \Omega$$

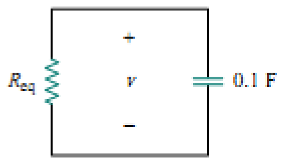


Figure 7.6 Equivalent circuit for the circuit in Fig. 7.5.

Hence, the equivalent circuit is as shown in Fig. 7.6, which is analogous to Fig. 7.1. The time constant is

$$\tau = R_{eq}C = 4(0.1) = 0.4\ \text{s}$$

Thus,

$$v = v(0)e^{-t/\tau} = 15e^{-t/0.4}\ \text{V}, \qquad v_C = v = 15e^{-2.5t}\ \text{V}$$

From Fig. 7.5, we can use voltage division to get v_x; so

$$v_x = \frac{12}{12+8}v = 0.6(15e^{-2.5t}) = 9e^{-2.5t}\ \text{V}$$

Finally,

$$i_x = \frac{v_x}{12} = 0.75e^{-2.5t}\ \text{A}$$

Example 7.2

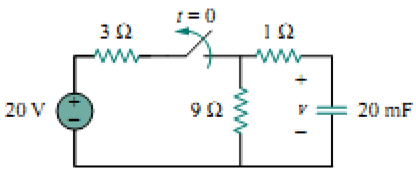


Figure 7.8 For Example 7.2.

The switch in the circuit in Fig. 7.8 has been closed for a long time, and it is opened at $t = 0$. Find $v(t)$ for $t \geq 0$. Calculate the initial energy stored in the capacitor.

Solution:

For $t < 0$, the switch is closed; the capacitor is an open circuit to dc, as represented in Fig. 7.9(a). Using voltage division

$$v_C(t) = \frac{9}{9+3}(20) = 15\ \text{V}, \qquad t < 0$$

Since the voltage across a capacitor cannot change instantaneously, the voltage across the capacitor at $t = 0^-$ is the same at $t = 0$, or

$$v_C(0) = V_0 = 15\ \text{V}$$

For $t > 0$, the switch is opened, and we have the RC circuit shown in Fig. 7.9(b). [Notice that the RC circuit in Fig. 7.9(b) is source free; the independent source in Fig. 7.8 is needed to provide V_0 or the initial energy in the capacitor.] The 1-Ω and 9-Ω resistors in series give

$$R_{eq} = 1 + 9 = 10\ \Omega$$

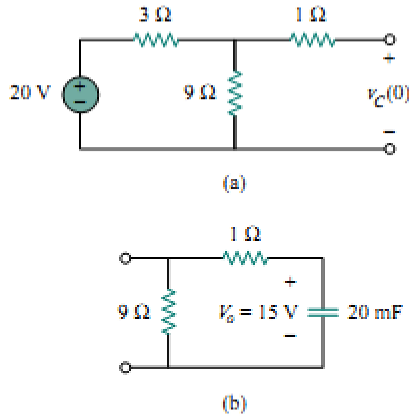


Figure 7.9 For Example 7.2: (a) $t < 0$, (b) $t > 0$.

The time constant is

$$\tau = R_{eq}C = 10 \times 20 \times 10^{-3} = 0.2\ \text{s}$$

Thus, the voltage across the capacitor for $t \geq 0$ is

$$v(t) = v_C(0)e^{-t/\tau} = 15e^{-t/0.2}\ \text{V}$$

or

$$v(t) = 15e^{-5t}\ \text{V}$$

The initial energy stored in the capacitor is

$$w_C(0) = \frac{1}{2}Cv_C^2(0) = \frac{1}{2} \times 20 \times 10^{-3} \times 15^2 = 2.25\ \text{J}$$

1. Alexander Practice problem 7.1,7.2

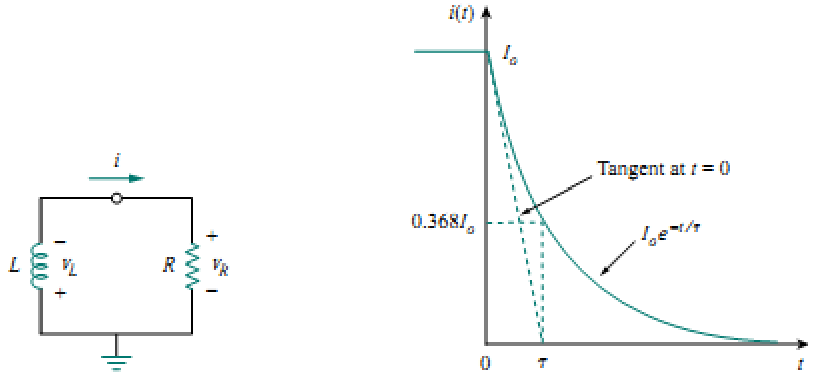
Source free RL circuit [Natural response]

Consider the series connection of a resistor and an inductor, as shown in Fig. 7. Our goal is to determine the circuit response, which we will assume to be the current i(t) through the inductor. We select the inductor current as the response in order to take advantage of the idea that the inductor current cannot change instantaneously. At t = 0, we assume that the inductor has an initial current I_0,or

$$i(0) = I_0 \qquad (6.13)$$

With the corresponding energy stored in the inductor as

$$w(0) = \frac{1}{2} L I_0^2 \qquad (6.14)$$



(a) (b)

Figure 3 (a) A source free RL circuit (b) its Current response

Applying KVL around the loop in figure 3

$$v_L + v_R = 0 \qquad (6.16)$$

But $v_l = L\, di/dt$ and $v_R = iR$. Thus,

$$L\frac{di}{dt} + Ri = 0$$

Or

$$\frac{di}{dt} + \frac{R}{L} i = 0 \qquad (6.16)$$

Rearranging terms and integrating gives

$$\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L} dt$$

$$\ln i \Big|_{I_0}^{i(t)} = -\frac{Rt}{L}\Big|_0^t \quad \Longrightarrow \quad \ln i(t) - \ln I_0 = -\frac{Rt}{L} + 0$$

Or

$$\ln \frac{i(t)}{I_0} = -\frac{Rt}{L} \qquad (6.17)$$

Taking the power of e, we have

$$i(t) = I_0 e^{-\frac{Rt}{L}} \qquad (6.18)$$

This shows that the natural response of the RL circuit is an exponential decay of the initial current. The current response is shown in Fig. 3(b). It is evident from Eq. (6.18) that the time constant for the RL circuit is

$$\boldsymbol{\tau} = \frac{\boldsymbol{L}}{\boldsymbol{R}} \qquad (6.19)$$

with τ again having the unit of seconds. Thus, Eq. (6.18) may be written as

$$\boldsymbol{i(t) = I_0 e^{-\frac{t}{\tau}}} \qquad (6.20)$$

With the current in Eq. (6.20), we can find the voltage across the resistor as

$$v_R(t) = iT = I_0 R e^{-\frac{t}{\tau}} \qquad (6.21)$$

The power dissipated by the resistor is

$$p = v_R i = I_0^2 R e^{-\frac{2t}{\tau}} \qquad (6.22)$$

The energy absorbed by the resistor is

$$w_R(t) = \int_0^t p\,dt = \int_0^t I_0^2 R e^{-2t/\tau}\,dt = -\frac{1}{2}\tau I_0^2 R e^{-2t/\tau}\Big|_0^t, \qquad \tau = \frac{L}{R}$$

