Instantaneous and Average Power

The instantaneous power p(t)absorbed by an element is the product of the instantaneous voltage v(t)across the element and the instantaneous current i(t)through it. Assuming the passive sign convention,

$$p(t) = v(t)i(t) \qquad (8.1)$$

The instantaneous power is the power at any instant of time. It is the rate at which an element absorbs energy.

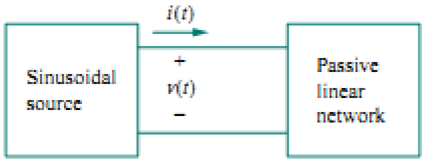

Figure 1 Sinusoidal source and passive linear circuit

Consider the general case of instantaneous power absorbed by an arbitrary combination of circuit elements under sinusoidal excitation, as shown in Fig. 1. Let the voltage and current at the terminals of the circuit be

$$v(t) = V_m \cos(\omega t + \theta_v) \qquad (8.2a)$$
$$i(t) = I_m \cos(\omega t + \theta_i) \qquad (8.2b)$$

where V_m and I_m are the amplitudes (or peak values), and θ_v and θ_i are the phase angles of the voltage and current, respectively. The instantaneous power absorbed by the circuit is

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i) \qquad (8.3)$$

Using trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)\,] \qquad (8.4)$$

To get

$$p(t) = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2}V_m I_m \cos(2\,\omega t + \theta_v + \theta_i) \qquad (8.5)$$

This shows us that the instantaneous power has two parts. The first part is constant or time independent. Its value depends on the phase difference between the voltage and the current. The second part is a sinusoidal function whose frequency is 2ω, which is twice the angular frequency of the voltage or current.

A sketch of the instantaneous power represented by equation 8.5 is shown in the figure 2 below.

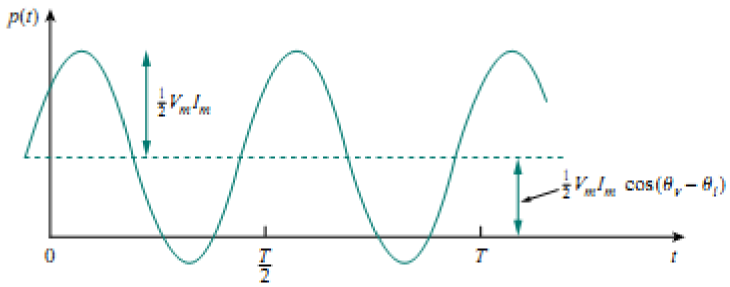

Figure 2 The instantaneous power p(t) entering a circuit.

We observe that p(t)is periodic, p(t) = p(t + T_0), and has a period of T_0 = T/2, since its frequency is twice that of voltage or current. We also observe that p(t)is positive for some part of each cycle and negative for the rest of the cycle. When p(t)is positive, power is absorbed by the circuit. When p(t)is negative, power is absorbed by the source; that is, power is transferred from the circuit to the source. This is possible because of the storage elements (capacitors and inductors) in the circuit.

The instantaneous power changes with time and is therefore difficult to measure. The *average* power is more convenient to measure. In fact, the wattmeter, the instrument for measuring power, responds to average power.

The average power is the average of the instantaneous power over one period.

Thus, the average power is given by

$$P = \frac{1}{T}\int_0^t p(t)dt \qquad (8.6)$$

Although Eq. (5.6) shows the averaging done over T, we would get the same result if we performed the integration over the actual period of p(t) which is $T_0 = T/2$.

Substituting p(t)in Eq. (8.5) into Eq. (8.6) gives

$$P = \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\,dt + \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)\,dt$$

$$= \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\frac{1}{T}\int_0^T dt + \frac{1}{2}V_m I_m \frac{1}{T}\int_0^T \cos(2\omega t + \theta_v + \theta_i)\,dt \qquad (8.7)$$

The first integrand is constant, and the average of a constant is the same constant. The second integrand is a sinusoid. We know that the average of a sinusoid over its period is zero because the area under the sinusoid during a positive half-cycle is canceled by the area under it during the following negative half-cycle. Thus, the second term in Eq. (8.7) vanishes and the average power becomes

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) \qquad (8.8)$$

Since $\cos(\theta_v - \theta_i) = \cos(\theta_i - \theta_v)$, what is important is the difference in the phase of the voltage and current.

Note that *p(t)* is time-varying while P does not depend on time. To find the instantaneous power, we must necessarily have *v(t)*and *i(t)* in the time domain. But we can find the average power when voltage and current are expressed in the time domain, as in Eq. (8.2), or when they are expressed in the frequency domain. The phasor forms of $v(t)$ and $i(t)$ in Eq. (8.2) are $V = V_m\angle\theta_v$ and $I = I_m\angle\theta_i$, respectively. P is calculated using Eq. (8.8) or using phasors V and I. To use phasors, we notice that

$$\frac{1}{2}\boldsymbol{VI}^* = \frac{1}{2}V_m I_m\angle(\theta_v - \theta_i) = \frac{1}{2}V_m I_m[\cos(\theta_v - \theta_i) + j\sin(\theta_v - \theta_i)] \qquad (8.9)$$

