Automatic Control Systems

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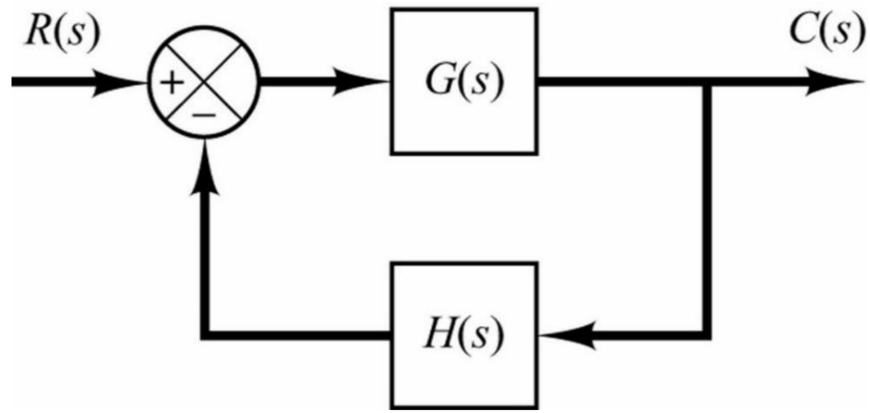
- **Reading: chapter 9**
  - Sections 9.1, 9.2, 9.3
  - Study **Table 9.6** on pages 704-711

Nyquist Stability Criterion

Stability in the frequency domain



- Open loop system $G(s)H(s)$

Characteristic equation of the Closed-loop system

$$\boxed{\boldsymbol{F(s) = 1 + G(s)H(s)}}$$

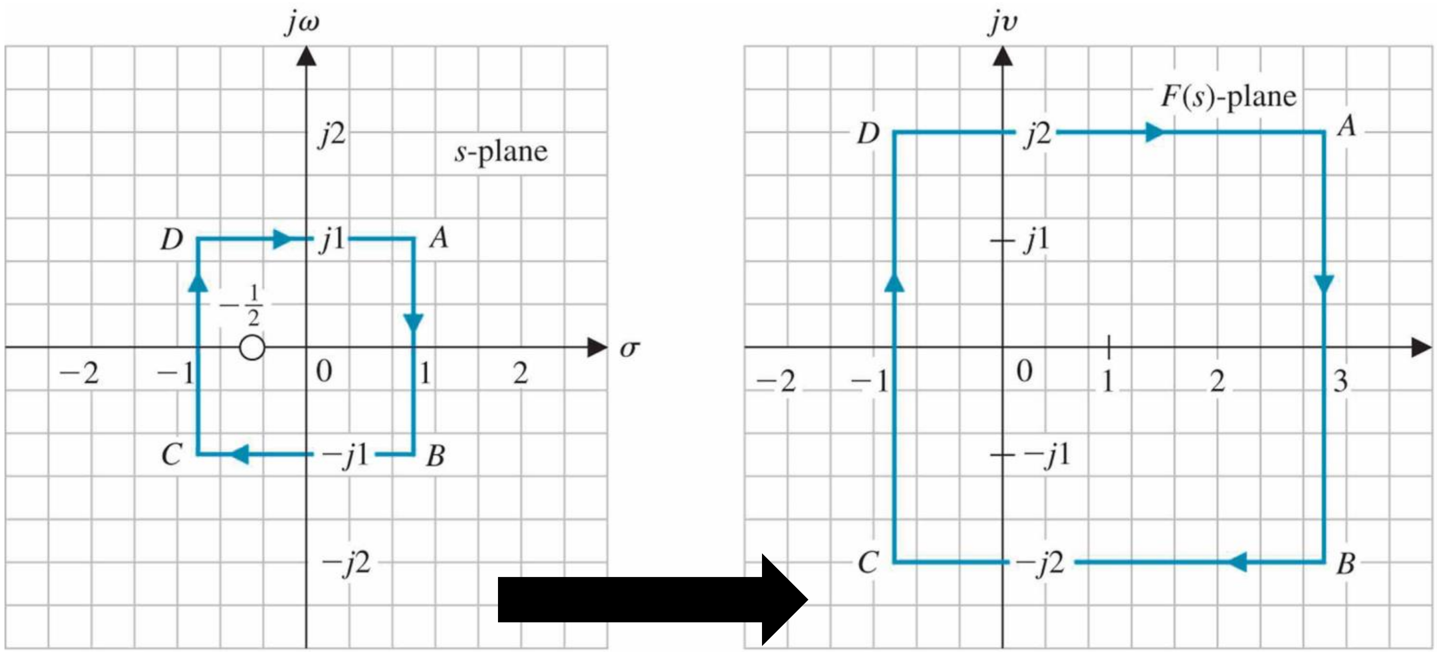
- The **Nyquist stability criterion**, a graphical method, determines the **stability of a closed-loop system** from its **open-loop system** without determining the closed-loop poles.

- The Nyquist stability criterion relates the open-loop frequency response $G(s)H(s)$ to the **number of zeros of the characteristic equation** that lies in the $1 + G(s)H(s)$ right-half plane

- The **Nyquist stability criterion** is based on **Cauchy's theorem** which is concerned with **mapping contours** in the s-plane

Contour Mapping

A contour map is a trajectory in one plane , s −plane, mapped into another plane, $F(s)$ − plane, by a relation $F(s)$. A **closed contour** in the s −plane produces a **closed contour** in the $F(s)$-plane



(a) (b)