Or

$$w_R(t) = \frac{1}{2} L I_0^2 \left(1 - e^{-\frac{2t}{\tau}}\right) \qquad (6.23)$$

Note that as $t \to \infty, wR(\infty) \to \frac{1}{2} L I_0^2$, which is the same as $w_L(0)$, the initial energy stored in the inductor as in Eq. (6.14). Again, the energy initially stored in the inductor is eventually dissipated in the resistor.

In summary, The Key to Working with a Source-free RL Circuit is to Find:

1. The initial current $i(0) = I_0$ through the inductor.
2. The time constant τ of the circuit.

With the two items, we obtain the response as the inductor current $i_L(t) = i(t) = i(0)e^{-\frac{t}{\tau}}$. Once we determine the inductor current iL, other variables (inductor voltage v_L, resistor voltage v_R, and resistor current i_R) can be obtained. Note that in general, R in Eq. (6.19) is the Thevenin resistance at the terminals of the inductor.

Example 7.3

Assuming that $i(0) = 10$ A, calculate $i(t)$ and $i_x(t)$ in the circuit in Fig. 7.13.

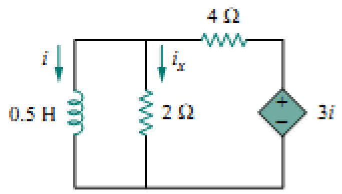


Figure 7.13 For Example 7.3.

Solution:

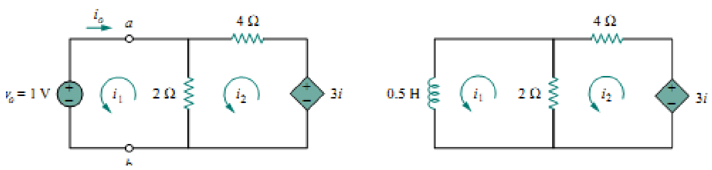
There are two ways we can solve this problem. One way is to obtain the equivalent resistance at the inductor terminals and then use Eq. (7.20). The other way is to start from scratch by using Kirchhoff's voltage law. Whichever approach is taken, it is always better to first obtain the inductor current.

METHOD 1 The equivalent resistance is the same as the Thevenin resistance at the inductor terminals. Because of the dependent source, we insert a voltage source with $v_o = 1$ V at the inductor terminals *a*-*b*, as in Fig. 7.14(a). (We could also insert a 1-A current source at the terminals.) Applying KVL to the two loops results in

$$2(i_1 - i_2) + 1 = 0 \quad \Longrightarrow \quad i_1 - i_2 = -\frac{1}{2} \qquad (7.3.1)$$

$$6i_2 - 2i_1 - 3i_1 = 0 \quad \Longrightarrow \quad i_2 = \frac{5}{6} i_1 \qquad (7.3.2)$$

Substituting Eq. (7.3.2) into Eq. (7.3.1) gives



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$$i_1 = -3 \text{ A}, \qquad i_o = -i_1 = 3 \text{ A}$$

Hence,

$$R_{eq} = R_{Th} = \frac{v_o}{i_o} = \frac{1}{3}\ \Omega$$

The time constant is

$$\tau = \frac{L}{R_{eq}} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}\text{ s}$$

Thus, the current through the inductor is

$$i(t) = i(0)e^{-t/\tau} = 10e^{-(2/3)t} \text{ A}, \qquad t > 0$$

METHOD 2 We may directly apply KVL to the circuit as in Fig. 7.14(b). For loop 1,

$$\frac{1}{2}\frac{di_1}{dt} + 2(i_1 - i_2) = 0$$

or

$$\frac{di_1}{dt} + 4i_1 - 4i_2 = 0 \qquad (7.3.3)$$

For loop 2,

$$6i_2 - 2i_1 - 3i_1 = 0 \qquad \Longrightarrow \qquad i_2 = \frac{5}{6}i_1 \qquad (7.3.4)$$

Substituting Eq. (7.3.4) into Eq. (7.3.3) gives

$$\frac{di_1}{dt} + \frac{2}{3}i_1 = 0$$

Rearranging terms,

$$\frac{di_1}{i_1} = -\frac{2}{3}dt$$

Since $i_1 = i$, we may replace i_1 with i and integrate:

$$\ln i \Big|_{i(0)}^{i(t)} = -\frac{2}{3}t\Big|_0^t$$

or

$$\ln\frac{i(t)}{i(0)} = -\frac{2}{3}t$$

Taking the powers of e, we finally obtain

$$i(t) = i(0)e^{-(2/3)t} = 10e^{-(2/3)t} \text{ A}, \qquad t > 0$$

which is the same as by Method 1.

The voltage across the inductor is

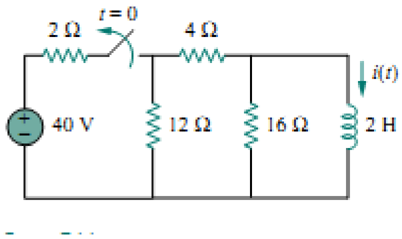
$$v = L\frac{di}{dt} = 0.5(10)\left(-\frac{2}{3}\right)e^{-(2/3)t} = -\frac{10}{3}e^{-(2/3)t} \text{ V}$$

Since the inductor and the 2-Ω resistor are in parallel,

$$i_x(t) = \frac{v}{2} = -1.667e^{-(2/3)t} \text{ A}, \qquad t > 0$$

Example 7.4

The switch in the circuit of Fig. 7.16 has been closed for a long time. At $t = 0$, the switch is opened. Calculate $i(t)$ for $t > 0$.