We recognize the real part of this expression as the average power P according to Eq. (8.8). Thus,

$$P = \frac{1}{2}RE[\boldsymbol{VI}^*] = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) \qquad (8.10)$$

Consider two special cases of Eq. (8.10). When $\theta_v = \theta_i$, the voltage and current are in phase. This implies a purely resistive circuit or resistive load R,and

$$P = \frac{1}{2}V_m I_m = \frac{1}{2}I_m^2 R = \frac{1}{2}|\mathbf{I}|^2 R \qquad (8.11)$$

Where $|\mathbf{I}| = \mathbf{I}\times\mathbf{I}^*$. Equation (8.11) shows that a purely resistive circuit absorbs power at all times. When $\theta_v - \theta_i = \pm 90°$,we have a purely reactive circuit, and

$$P = \frac{1}{2}V_m I_m \cos 90° = 0 \qquad (8.12)$$

showing that a purely reactive circuit absorbs no average power. In summary,

A resistive load (R) absorbs power at all times, while a reactive load (L or C) absorbs zero average power.

Maximum Average Power Transfer

In previous lecture we solved problems for maximizing power delivery by a power-supplying resistive network to a load R_L. Representing the circuit by its Thevenin equivalent, we proved that the maximum power would be delivered to the load if the load resistance is equal to the Thevenin resistance $R_L = R_{Th}$. We now extend that result to ac circuits.

Consider the circuit in Fig. 3, where an ac circuit is connected to a load $\mathbf{Z_L}$ and is represented by its Thevenin equivalent. The load is usually represented by impedance, which may model an electric motor, an antenna, a TV, and so forth. In rectangular form, the Thevenin impedance $\mathbf{Z_{Th}}$ and the load impedance $\mathbf{Z_L}$ are

$$Z_{Th} = R_{Th} + jX_{Th} \qquad (8.13a)$$
$$Z_L = R_L + jX_L \qquad (8.13b)$$

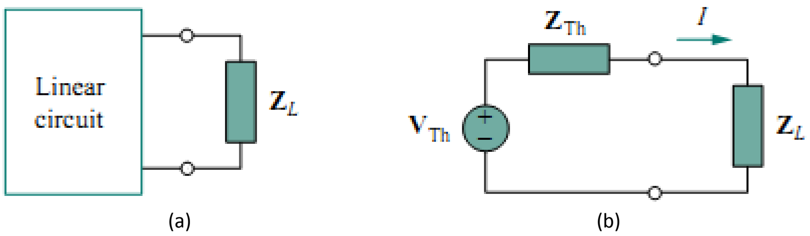

Figure 3 Finding the maximum average power transfer: (a) circuit with a load, (b) the Thevenin equivalent.

The current through the load is

$$I = \frac{V_{Th}}{Z_{th} + Z_L} = \frac{V_{Th}}{(R_{Th} + jX_{Th}) + (R_L + jX_L)} \qquad (8.14)$$

From Eq. (8.11), the average power delivered to the load is

$$P = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{|V_{Th}|^2 R_L/2}{(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2} \qquad (8.15)$$

Our objective is to adjust the load parameters R_L and X_L so that P is maximum. To do this we set $\partial P/\partial R_L$ and $\partial P/\partial X_L$ equal to zero. From Eq. (8.15), we obtain

$$\frac{\delta P}{\delta X_L} = -\frac{|V_{Th}|^2 R_L (X_{Th} + X_L)}{[(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2]^2} \qquad (8.16a)$$
$$\frac{\delta P}{\delta R_L} = -\frac{|V_{Th}|^2 [(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2 - 2R_L(R_{Th} + R_L)]}{2[(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2]^2} \qquad (8.16b)$$

Setting ∂P/∂XL to zero gives

$$X_L = -X_{Th} \qquad (8.17)$$

and setting ∂P/∂RL to zero results in

$$R_L = \sqrt{R_{Th}^2 + (X_{Th} + X_L)^2} \qquad (8.18)$$

Combining Eqs. (8.17) and (8.18) leads to the conclusion that for maximum average power transfer, ZL must be selected so that $X_L = -X_{Th}$ and $R_L = R_{Th}$, i.e.,

$$Z_L = R_L + jX_L = R_{Th} - jX_{Th} = \mathbf{Z}_{Th}^* \qquad (8.19)$$

For maximum average power transfer, the load impedance Z_L must be equal to the complex conjugate of the Thevenin impedance Z_{Th}.

This result is known as the maximum average power transfer theorem for the sinusoidal steady state. Setting $R_L = R_{Th}$ and $X_L = -X_{Th}$ in Eq. (8.15) gives us the maximum average power as

$$P_{max} = \frac{|V_{Th}|^2}{8R_{Th}} \qquad (8.20)$$

In a situation in which the load is purely real, the condition for maximum power transfer is obtained from Eq. (8.18) by setting $X_L = 0$; that is,

$$R_L = \sqrt{R_{Th}^2 + X_{Th}^2} = |Z_{Th}| \qquad (8.21)$$

This means that for maximum average power transfer to a purely resistive load, the load impedance (or resistance) is equal to the magnitude of the Thevenin impedance.