- Mapping function

$$\boldsymbol{F(s) = 2s + 1}$$

- Change of variables

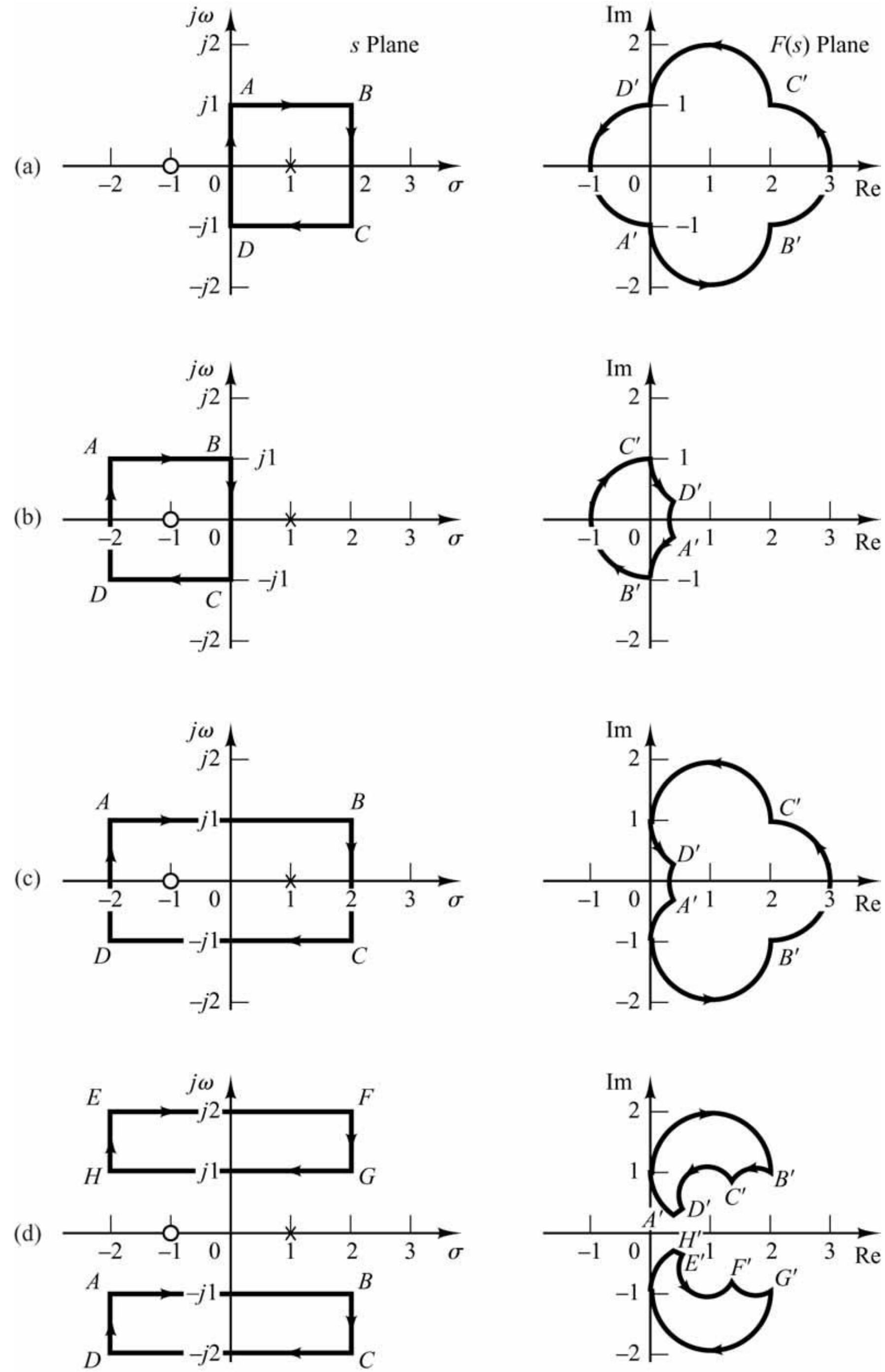
$$u + jv = F(\sigma + j\omega) = 2(\sigma + j\omega) + 1 = (2\sigma + 1) + j\omega$$

$$\boldsymbol{u = 2\sigma + 1, \qquad v = 2\omega}$$

- **example**

Mapping function:

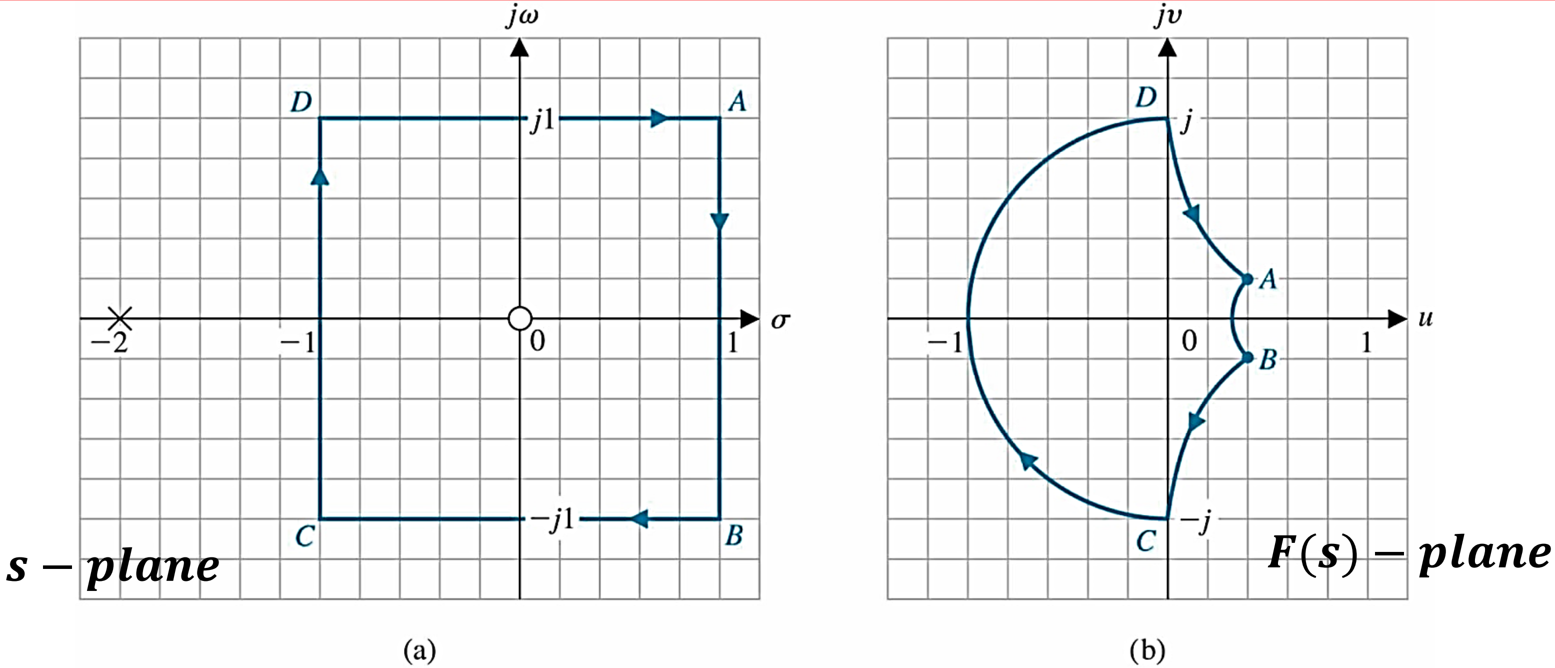
$$\boxed{F(s)=\frac{(s+1)}{(s-1)}}$$



- *example*

$$\boxed{F(s) = \frac{s}{(s+2)}}$$

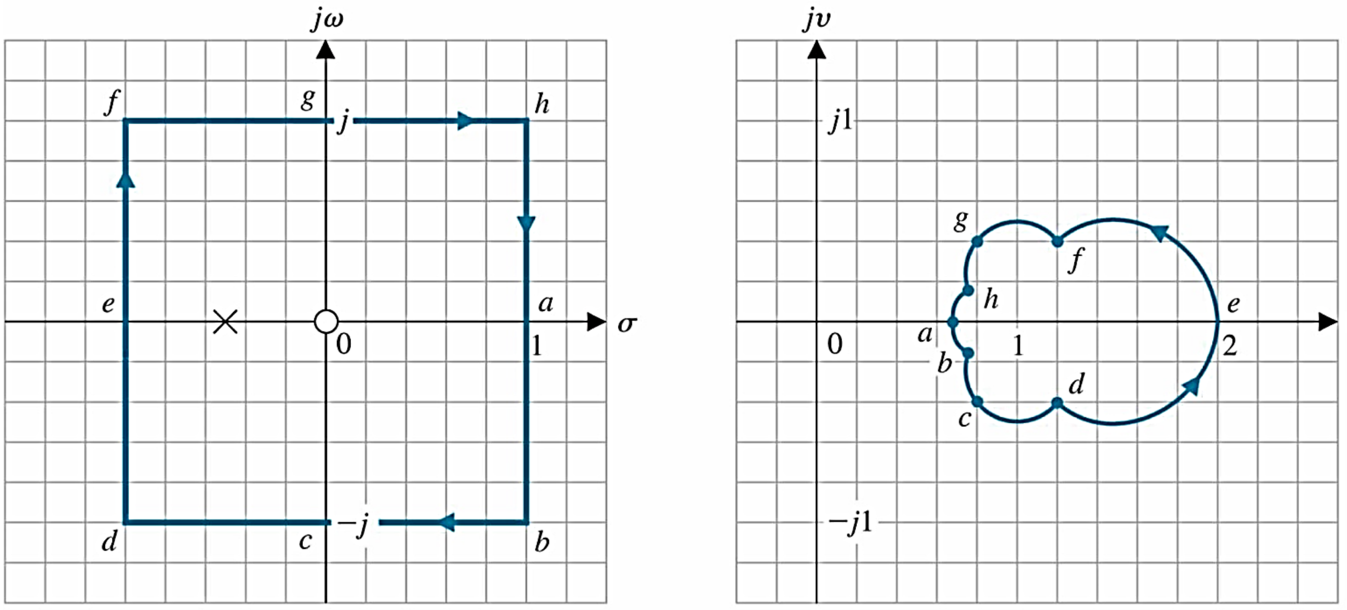
- **Define the positive direction is the <u>clock-wise</u> direction and the enclosed area is <u>on the right</u> "CLOCK-WISE AND FACING RIGHT"**



The contour in the $F(s)$ −plane ***<u>encloses (encircles) the origin</u>*** *of the* ***$F(s)$ −plane***

- ***example***

$$\boxed{F(s) = \frac{s}{(s+0.5)}}$$



$s-plane$ (a)

(b) $F(s)-plane$

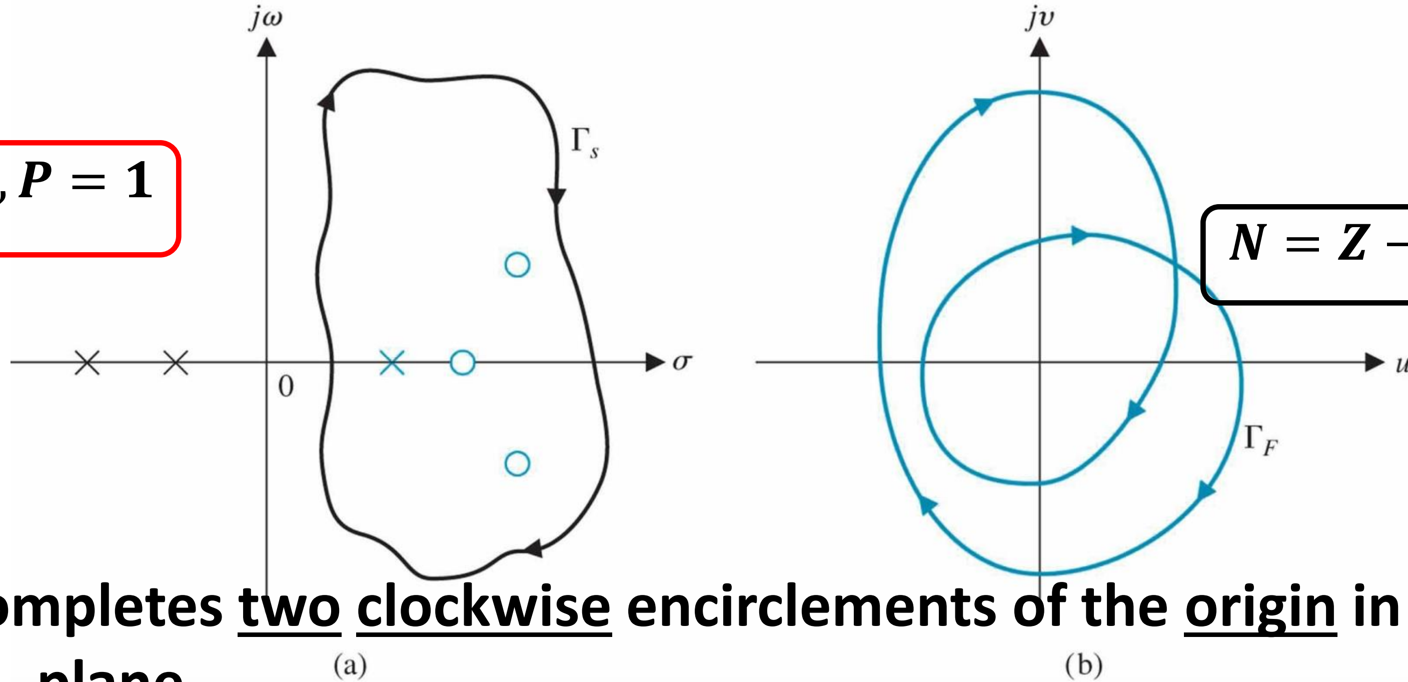
The contour in the $F(s)$ −plane ***<u>does not enclose (encircle)</u>*** *the origin of the* ***$F(s)$ −plane***