Solution:

When $t < 0$, the switch is closed, and the inductor acts as a short circuit to dc. The 16-Ω resistor is short-circuited; the resulting circuit is shown in Fig. 7.17(a). To get i_1 in Fig. 7.17(a), we combine the 4-Ω and 12-Ω resistors in parallel to get

$$\frac{4 \times 12}{4 + 12} = 3\ \Omega$$

Hence,

$$i_1 = \frac{40}{2 + 3} = 8\ \text{A}$$

We obtain $i(t)$ from i_1 in Fig. 7.17(a) using current division, by writing

$$i(t) = \frac{12}{12 + 4} i_1 = 6\ \text{A}, \qquad t < 0$$

Since the current through an inductor cannot change instantaneously,

$$i(0) = i(0^-) = 6\ \text{A}$$

When $t > 0$, the switch is open and the voltage source is disconnected. We now have the RL circuit in Fig. 7.17(b). Combining the resistors, we have

$$R_{eq} = (12 + 4) \parallel 16 = 8\ \Omega$$

The time constant is

$$\tau = \frac{L}{R_{eq}} = \frac{2}{8} = \frac{1}{4}\ \text{s}$$

Thus,

$$i(t) = i(0)e^{-t/\tau} = 6e^{-4t}\ \text{A}$$

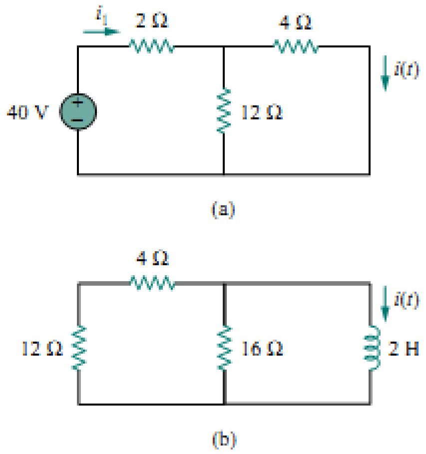


Figure 7.17 Solving the circuit of Fig. 7.16: (a) for $t < 0$, (b) for $t > 0$.

2. Alexander example 7.5
3. Alexander Practice problem 7.3, 7.4, 7.5

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Singularity functions

Before going on to analyze different responses we need to first consider some mathematical concepts that will aid our understanding of transient analysis

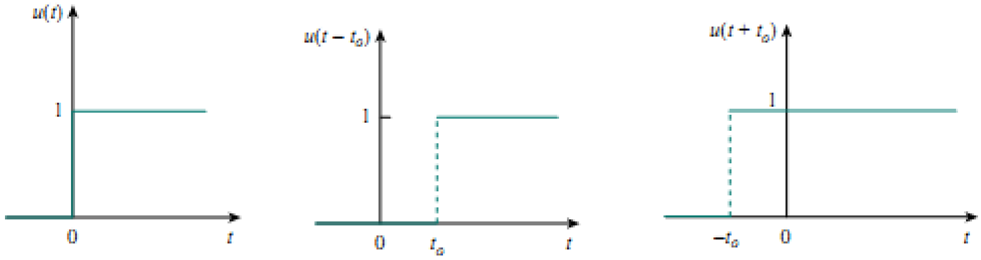
Singularity functions (also called switching functions) are very useful in circuit analysis. They serve as good approximations to the switching signals that arise in circuits with switching operations. They are helpful in the neat, compact description of some circuit phenomena, especially the step response of RC or RL circuits to be discussed in the next sections. By definition,

Singularity functions are functions that either are discontinuous or have discontinuous derivatives.

The three most widely used singularity functions in circuit analysis are the ***unit step***, the ***unit impulse***, and the ***unit ramp*** functions.

Unit Step function

The unit step function u(t) is 0 for negative values of t and 1 for positive values of t.



(a) (b) (c)

Figure 4 (a) The unit Step Function (b) The unit step function delayed by t_0 , (c) the unit step advanced by t_0.

In mathematical terms,

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \qquad (6.24)$$

The unit step function is undefined at $t = 0$, where it changes abruptly from 0 to 1. It is dimensionless, like other mathematical functions such as sine and cosine. Figure 4(a) depicts the unit step function. If the abrupt change occurs at $t = t_0$ (where $t_0 > 0$) instead of $t = 0$, the unit step function becomes

$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases} \qquad (6.25)$$

which is the same as saying that u(t)is delayed by t0 seconds, as shown in Fig. 4(b). To get Eq. (6.25) from Eq. (6.24), we simply replace every t by $t - t_0$. If the change is at $t =- t_0$, the unit step function becomes

$$u(t + t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases} \qquad (6.26)$$

meaning that u(t) is advanced by t_0 seconds, as shown in Fig. 4(c). We use the step function to represent an abrupt change in voltage or current, like the changes that occur in the circuits of control systems and digital computers. For example, the voltage

$$v(t) = \begin{cases} 0, & t < t_0 \\ V_0, & t > t_0 \end{cases} \qquad (6.27)$$

may be expressed in terms of the unit step function as

$$v(t) = V_0 u(t - t_0) \qquad (6.28)$$

If we let $t_0 = 0$, then v(t) is simply the step voltage $V_0u(t)$. A voltage source of $V_0u(t)$ is shown in Fig. 5(a); its equivalent circuit is shown in Fig. 5(b). It is evident in Fig. 5(b) that terminals a-b are short--

circuited (v = 0) for t< 0 and that v = V0 appears at the terminals for t> 0. Similarly, a current source of $I_0u(t)$ is shown in Fig. 6(a), while its equivalent circuit is in Fig. 6(b). Notice that for t< 0, there is an open circuit (i = 0), and that i = I_0 flow for t> 0.

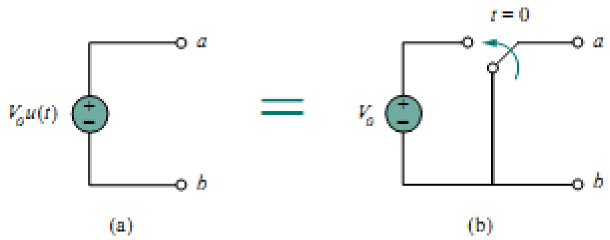


Figure 5 (a) Voltage source of V_0 u(t), (b) its equivalent circuit.

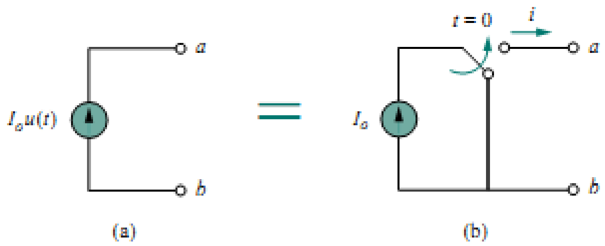


Figure 6 (a) Current source of I_0u(t), (b) its equivalent circuit.

Unit Impulse function

he derivative of the unit step function u(t) is the unit impulse function δ(t), which we write as

$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0, & t < 0 \\ \text{Undefined}, & t = 0 \\ 0, & t > 0 \end{cases} \qquad (6.29)$$

The unit impulse function—also known as the *delta* function—is shown in Fig. 7(a).