Effective or RMS Value

The idea of *effective value* arises from the need to measure the effectiveness of a voltage or current source in delivering power to a resistive load.

The effective value of a periodic current is the dc current that delivers the same average power to a resistor as the periodic current.

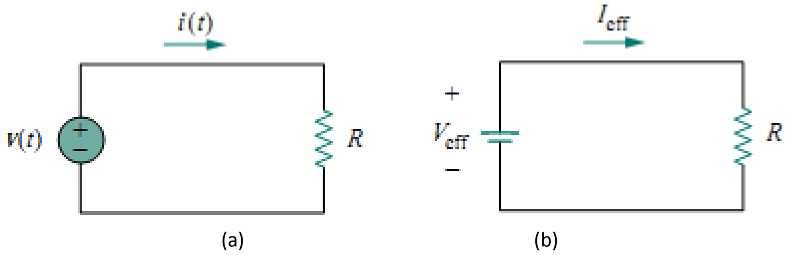

Figure 4 Finding the effective current: (a) ac circuit, (b) dc circuit.

In Fig. 4, the circuit in (a) is ac while that of (b) is dc. Our objective is to find I_{eff} that will transfer the same power to resistor R as the sinusoid i. The average power absorbed by the resistor in the ac circuit is

$$P = \frac{1}{T}\int_0^T i^2 R\, dt = \frac{R}{T}\int_0^T i^2 dt \qquad (8.22)$$

while the power absorbed by the resistor in the dc circuit is

$$P = I_{eff}^2 R \qquad (8.23)$$

Equating the expressions in Eqs. (8.22) and (8.23) and solving for I_{eff} , we obtain

$$I_{eff} = \sqrt{\frac{1}{T}\int_0^T i^2 dt} \qquad (8.24)$$

The effective value of the voltage is found in the same way as current; that is,

$$V_{eff} = \sqrt{\frac{1}{T}\int_0^T v^2 dt} \qquad (8.25)$$

This indicates that the effective value is the (square) root of the mean (or average) of the square of the periodic signal. Thus, the effective value is often known as the root-mean-square value, or rms value for short; and we write

$$I_{eff} = I_{rms}, \qquad V_{eff} = V_{rms} \qquad (8.26)$$

For any periodic function x(t)in general, the rms value is given by

$$X_{eff} = \sqrt{\frac{1}{T}\int_0^T x^2 dt} \qquad (8.27)$$

The effective value of a periodic signal is its root mean square (rms) value.

Equation 8.27 states that to find the rms value of x(t),we first find its square x^2 and then find the mean of that, or

$$\frac{1}{T}\int_0^T x^2 dt$$

And then the square root (√) of that mean. The rms value of a constant is the constant itself. For the sinusoid $i(t) = I_m \cos \omega t$, the effective or rms value is

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T I_m^2 \cos^2 \omega t\, dt} = \sqrt{\frac{I_m^2}{T}\int_0^T \frac{1}{2}(1 + \cos 2\omega t)\, dt} = \frac{I_m}{\sqrt{2}} \qquad (8.28)$$

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Similarly, for $v(t) = V_m cos\ \omega t$,

$$V_{rms} = \frac{V_m}{\sqrt{2}} \qquad (8.29)$$

Keep in mind that Eqs. (8.28) and (8.29) are only valid for sinusoidal signals.

The average power in Eq. (8.8) can be written in terms of the rms values.

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos(\theta_v - \theta_i) = V_{rms} I_{rms} \cos(\theta_v - \theta_i) \qquad (8.30)$$

Similarly, the average power absorbed by a resistor R in Eq. (8.11) can be written as

$$P = I_{rms}^2 R = \frac{V_{rms}^2}{R} \qquad (8.31)$$

When a sinusoidal voltage or current is specified, it is often in terms of its maximum (or peak) value or its rms value, since its average value is zero. The power industries specify phasor magnitudes in terms of their rms values rather than peak values. For instance, the 110 V available at every household is the rms value of the voltage from the power company. It is convenient in power analysis to express voltage and current in their rms values. Also, analog voltmeters and ammeters are designed to read directly the rms value of voltage and current, respectively.

Apparent Power and Power Factor

We saw that if the voltage and current at the terminals of a circuit are

$$v(t) = V_m \cos(\omega t + \theta_v) \quad \text{and} \quad i(t) = I_m \cos(\omega t + \theta_i) \qquad (8.32)$$

Or in phasor form, $V = V_m \angle \theta_v \ \ and\ I = I_m \angle \theta_i$, the average power is

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \qquad (8.33)$$

We further saw that

$$P = V_{rms} I_{rms} \cos(\theta_v - \theta_i) = S \cos(\theta_v - \theta_i) \qquad (8.34)$$

Equation equations 8.33 and 8.34 we get

$$S = V_{rms} I_{rms} \qquad (8.35)$$

The average power is a product of two terms. The product $V_{rms}I_{rms}$ is known as the *apparent power* S. The factor $\cos(\theta_v - \theta_i)$ is called the power factor (pf).