Cauchy's Theorem

"Cauchy's Argument Principle"

"if a contour Γ_s in the s-plane encircles Z zeros and P poles of $F(s)$ and does not pass through any poles or zeros of $F(s)$ and the transversal is in the clockwise direction along the contour, the corresponding contour Γ_F in the $F(s)$ -plane encircles the origin of the $F(s)$ –plane $N = Z - P$ times in the clockwise direction"

$\boxed{Z = 3, P = 1}$



$\boxed{N = Z - P = +2}$

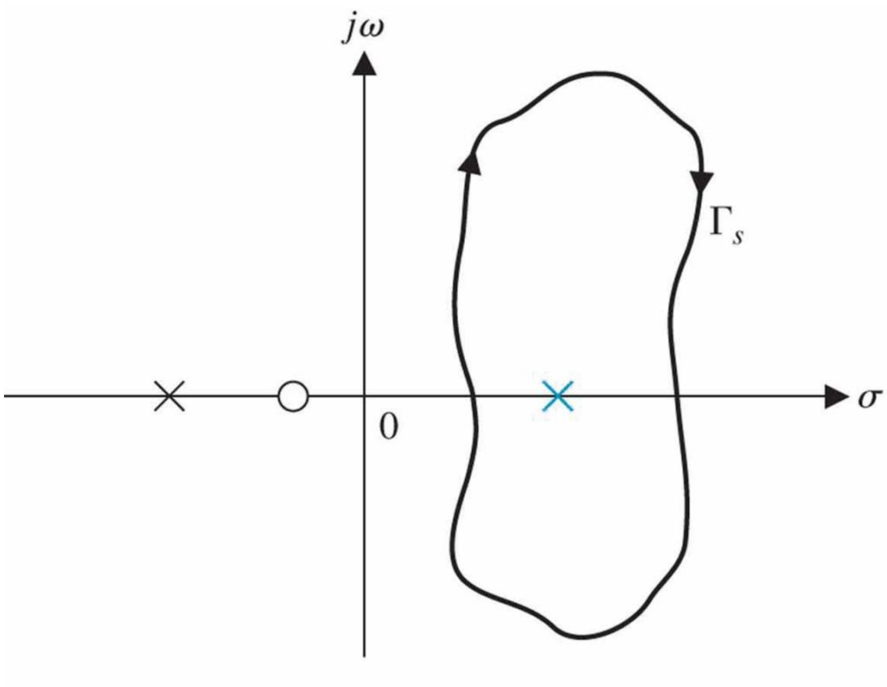
Γ_F completes two clockwise encirclements of the origin in the $F(s)$ - plane

- *example*

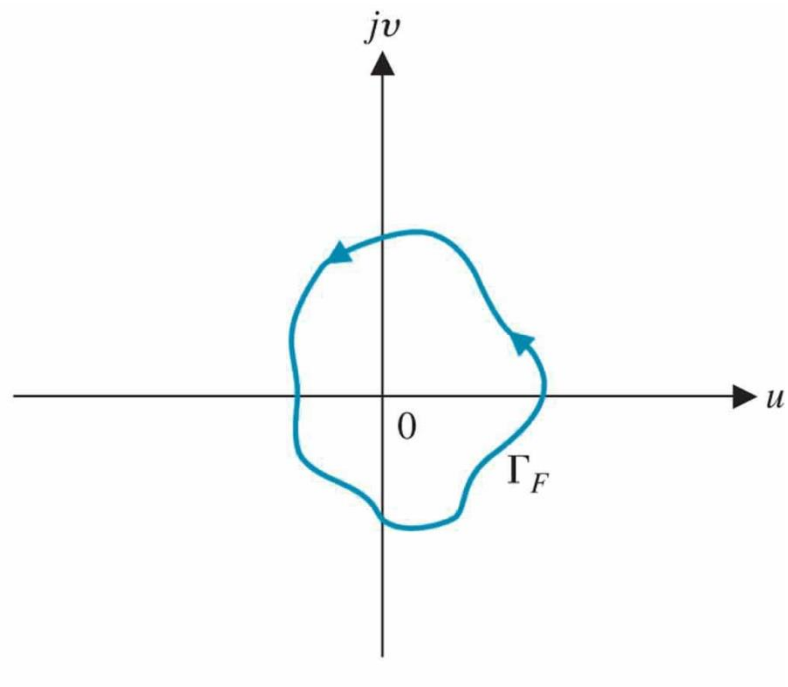
The contour $\boldsymbol{\Gamma_F}$ encircles the <u>origin</u> in the $\boldsymbol{F(s)}$-plane <u>once</u> in the <u>counterclockwise</u> direction

$$\boxed{Z = 0, P = 1}$$

$$\boxed{N = Z - P = -1}$$



(a)

