Thus far our analysis has been limited to using DC only. In everyday life the mains supply is AC, so it is vital to know about AC. This lecture mainly focuses on AC.

A ***direct current*** (dc) is a current that remains constant with time. E.g. Batteries, solar cell, etc

An ***alternating current*** (ac) is a current that varies sinusoidally with time. E.g. mains supply, generator etc.

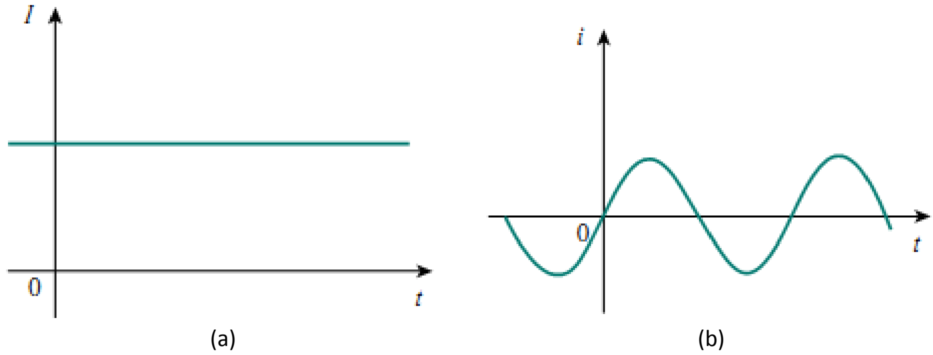

Figure 1 Two common types of current: (a) direct current (dc), (b) alternating current (ac).

We now begin the analysis of circuits in which the source voltage or current is time-varying. In this chapter, we are particularly interested in sinusoidally time-varying excitation or simply, excitation by a sinusoid.

A sinusoid is a signal that has the form of the sine or cosine function.

A sinusoidal current is usually referred to as alternating current (ac). Such a current reverses at regular time intervals and has alternately positive and negative values. Circuits driven by sinusoidal current or voltage sources are called ac circuits.

We begin with a basic discussion of sinusoids and phasors. We then introduce the concepts of impedance and admittance. The basic circuit laws, Kirchhoff's and Ohm's, introduced for dc circuits, will be applied to ac circuits.

SINUSOIDS

Consider the sinusoidal voltage

$$v(t) = V_m \, sin\, \omega t \qquad (7.1)$$

Here
V_m = The amplitude of the sinusoid
ω = The angular frequency in radians/s
ωt = The argument of the sinusoid

The sinusoid is shown in Fig. 9.1(a) as a function of its argument and in Fig. 2(b) as a function of time. It is evident that the sinusoid repeats itself every T seconds; thus, T is called the period of the sinusoid. From the two plots in Fig. 2, we observe that ωT = 2π,

$$T = \frac{2\pi}{\omega} \qquad (7.2)$$

The fact that V(t) repeats itself every T seconds is shown by replacing t by t + T in Eq. (7.1). We get

$$v(t+T) = V_m \sin\omega(t+T) = V_m \sin\omega\left(t+\frac{2\pi}{\omega}\right) = V_m \sin(\omega t + 2\pi) = V_m \sin\omega t = v(t) \qquad (7.3)$$

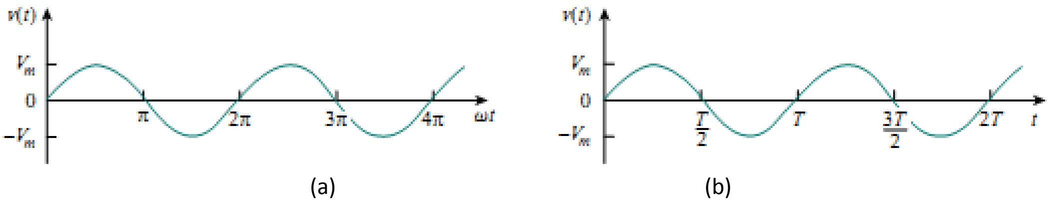

Figure 2 A sketch of V_m sin ωt: (a) as a function of ωt, (b) as a function of t.

Hence,

$$v(t+T) = v(t) \qquad (7.4)$$

I.e. , v has the same value at t + T as it does at t and v(t)is said to be periodic. In general,

A periodic function is one that satisfies f (t) = f (t + nT), for all t and for all integers n.

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As mentioned, the period T of the periodic function is the time of one complete cycle or the number of seconds per cycle. The reciprocal of this quantity is the number of cycles per second, known as the cyclic frequency f of the sinusoid. Thus,

$$f = \frac{1}{T} \qquad (7.5)$$

From Eqs. (7.2) and (7.5), it is clear that

$$\omega = 2\pi f \qquad (7.6)$$

While ω is in radians per second (rad/s), f is in hertz (Hz).

Let us now consider a more general expression for the sinusoid,

$$v(t) = V_m \sin(\omega t + \varphi) \qquad (7.7)$$

where $(\omega t + \varphi)$ is the argument and φ is the phase. Both argument and phase can be in radians or degrees.

Let us examine the two sinusoids

$$v_1(t) = V_m \sin \omega t \quad and \quad v_2(t) = V_m \sin(\omega t + \varphi) \qquad (7.8)$$

shown in Fig. 3. The starting point of v_2 in Fig. 3 occurs first in time. Therefore, we say that v_2 leads v_1 by φ or that v_1 lags v_2 by φ.If $\varphi \neq 0$, we also say that v_1 and v_2 are out of phase. If $\varphi = 0$, then v_1 and v_2 are said to be in phase; they reach their minima and maxima at exactly the same time. We can compare v_1 and v_2 in this manner because they operate at the same frequency; they do not need to have the same amplitude.

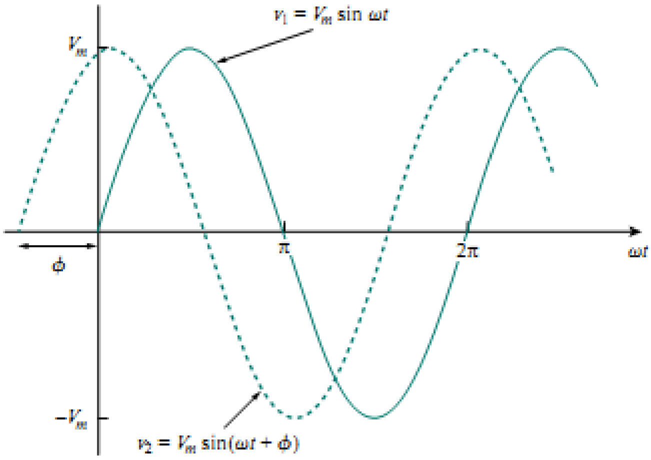

Figure 3 Two sinusoids with different phases

A sinusoid can be expressed in either sine or cosine form. When comparing two sinusoids, it is expedient to express both as either sine or cosine with positive amplitudes. This is achieved by using the following trigonometric identities:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \qquad (7.9)$$

With these identities, it is easy to show that

$$\sin(\omega t \pm 180^\circ) = -\sin \omega t$$
$$\cos(\omega t \pm 180^\circ) = -\cos \omega t$$
$$\sin(\omega t \pm 90^\circ) = \pm \cos \omega t$$
$$\cos(\omega t \pm 90^\circ) = \mp \sin \omega t \qquad (7.10)$$

Using these relationships, we can transform a sinusoid from sine form to cosine form or vice versa.