The apparent power (in VA) is the product of the rms values of voltage and current.

The apparent power is so called because it seems apparent that the power should be the voltage-current product, by analogy with dc resistive circuits. It is measured in volt-amperes or VA to distinguish it from the average or real power, which is measured in watts. The power factor is dimensionless, since it is the ratio of the average power to the apparent power,

$$\boldsymbol{pf = \frac{P}{S} = \cos(\theta_v - \theta_i)} \qquad (8.36)$$

The angle $\theta_v - \theta_i$ is called the power factor angle, since it is the angle whose cosine is the power factor. The power factor angle is equal to the angle of the load impedance if V is the voltage across the load and I is the current through it. This is evident from the fact that

$$\boldsymbol{Z} = \frac{\boldsymbol{V}}{\boldsymbol{I}} = \frac{V_m \angle \theta_v}{I_m \angle \theta_i} = \frac{V_m}{I_m} \angle(\theta_v - \theta_i) \qquad (8.37)$$

Alternatively, since

$$V_{rms} = \frac{\boldsymbol{V}}{\sqrt{2}} = V_{rms} \angle \theta_v \qquad (8.38a)$$

And

$$I_{rms} = \frac{\boldsymbol{I}}{\sqrt{2}} = I_{rms} \angle \theta_i \qquad (8.38a)$$

The impedance is

$$Z = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V_{rms}}}{\mathbf{I_{rms}}} = \frac{V_{rms}}{I_{rms}} \angle(\theta_v - \theta_i) \qquad (8.39)$$

The power factor is the cosine of the phase difference between voltage and current. It is also the cosine of the angle of the load impedance.

From Eq. (8.36), the power factor may be seen as that factor by which the apparent power must be multiplied to obtain the real or average power. The value of pf ranges between zero and unity. For a purely resistive load, the voltage and current are in phase, so that $\theta_v - \theta_i = 0$ and pf = 1. This implies that the apparent power is equal to the average power. For a purely reactive load, $\theta_v - \theta_i = \pm 90°$ and pf = 0. In this case the average power is zero. In between these two extreme cases, pf is said to be leading or lagging. **Leading power factor** means that **current leads voltage**, which implies a **capacitive load**. *Lagging power factor* means that *current lags voltage*, implying an *inductive load*. Power factor affects the electric bills consumers pay the electric utility companies.

Complex Power

Considerable effort has been expended over the years to express power relations as simply as possible. Power engineers have coined the term *complex power*, which they use to find the total effect of parallel loads. Complex power is important in power analysis because it contains all the information pertaining to the power absorbed by a given load.

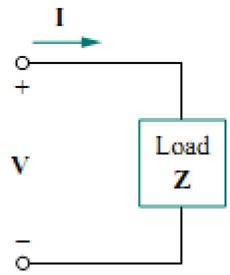

Figure 5 Voltage and current phasor associated with a load

Consider the ac load in Fig. 5. Given the phasor form $V = V_m \angle\theta_v \ and \ I = I_m \angle\theta_i$ of voltage v(t) and current i(t), the complex power S absorbed by the ac load is the product of the voltage and the complex conjugate of the current, or

$$S = \frac{1}{2}\mathbf{VI}^* \qquad (8.40)$$

Assuming the passive sign convention (see Fig. 5). In terms of the rms values,

$$\mathbf{S} = \mathbf{V_{rms}I_{rms}^*} \qquad (8.41)$$

Where

$$V_{rms} = \frac{\boldsymbol{V}}{\sqrt{2}} = V_{rms}\angle\theta_v \qquad (8.42)$$

And

$$I_{rms} = \frac{\boldsymbol{I}}{\sqrt{2}} = I_{rms}\angle\theta_i \qquad (8.43)$$

We can write equation 8.41 as

$$\mathbf{S} = V_{rms} I_{rms} \angle(\theta_v - \theta_i) = V_{rms} I_{rms} \cos(\theta_v - \theta_i) + j\, V_{rms} I_{rms} \sin(\theta_v - \theta_i) \qquad (8.44)$$

This equation can also be obtained from Eq. (8.9). We notice from Eq. (11.44) that the magnitude of the complex power is the apparent power; hence, the complex power is measured in volt-amperes (VA). Also, we notice that the angle of the complex power is the power factor angle.

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The complex power may be expressed in terms of the load impedance Z. From Eq. (8.37), the load impedance Z may be written as

$$\boldsymbol{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V}_{\boldsymbol{rms}}}{\mathbf{I}_{\boldsymbol{rms}}} = \frac{V_{rms}}{I_{rms}} \angle(\theta_v - \theta_i) \qquad (8.45)$$

Thus, $\mathbf{V}_{rms} = \mathbf{Z}\mathbf{I}_{rms}$. Substituting this into Eq. (8.41) gives

$$S = I_{rms}^2 \boldsymbol{Z} = \frac{V_{rms}^2}{\boldsymbol{Z}^*} \qquad (8.46)$$

Since $\boldsymbol{Z} = R + jX$, Eq. (8.46) becomes

$$\boldsymbol{S} = I_{rms}^2 (R + jX) = P + JQ \qquad (8.47)$$

where P and Q are the real and imaginary parts of the complex power; that is,

$$P = \mathrm{RE}(S) = I_{rms}^2 R \qquad (8.48)$$

$$Q = \mathrm{Im}(S) = I_{rms}^2 X \qquad (8.48)$$

P is the average or real power and it depends on the load's resistance R. Q depends on the load's reactance X and is called the *reactive* (or quadrature) power.