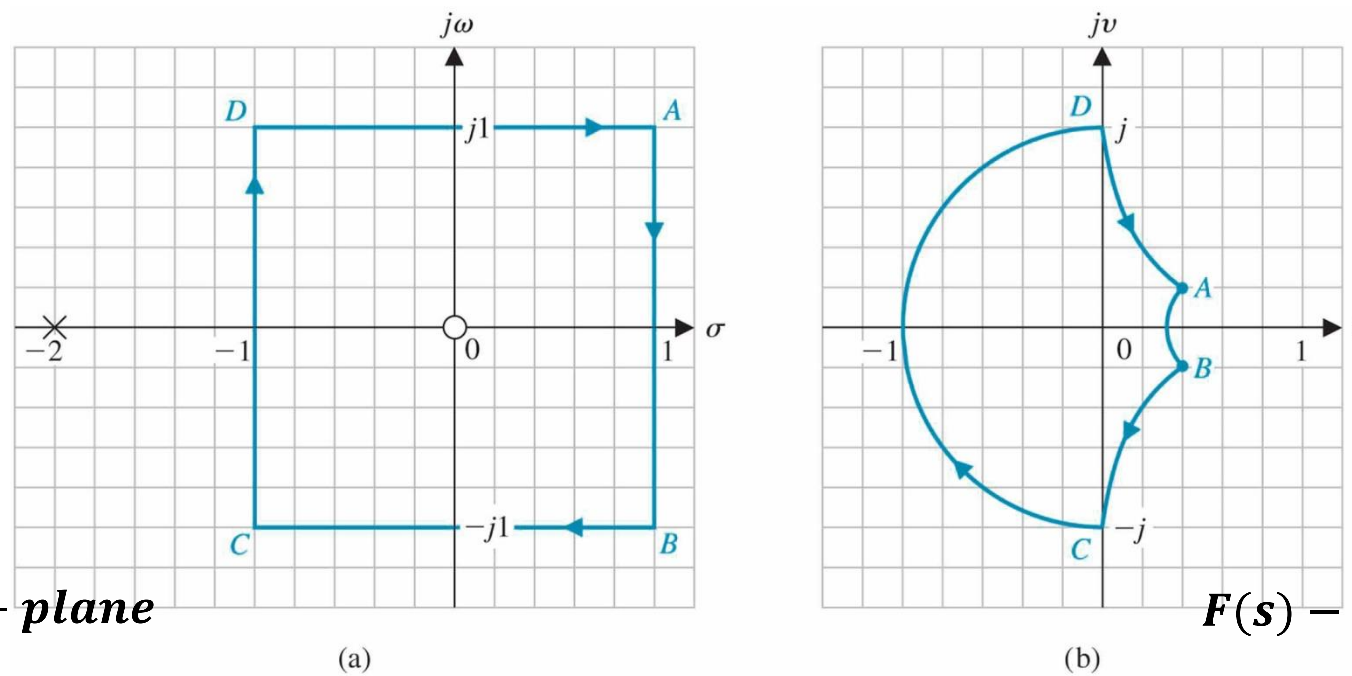


(b)

- *example*

$$\boxed{F(s) = \frac{s}{(s+2)}}$$

$$\boxed{Z = 1, P = 0} \quad \Longrightarrow \quad \boxed{N = Z - P = +1}$$



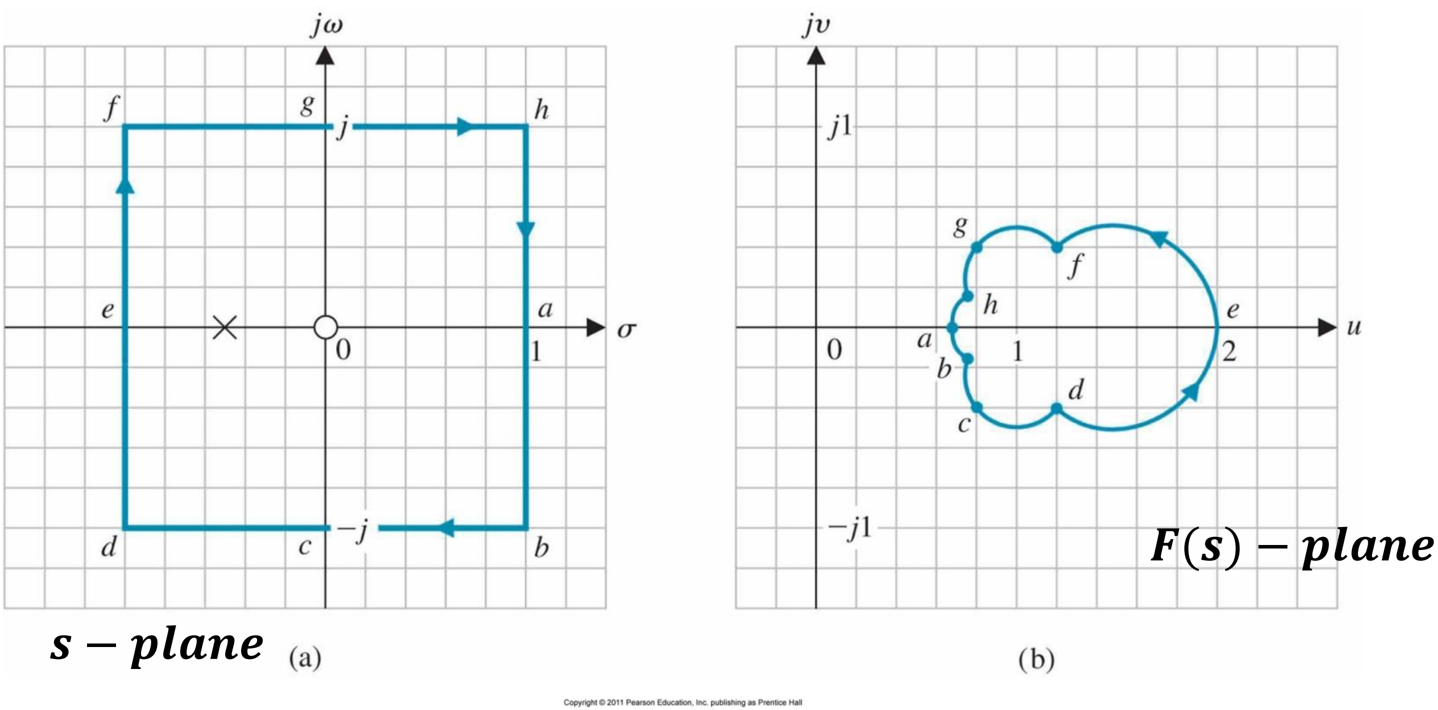
$s - plane$ $\qquad$ $F(s) - plane$

- **The contour in the $F(s)$ −plane <u>encircles the origin</u> of the $F(s)$ −plane *once* <u>in the *clockwise* direction</u>**

