Queueing Systems

Ahmed Ali Hasan —— 22010538
Ibraheem Abdelhalim Mohammed —— 22010474
Moamen Hesham Mohammed —— 22011063

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1. Introduction

The M/M/1 queue is a single-server queueing system in which arrivals follow a *Poisson process* and service times follow an *exponential distribution*.

In this work, we explore the theoretical characteristics of M/M/1 queues, analyze their performance metrics, and compare these results with outcomes from simulation runs.

- Theoretical calculations are based on the following parameters:
  - Arrival rate (λ)
  - Service rate (μ)
- Simulation runs are conducted with the same parameters, generating random arrival and service times.

2. Theoretical Calculations

Arrival Rate (λ) : 4 customers per hour
Service Rate (μ) : 12 customers per hour

System Utilization (ρ)

$$\begin{aligned}\rho &= \frac{\lambda}{\mu} \\ &= \frac{4}{12} \\ &= \mathbf{0.3333}\end{aligned}$$

Average Wait Time in Queue (W_q)

$$\begin{aligned}W_q &= \frac{\lambda}{\mu(\mu-\lambda)} \\ &= \frac{4}{12(12-4)} \\ &= \frac{4}{96} \\ &= \frac{1}{24} \\ &\Rightarrow W_q = \frac{1}{24} \times 60 = \mathbf{2.5} \text{ min}\end{aligned}$$

Average Time in System (W_s)

$$\begin{aligned}W_s &= W_q + \frac{1}{\mu} \\ &= \frac{1}{24} + \frac{1}{12} \\ &= \frac{1+2}{24} \\ &= \frac{3}{24} \\ &\Rightarrow W_s = \frac{3}{24} \times 60 = \mathbf{7.5} \text{ min}\end{aligned}$$

Time-Averaged Customers in System (L)

$$\begin{aligned} L &= \lambda W_s \\ &= 4 \times \frac{3}{24} \\ &= \frac{12}{24} \\ &= \frac{1}{2} \\ &= \mathbf{0.5} \end{aligned}$$

Time-Averaged Customers in Queue (L_q)

$$\begin{aligned} L_q &= \lambda W_q \\ &= 4 \times \frac{1}{24} \\ &= \frac{4}{24} \\ &= \frac{1}{6} \\ &= \mathbf{0.1667} \end{aligned}$$

3. Theoretical & Simulation Comparison

Table 3.1: $\mu = 12$, $\lambda = 4$ (Scenario 0)

Performance Metrics	Theoretical	Simulation
Server Utilization ($\boldsymbol{\rho}$)	0.3333	0.3344
Average Wait Time in Queue ($\boldsymbol{W_q}$)	2.5000 min	2.5964 min
Average Time in System ($\boldsymbol{W_s}$)	7.5000 min	7.5841 min
Time Averaged Customers in System ($\boldsymbol{L}$)	0.5000	0.5085
Time Averaged Customers in Queue ($\boldsymbol{L_q}$)	0.1667	0.1741
Probability Distribution		
$\boldsymbol{P_0}$ (System has 0 customers)	0.6667	0.6656
$\boldsymbol{P_1}$ (System has 1 customer)	0.2222	0.2205
$\boldsymbol{P_2}$ (System has 2 customers)	0.0741	0.0750
$\boldsymbol{P_3}$ (System has 3 customers)	0.0247	0.0250

Table 3.2: $\mu = 12$, $\lambda = 6$ (Scenario 1)

Performance Metrics	Theoretical	Simulation
Server Utilization ($\boldsymbol{\rho}$)	0.5000	0.5054
Average Wait Time in Queue ($\boldsymbol{W_q}$)	5.0000 min	4.9949 min
Average Time in System ($\boldsymbol{W_s}$)	10.0000 min	10.0486 min
Time Averaged Customers in System ($\boldsymbol{L}$)	1.0000	1.0050
Time Averaged Customers in Queue ($\boldsymbol{L_q}$)	0.5000	0.4996
Probability Distribution		
$\boldsymbol{P_0}$ (System has 0 customers)	0.5000	0.4946
$\boldsymbol{P_1}$ (System has 1 customer)	0.2500	0.2528
$\boldsymbol{P_2}$ (System has 2 customers)	0.1250	0.1270
$\boldsymbol{P_3}$ (System has 3 customers)	0.0625	0.0641

Table 3.3: $\mu = 12$, $\lambda = 10$ (Scenario 2)

Performance Metrics	**Theoretical**	**Simulation**
Server Utilization ($\boldsymbol{\rho}$)	0.8333	0.8313
Average Wait Time in Queue ($\boldsymbol{W_q}$)	25.0000 min	25.8820 min
Average Time in System ($\boldsymbol{W_s}$)	30.0000 min	30.8861 min
Time Averaged Customers in System ($\boldsymbol{L}$)	5.0000	5.1311
Time Averaged Customers in Queue ($\boldsymbol{L_q}$)	4.1667	4.2998
Probability Distribution		
$\boldsymbol{P_0}$ (System has 0 customers)	0.1667	0.1687
$\boldsymbol{P_1}$ (System has 1 customer)	0.1389	0.1374
$\boldsymbol{P_2}$ (System has 2 customers)	0.1157	0.1124
$\boldsymbol{P_3}$ (System has 3 customers)	0.0965	0.0948

4. Results Discrepancy Analysis

The simulation results are closely aligned with the theoretical with minor offsets. This difference can be attributed to the randomness nature and the finite time span of the simulation process. Though the simulation results can be more accurate with increasing the time span, the current results are sufficient for practical applications.