Comparing Eq. (8.44) with Eq. (8.47), we notice that

$$P = V_{rms} I_{rms} \cos(\theta_v - \theta_i), \qquad Q = V_{rms} I_{rms} \sin(\theta_v - \theta_i) \qquad (8.50)$$

The real power P is the average power in watts delivered to a load; it is the only useful power. It is the actual power dissipated by the load. The reactive power Q is a measure of the energy exchange between the source and the reactive part of the load. The unit of Q is the *volt-ampere reactive* (VAR) to distinguish it from the real power, whose unit is the watt. We know from Chapter 6 that energy storage elements neither dissipate nor supply power, but exchange power back and forth with the rest of the network. In the same way, the reactive power is being transferred back and forth between the load and the source. It represents a lossless interchange between the load and the source. Notice that:

1. $Q = 0$ for resistive loads (unity pf).
2. $Q < 0$ for capacitive loads (leading pf).
3. $Q > 0$ for inductive loads (lagging pf).

Thus,

Complex power (in VA) is the product of the rms voltage phasor and the complex conjugate of the rms current phasor. As a complex quantity, its real part is real power P and its imaginary part is reactive power Q.

Introducing the complex power enables us to obtain the real and reactive powers directly from voltage and current phasors.

$$\boxed{\begin{aligned} Complex\ Power &= \mathbf{S} = P + jQ = \frac{1}{2}\mathbf{V}\mathbf{I}^* \\ &= V_{rms} I_{rms} \angle\, (\theta_v - \theta_i) \\ Apparent\ power &= S = |S| = V_{rms} I_{rms} = \sqrt{P^2 + Q^2} \\ Real\ Power &= P = \mathrm{Re}(\mathbf{S}) = S\cos(\theta_v - \theta_i) \\ Reactive\ Power &= Q = \mathrm{Im}(\mathbf{S}) = S\sin(\theta_v - \theta_i) \\ Power\ Factor &= \frac{P}{S} = cos\ (\theta_v - \theta_i) \end{aligned}} \qquad (8.51)$$

This shows how the complex power contains all the relevant power information in a given load.

It is a standard practice to represent S, P, and Q in the form of a triangle, known as the power triangle, shown in Fig. 6(a). This is similar to the impedance triangle showing the relationship between Z, R, and X, illustrated in Fig. 6(b). The power triangle has four items—the apparent/complex power, real power, reactive power, and the power factor angle. Given two of these items, the other two can easily be obtained from the triangle. As shown in Fig. 7, when S lies in the first quadrant, we have an inductive load and a lagging pf. When S lies in the fourth quadrant, the load is capacitive and the pf is leading. It is also possible for the complex power to lie in the second or third quadrant. This requires that the load impedance have a negative resistance, which is possible with active circuits.

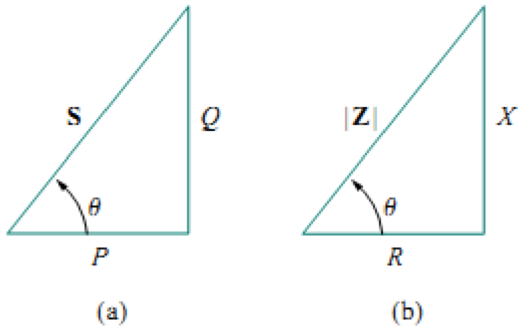


Figure 6 (a) Power Triangle (b) impedance triangle

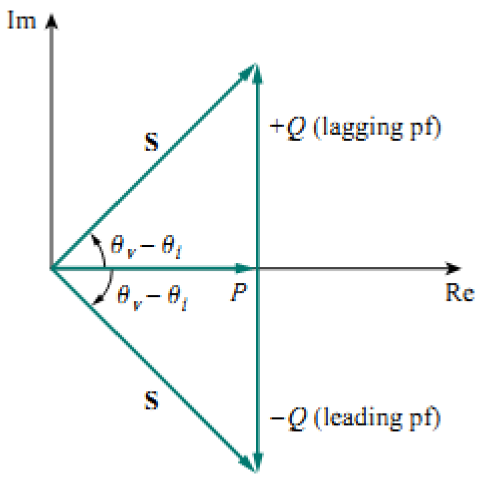


Figure 7 Power triangle illustrating leading and lagging power factor

It is to be noted that the total complex power in a network is the sum of the complex powers of the individual components.

The complex, real, and reactive powers of the sources equal the respective sums ofthe complex, real, and reactive powers of the individual loads.

Example 11.1

Given that

$v(t) = 120\cos(377t + 45^\circ)$ V and $i(t) = 10\cos(377t - 10^\circ)$ A

find the instantaneous power and the average power absorbed by the passive linear network of Fig. 11.1.