A graphical approach may be used to relate or compare sinusoids as an alternative to using the trigonometric identities in Eqs. (7.9) and (7.10). Consider the set of axes shown in Fig. 4(a). The horizontal axis represents the magnitude of cosine, while the vertical axis (pointing down) denotes the magnitude of sine. Angles are measured positively counterclockwise from the horizontal, as usual in polar coordinates. This graphical technique can be used to relate two sinusoids. For example, we see in Fig. 4(a) that subtracting 90° from the argument of $cos\ \omega t$ gives $sin\ \omega t$, or $\boldsymbol{cos(\omega t - 90°) = sin\ \omega t}$. Similarly, adding 180◦ to the argument of $sin\ \omega t$ gives $-\ sin\ \omega t$, or $\boldsymbol{sin(\omega t - 180°) = -\ sin\ \omega t}$, as shown in Fig. 4(b).

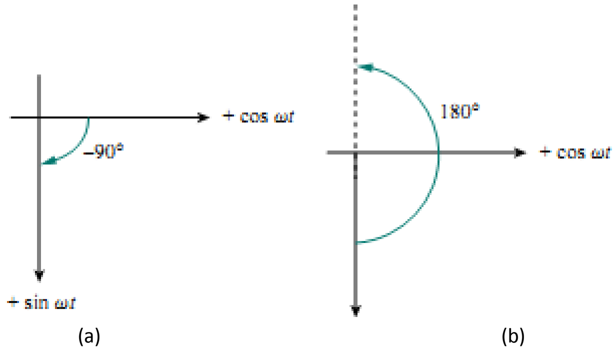

Figure 4 A graphical means of relating cosine and sine: (a) cos(ωt − 90◦) = sin ωt, (b) sin(ωt + 180◦) =− sin ωt.

The graphical technique can also be used to add two sinusoids of the same frequency when one is in sine form and the other is in cosine form. To add $Acos\ \omega t$ and $Bsin\ \omega t$, we note that A is the magnitude of $cos\ \omega t$ while B is the magnitude of $sin\ \omega t$, as shown in Fig. 5(a). The magnitude and argument of the resultant sinusoid in cosine form is readily obtained from the triangle. Thus,

$$Acos\ \omega t + Bsin\ \omega t = Ccos(\omega t - \theta) \qquad (7.11)$$

Where

$$C = \sqrt{A^2 + B^2}, \qquad \theta = \tan^{-1}\frac{B}{A} \qquad (7.12)$$

For example, we may add 3 $cos\ \omega t\ and\ -4\ sin\ \omega t$ as shown in Fig. 5(b) and obtain

$$3\cos\omega t - 4\sin\omega t = 5\ cos(\omega t + 53.1°) \qquad (7.13)$$

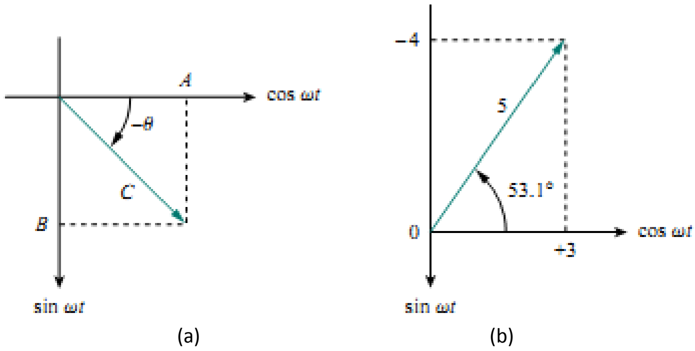

Figure 5 (a) Adding $Acos\ \omega t$ and $Bsin\ \omega t$,(b) adding 3 $cos\ \omega t$ and $-4\ sin\ \omega t$.

Compared with the trigonometric identities in Eqs. (7.9) and (7.10), the graphical approach eliminates memorization. However, we must not confuse the sine and cosine axes with the axes for complex numbers to be discussed in the next section. Something else to note in Figs. 4 and 5 is that although the natural tendency is to have the vertical axis point up, the positive direction of the sine function is down in the present case.

Another important characteristic of the sinusoidal voltage (or current) is its rms value. The rms value of a periodic function is defined as the square root of the mean value of the squared function. Hence, if $v = V_m cos\ (\omega t + \varphi)$, the rms value of v is

$$V_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2\cos^2(\omega t + \varphi)\,dt}$$

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Note from the equation that we obtain the mean value of the squared voltage by integrating v^2 over one period (that is, from t_0 to $t_0 + T$) and then dividing by the range of integration, T. Note further that the starting point for the integration t_0 is arbitrary.

THe quantity under the radical sign is reduced to $V_m^2/2$. Hence the RMS value of v is

$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

Example 9.1

Find the amplitude, phase, perios, and frequency of the sinusoid $v(t) = 12\cos(50t + 10°)$

Solution:

The amplitude is $V_m = 12$ V.

The phase is $\phi = 10°$.

The angular frequency is $\omega = 50$ rad/s.

The period $T = \frac{2\pi}{\omega} = \frac{2\pi}{50} = 0.1257$ s.

The frequency is $f = \frac{1}{T} = 7.958$ Hz.

Example 9.2

Calculate the phase angle between $v_1 = -10\cos(\omega t + 50°)$ and $v_2 = 12\sin(\omega t - 10°)$. State which sinusoid is leading.

Solution:

Let us calculate the phase in three ways. The first two methods use trigonometric identities, while the third method uses the graphical approach.

METHOD 1 In order to compare v_1 and v_2, we must express them in the same form. If we express them in cosine form with positive amplitudes,

$$v_1 = -10\cos(\omega t + 50°) = 10\cos(\omega t + 50° - 180°)$$
$$v_1 = 10\cos(\omega t - 130°) \quad \text{or} \quad v_1 = 10\cos(\omega t + 230°) \tag{9.2.1}$$

and

$$v_2 = 12\sin(\omega t - 10°) = 12\cos(\omega t - 10° - 90°)$$
$$v_2 = 12\cos(\omega t - 100°) \tag{9.2.2}$$

It can be deduced from Eqs. (9.2.1) and (9.2.2) that the phase difference between v_1 and v_2 is 30°. We can write v_2 as

$$v_2 = 12\cos(\omega t - 130° + 30°) \quad \text{or} \quad v_2 = 12\cos(\omega t + 260°) \tag{9.2.3}$$

Comparing Eqs. (9.2.1) and (9.2.3) shows clearly that v_2 leads v_1 by 30°.

METHOD 2 Alternatively, we may express v_1 in sine form:

$$v_1 = -10\cos(\omega t + 50°) = 10\sin(\omega t + 50° - 90°)$$
$$= 10\sin(\omega t - 40°) = 10\sin(\omega t - 10° - 30°)$$

But $v_2 = 12\sin(\omega t - 10°)$. Comparing the two shows that v_1 lags v_2 by 30°. This is the same as saying that v_2 leads v_1 by 30°.