5. Design and Implementation

5.1 Simulation

- A queue is used to track customers waiting for service.
- The simulation runs by default for 300 thousand minutes.
- Random arrival and service times based on λ and μ are generated.
- The simulation current running time traverses to the next customer arrival or service completion times (the closer event).
- After the simulation current time reaches the end, arrival generation stops, and the service continues until all customers are served (queue gets empty).
- The simulation data:
  - Total number of customers served.
  - Total time the server was busy.
  - List containing each customer time portion spent in the **queue**.
  - List containing each customer time portion spent in the **system**.
  - Total waiting time in the **queue**.
  - Total time spent in the **system**.
  - Time-averaged customers in the **queue**.
  - Time-averaged customers in the **system**.
  - Number of customers in the **queue** mapped to a list of durations during which that number of customers persisted.
  - Number of customers in the **system** mapped to a list of durations during which that number of customers persisted.
- The queueing metrics are calculated based on the simulation data:
  - $\rho = \frac{\text{Total time the server was busy}}{\text{Total Simulation time}}$
  - $W_q = \frac{\text{Total waiting time in the queue}}{\text{Total customers served}}$
  - $W_s = \frac{\text{Total time spent in the system}}{\text{Total customers served}}$
  - $L = \text{Time-averaged customers in the system}$
  - $L_q = \text{Time-averaged customers in the queue}$
  - $P\{\text{customers} = n\} = \frac{\text{Total duration the system had n customers}}{\text{Total Simulation time}}$

5.2 Theoretical

- ρ is calculated as $\frac{\lambda}{\mu}$
- W_q is calculated as $\frac{\lambda}{\mu \cdot (\mu - \lambda)}$
- W_s is calculated as $W_q + \frac{1}{\mu}$
- L is calculated as $\lambda \cdot W_s$
- L_q is calculated as $\lambda \cdot W_q$
- The probability distribution is calculated using the formula $P_n = (1 - \rho) \cdot \rho^n$ for $n \geq 0$

5.3 Plot

- 20 well-spreaded data points are generated for ρ ranging from 0 to 1.
- For each ρ, Theoretical values of W_q are calculated as follows:
  - $\lambda = \rho \cdot \mu$
  - $W_q = \frac{\lambda}{\mu \cdot (\mu - \lambda)}$
- For each ρ, Simulation values of W_q are calculated as follows:
  - $\lambda = \rho \cdot \mu$
  - A simulation run for λ and μ is performed.
  - We then obtain W_q from the simulation data.