Solution:

The instantaneous power is given by

$$p = vi = 1200\cos(377t + 45^\circ)\cos(377t - 10^\circ)$$

Applying the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

gives

$$p = 600[\cos(754t + 35^\circ) + \cos 55^\circ]$$

or

$$p(t) = 344.2 + 600\cos(754t + 35^\circ)\ \text{W}$$

The average power is

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}120(10)\cos[45^\circ - (-10^\circ)]$$

$$= 600\cos 55^\circ = 344.2\ \text{W}$$

which is the constant part of $p(t)$ above.

Example 11.2

Calculate the average power absorbed by an impedance $\mathbf{Z} = 30 - j70\ \Omega$ when a voltage $\mathbf{V} = 120\angle 0^\circ$ is applied across it.

Solution:

The current through the impedance is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{120\angle 0^\circ}{30 - j70} = \frac{120\angle 0^\circ}{76.16\angle -66.8^\circ} = 1.576\angle 66.8^\circ\ \text{A}$$

The average power is

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}(120)(1.576)\cos(0 - 66.8^\circ) = 37.24\ \text{W}$$

Example 11.3

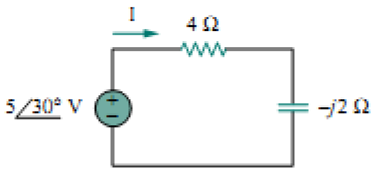


Figure 11.3 For Example 11.3.

For the circuit shown in Fig. 11.3, find the average power supplied by the source and the average power absorbed by the resistor.

Solution:

The current $\mathbf{I}$ is given by

$$\mathbf{I} = \frac{5\angle 30^\circ}{4 - j2} = \frac{5\angle 30^\circ}{4.472\angle -26.57^\circ} = 1.118\angle 56.57^\circ\ \text{A}$$

The average power supplied by the voltage source is

$$P = \frac{1}{2}(5)(1.118)\cos(30^\circ - 56.57^\circ) = 2.5\ \text{W}$$

The current through the resistor is

$$\mathbf{I} = \mathbf{I}_R = 1.118\angle 56.57^\circ\ \text{A}$$

and the voltage across it is

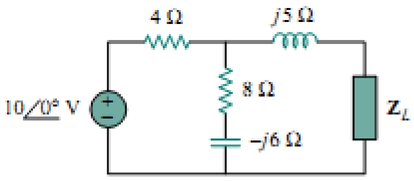
$$\mathbf{V}_R = 4\mathbf{I}_R = 4.472\angle 56.57^\circ\ \text{V}$$

The average power absorbed by the resistor is

$$P = \frac{1}{2}(4.472)(1.118) = 2.5\ \text{W}$$

which is the same as the average power supplied. Zero average power is absorbed by the capacitor.

Example 11.5



Determine the load impedance $\mathbf{Z}_L$ that maximizes the average power drawn from the circuit of Fig. 11.8. What is the maximum average power?

Solution:

First we obtain the Thevenin equivalent at the load terminals. To get $\mathbf{Z}_{Th}$, consider the circuit shown in Fig. 11.9(a). We find

$$\mathbf{Z}_{Th} = j5 + 4 \parallel (8 - j6) = j5 + \frac{4(8 - j6)}{4 + 8 - j6} = 2.933 + j4.467\ \Omega$$

To find $\mathbf{V}_{Th}$, consider the circuit in Fig. 11.8(b). By voltage division,

$$\mathbf{V}_{Th} = \frac{8 - j6}{4 + 8 - j6}(10) = 7.454\underline{/-10.3^\circ}\ \text{V}$$

The load impedance draws the maximum power from the circuit when

$$\mathbf{Z}_L = \mathbf{Z}_{Th}^* = 2.933 - j4.467\ \Omega$$

According to Eq. (11.20), the maximum average power is

$$P_{max} = \frac{|\mathbf{V}_{Th}|^2}{8R_{Th}} = \frac{(7.454)^2}{8(2.933)} = 2.368\ \text{W}$$

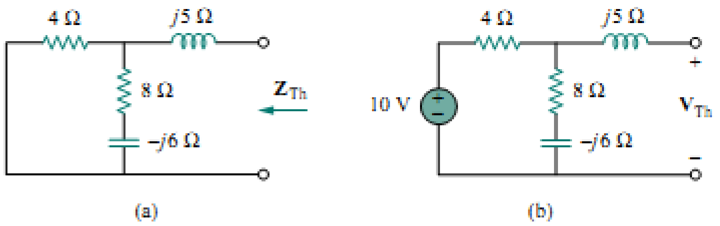


Figure 11.9 Finding the Thevenin equivalent of the circuit in Fig. 11.8.

Example 11.6

In the circuit in Fig. 11.11, find the value of R_L that will absorb the maximum average power. Calculate that power.