METHOD 3 We may regard v_1 as simply $-10\cos\omega t$ with a phase shift of +50°. Hence, v_1 is as shown in Fig. 9.5. Similarly, v_2 is $12\sin\omega t$ with a phase shift of −10°, as shown in Fig. 9.5. It is easy to see from Fig. 9.5 that v_2 leads v_1 by 30°, that is, 90° − 50° − 10°.

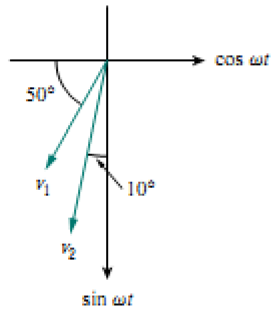

Figure 9.5 For Example 9.2.

1. Alexander Practice problem 9.1,9.2

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Phasors

Sinusoids are easily expressed in terms of phasors, which are more convenient to work with than sine and cosine functions.

A phasor is a complex number that represents the amplitude and phase of a sinusoid.

A complex number z can be written in rectangular form as

$$z = x + jy \qquad (7.14a)$$

where j = $\sqrt{-1}$; x is the real part of z; y is the imaginary part of z. In this context, the variables x and y do not represent a location as in two-dimensional vector analysis but rather the real and imaginary parts of z in the complex plane. Nevertheless, we note that there are some resemblances between manipulating complex numbers and manipulating two-dimensional vectors.

The complex number z can also be written in polar or exponential form as

$$z = r\angle\phi = re^{j\phi} \qquad (7.14b)$$

where r is the magnitude of z, and ϕ is the phase of z. We notice that z can be represented in three ways:

$z = x + jy$	Rectangular form	
$z = r\angle\phi = re^{j\phi}$	Polar form	(7.15)
$z = re^{j\varphi}$	Exponential form	

The relationship between the rectangular form and the polar form is shown in Fig. 6, where the x axis represents the real part and the y axis represents the imaginary part of a complex number. Given x and y, we can get r and φ as

$$r = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1}\frac{y}{x} \qquad (7.16a)$$

On the other hand, if we know r and φ, we can obtain x and y as

$$x = rcos\,\varphi, \qquad y = rsin\,\varphi \qquad (7.16b)$$

Thus, z may be written as

$$z = x + jy = r\angle\varphi = r(cos\,\varphi + j\,sin\,\varphi) \qquad (7.17)$$

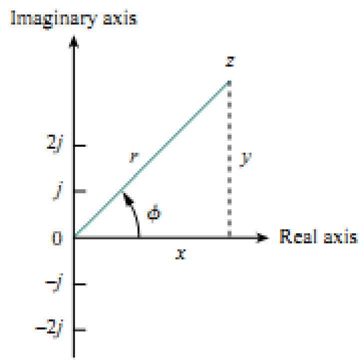

Figure 6 Representation of a complex number $z = x + jy = r\angle\phi$

Addition and subtraction of complex numbers are better performed in rectangular form; multiplication and division are better done in polar form. Given the complex numbers

$$z = x + jy = r\angle\phi, \qquad z_1 = x_1 + jy_1 = r_1\angle\phi_1$$
$$z_2 = x_2 + jy_2 = r_2\angle\phi_2$$

The following operations are important

Addition	$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$	*(7.18a)*
Subtraction	$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$	*(7.18b)*
Multiplication	$z_1 z_2 = r_1 r_2 \angle\phi_1 + \phi_2$	*(7.18c)*
Division	$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle\phi_1 - \phi_2$	*(7.18d)*

Reciprocal $\frac{1}{z} = \frac{1}{r}\angle -\phi$ *(7.18e)*

Square root $\sqrt{z} = \sqrt{r}\angle \phi/2$ *(7.18f)*

Conjugate $z^* = x - jy = r\angle -\phi = re^{-j\phi}$ *(7.18g)*

Note that from equation 7.18e,

$$\frac{1}{j} = -j \qquad (7.18h)$$

These are the basic properties of complex numbers we need.

The idea of phasor representation is based on Euler's identity. In general,

$$e^{\pm j\varphi} = cos\ \varphi \pm j\ sin\ \varphi \qquad (7.19)$$

which shows that we may regard $cos\ \varphi$ and $sin\ \varphi$ as the real and imaginary parts of ejϕ; we may write

$$cos\ \varphi = Re(e^{j\varphi}) \qquad (7.20a)$$

$$sin\ \varphi = Im(e^{j\varphi}) \qquad (7.20b)$$

where Re and Im stand for the real part of and the imaginary part of. Given a sinusoid $v(t) = V_m\ cos(\omega t + \varphi)$, we use Eq. (7.20a) to express $v(t)$as

$$v(t) = V_m\ cos(\omega t + \varphi) = Re(V_m e^{j(\omega t+\varphi)} \qquad (7.21)$$

Or

$$v(t) = Re\left(V_m e^{j\varphi}\ e^{j\omega t}\right) \qquad (7.22)$$

Thus,

$$v(t) = Re\left(Ve^{j\omega t}\right) \qquad (7.23)$$

Where

$$V = V_m e^{j\varphi} = V_m\angle\phi \qquad (7.24)$$

V is thus the phasor representation of the sinusoid $v(t)$, as we said earlier. In other words, a phasor is a complex representation of the magnitude and phase of a sinusoid. Either Eq. (7.20a) or Eq. (7.20b) can be used to develop the phasor, but the standard convention is to use Eq. (7.20a).

A phasor may be regarded as a mathematical equivalent of a sinusoid with the time dependence dropped.

One way of looking at Eqs. (7.23) and (7.24) is to consider the plot of the sinor $Ve^{j\omega t} = V_m e^{j(\omega t+\varphi)}$ on the complex plane. As time increases, the sinor rotates on a circle of radius V_m at an angular velocity ω in the counterclockwise direction, as shown in Fig. 7(a). In other words, the entire complex plane is rotating at an angular velocity of ω. We may regard v(t)as the projection of the sinor Vejωt on the real axis, as shown in Fig. 7(b). The value of the sinor at time t = 0 is the phasor V of the sinusoid v(t). The sinor may be regarded as a rotating phasor. Thus, whenever a sinusoid is expressed as a phasor, the term $e^{j\omega t}$ is implicitly present. It is therefore important, when dealing with phasors, to keep in mind the frequency ω of the phasor; otherwise we can make serious mistakes.

If we use sine for the phasor instead of cosine, then $v\ (t\ ) = V_m sin\ (\omega t + \varphi) = I_m\,(V_m e^{j(\omega t+\varphi)})$ and the corresponding phasor is the same as that in Eq. (7.24).