- *example*

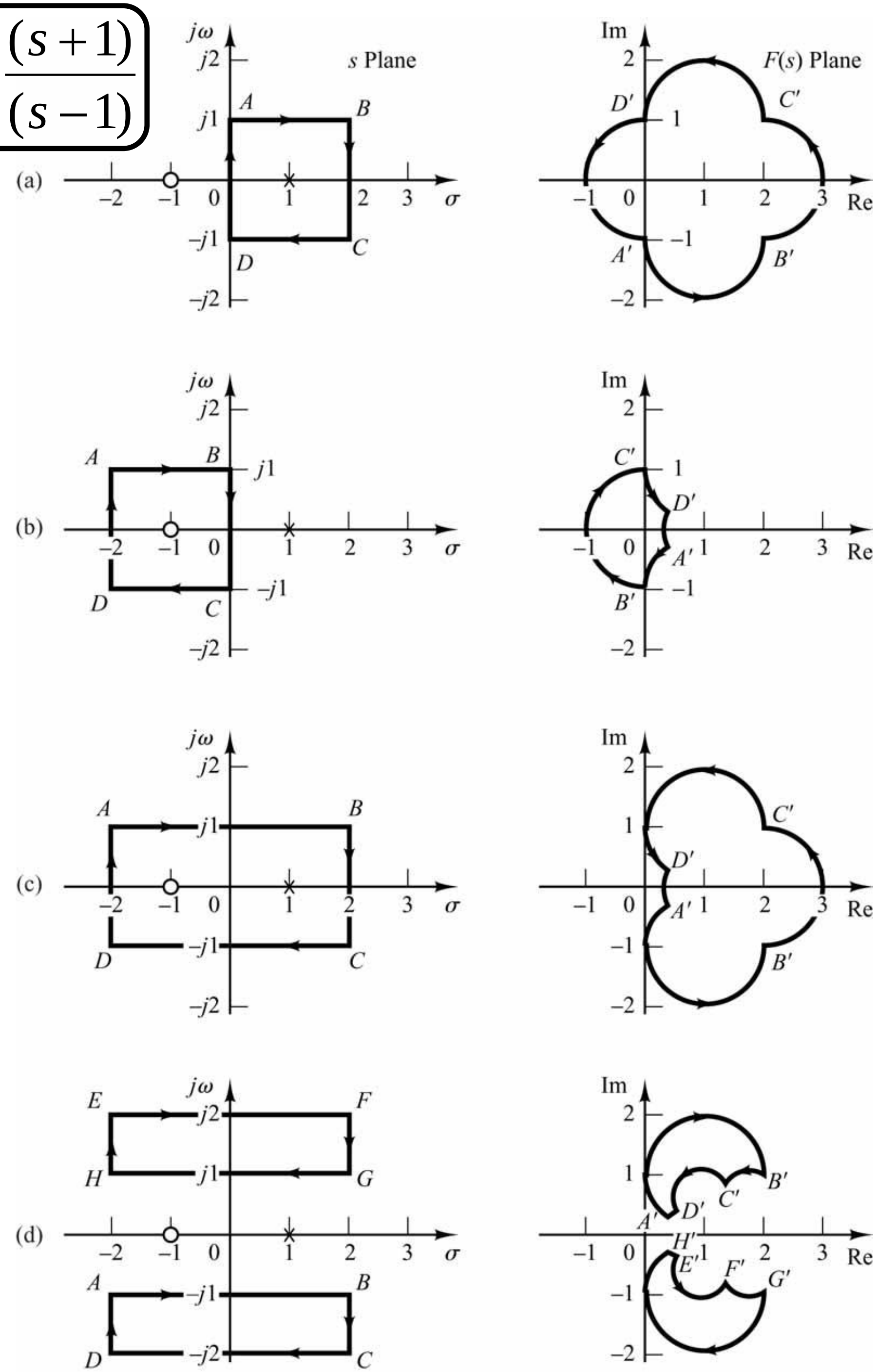
$$\boxed{F(s) = \frac{s}{(s+0.5)}}$$

$$\boxed{Z = 1, P = 1} \quad \Longrightarrow \quad \boxed{N = Z - P = 0}$$



The contour in the $F(s)$ −plane ***<u>does not encircle</u>*** **the <u>origin</u> of the** $F(s)$ −plane

- ***example*** $\boxed{F(s)=\dfrac{(s+1)}{(s-1)}}$



- $\boldsymbol{P=1, Z=0}$ $\boldsymbol{N=Z-P=-1}$
- $\boldsymbol{P=0, Z=1}$ $\boldsymbol{N=Z-P=1}$
- $\boldsymbol{P=1, Z=1}$ $\boldsymbol{N=Z-P=0}$
- $\boldsymbol{P=0, Z=0}$ $\boldsymbol{N=Z-P=0}$

The Nyquist Stability Criterion

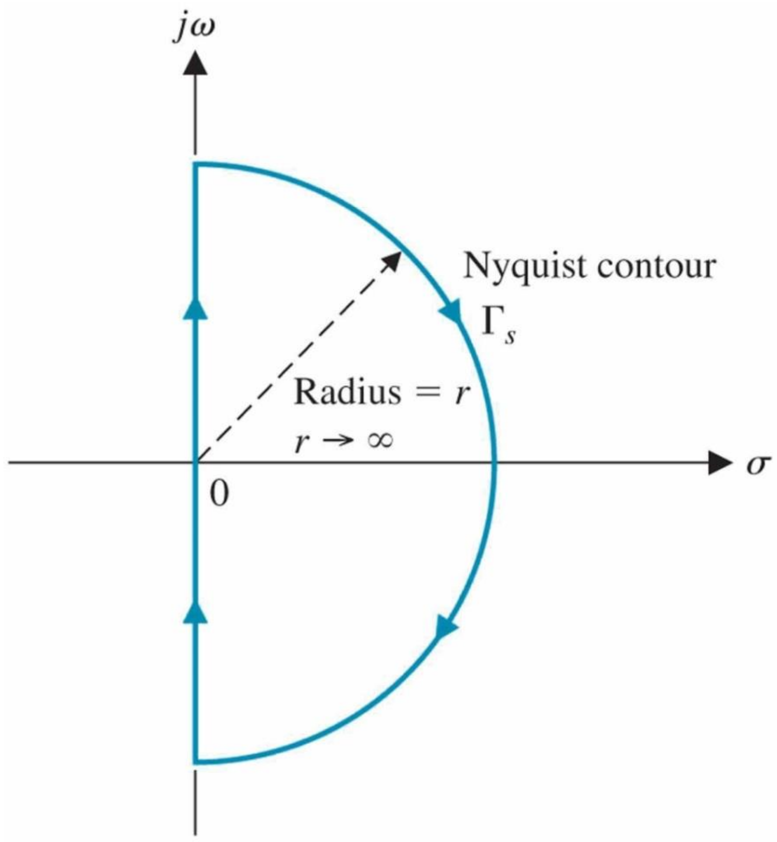
- To ensure stability, we must ascertain that all the **zeros** of the characteristic equation $\boldsymbol{F(s) = 1 + G(s)H(s)}$ would lie in the **left-half-plane (LHP)** of the s-plane, i.e., the left of the $j\omega$ – axis.