The unit impulse function δ(t) is zero everywhere except at t = 0, where it is undefined.

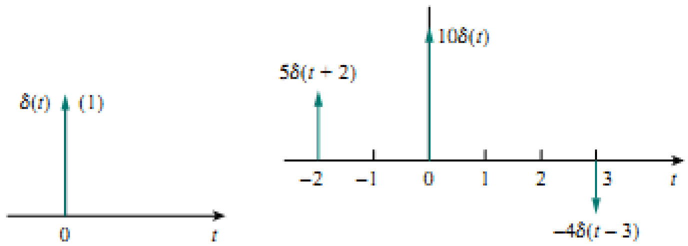


Figure 7 (a) The unit impulse functions (b) different impulse functions

Impulsive currents and voltages occur in electric circuits as a result of switching operations or impulsive sources. Although the unit impulse function is not physically realizable (just like ideal sources, ideal resistors, etc.), it is a very useful mathematical tool.

The unit impulse may be regarded as an applied or resulting shock. It may be visualized as a very short duration pulse of unit area. This may be expressed mathematically as

$$\int_{0-}^{0+} \delta(t)\, dt = 1 \qquad (6.30)$$

where t = 0− denotes the time just before t = 0 and t = 0+ is the time just after t = 0. For this reason, it is customary to write 1 (denoting unit area) beside the arrow that is used to symbolize the unit impulse function, as in Fig. 7.27. The unit area is known as the strength of the impulse function. When an impulse function has a strength other than unity, the area of the impulse is equal to its strength. For example, an impulse function 10δ(t) has an area of 10. Figure 7(b) shows the impulse functions 5δ(t + 2),10δ(t), and −4δ(t − 3).

To illustrate how the impulse function affects other functions, let us evaluate the integral

$$\int_a^b f(t)\delta(t-t_0)\,dt \qquad (6.31)$$

where $a < t_0 < b$. Since $\delta(t - t_0) = 0$ except at $t = t_0$, the integrand is zero except at t_0. Thus,

$$\int_a^b f(t)\delta(t-t_0)\,dt = \int_a^b f(t_0)\delta(t-t_0)\,dt$$

$$= f(t_0)\int_a^b \delta(t-t_0)\,dt = f(t_0)$$

Or

$$\int_a^b f(t)\delta(t-t_0)\,dt = f(t_0) \qquad (6.32)$$

This shows that when a function is integrated with the impulse function, we obtain the value of the function at the point where the impulse occurs. This is a highly useful property of the impulse function known as the sampling or sifting property. The special case of Eq. (6.31) is for $t_0 = 0$. Then Eq. (6.32) becomes

$$\int_{0-}^{0+} f(t)\delta(t)\,dt = f(0) \qquad (6.33)$$

Unit Ramp Function

Integrating the unit step function u(t) results in the unit ramp function r(t); we write

$$r(t) = \int_{-\infty}^{t} u(t)\,dt = tu(t) \qquad (6.34)$$

Or

$$r(t) = \begin{cases} 0, & t \le 0 \\ t, & t \ge 0 \end{cases} \qquad (6.35)$$

The unit ramp function is zero for negative values of t and has a unit slope for positive values of t. Figure 8(a) shows the unit ramp function. In general, a ramp is a function that changes at a constant rate.

The unit ramp function may be delayed or advanced as shown in Fig. 8(b) and 8(c). For the delayed unit ramp function,

$$r(t-t_0) = \begin{cases} 0, & t \le t_0 \\ t-t_0, & t \ge t_0 \end{cases} \qquad (6.36)$$

And for the advanced unit ramp function,

$$r(t+t_0) = \begin{cases} 0, & t \le -t_0 \\ t-t_0, & t \ge -t_0 \end{cases} \qquad (6.37)$$

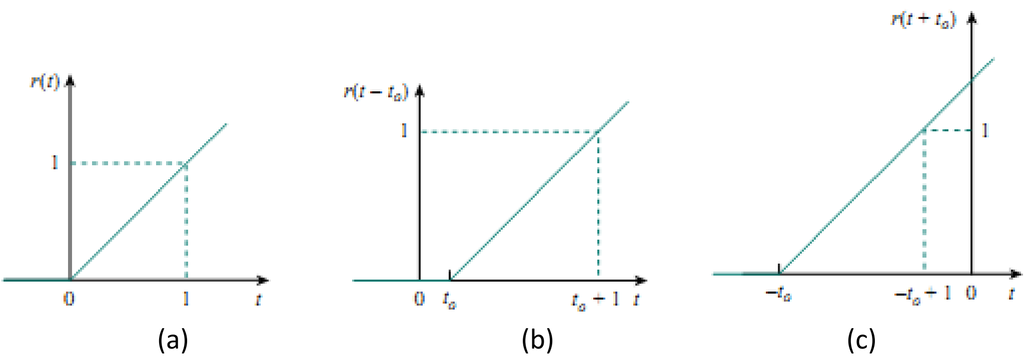


Figure 8 (a) A unit ramp function (b) A unit ramp function delayed by t_0 (c) A unit ramp function advanced by t_0

We should keep in mind that the three singularity functions (impulse, step, and ramp) are related by differentiation as

$$\delta(t) = \frac{du(t)}{dt}, \qquad u(t) = \frac{dr(t)}{dt} \tag{6.38}$$

Or by integration as

$$u(t) = \int_{-\infty}^{t} \delta(t)\,dt, \qquad r(t) = \int_{-\infty}^{t} u(t)\,dt \tag{6.39}$$

Example 7.6

Express the voltage pulse in Fig. 7.31 in terms of the unit step. Calculate its derivative and sketch it.

Solution:

The type of pulse in Fig. 7.31 is called the *gate function*. It may be regarded as a step function that switches on at one value of t and switches off at another value of t. The gate function shown in Fig. 7.31 switches on at $t = 2$ s and switches off at $t = 5$ s. It consists of the sum of two unit step functions as shown in Fig. 7.32(a). From the figure, it is evident that

$$v(t) = 10u(t-2) - 10u(t-5) = 10[u(t-2) - u(t-5)]$$

Taking the derivative of this gives

$$\frac{dv}{dt} = 10[\delta(t-2) - \delta(t-5)]$$

which is shown in Fig. 7.32(b). We can obtain Fig. 7.32(b) directly from Fig. 7.31 by simply observing that there is a sudden increase by 10 V at $t = 2$ s leading to $10\delta(t-2)$. At $t = 5$ s, there is a sudden decrease by 10 V leading to -10 V $\delta(t-5)$.

Gate functions are used along with switches to pass or block another signal.