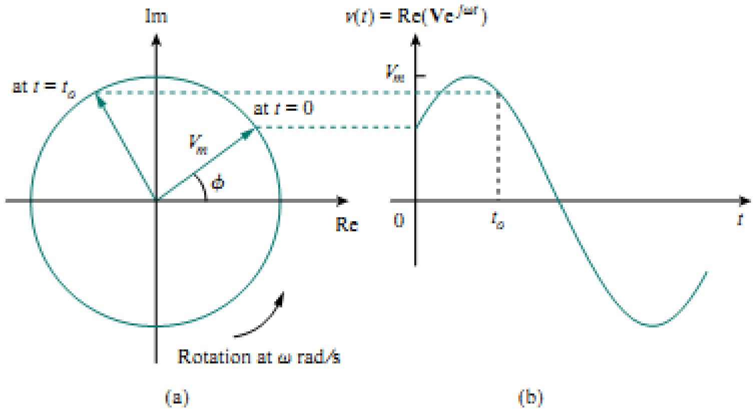

Figure 7 Representation of $Ve^{j\omega t}$: (a) sinor rotating counterclockwise, (b) its projection on the real axis, as a function of time.

Equation (7.23) states that to obtain the sinusoid corresponding to a given phasor V, multiply the phasor by the time factor $e^{j\omega t}$ and take the real part. As a complex quantity, a phasor may be expressed in rectangular form, polar form, or exponential form. Since a phasor has magnitude and phase ("direction"), it behaves as a vector and is printed in boldface. For example, phasors $\boldsymbol{V = V_m\angle\varphi}$ ***and*** $\boldsymbol{I = I_m\angle -\theta}$ are graphically represented in Fig. 8. Such a graphical representation of phasors is known as a ***phasor diagram***.

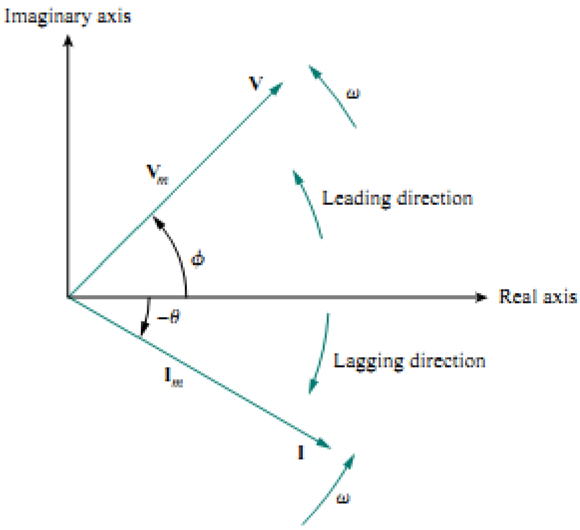

Figure 8 A phasor diagram showing $V = V_m\angle\varphi$ *and* $\boldsymbol{I = I_m\angle -\theta}$.

Equations (7.21) through (7.23) reveal that to get the phasor corresponding to a sinusoid, we first express the sinusoid in the cosine form so that the sinusoid can be written as the real part of a complex number. Then we take out the time factor $e^{j\omega t}$, and whatever is left is the phasor corresponding to the sinusoid. By suppressing the time factor, we transform the sinusoid from the time domain to the phasor domain. This transformation is summarized as follows:

$$\underset{\text{(Time-domain representation)}}{v(t) = V_m\cos(\omega t + \phi)} \quad \Longleftrightarrow \quad \underset{\text{(Phasor-domain representation)}}{\mathbf{V} = V_m\angle\phi} \qquad (7.25)$$

Given a sinusoid v(t) = Vm cos(ωt + ϕ), we obtain the corresponding phasor as $V = Vm\angle\varphi$. Equation (7.25) is also demonstrated in Table 1, where the sine function is considered in addition to the cosine function. From Eq. (7.25), we see that to get the phasor representation of a sinusoid, we express it in cosine form and take the magnitude and phase. Given a phasor, we obtain the time-domain representation as the cosine function with the same magnitude as the phasor and the argument as ωt plus the phase of the phasor. The idea of expressing information in alternate domains is fundamental to all areas of engineering.

Table 1 Sinusoid Phasor Transformation

Time-domain representation	Phasor-domain representation
$V_m \cos(\omega t + \phi)$	$V_m \angle \phi$
$V_m \sin(\omega t + \phi)$	$V_m \angle \phi - 90°$
$I_m \cos(\omega t + \theta)$	$I_m \angle \theta$
$I_m \sin(\omega t + \theta)$	$I_m \angle \theta - 90°$

Note that in Eq. (7.25) the frequency (or time) factor $e^{j\omega t}$ is suppressed, and the frequency is not explicitly shown in the phasor-domain representation because ω is constant. However, the response depends on ω. For this reason, the phasor domain is also known as the ***frequency domain***.

From Eqs. (7.23) and (7.24), $v(t) = Re(Ve^{j\omega t}) = V_m cos\,(\omega t + \varphi)$, so that

$$\frac{dv}{dt} = -\omega V_m \sin(\omega t + \varphi) = \omega V_m \cos(\omega t + \varphi + 90°) = \mathrm{Re}\big(\omega V_m e^{j\omega t} e^{j\varphi} e^{j90°}\big)$$
$$= \mathrm{Re}\big(j\omega V e^{j\omega t}\big) \qquad (7.26)$$

This shows that the derivative $v(t)$ is transformed to the phasor domain as **jωV**

$$\underset{\text{(Time domain)}}{\frac{dv}{dt}} \quad \Longleftrightarrow \quad \underset{\text{(Phasor domain)}}{j\omega \mathbf{V}} \qquad (7.27)$$

Similarly, the integral of $v(t)$ is transformed to the phasor domain as $V/j\omega$

$$\underset{\text{(Time domain)}}{\int v\,dt} \quad \Longleftrightarrow \quad \underset{\text{(Phasor domain)}}{\frac{\mathbf{V}}{j\omega}} \qquad (7.28)$$

Equation (7.27) allows the replacement of a derivative with respect to time with multiplication of jω in the phasor domain, whereas Eq. (7.28) allows the replacement of an integral with respect to time with division by jω in the phasor domain. Equations (7.27) and (7.28) are useful in finding the steady-state solution, which does not require knowing the initial values of the variable involved. This is one of the important applications of phasors.

Besides time differentiation and integration, another important us of phasors is found in summing sinusoids of the same frequency. This best illustrated with an example, and Example 9.6 provides one.

The differences between v(t)and V should be emphasized:

1. $v(t)$is the instantaneous or time-domain representation, while V is the frequency or phasor-domain representation.
2. $v(t)$is time dependent, while V is not. (This fact is often forgotten by students.)
3. $v(t)$is always real with no complex term, while V is generally complex.