6. User Manual

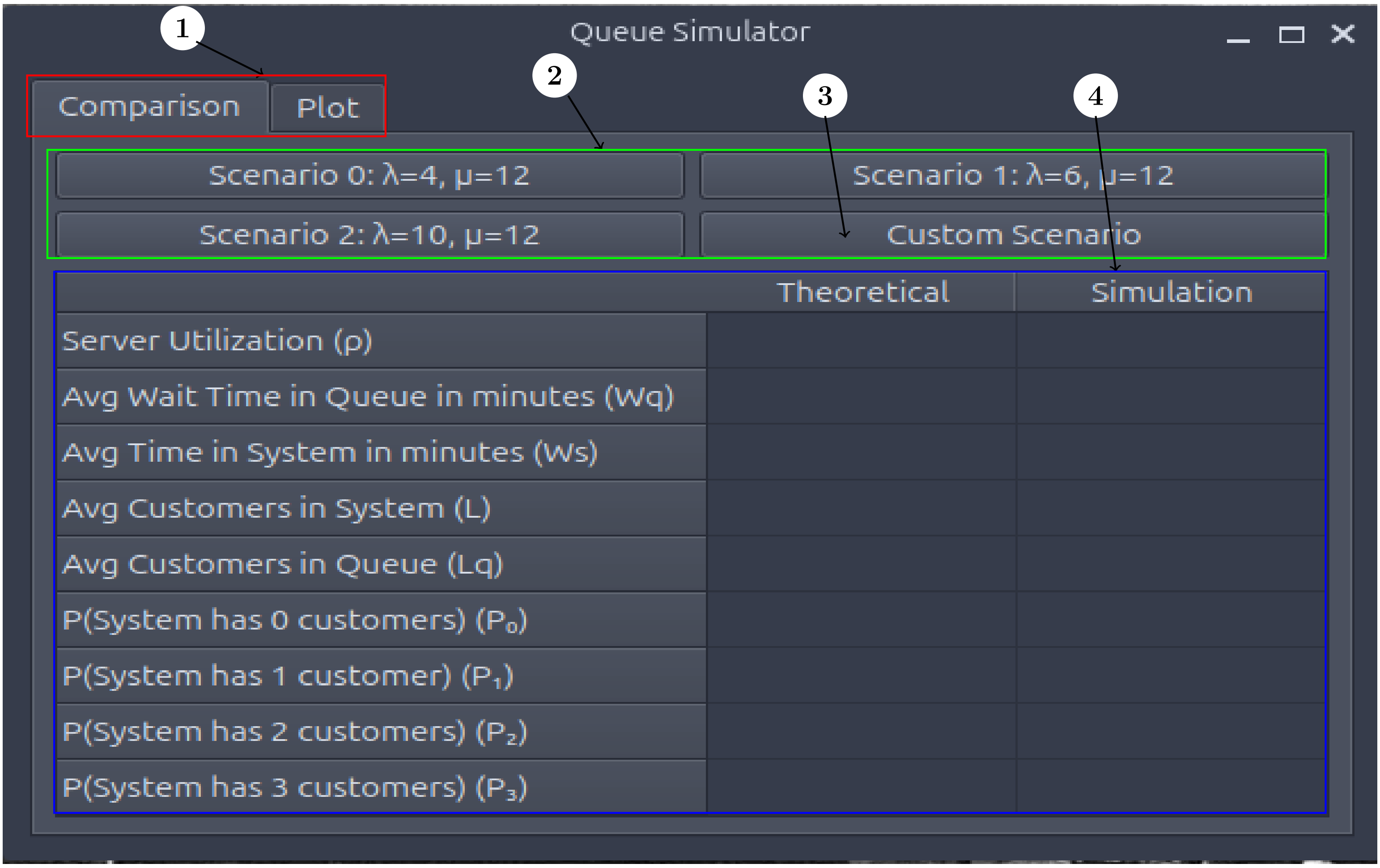


Figure 6.1: GUI Main Window

Fig (1) Descriptions

1. **Tabs switching between Plot and Simulation run**:
2. **Different Scenarios for Simulation Runs**
3. **Custom Scenario Choice**:

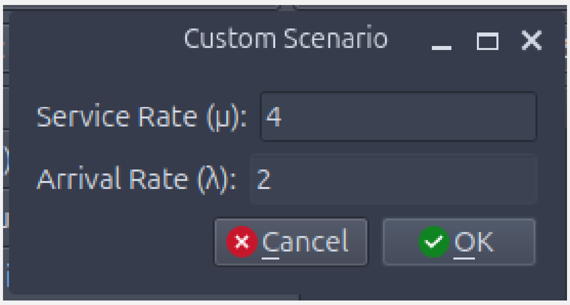
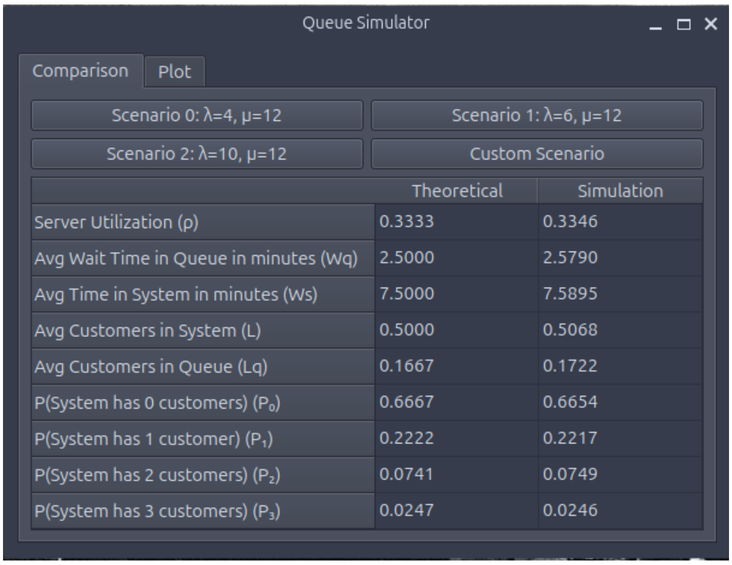


Figure 6.2: Custom Scenario Window

4. **Comparison Table between Theoretical and Simulation Results**

7. Sample Runs

7.1 Scenario #0



Queue Simulator

Comparison | Plot

Scenario 0: λ=4, μ=12 | Scenario 1: λ=6, μ=12 | Scenario 2: λ=10, μ=12 | Custom Scenario

	Theoretical	Simulation
Server Utilization (ρ)	0.3333	0.3346
Avg Wait Time in Queue in minutes (Wq)	2.5000	2.5790
Avg Time in System in minutes (Ws)	7.5000	7.5895
Avg Customers in System (L)	0.5000	0.5068
Avg Customers in Queue (Lq)	0.1667	0.1722
P(System has 0 customers) (P_0)	0.6667	0.6654
P(System has 1 customer) (P_1)	0.2222	0.2217
P(System has 2 customers) (P_2)	0.0741	0.0749
P(System has 3 customers) (P_3)	0.0247	0.0246