Solution:

We first find the Thevenin equivalent at the terminals of R_L.

$$\mathbf{Z}_{Th} = (40 - j30) \parallel j20 = \frac{j20(40 - j30)}{j20 + 40 - j30} = 9.412 + j22.35\ \Omega$$

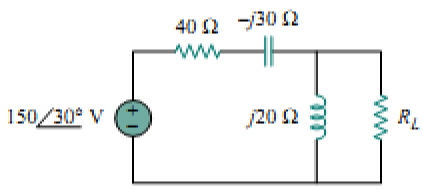


Figure 11.11 For Example 11.6.

By voltage division,

$$\mathbf{V}_{Th} = \frac{j20}{j20 + 40 - j30}(150\underline{/30^\circ}) = 72.76\underline{/134^\circ}\ \text{V}$$

The value of R_L that will absorb the maximum average power is

$$R_L = |\mathbf{Z}_{Th}| = \sqrt{9.412^2 + 22.35^2} = 24.25\ \Omega$$

The current through the load is

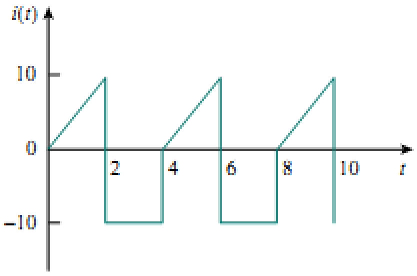
$$\mathbf{I} = \frac{\mathbf{V}_{Th}}{\mathbf{Z}_{Th} + R_L} = \frac{72.76\underline{/134^\circ}}{33.39 + j22.35} = 1.8\underline{/100.2^\circ}\ \text{A}$$

The maximum average power absorbed by R_L is

$$P_{max} = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{1}{2}(1.8)^2(24.25) = 39.29\ \text{W}$$

Example 11.7

Determine the rms value of the current waveform in Fig. 11.14. If the current is passed through a 2-Ω resistor, find the average power absorbed by the resistor.



Solution:

The period of the waveform is $T = 4$. Over a period, we can write the current waveform as

$$i(t) = \begin{cases} 5t, & 0 < t < 2 \\ -10, & 2 < t < 4 \end{cases}$$

The rms value is

$$I_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T i^2\,dt} = \sqrt{\frac{1}{4}\left[\int_0^2 (5t)^2\,dt + \int_2^4 (-10)^2\,dt\right]}$$

$$= \sqrt{\frac{1}{4}\left[25\frac{t^3}{3}\bigg|_0^2 + 100t\bigg|_2^4\right]} = \sqrt{\frac{1}{4}\left(\frac{200}{3} + 200\right)} = 8.165 \text{ A}$$

The power absorbed by a 2-Ω resistor is

$$P = I_{\text{rms}}^2 R = (8.165)^2(2) = 133.3 \text{ W}$$

Example 11.8

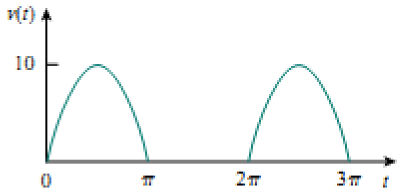


Figure 11.16 For Example 11.8.

The waveform shown in Fig. 11.16 is a half-wave rectified sine wave. Find the rms value and the amount of average power dissipated in a 10-Ω resistor.

Solution:

The period of the voltage waveform is $T = 2\pi$, and

$$v(t) = \begin{cases} 10\sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

The rms value is obtained as

$$V_{\text{rms}}^2 = \frac{1}{T}\int_0^T v^2(t)\,dt = \frac{1}{2\pi}\left[\int_0^\pi (10\sin t)^2\,dt + \int_\pi^{2\pi} 0^2\,dt\right]$$

But $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$. Hence

$$V_{\text{rms}}^2 = \frac{1}{2\pi}\int_0^\pi \frac{100}{2}(1 - \cos 2t)\,dt = \frac{50}{2\pi}\left(t - \frac{\sin 2t}{2}\right)\bigg|_0^\pi$$

$$= \frac{50}{2\pi}\left(\pi - \frac{1}{2}\sin 2\pi - 0\right) = 25, \qquad V_{\text{rms}} = 5 \text{ V}$$

The average power absorbed is

$$P = \frac{V_{\text{rms}}^2}{R} = \frac{5^2}{10} = 2.5 \text{ W}$$

Example 11.9

A series-connected load draws a current $i(t) = 4\cos(100\pi t + 10°)$ A when the applied voltage is $v(t) = 120\cos(100\pi t - 20°)$ V. Find the apparent power and the power factor of the load. Determine the element values that form the series-connected load.

Solution:

The apparent power is

$$S = V_{\text{rms}} I_{\text{rms}} = \frac{120}{\sqrt{2}}\frac{4}{\sqrt{2}} = 240 \text{ VA}$$

The power factor is

$$\text{pf} = \cos(\theta_v - \theta_i) = \cos(-20^\circ - 10^\circ) = 0.866 \qquad \text{(leading)}$$

The pf is leading because the current leads the voltage. The pf may also be obtained from the load impedance.