Finally, **we should bear in mind that phasor analysis applies only when frequency is constant; it applies in manipulating two or more sinusoidal signals only if they are of the same frequency.**

Example 9.3

Evaluate these complex numbers:

(a) $(40\angle 50° + 20\angle -30°)^{1/2}$

(b) $\dfrac{10\angle -30° + (3 - j4)}{(2 + j4)(3 - j5)^*}$

Solution:

(a) Using polar to rectangular transformation,

$$40\angle 50° = 40(\cos 50° + j\sin 50°) = 25.71 + j30.64$$

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$$20\angle -30^\circ = 20[\cos(-30^\circ) + j\sin(-30^\circ)] = 17.32 - j10$$

Adding them up gives

$$40\angle 50^\circ + 20\angle -30^\circ = 43.03 + j20.64 = 47.72\angle 25.63^\circ$$

Taking the square root of this,

$$(40\angle 50^\circ + 20\angle -30^\circ)^{1/2} = 6.91\angle 12.81^\circ$$

(b) Using polar-rectangular transformation, addition, multiplication, and division,

$$\frac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*} = \frac{8.66 - j5 + (3 - j4)}{(2 + j4)(3 + j5)}$$

$$= \frac{11.66 - j9}{-14 + j22} = \frac{14.73\angle -37.66^\circ}{26.08\angle 122.47^\circ}$$

$$= 0.565\angle -160.31^\circ$$

Example 9.4

Transform these sinusoids to phasors:

(a) $v = -4\sin(30t + 50^\circ)$

(b) $i = 6\cos(50t - 40^\circ)$

Solution:

(a) Since $-\sin A = \cos(A + 90^\circ)$,

$$v = -4\sin(30t + 50^\circ) = 4\cos(30t + 50^\circ + 90^\circ)$$
$$= 4\cos(30t + 140^\circ)$$

The phasor form of v is

$$\mathbf{V} = 4\angle 140^\circ$$

(b) $i = 6\cos(50t - 40^\circ)$ has the phasor

$$\mathbf{I} = 6\angle -40^\circ$$

Example 9.5

Find the sinusoids represented by these phasors:

(a) $\mathbf{V} = j8e^{-j20^\circ}$

(b) $\mathbf{I} = -3 + j4$

Solution:

(a) Since $j = 1\angle 90^\circ$,

$$\mathbf{V} = j8\angle -20^\circ = (1\angle 90^\circ)(8\angle -20^\circ)$$
$$= 8\angle 90^\circ - 20^\circ = 8\angle 70^\circ \text{ V}$$

Converting this to the time domain gives

$$v(t) = 8\cos(\omega t + 70^\circ) \text{ V}$$

(b) $\mathbf{I} = -3 + j4 = 5\angle 126.87^\circ$. Transforming this to the time domain gives

$$i(t) = 5\cos(\omega t + 126.87^\circ) \text{ A}$$

Example 9.6

Given $i_1(t) = 4\cos(\omega t + 30^\circ)$ and $i_2(t) = 5\sin(\omega t - 20^\circ)$, find their sum.

Solution:

Here is an important use of phasors—for summing sinusoids of the same frequency. Current $i_1(t)$ is in the standard form. Its phasor is

$$\mathbf{I}_1 = 4\angle 30^\circ$$

We need to express $i_2(t)$ in cosine form. The rule for converting sine to cosine is to subtract 90°. Hence,

$$i_2 = 5\cos(\omega t - 20^\circ - 90^\circ) = 5\cos(\omega t - 110^\circ)$$

and its phasor is

$$\mathbf{I}_2 = 5\underline{/-110^\circ}$$

If we let $i = i_1 + i_2$, then

$$\begin{aligned}\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 &= 4\underline{/30^\circ} + 5\underline{/-110^\circ}\\ &= 3.464 + j2 - 1.71 - j4.698 = 1.754 - j2.698\\ &= 3.218\underline{/-56.97^\circ}\text{ A}\end{aligned}$$

Transforming this to the time domain, we get

$$i(t) = 3.218\cos(\omega t - 56.97^\circ)\text{ A}$$

Of course, we can find $i_1 + i_2$ using Eqs. (9.9), but that is the hard way.

Example 9.7

Using the phasor approach, determine the current $i(t)$ in a circuit described by the integrodifferential equation

$$4i + 8\int i\,dt - 3\frac{di}{dt} = 50\cos(2t + 75^\circ)$$

Solution:

We transform each term in the equation from time domain to phasor domain. Keeping Eqs. (9.27) and (9.28) in mind, we obtain the phasor form of the given equation as

$$4\mathbf{I} + \frac{8\mathbf{I}}{j\omega} - 3j\omega\mathbf{I} = 50\underline{/75^\circ}$$

But $\omega = 2$, so

$$\mathbf{I}(4 - j4 - j6) = 50\underline{/75^\circ}$$

$$\mathbf{I} = \frac{50\underline{/75^\circ}}{4 - j10} = \frac{50\underline{/75^\circ}}{10.77\underline{/-68.2^\circ}} = 4.642\underline{/143.2^\circ}\text{ A}$$

Converting this to the time domain,

$$i(t) = 4.642\cos(2t + 143.2^\circ)\text{ A}$$

Keep in mind that this is only the steady-state solution, and it does not require knowing the initial values.

2. Alexander Practice problem 9.3-9.7

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Phasor relationships for circuit elements

Now that we know how to represent a voltage or current in the phasor or frequency domain, one may legitimately ask how we apply this to circuits involving the passive elements R, L, and C. What we need to do is to transform the voltage-current relationship from the time domain to the frequency domain for each element. Again, we will assume the passive sign convention.

Resistor

We begin with the resistor. If the current through a resistor R is $i = I_m cos(\omega t + \varphi)$, the voltage across it is given by Ohm's law as

$$v = iR = R\, I_m\, cos(\omega t + \varphi) \qquad (7.29)$$

The phasor form of this voltage is

$$\boldsymbol{V} = RI_m \angle\, \varphi \qquad (7.30)$$

But the phasor relationship of the current is $\boldsymbol{I} = I_m \angle\varphi$. Hence,

$$\boldsymbol{V} = R\boldsymbol{I} \qquad (7.31)$$

showing that the voltage-current relation for the resistor in the phasor domain continues to be Ohm's law, as in the time domain. Figure 9(a and b) illustrates the voltage-current relations of a resistor. We should note from Eq. (7.31) that voltage and current are in phase, as illustrated in the phasor diagram in Fig. 9(c).

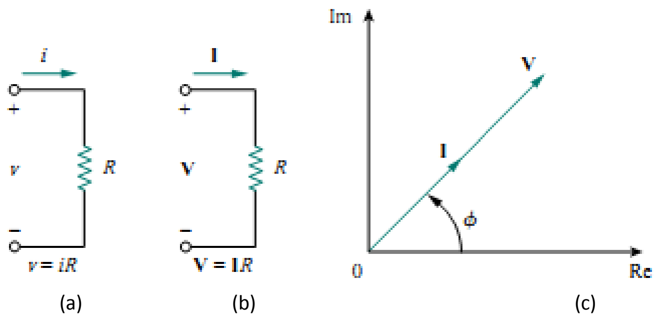


Figure 9 Voltage-current relations for a resistor in the: (a) time domain, (b) frequency domain. (c) Phasor diagram for the resistor

Inductor

For the inductor L, assume the current through it is $i = I_m cos(\omega t + \varphi)$. The voltage across the inductor is