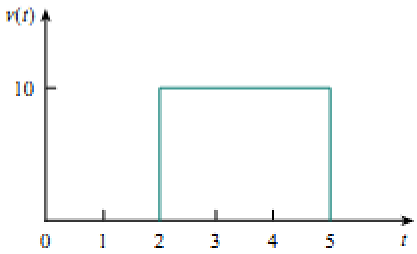
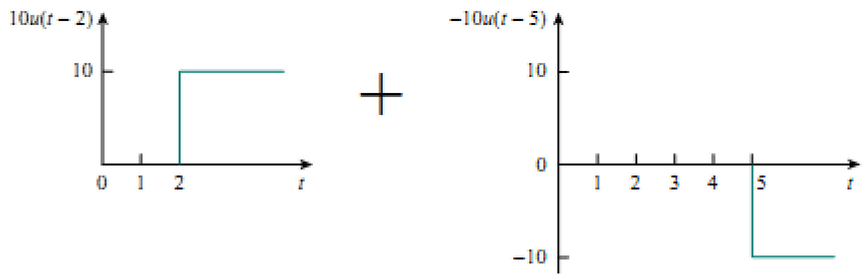
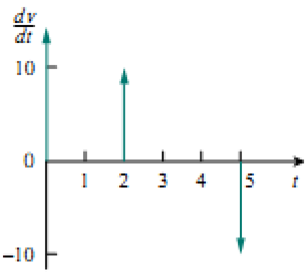


Figure 7.31 For Example 7.6.



Original



differentiated

Example 7.8

Given the signal

$$g(t) = \begin{cases} 3, & t < 0 \\ -2, & 0 < t < 1 \\ 2t - 4, & t > 1 \end{cases}$$

express $g(t)$ in terms of step and ramp functions.

Solution:

The signal $g(t)$ may be regarded as the sum of three functions specified within the three intervals $t < 0$, $0 < t < 1$, and $t > 1$.

For $t < 0$, $g(t)$ may be regarded as 3 multiplied by $u(-t)$, where $u(-t) = 1$ for $t < 0$ and 0 for $t > 0$. Within the time interval $0 < t < 1$, the function may be considered as -2 multiplied by a gated function $[u(t) - u(t-1)]$. For $t > 1$, the function may be regarded as $2t - 4$ multiplied by the unit step function $u(t-1)$. Thus,

$$\begin{aligned} g(t) &= 3u(-t) - 2[u(t) - u(t-1)] + (2t-4)u(t-1) \\ &= 3u(-t) - 2u(t) + (2t - 4 + 2)u(t-1) \\ &= 3u(-t) - 2u(t) + 2(t-1)u(t-1) \\ &= 3u(-t) - 2u(t) + 2r(t-1) \end{aligned}$$

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Step Response of an RC circuit

When the dc source of an RC circuit is suddenly applied, the voltage or current source can be modeled as a step function, and the response is known as a ***step response***.

The step response of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.

The step response is the response of the circuit due to a sudden application of a dc voltage or current source.

Consider the RC circuit in Fig. 9(a) which can be replaced by the circuit in Fig. 9(b), where V_S is a constant, dc voltage source. Again, we select the capacitor voltage as the circuit response to be determined. We assume an initial voltage V_0 on the capacitor, although this is not necessary for the step response.

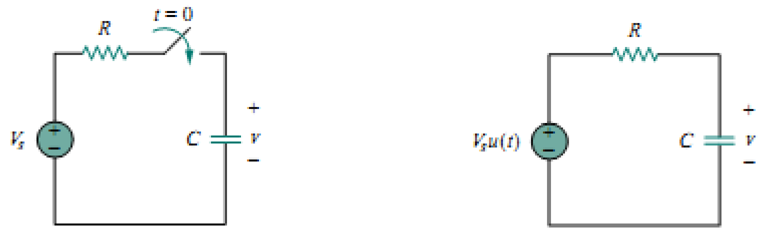


Figure 9 An RC circuit with voltage step input

Since the voltage of a capacitor cannot change instantaneously,

$$v(0^-) = v(0^+) = V_0 \qquad (6.40)$$

where $v(0^-)$ is the voltage across the capacitor just before switching and $v(0^+)$is its voltage immediately after switching. Applying KCL, we have

$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0$$

Or

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}u(t) \qquad (6.41)$$

where v is the voltage across the capacitor. For t> 0, Eq. (6.41) becomes

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC} \qquad (6.42)$$

Rearranging terms gives

$$\frac{dv}{dt} = -\frac{v - V_S}{RC}$$

Or

$$\frac{dv}{v - V_S} = -\frac{dt}{RC} \qquad (6.43)$$

Integrating both sides and introducing the initial conditions,

$$\ln(v - V_s)\Big|_{V_0}^{v(t)} = -\frac{t}{RC}\Big|_0^t$$

$$\ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

or

$$\ln\left(\frac{v - V_S}{V_0 - V_S}\right) = -\frac{t}{RC} \qquad (6.44)$$

Taking exponential on both sides

$$\frac{v - V_S}{V_0 - V_S} = e^{-\left(\frac{t}{\tau}\right)}, \qquad \tau = RC$$

$$v - V_S = (V_0 - V_S)e^{-\left(\frac{t}{\tau}\right)}$$

or

$$v(t) = V_S + (V_0 - V_S)e^{-\frac{t}{\tau}} \qquad (7.45)$$

Thus,

$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases} \qquad (6.46)$$

This is known as the complete response of the RC circuit to a sudden application of a dc voltage source, assuming the capacitor is initially charged. The reason for the term “complete” will become evident a little later. Assuming that $V_s > V_0$, a plot of $v(t)$ is shown in Fig. 10.

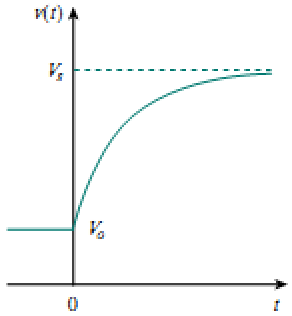


Figure 10 Response of an RC ccircuit with initially charged capacitor

If we assume that the capacitor is uncharged initially, we set $V_0 = 0$ in Eq. (6.46) so that

$$v(t) = \begin{cases} 0, & t < 0 \\ V_s(1 - e^{-t/\tau}), & t > 0 \end{cases} \qquad (6.47)$$