- Nyquist chose the contour Γ_s in the s-plane that **encloses the entire right-half s-plane (RHP)**.

- Use Cauchy's theorem to determine if there are any zeros of $\boldsymbol{F(s)}$ that would lie within Γ_s.

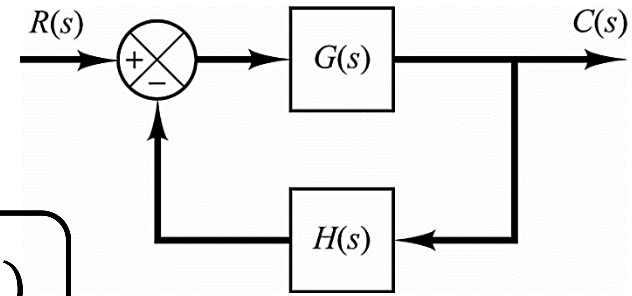
- Plot Γ_F in the $F(s)$-plane and determine the number of encirclements of the **origin** N

- Nyquist plot is a **POLAR** plot



The Nyquist Stability Criterion

- **Characteristic equation $\boldsymbol{F(s)}$** of the system

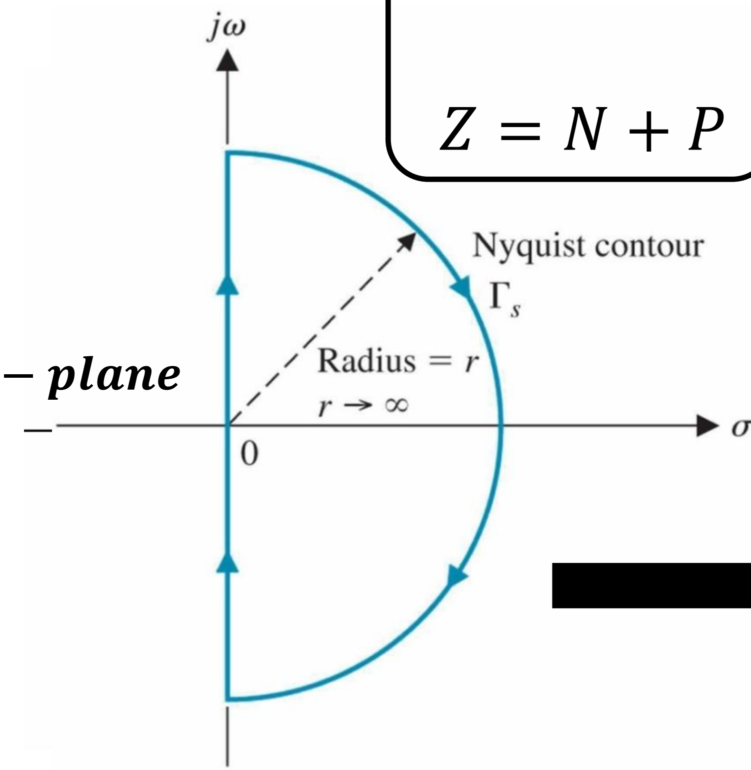


$$\boxed{F(s) = 1 + G(s)H(s)}$$

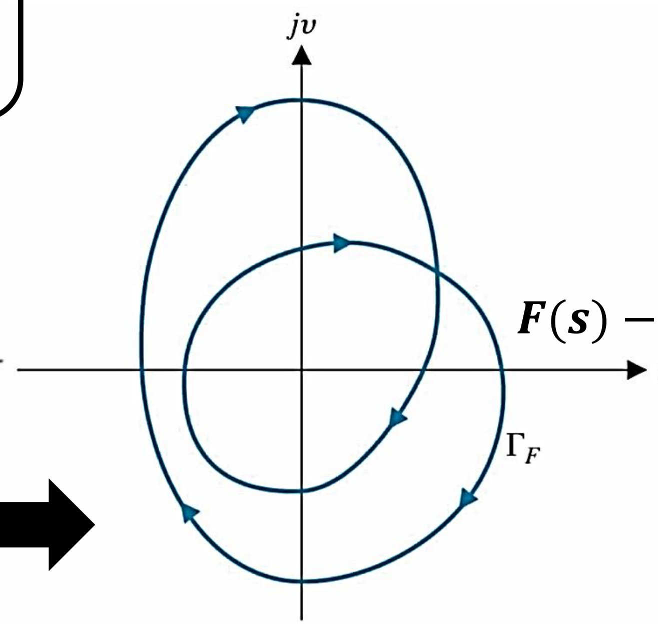
$$N = Z - P$$

$$Z = N + P$$

$\boldsymbol{s-plane}$



$\boldsymbol{F(s)-plane}$

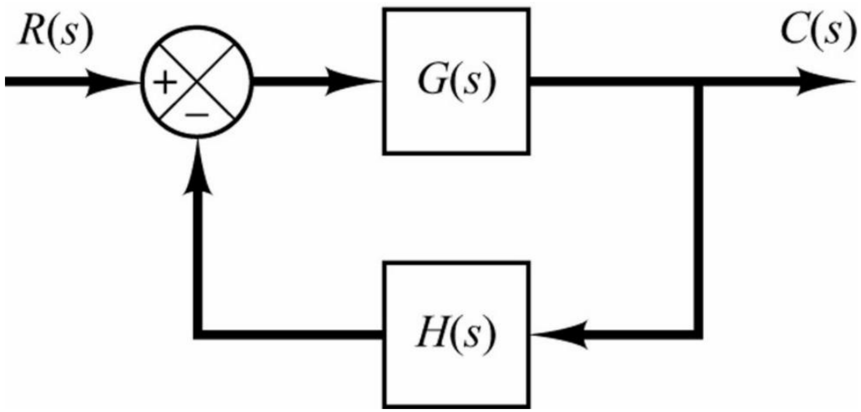


(b)