Figure 7.1: Comparison of Theoretical and Simulation Results for Scenario 0

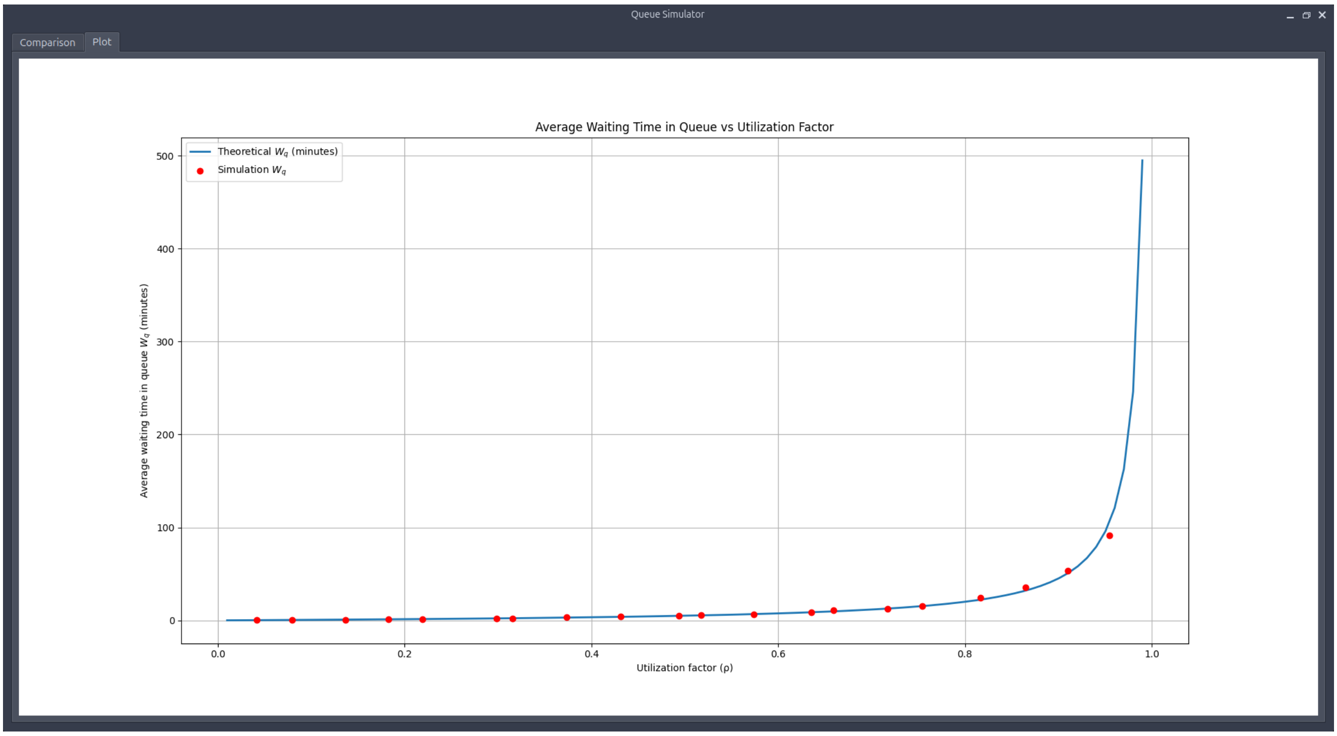
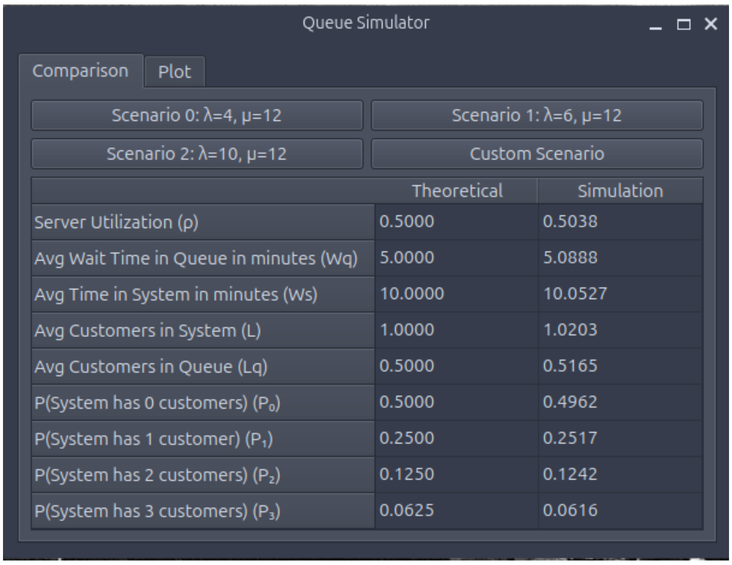


Figure 7.2: Plot of W_q for Scenario 0

7.2 Scenario #1



	Theoretical	Simulation
Server Utilization (ρ)	0.5000	0.5038
Avg Wait Time in Queue in minutes (Wq)	5.0000	5.0888
Avg Time in System in minutes (Ws)	10.0000	10.0527
Avg Customers in System (L)	1.0000	1.0203
Avg Customers in Queue (Lq)	0.5000	0.5165
P(System has 0 customers) (P_0)	0.5000	0.4962
P(System has 1 customer) (P_1)	0.2500	0.2517
P(System has 2 customers) (P_2)	0.1250	0.1242
P(System has 3 customers) (P_3)	0.0625	0.0616