which can be written alternatively as

$$v(t) = V_S(1 - e^{-\frac{t}{\tau}} u(t) \qquad (6.48)$$

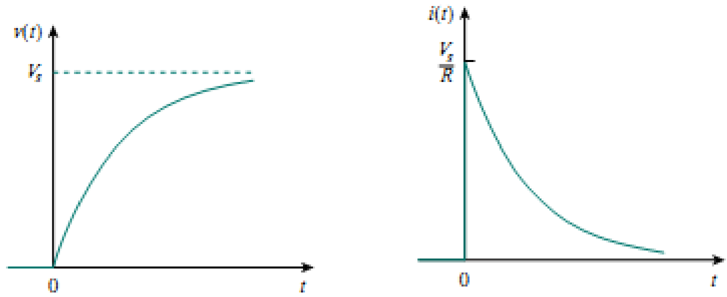
This is the complete step response of the RC circuit when the capacitor is initially uncharged. The current through the capacitor is obtained from Eq. (6.47) using $i(t) = Cdv/dt$. We get

$$i(t) = C\frac{dv}{dt} = \frac{C}{\tau}V_S e^{-\frac{t}{\tau}}, \qquad \tau = RC, \quad t > 0$$

or

$$i(t) = \frac{V_S}{R}e^{-\frac{t}{\tau}}u(t) \qquad (6.49)$$

Figure 11 shows the plots of capacitor voltage v(t)and capacitor current i(t)



(a) (b)

Figure 11 Step response of an RC circuit with initially uncharged capacitor: (a) voltage response, (b) current response.

Rather than going through the derivations above, there is a systematic approach—or rather, a short-cut method—for finding the step response of an RC or RL circuit. Let us reexamine Eq. (6.45), which is more general than Eq. (6.48). It is evident that $v(t)$ has two components. Thus, we may write

$$v = v_f + v_n \qquad (6.50)$$

where

$$v_f = V_S \qquad (6.51)$$

and

$$v_n = (V_0 - V_S)e^{-\frac{t}{\tau}} \qquad (6.52)$$

We know that vn is the natural response of the circuit, as discussed in the start of this lecture. Since this part of the response will decay to almost zero after five time constants, it is also called the *transient* response because it is a temporary response that will die out with time. Now, v_f is known as the *forced* response because it is produced by the circuit when an external "force" is applied (a voltage source in this case). It represents what the circuit is forced to do by the input excitation. It is also known as the *steady-state response*, because it remains a long time after the circuit is excited.

The natural response or transient response is the circuit's temporary response that will die out with time.

The forced response or steady-state response is the behavior of the circuit a long time after an external excitation is applied.

The complete response of the circuit is the sum of the natural response and the forced response. Therefore, we may write Eq. (6.45) as

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-\frac{t}{\tau}} \qquad (6.53)$$

where v(0) is the initial voltage at t = 0+ and v(∞) is the final or steady-state value. Thus, to find the step response of an RC circuit requires three things:

1. *The initial capacitor voltage* $v(0)$.
2. *The final capacitor voltage* $v(\infty)$.
3. *The time constant* τ.

We obtain item 1 from the given circuit for $t < 0$ and items 2 and 3 from the circuit for $t > 0$. Once these items are determined, we obtain the response using Eq. (6.53). This technique equally applies to RL circuits, as we shall see in the next part of the lecture.

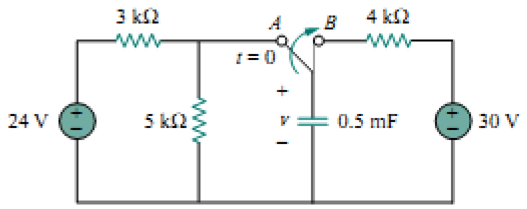
Note that if the switch changes position at time $t = t_0$ instead of at t = 0, there is a time delay in the response so that Eq. (6.53) becomes

$$v(t) = v(\infty) + [v(t_0) - v(\infty)]e^{-\frac{t-t_0}{\tau}} \qquad (6.54)$$

where $v(t_0)$ is the initial value at $t = t_0^+$. Keep in mind that Eq. (6.53) or (6.54) applies only to step responses, that is, when the input excitation is constant.

Example 7.10

The switch in Fig. 7.43 has been in position A for a long time. At $t = 0$, the switch moves to B. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1$ s and 4 s.



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Solution:

For $t < 0$, the switch is at position A. Since v is the same as the voltage across the 5-kΩ resistor, the voltage across the capacitor just before $t = 0$ is obtained by voltage division as

$$v(0^-) = \frac{5}{5+3}(24) = 15 \text{ V}$$

Using the fact that the capacitor voltage cannot change instantaneously,

$$v(0) = v(0^-) = v(0^+) = 15 \text{ V}$$

For $t > 0$, the switch is in position B. The Thevenin resistance connected to the capacitor is $R_{\text{Th}} = 4$ kΩ, and the time constant is

$$\tau = R_{\text{Th}}C = 4 \times 10^3 \times 0.5 \times 10^{-3} = 2 \text{ s}$$

Since the capacitor acts like an open circuit to dc at steady state, $v(\infty) =$ 30 V. Thus,

$$\begin{aligned} v(t) &= v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} \\ &= 30 + (15 - 30)e^{-t/2} = (30 - 15e^{-0.5t}) \text{ V} \end{aligned}$$

At $t = 1$,

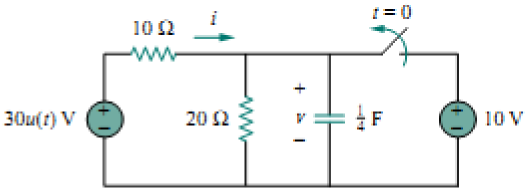
$$v(1) = 30 - 15e^{-0.5} = 20.902 \text{ V}$$

At $t = 4$,

$$v(4) = 30 - 15e^{-2} = 27.97 \text{ V}$$

Example 7.11

In Fig. 7.45, the switch has been closed for a long time and is opened at $t = 0$. Find i and v for all time.



Solution:

The resistor current i can be discontinuous at $t = 0$, while the capacitor voltage v cannot. Hence, it is always better to find v and then obtain i from v.

By definition of the unit step function,

$$30u(t) = \begin{cases} 0, & t < 0 \\ 30, & t > 0 \end{cases}$$

For $t < 0$, the switch is closed and $30u(t) = 0$, so that the $30u(t)$ voltage source is replaced by a short circuit and should be regarded as contributing nothing to v. Since the switch has been closed for a long time, the capacitor voltage has reached steady state and the capacitor acts like an open circuit. Hence, the circuit becomes that shown in Fig. 7.46(a) for $t < 0$. From this circuit we obtain

$$v = 10 \text{ V}, \qquad i = -\frac{v}{10} = -1 \text{ A}$$

Since the capacitor voltage cannot change instantaneously,

$$v(0) = v(0^-) = 10 \text{ V}$$

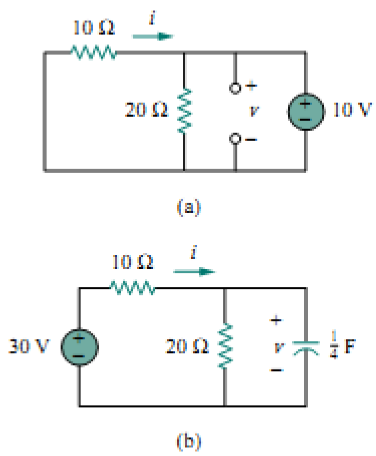


Figure 7.46 Solution of Example 7.11: (a) for $t < 0$, (b) for $t > 0$.