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \varphi) \qquad (7.32)$$

Recall from Eq. (7.10) that $-\, sin\, A = cos(A + 90°)$. We can write the voltage as

$$v = \omega L\, I_m \cos(\omega t + \varphi + 90°) \qquad (7.33)$$

Which transforms to the phasor

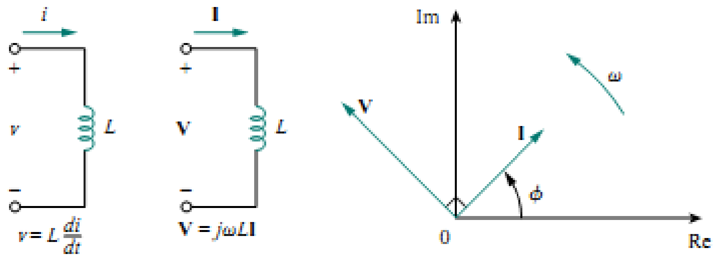
$$\boldsymbol{V} = \omega L I_m e^{j(\varphi+90°)} = \omega L I_m e^{j\varphi} e^{j90°} = \omega L I_m \angle\left(\varphi e^{j90°}\right) \qquad (7.34)$$

But $I_m \angle\varphi = 1$, and from Eq. (7.19) $e^{j90°} = j$. Thus,

$$\boldsymbol{V} = j\omega L\boldsymbol{I} \qquad (7.35)$$

showing that the voltage has a magnitude of ωLIm and a phase of $\varphi + 90°$. The voltage and current are 90◦ out of phase. Specifically, the current lags the voltage by 90◦ . Figure 10a and 10b shows the voltage-current relations for the inductor. Figure 10c shows the phasor diagram.

Although it is equally correct to say that the inductor voltage leads the current by 90◦ , convention gives the current phase relative to the voltage.



(a) (b) (c)

Figure 10 Voltage-current relations for a inductor in the: (a) time domain, (b) frequency domain. (c) Phasor diagram for the inductor; I lags V

Capacitor

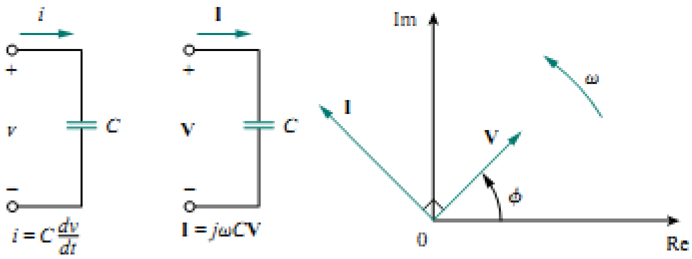
For the capacitor C, assume the voltage across it is $v = V_m cos(\omega t + \varphi)$. The current through the capacitor is

$$i = C\frac{dv}{dt} \qquad (7.36)$$

By following the same steps as we took for the inductor or by applying Eq. (7.27) on Eq. (7.36), we obtain

$$\boldsymbol{I} = j\omega C\boldsymbol{V} \qquad \Rightarrow \boldsymbol{V} = \frac{\boldsymbol{I}}{j\omega C} \qquad (7.37)$$

showing that the current and voltage are 90◦ out of phase. To be specific, the current leads the voltage by 90◦ . Figure 11a and 11b shows the voltage-current relations for the capacitor; Fig. 11c gives the phasor diagram.



(a) (b) (c)

Figure 11 Voltage-current relations for a Capacitor in the: (a) time domain, (b) frequency domain. (c) Phasor diagram for the Capacitor; I leads V

Table 2 summarizes the time-domain and phasor-domain representations of the circuit elements.

Table 2 Summary of voltage-current relationships

Element	Time domain	Frequency domain
R	$v = Ri$	$\boldsymbol{V} = R\boldsymbol{I}$
L	$v = L\frac{di}{dt}$	$\boldsymbol{V} = j\omega L\boldsymbol{I}$
C	$i = C\frac{dv}{dt}$	$\boldsymbol{V} = \frac{\boldsymbol{I}}{j\omega C}$

Example 9.8

The voltage $v = 12\cos(60t + 45^\circ)$ is applied to a 0.1-H inductor. Finc the steady-state current through the inductor.

Solution:

For the inductor, $\mathbf{V} = j\omega L\mathbf{I}$, where $\omega = 60$ rad/s and $\mathbf{V} = 12\underline{/45^\circ}$ V Hence

$$\mathbf{I} = \frac{\mathbf{V}}{j\omega L} = \frac{12\underline{/45^\circ}}{j60 \times 0.1} = \frac{12\underline{/45^\circ}}{6\underline{/90^\circ}} = 2\underline{/-45^\circ}\text{ A}$$

Converting this to the time domain,

$$i(t) = 2\cos(60t - 45^\circ)\text{ A}$$

3. Alexander Practice problem 9.8-9.

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Impedance and admittance

Previously we found the voltage current relationship for three passive components

$$\boldsymbol{V} = R\boldsymbol{I}, \quad \boldsymbol{V} = j\omega L\boldsymbol{I}, \quad \boldsymbol{V} = \frac{\boldsymbol{I}}{j\omega C} \qquad (7.38)$$

These equations may be written in terms of the ratio of the phasor voltage to the phasor current as

$$\frac{\boldsymbol{V}}{\boldsymbol{I}} = R, \qquad \frac{\boldsymbol{V}}{\boldsymbol{I}} = j\omega L\,, \qquad \frac{\boldsymbol{V}}{\boldsymbol{I}} = \frac{1}{j\omega C} \qquad (7.39)$$

From these three expressions, we obtain Ohm's law in phasor form for any type of element as

$$Z = \frac{\boldsymbol{V}}{\boldsymbol{I}} \quad or \quad V = ZI \qquad (7.40)$$

Where Z is a frequency-dependent quantity known as impedance, measured in ohms.

The impedance Z of a circuit is the ratio of the phasor voltage V to the phasor current I, measured in ohms (Ω).

The impedance represents the opposition which the circuit exhibits to the flow of sinusoidal current. Although the impedance is the ratio of two phasors, it is not a phasor, because it does not correspond to a sinusoidally varying quantity.

The impedances of resistors, inductors, and capacitors can be readily obtained from Eq. (7.39). Table 3 summarizes their impedances and admittance. From the table we notice that $\boldsymbol{Z_L = j\omega L}$ and $Z_C = -j/\omega C$. Consider two extreme cases of angular frequency. When ω = 0 (i.e., for dc sources), $Z_L = 0$ and $Z_C \rightarrow \infty$,confirming what we already know—that the inductor acts like a short circuit, while the capacitor acts like an open circuit. When $\omega \rightarrow \infty$ (i.e., for high frequencies), $Z_L \rightarrow \infty$ and $Z_C = 0$, indicating that the inductor is an open circuit to high frequencies, while the capacitor is a short circuit. Figure 12 illustrates this.

Table 3 Impedances and admittances of passive elements.

Element	Impedance	Admittance
R	$Z = R$	$Y = \frac{1}{R}$
L	$Z = j\omega L$	$Y = \frac{1}{j\omega L}$
C	$Z = \frac{1}{j\omega C}$	$Y = j\omega C$

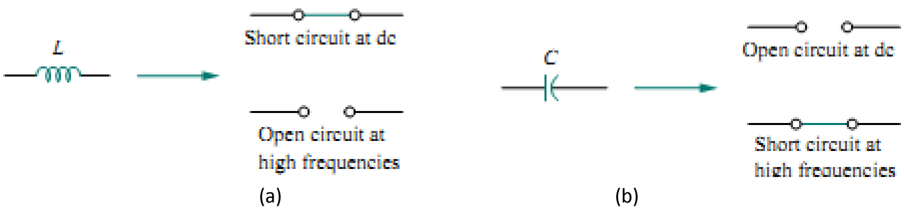


Figure 12 Equivalent circuits at dc and high frequencies: (a) inductor, (b) capacitor.