Figure 7.3: Comparison of Theoretical and Simulation Results for Scenario 1

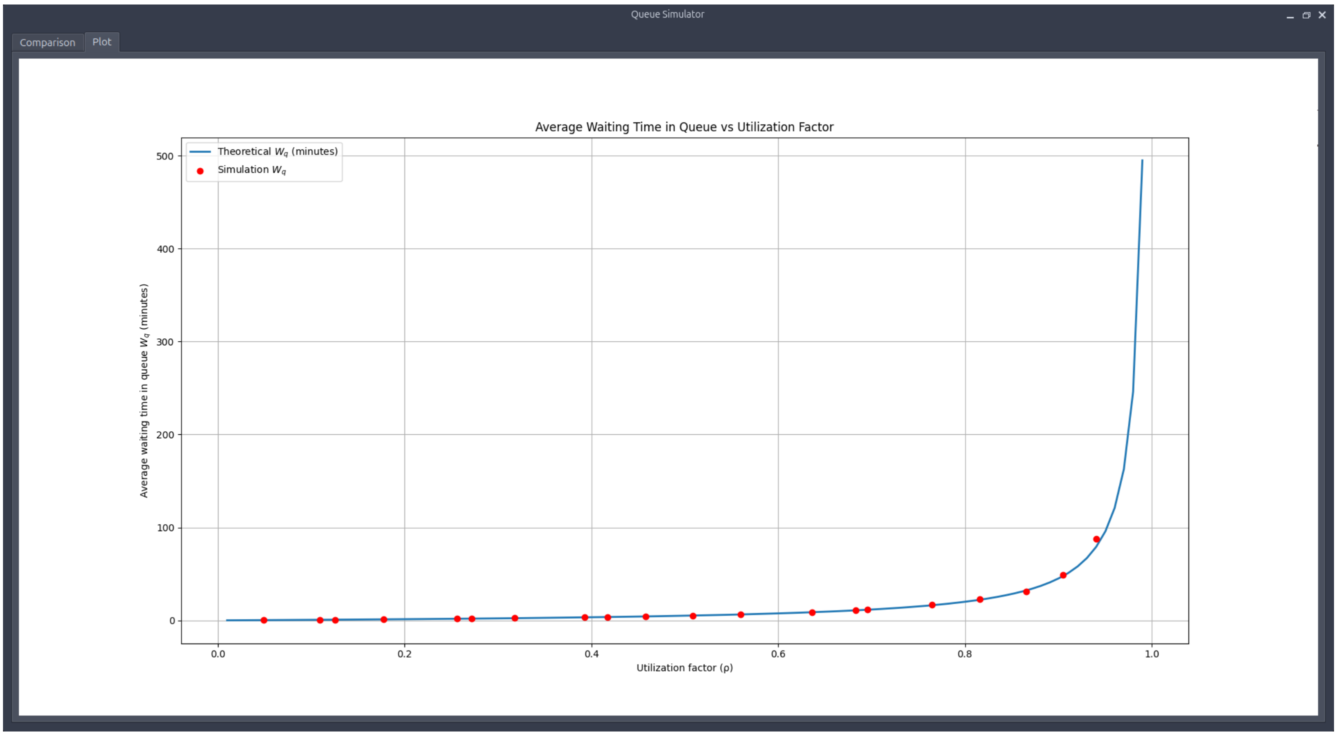


Figure 7.4: Plot of W_q for Scenario 1

7.3 Scenario #2