For $t > 0$, the switch is opened and the 10-V voltage source is disconnected from the circuit. The $30u(t)$ voltage source is now operative, so the circuit becomes that shown in Fig. 7.46(b). After a long time, the circuit reaches steady state and the capacitor acts like an open circuit again. We obtain $v(\infty)$ by using voltage division, writing

$$v(\infty) = \frac{20}{20+10}(30) = 20 \text{ V}$$

The Thevenin resistance at the capacitor terminals is

$$R_{\text{Th}} = 10 \parallel 20 = \frac{10 \times 20}{30} = \frac{20}{3}\ \Omega$$

and the time constant is

$$\tau = R_{\text{Th}}C = \frac{20}{3} \cdot \frac{1}{4} = \frac{5}{3} \text{ s}$$

Thus,

$$\begin{aligned} v(t) &= v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} \\ &= 20 + (10 - 20)e^{-(3/5)t} = (20 - 10e^{-0.6t}) \text{ V} \end{aligned}$$

To obtain i, we notice from Fig. 7.46(b) that i is the sum of the currents through the 20-Ω resistor and the capacitor; that is,

$$\begin{aligned} i &= \frac{v}{20} + C\frac{dv}{dt} \\ &= 1 - 0.5e^{-0.6t} + 0.25(-0.6)(-10)e^{-0.6t} = (1 + e^{-0.6t}) \text{ A} \end{aligned}$$

Notice from Fig. 7.46(b) that $v + 10i = 30$ is satisfied, as expected. Hence,

$$v = \begin{cases} 10 \text{ V}, & t < 0 \\ (20 - 10e^{-0.6t}) \text{ V}, & t \geq 0 \end{cases}$$

$$i = \begin{cases} -1 \text{ A}, & t < 0 \\ (1 + e^{-0.6t}) \text{ A}, & t > 0 \end{cases}$$

Notice that the capacitor voltage is continuous while the resistor current is not.

4. Alexander practice problem 7.10, 7.11

Step Response of an RL circuit

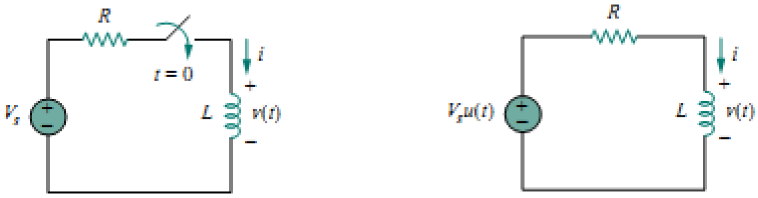
Consider the *RL* circuit in Fig. 12(a), which may be replaced by the circuit in Fig. 12(b). Again, our goal is to find the inductor current i as the circuit response. Rather than apply Kirchhoff's laws, we will use the simple technique in Eqs. (6.50) through (6.53). Let the response be the sum of the natural current and the forced current,

$$i = i_n + i_f \qquad (6.55)$$

We know that the natural response is always a decaying exponential, that is,

$$i_n = Ae^{-\frac{t}{\tau}}, \qquad \tau = \frac{L}{R} \qquad (6.56)$$

Where A is a constant to be determinbed.



(a) (b)

Figure 12 An RL circuit with a step input voltage

The forced response is the value of the current a long time after the switch in Fig. 12(a) is closed. We know that the natural response essentially dies out after five time constants. At that time, the inductor becomes a short circuit, and the voltage across it is zero. The entire source voltage V_S appears across R. Thus, the forced response is

$$i_f = \frac{V_S}{R} \qquad (6.57)$$

Substituting Eqs. (6.56) and (6.57) into Eq. (6.55) gives

$$i = Ae^{-\frac{t}{\tau}} + \frac{V_S}{R} \qquad (6.58)$$

We now determine the constant A from the initial value of i. Let I_0 be the initial current through the inductor, which may come from a source other than V_S. Since the current through the inductor cannot change instantaneously,

$$i(0^+) = i(0^-) = I_0 \qquad (6.59)$$

Thus at $t = 0$, Eq. 6.58 becomes

$$I_0 = A + \frac{V_S}{R}$$

From this, we obtain A as

$$A = I_0 - \frac{V_S}{R}$$

Substituting for A in eq. 6.58, we get

$$i(t) = \frac{V_S}{R} + \left(I_0 - \frac{V_S}{R}\right)e^{-\frac{t}{\tau}} \qquad (6.60)$$

This is the complete response of the RL circuit. It is illustrated in Fig. 13.

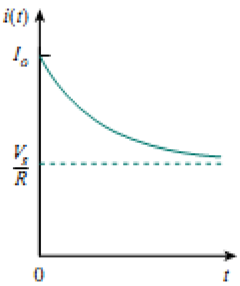


Figure 13 The total Response of the RL circuit with initial inductor current I_0

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The response in Eq. (6.60) may be written as

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-\frac{t}{\tau}} \qquad (6.61)$$

where $i(0)$ *and* $i(\infty)$ are the initial and final values of i. Thus, to find the step response of an RL circuit requires three things:

1. *The initial inductor current i(0) at t = 0^+ .*
2. *The final inductor current $i(\infty)$.*
3. *The time constant τ.*

We obtain item 1 from the given circuit for t< 0 and items 2 and 3 from the circuit for t> 0. Once these items are determined, we obtain the response using Eq. (6.61). Keep in mind that this technique applies only for step responses.

Again, if the switching takes place at time t = t_0 instead of t = 0, Eq. (6.61) becomes

$$i(t) = i(\infty) + [i(t_0) - i(\infty)]e^{-\frac{t-t_0}{\tau}} \qquad (6.62)$$

If $I_0 = 0$, then

$$i(t) = \begin{cases} 0, & t < 0 \\ \frac{V_s}{R}(1 - e^{-t/\tau}), & t > 0 \end{cases} \qquad (6.63a)$$

Or

$$i(t) = \frac{V_s}{R}(1 - e^{-t/\tau})u(t) \qquad (6.63b)$$

This is the step response of the RLcircuit. The voltage across the inductor is obtained from Eq. (6.63) using $v = L\, di/dt$. We get

$$v(t) = L\frac{di}{dt} = \frac{V_S L}{\tau R}e^{-\frac{t}{\tau}}, \qquad \tau = \frac{L}{R}, \qquad t > 0$$

Or

$$v(t) = V_S e^{-\frac{t}{\tau}}u(t) \qquad (6.64)$$

Figure 14 shows the step responses in Eqs. (6.63) and (6.64).