As a complex quantity, the impedance may be expressed in rectangular form as

$$Z = R + jX \qquad (7.41)$$

Where $R = \mathrm{Re}\,Z$ is the resistance and $X = \mathrm{Im}\,Z$ is the reactance. The reactance X may be positive or negative. We say that the impedance is **inductive when X is positive** or **capacitive when X is negative**. Thus, impedance $Z = R + jX$ is said to be inductive or lagging since current lags voltage, while impedance $Z = R - jX$ is capacitive or leading because current leads voltage. The impedance, resistance, and reactance are all measured in ohms. The impedance may also be expressed in polar form as

$$Z = |Z|\angle\theta \qquad (7.42)$$

Comparing Eqs. (7.41) and (7.42), we infer that

$$Z = R + jX = |Z|\angle\theta \qquad (7.43)$$

Where

$$|Z| = \sqrt{R^2 + X^2}, \qquad \theta = \tan^{-1}\frac{X}{R} \qquad (7.44)$$

And

$$R = |Z|\cos\theta, \qquad X = |Z|\sin\theta \qquad (7.45)$$

It is sometimes convenient to work with the reciprocal of impedance, known as admittance.

The admittance Y is the reciprocal of impedance, measured in siemens (S).

The admittance Y of an element (or a circuit) is the ratio of the phasor current through it to the phasor voltage across it, or

$$Y = \frac{1}{Z} = \frac{1}{V} \qquad (7.46)$$

The admittances of resistors, inductors, and capacitors can be obtained from Eq. (7.39). They are also summarized in Table 3.

As a complex quantity, we may write Y as

$$Y = G + jB \qquad (7.47)$$

where $G = \mathrm{Re}Y$ is called the conductance and $B = \mathrm{Im}Y$ is called the susceptance. Admittance, conductance, and susceptance are all expressed in the unit of siemens (or mhos). From Eqs. (7.41) and (7.47),

$$G + jB = \frac{1}{R + jX} \qquad (7.48)$$

By rationalization,

$$G + jB = \frac{1}{R + jX} \cdot \frac{R - jX}{R - jX} = \frac{R - jX}{R^2 + x^2} \qquad (7.49)$$

Equating the real and imaginary parts gives

$$G = \frac{R}{R^2 + X^2}, \qquad B = -\frac{X}{R^2 + X^2} \qquad (7.50)$$

showing that $G \neq 1/R$ as it is in resistive circuits. Of course, if X = 0, then $G = 1/R$.

Kirchhoff's Law work in AC circuit similar to DC circuit. This topic is kept as self study. You can find this Topic in section 9.5 in Alexander's book.

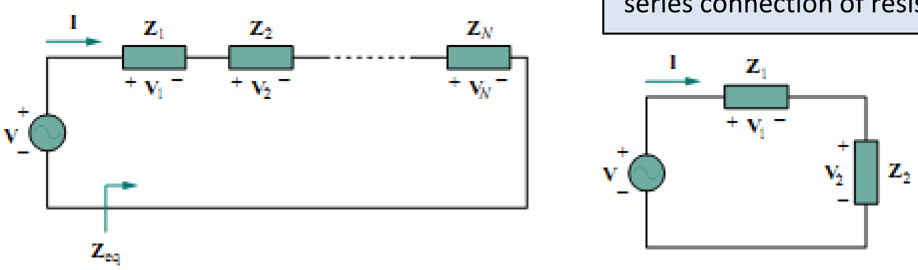
Impedance combinations

Consider the N series-connected impedances shown in Fig. 13a. The same current I flows through the impedances. Applying KVL around the loop gives

$$V = V_1 + V_2 + \cdots + V_N = I(Z_1 + Z_2 + \cdots + Z_N) \qquad (7.51$$

$$Z_{eq} = \frac{V}{I} = Z_1 + Z_2 + \cdots + Z_N \qquad (7.52)$$

The total or equivalent impedance of series-connected impedances is the sum of the individual impedances. This is similar to the series connection of resistances.



(a) (b)

Figure 13 (a) N impedance in Series (b)Voltage division

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If N = 2, as shown in Fig. 13b, the current through the impedances is

$$I = \frac{V}{Z_1 + Z_2} \qquad (7.53)$$

Since $V_1 = ZI$ and $V_2 = Z_2 I$, then

$$V_1 = \frac{Z_1}{Z_1 + Z_2}V, \quad V_2 = \frac{Z_2}{Z_1 + Z_2}V$$

This is the voltage division relationship, which is similar to DC circuit except that the resistance is replaced by the impedance in the formula.

In the same manner, we can obtain the equivalent impedance or admittance of the N parallel-connected impedances shown in Fig. 14a. The voltage across each impedance is the same. Applying KCL at the top node,

$$I = I_1 + I_2 + \cdots + I_N = V\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_N}\right) \qquad (7.54)$$

The equivalent impedance is

$$\frac{1}{Z_{eq}} = \frac{I}{V} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_N} \qquad (7.55)$$

$$Z_{eq} = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_N}} \qquad (7.56)$$

And the equivalent admittance is

$$Y_{eq} = Y_1 + Y_2 + \cdots + Y_N$$

The equivalent admittance of a parallel connection of admittances is the sum of the individual admittances.

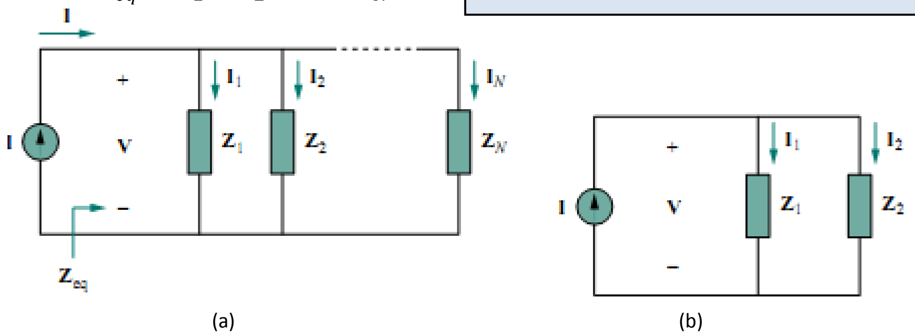

Figure 14 (a) N impedance in Parallel (b)Current division

When N = 2, as shown in Fig. 14b, the equivalent impedance becomes

$$Z_{eq} = \frac{1}{Y_{eq}} = \frac{1}{Y_1 + Y_2} = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}} = \frac{Z_1 Z_2}{Z_1 + Z_2} \qquad (7.58)$$

Also, since

$$V = IZ_{eq} = I_1 Z_1 = I_2 Z_2$$

The currents in the impedances are

$$I_1 = \frac{Z_2}{Z_1 + Z_2}I, \qquad I_2 = \frac{Z_1}{Z_1 + Z_2}I \qquad (7.59)$$

This is current division principle

Wye-Delta Transformation (Y-Δ)

Y-Δ conversion for ac circuit is similar to that of dc circuit. The only difference is that in the formula the resistance is replaced by impedance. For the circuit equivalence in figure 15.