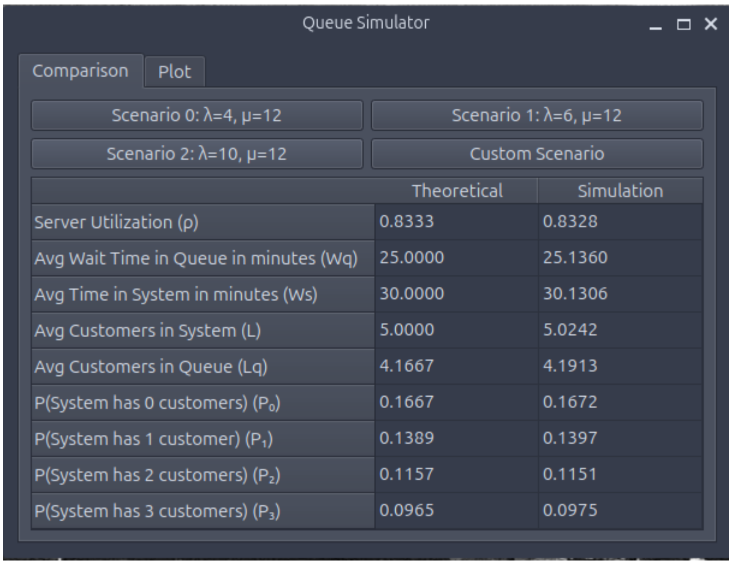


Figure 7.5: Comparison of Theoretical and Simulation Results for Scenario 2

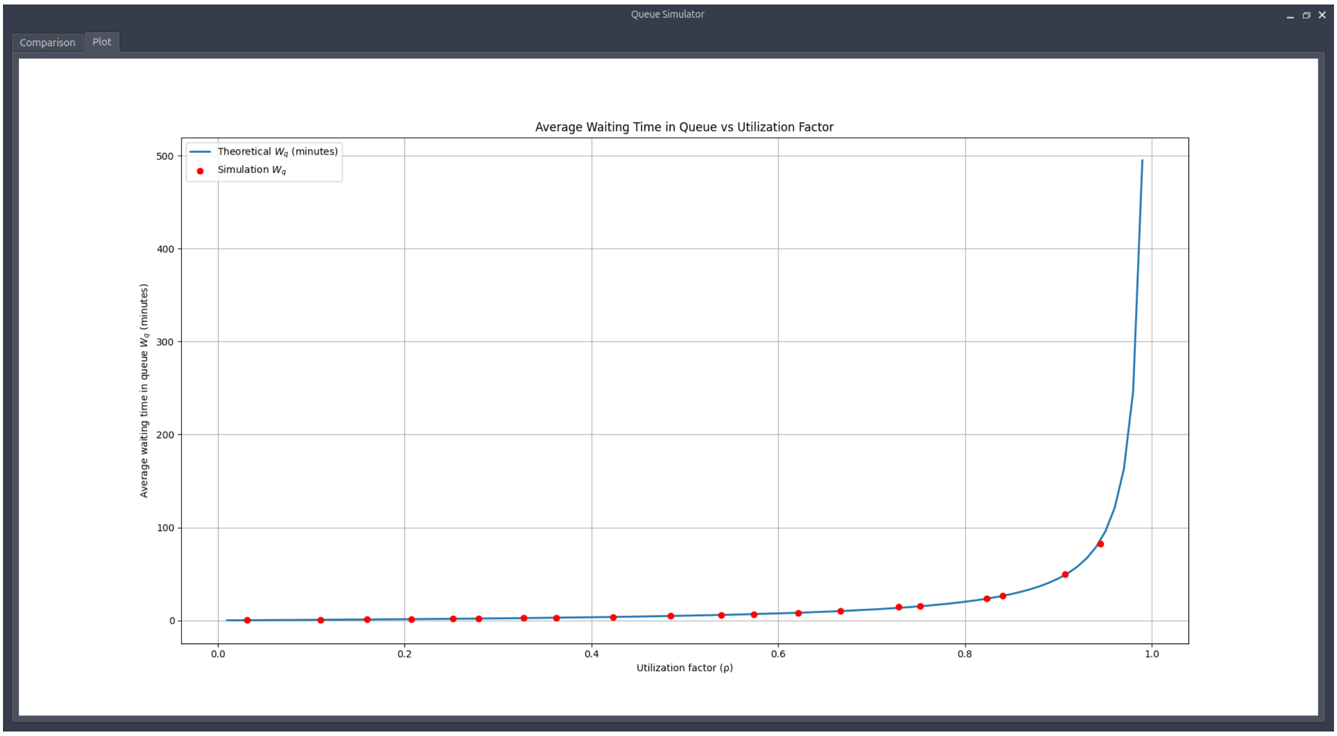
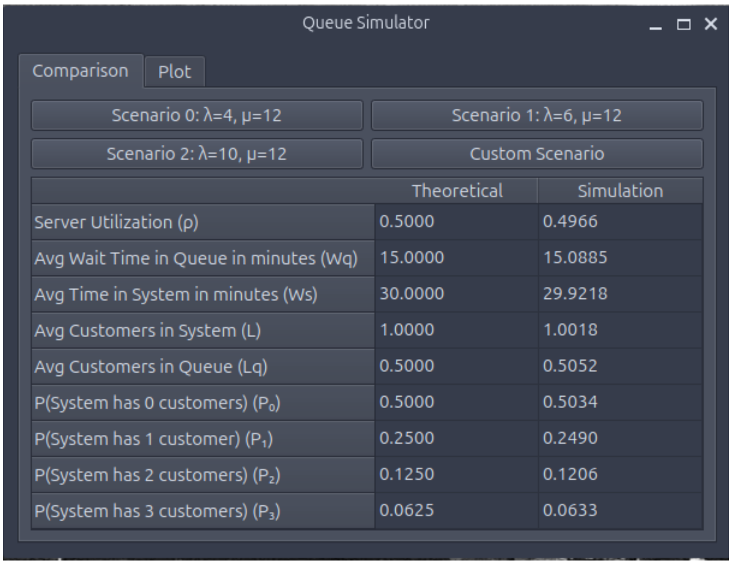