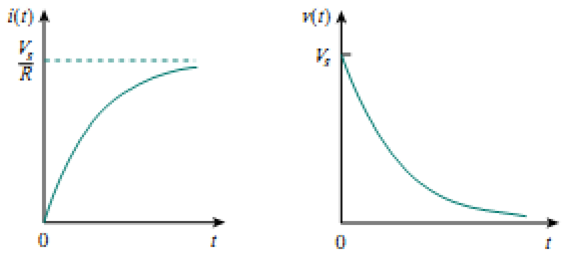

Figure 14 Step responses of an RL circuit with no initial inductor current: (a) current response, (b) voltage response.

Example 7.12

Find $i(t)$ in the circuit in Fig. 7.51 for $t > 0$. Assume that the switch has been closed for a long time.

Solution:

When $t < 0$, the 3-Ω resistor is short-circuited, and the inductor acts like a short circuit. The current through the inductor at $t = 0^-$ (i.e., just before $t = 0$) is

$$i(0^-) = \frac{10}{2} = 5 \text{ A}$$

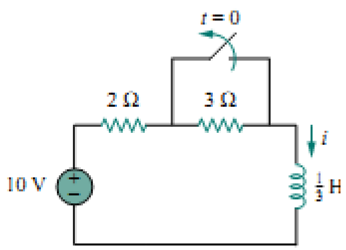

Figure 7.51 For Example 7.12.

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Since the inductor current cannot change instantaneously,

$$i(0) = i(0^+) = i(0^-) = 5 \text{ A}$$

When $t > 0$, the switch is open. The 2-Ω and 3-Ω resistors are in series, so that

$$i(\infty) = \frac{10}{2+3} = 2 \text{ A}$$

The Thevenin resistance across the inductor terminals is

$$R_{Th} = 2 + 3 = 5\ \Omega$$

For the time constant,

$$\tau = \frac{L}{R_{Th}} = \frac{\frac{1}{3}}{5} = \frac{1}{15} \text{ s}$$

Thus,

$$\begin{aligned} i(t) &= i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} \\ &= 2 + (5-2)e^{-15t} = 2 + 3e^{-15t} \text{ A}, \qquad t > 0 \end{aligned}$$

Check: In Fig. 7.51, for $t > 0$, KVL must be satisfied; that is,

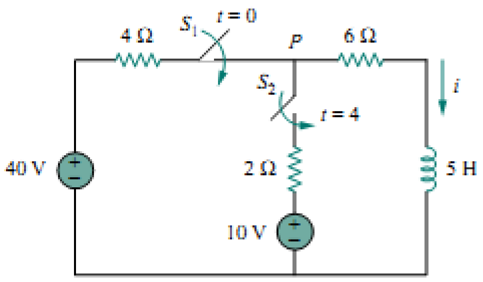
$$10 = 5i + L\frac{di}{dt}$$

$$5i + L\frac{di}{dt} = [10 + 15e^{-15t}] + \left[\frac{1}{3}(3)(-15)e^{-15t}\right] = 10$$

This confirms the result.

Example 7.13

At $t = 0$, switch 1 in Fig. 7.53 is closed, and switch 2 is closed 4 s later. Find $i(t)$ for $t > 0$. Calculate i for $t = 2$ s and $t = 5$ s.



Solution:

We need to consider the three time intervals $t \le 0$, $0 \le t \le 4$, and $t \ge 4$ separately. For $t < 0$, switches S_1 and S_2 are open so that $i = 0$. Since the inductor current cannot change instantly,

$$i(0^-) = i(0) = i(0^+) = 0$$

For $0 \le t \le 4$, S_1 is closed so that the 4-Ω and 6-Ω resistors are in series. Hence, assuming for now that S_1 is closed forever,

$$i(\infty) = \frac{40}{4+6} = 4 \text{ A}, \qquad R_{Th} = 4 + 6 = 10\ \Omega$$

$$\tau = \frac{L}{R_{Th}} = \frac{5}{10} = \frac{1}{2} \text{ s}$$

Thus,

$$\begin{aligned} i(t) &= i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} \\ &= 4 + (0-4)e^{-2t} = 4(1 - e^{-2t}) \text{ A}, \qquad 0 \le t \le 4 \end{aligned}$$

For $t \geq 4$, S_2 is closed; the 10-V voltage source is connected, and the circuit changes. This sudden change does not affect the inductor current because the current cannot change abruptly. Thus, the initial current is

$$i(4) = i(4^-) = 4(1 - e^{-8}) \simeq 4 \text{ A}$$

To find $i(\infty)$, let v be the voltage at node P in Fig. 7.53. Using KCL,

$$\frac{40 - v}{4} + \frac{10 - v}{2} = \frac{v}{6} \quad \Longrightarrow \quad v = \frac{180}{11} \text{ V}$$

$$i(\infty) = \frac{v}{6} = \frac{30}{11} = 2.727 \text{ A}$$

The Thevenin resistance at the inductor terminals is

$$R_{\text{Th}} = 4 \parallel 2 + 6 = \frac{4 \times 2}{6} + 6 = \frac{22}{3} \, \Omega$$

and

$$\tau = \frac{L}{R_{\text{Th}}} = \frac{5}{\frac{22}{3}} = \frac{15}{22} \text{ s}$$

Hence,

$$i(t) = i(\infty) + [i(4) - i(\infty)]e^{-(t-4)/\tau}, \qquad t \geq 4$$

We need $(t - 4)$ in the exponential because of the time delay. Thus,

$$i(t) = 2.727 + (4 - 2.727)e^{-(t-4)/\tau}, \qquad \tau = \frac{15}{22}$$
$$= 2.727 + 1.273e^{-1.4667(t-4)}, \qquad t \geq 4$$

Putting all this together,

$$i(t) = \begin{cases} 0, & t \leq 0 \\ 4(1 - e^{-2t}), & 0 \leq t \leq 4 \\ 2.727 + 1.273e^{-1.4667(t-4)}, & t \geq 4 \end{cases}$$

At $t = 2$,

$$i(2) = 4(1 - e^{-4}) = 3.93 \text{ A}$$

At $t = 5$,

$$i(5) = 2.727 + 1.273e^{-1.4667} = 3.02 \text{ A}$$

5. Alexander Practice problems 7.12, 7.13