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{120\underline{/-20^\circ}}{4\underline{/10^\circ}} = 30\underline{/-30^\circ} = 25.98 - j15\ \Omega$$

$$\text{pf} = \cos(-30^\circ) = 0.866 \qquad \text{(leading)}$$

The load impedance $\mathbf{Z}$ can be modeled by a 25.98-Ω resistor in series with a capacitor with

$$X_C = -15 = -\frac{1}{\omega C}$$

or

$$C = \frac{1}{15\omega} = \frac{1}{15 \times 100\pi} = 212.2\ \mu\text{F}$$

Example 11.10

Determine the power factor of the entire circuit of Fig. 11.18 as seen by the source. Calculate the average power delivered by the source.

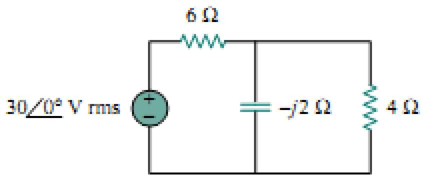


Figure 11.18 For Example 11.10.

Solution:

The total impedance is

$$\mathbf{Z} = 6 + 4 \parallel (-j2) = 6 + \frac{-j2 \times 4}{4 - j2} = 6.8 - j1.6 = 7\underline{/-13.24}\ \Omega$$

The power factor is

$$\text{pf} = \cos(-13.24) = 0.9734 \qquad \text{(leading)}$$

since the impedance is capacitive. The rms value of the current is

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{Z}} = \frac{30\underline{/0^\circ}}{7\underline{/-13.24^\circ}} = 4.286\underline{/13.24^\circ}\ \text{A}$$

The average power supplied by the source is

$$P = V_{\text{rms}} I_{\text{rms}}\ \text{pf} = (30)(4.286)0.9734 = 125\ \text{W}$$

or

$$P = I_{\text{rms}}^2 R = (4.286)^2(6.8) = 125\ \text{W}$$

where R is the resistive part of $\mathbf{Z}$.

Example 11.11

The voltage across a load is $v(t) = 60\cos(\omega t - 10^\circ)$ V and the current through the element in the direction of the voltage drop is $i(t) = 1.5\cos(\omega t + 50^\circ)$ A. Find: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

Solution:

(a) For the rms values of the voltage and current, we write

$$\mathbf{V}_{\text{rms}} = \frac{60}{\sqrt{2}}\underline{/-10^\circ}, \qquad \mathbf{I}_{\text{rms}} = \frac{1.5}{\sqrt{2}}\underline{/+50^\circ}$$

The complex power is

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}}^* = \left(\frac{60}{\sqrt{2}}\underline{/-10^\circ}\right)\left(\frac{1.5}{\sqrt{2}}\underline{/-50^\circ}\right) = 45\underline{/-60^\circ}\ \text{VA}$$

The apparent power is

$$S = |\mathbf{S}| = 45\ \text{VA}$$

(b) We can express the complex power in rectangular form as

$$\mathbf{S} = 45\angle -60^\circ = 45[\cos(-60^\circ) + j\sin(-60^\circ)] = 22.5 - j38.97$$

Since $\mathbf{S} = P + jQ$, the real power is

$$P = 22.5 \text{ W}$$

while the reactive power is

$$Q = -38.97 \text{ VAR}$$

(c) The power factor is

$$\text{pf} = \cos(-60^\circ) = 0.5 \text{ (leading)}$$

It is leading, because the reactive power is negative. The load impedance is

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{60\angle -10^\circ}{1.5\angle +50^\circ} = 40\angle -60^\circ\ \Omega$$

which is a capacitive impedance.

Example 11.12

A load $\mathbf{Z}$ draws 12 kVA at a power factor of 0.856 lagging from a 120-V rms sinusoidal source. Calculate: (a) the average and reactive powers delivered to the load, (b) the peak current, and (c) the load impedance.

Solution:

(a) Given that $\text{pf} = \cos\theta = 0.856$, we obtain the power angle as $\theta = \cos^{-1} 0.856 = 31.13^\circ$. If the apparent power is $S = 12{,}000$ VA, then the average or real power is

$$P = S\cos\theta = 12{,}000 \times 0.856 = 10.272 \text{ kW}$$

while the reactive power is

$$Q = S\sin\theta = 12{,}000 \times 0.517 = 6.204 \text{ kVA}$$

(b) Since the pf is lagging, the complex power is

$$\mathbf{S} = P + jQ = 10.272 + j6.204 \text{ kVA}$$

From $\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}}$, we obtain

$$\mathbf{I}^*_{\text{rms}} = \frac{\mathbf{S}}{\mathbf{V}_{\text{rms}}} = \frac{10{,}272 + j6204}{120\angle 0^\circ} = 85.6 + j51.7 \text{ A} = 100\angle 31.13^\circ \text{ A}$$

Thus $\mathbf{I}_{\text{rms}} = 100\angle -31.13^\circ$ and the peak current is

$$I_m = \sqrt{2}I_{\text{rms}} = \sqrt{2}(100) = 141.4 \text{ A}$$

(c) The load impedance

$$\mathbf{Z} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{I}_{\text{rms}}} = \frac{120\angle 0^\circ}{100\angle -31.13^\circ} = 1.2\angle 31.13^\circ\ \Omega$$

which is an inductive impedance.

1. Alexander practice problems 11.1-11.3
2. Alexander practice problems 11.5-11.12