Figure 7.6: Plot of W_q for Scenario 2

7.4 Custom Scenario ($\lambda = 2$, $\mu = 4$)



	Theoretical	Simulation
Server Utilization (ρ)	0.5000	0.4966
Avg Wait Time in Queue in minutes (Wq)	15.0000	15.0885
Avg Time in System in minutes (Ws)	30.0000	29.9218
Avg Customers in System (L)	1.0000	1.0018
Avg Customers in Queue (Lq)	0.5000	0.5052
P(System has 0 customers) (P_0)	0.5000	0.5034
P(System has 1 customer) (P_1)	0.2500	0.2490
P(System has 2 customers) (P_2)	0.1250	0.1206
P(System has 3 customers) (P_3)	0.0625	0.0633

Figure 7.7: Comparison of Theoretical and Simulation Results for Custom Scenario

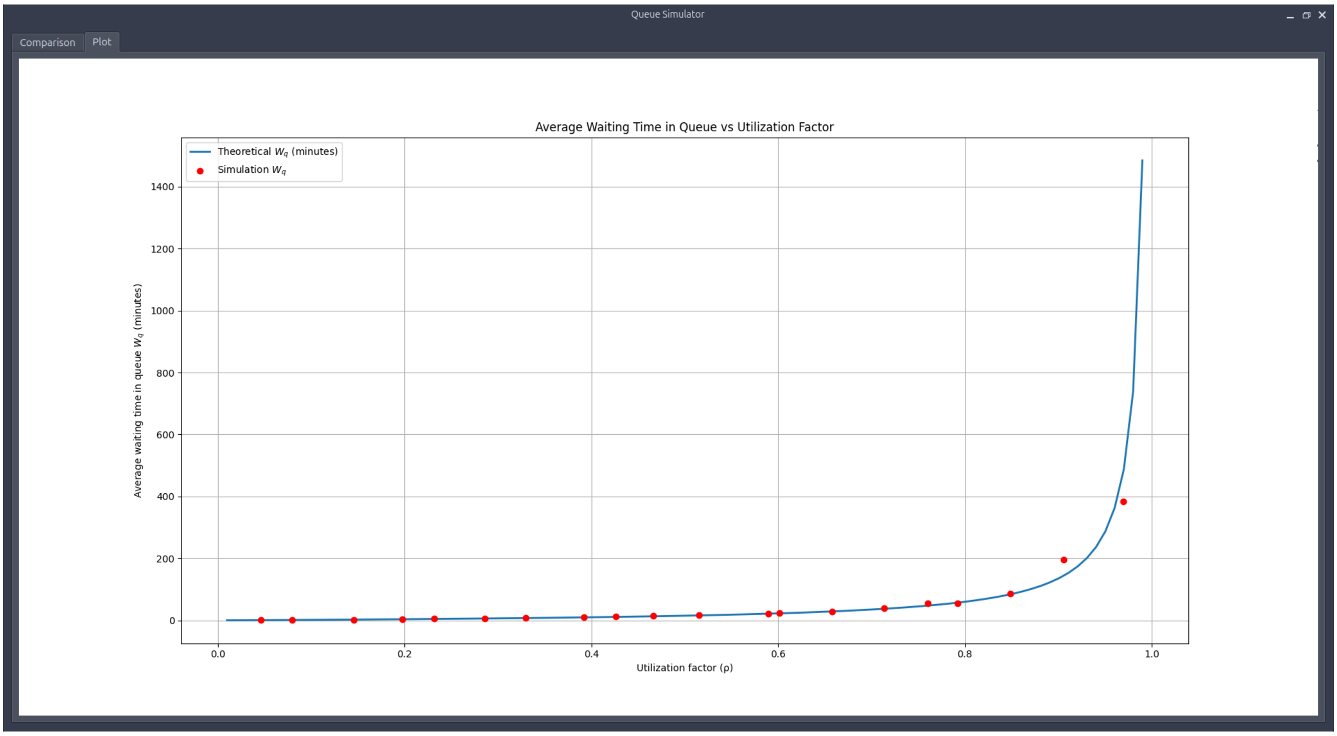


Figure 7.8: Plot of W_q for Custom Scenario

7.5 Error Handling

7.5.1 Empty Input Test Scenario

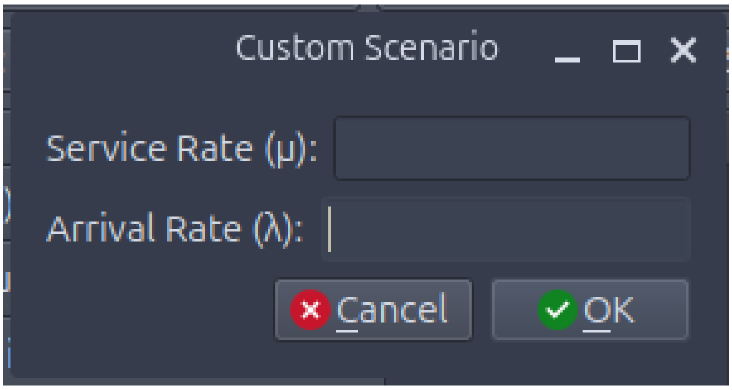


Figure 7.9: Input Window for Empty Input Test Scenario

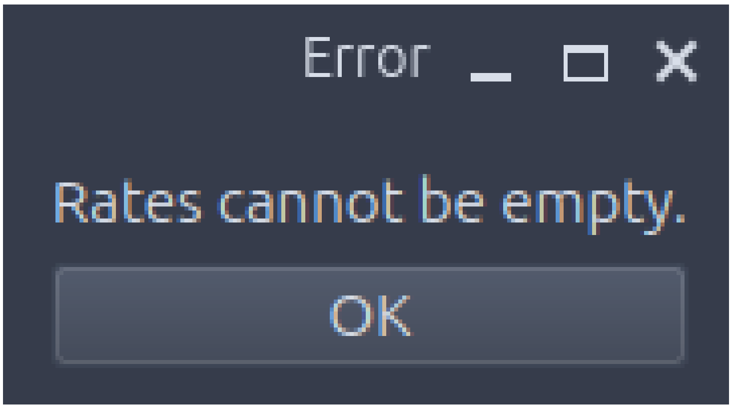