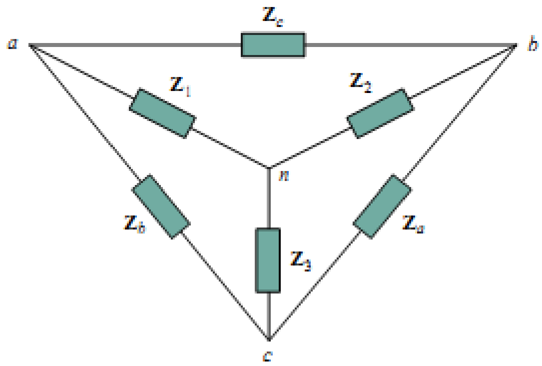

Figure 15 Superimposed Y-Δ network

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Δ to Y conversion

$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c}$$
$$Z_2 = \frac{Z_a Z_c}{Z_a + Z_b + Z_c}$$
$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}$$

Y to Δ Conversion

$$Z_a = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_1}$$
$$Z_b = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_2}$$
$$Z_c = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_3}$$

When a '-Y circuit is balanced, the above equations become

$$\boldsymbol{Z_\Delta = 3Z_Y \text{ or } Z_Y = \frac{1}{3}Z_\Delta}$$

Example 9.9

Find $v(t)$ and $i(t)$ in the circuit shown in Fig. 9.16.

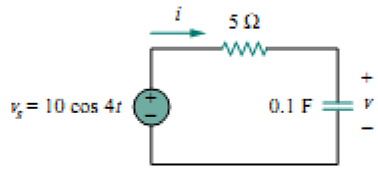

Figure 9.16 For Example 9.9.

Solution:

From the voltage source $10 \cos 4t$, $\omega = 4$,

$$\mathbf{V}_s = 10\angle 0^\circ \text{ V}$$

The impedance is

$$\mathbf{Z} = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4 \times 0.1} = 5 - j2.5\ \Omega$$

Hence the current

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{10\angle 0^\circ}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2}$$
$$= 1.6 + j0.8 = 1.789\angle 26.57^\circ \text{ A} \tag{9.9.1}$$

The voltage across the capacitor is

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_C = \frac{\mathbf{I}}{j\omega C} = \frac{1.789\angle 26.57^\circ}{j4 \times 0.1}$$
$$= \frac{1.789\angle 26.57^\circ}{0.4\angle 90^\circ} = 4.47\angle -63.43^\circ \text{ V} \tag{9.9.2}$$

Converting $\mathbf{I}$ and $\mathbf{V}$ in Eqs. (9.9.1) and (9.9.2) to the time domain, we get

$$i(t) = 1.789 \cos(4t + 26.57^\circ) \text{ A}$$
$$v(t) = 4.47 \cos(4t - 63.43^\circ) \text{ V}$$

Notice that $i(t)$ leads $v(t)$ by 90° as expected.

Example 9.10

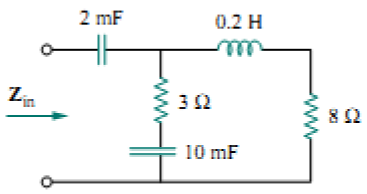

Figure 9.23 For Example 9.10.

Find the input impedance of the circuit in Fig. 9.23. Assume that the circuit operates at $\omega = 50$ rad/s.

Solution:

Let

$\mathbf{Z}_1$ = Impedance of the 2-mF capacitor

$\mathbf{Z}_2$ = Impedance of the 3-Ω resistor in series with the 10-mF capacitor

$\mathbf{Z}_3$ = Impedance of the 0.2-H inductor in series with the 8-Ω resistor

Then

$$\mathbf{Z}_1 = \frac{1}{j\omega C} = \frac{1}{j50 \times 2 \times 10^{-3}} = -j10\ \Omega$$

$$\mathbf{Z}_2 = 3 + \frac{1}{j\omega C} = 3 + \frac{1}{j50 \times 10 \times 10^{-3}} = (3 - j2)\ \Omega$$

$$\mathbf{Z}_3 = 8 + j\omega L = 8 + j50 \times 0.2 = (8 + j10)\ \Omega$$

The input impedance is

$$\mathbf{Z}_{in} = \mathbf{Z}_1 + \mathbf{Z}_2 \parallel \mathbf{Z}_3 = -j10 + \frac{(3 - j2)(8 + j10)}{11 + j8}$$

$$= -j10 + \frac{(44 + j14)(11 - j8)}{11^2 + 8^2} = -j10 + 3.22 - j1.07\ \Omega$$

Thus,

$$\mathbf{Z}_{in} = 3.22 - j11.07\ \Omega$$

Example 9.11

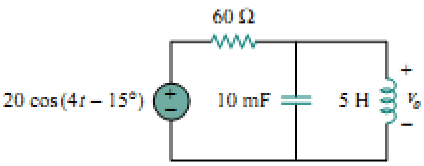

Figure 9.25 For Example 9.11.

Determine $v_o(t)$ in the circuit in Fig. 9.25.

Solution:

To do the analysis in the frequency domain, we must first transform the time-domain circuit in Fig. 9.25 to the phasor-domain equivalent in Fig. 9.26. The transformation produces

$$v_s = 20\cos(4t - 15°) \quad \Longrightarrow \quad \mathbf{V}_s = 20\underline{/-15°}\ \text{V}, \qquad \omega = 4$$

$$10\ \text{mF} \quad \Longrightarrow \quad \frac{1}{j\omega C} = \frac{1}{j4 \times 10 \times 10^{-3}} = -j25\ \Omega$$

$$5\ \text{H} \quad \Longrightarrow \quad j\omega L = j4 \times 5 = j20\ \Omega$$

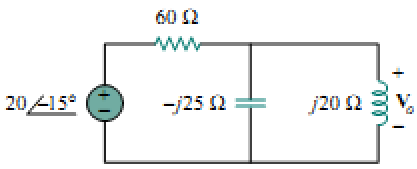

Figure 9.26 The frequency-domain equivalent of the circuit in Fig. 9.25.

Let

$\mathbf{Z}_1$ = Impedance of the 60-Ω resistor

$\mathbf{Z}_2$ = Impedance of the parallel combination of the 10-mF capacitor and the 5-H inductor

Then $\mathbf{Z}_1 = 60\ \Omega$ and

$$\mathbf{Z}_2 = -j25 \parallel j20 = \frac{-j25 \times j20}{-j25 + j20} = j100\ \Omega$$

By the voltage-division principle,

$$\mathbf{V}_o = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}_s = \frac{j100}{60 + j100}(20\underline{/-15°})$$

$$= (0.8575\underline{/30.96°})(20\underline{/-15°}) = 17.15\underline{/15.96°}\ \text{V}.$$

We convert this to the time domain and obtain

$$v_o(t) = 17.15\cos(4t + 15.96°)\text{V}$$

4. Alexander Exercise problems 9.9-9.12
5. Alexander Example 9.12