Figure 7.10: Error Message for Empty Input Test Scenario

7.5.2 Invalid Input Test Scenario ($\lambda = \mu$)

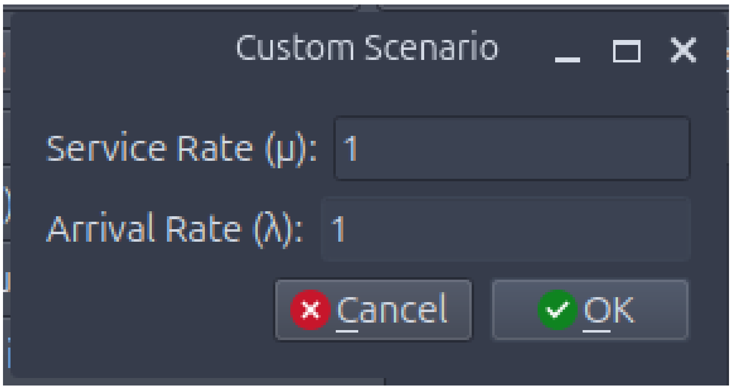


Figure 7.11: Input Window for Invalid Input Test Scenario

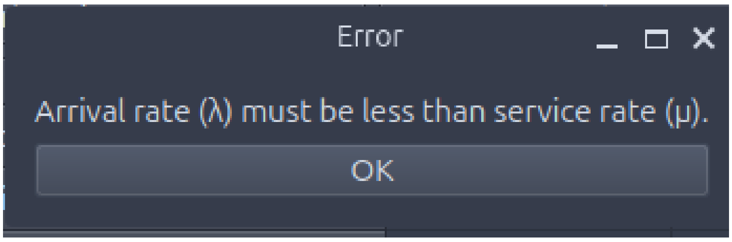


Figure 7.12: Error Message for Invalid Input Test Scenario

7.5.3 Invalid Input Test Scenario (λ or $\mu \leq 0$)

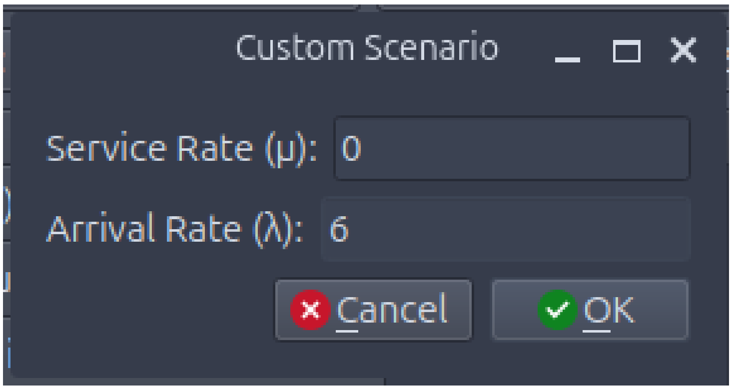


Figure 7.13: Input Window for Invalid Input Test Scenario

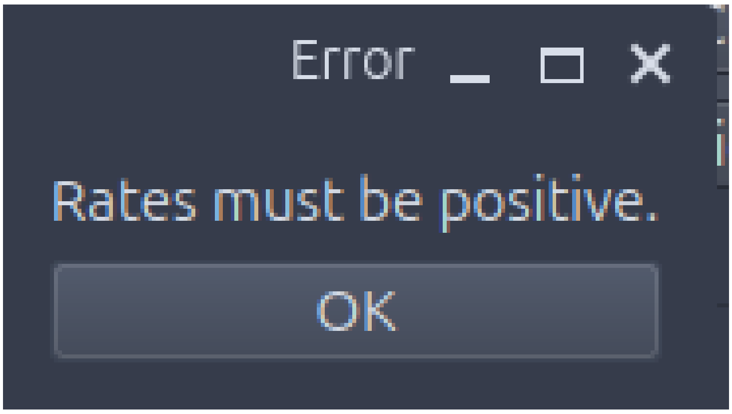


Figure 7.14: Error Message for Invalid Input Test Scenario

8. Assumptions and Notes

- In the simulation, once the planned time span is reached, no new arrivals are generated. However, service continues until all remaining customers in the queue are served.
- The values of W_q and W_s are expressed in minutes for clarity.
- The input rates for λ and μ are assumed to be given in customers per hour.