The Nyquist Stability Criterion

- In order to determine the stability of a closed-loop system we must investigate the **characteristic equation** $\boldsymbol{F(s)}$ of the system

- $$\boxed{F(s) = 1 + G(s)H(s)}$$



$$\boxed{\begin{array}{l} N = Z - P \\ \\ Z = N + P \end{array}}$$

✓ $\boldsymbol{Z}$ = number of **zeros** of $\boldsymbol{1 + G(s)H(s)}$ in the ***right-half s-plane***

➢ $\boldsymbol{P}$ = *number of* ***poles*** *of* $1 + \boldsymbol{G(s)H(s)}$ *in the* ***right-half s-plane*** *!!*

- $\boldsymbol{N}$ = number of ***clockwise*** encirclement of the origin in the $\boldsymbol{F(s)}$**-plane**

- Then, the number of zeros of the **characteristic equation** $\boldsymbol{F(s)}$ in the **RHP** $\boxed{\boldsymbol{Z = N + P}}$

- Note that, $\underline{\boldsymbol{P = the\ number\ of\ OPEN - LOOP\ poles\ in\ the\ RHP}}$ **WHY**?

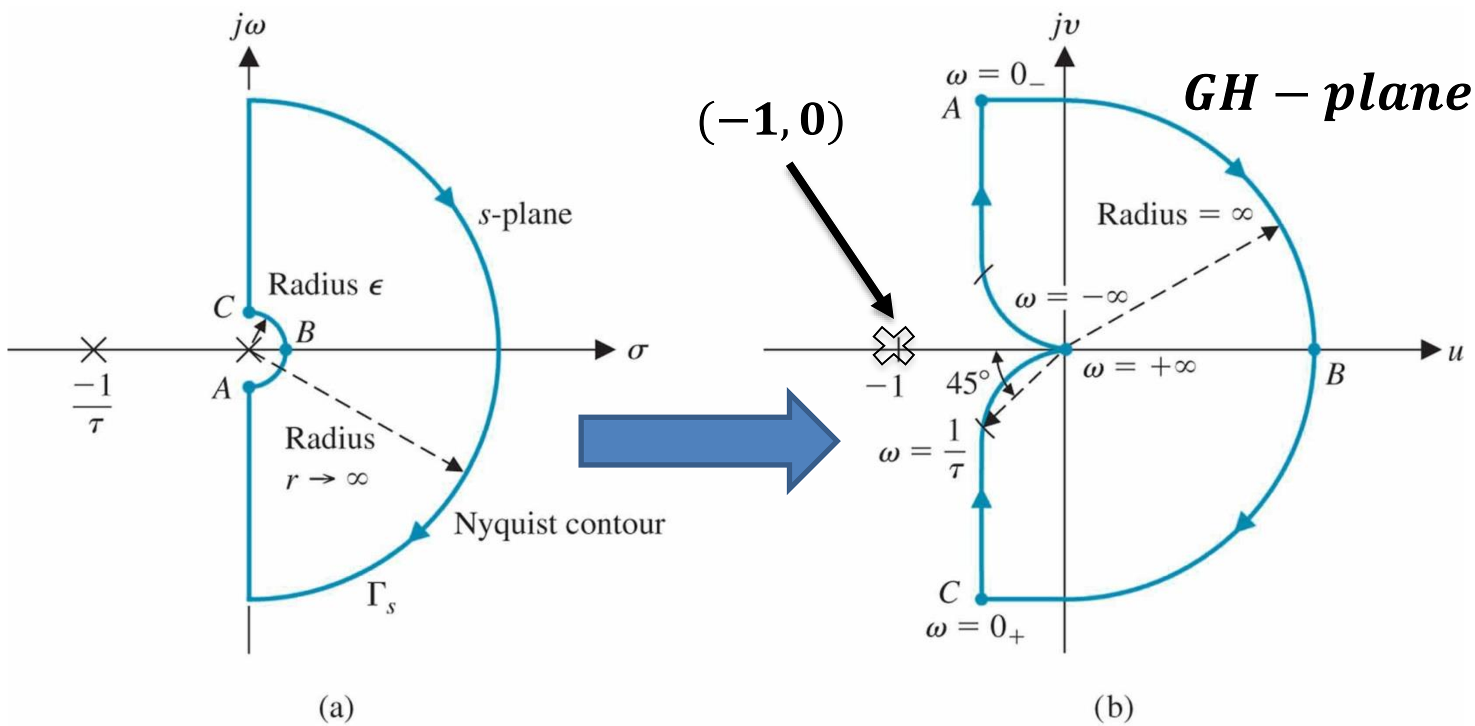
$$\boxed{\begin{aligned} F(s) &= 1 + G(s)H(s) \\ G(s)H(s) &= \frac{N(s)}{D(s)} \\ F(s) &= 1 + \frac{N(s)}{D(s)} = \frac{D(s) + N(s)}{D(s)} \end{aligned}}$$

- $\boldsymbol{P}$: **<u>OPEN-LOOP</u> POLES** *(poles of $G(s)H(s)$)*
  - *Case 1:* $P = 0 \Rightarrow Z = N$
  - *Case 2:* $P \neq 0 \Rightarrow Z = N + P$
- $\boldsymbol{Z}$: **<u>CHARACTERISTIC EQUATION</u> ZEROS** *(roots of $1 + G(s)H(s)$)*

- Because $G(s)H(s)$ is typically available in factored form, consider

Mapping function	Plane	Encirclement
$F(s) = 1 + G(s)H(s)$	$1 + G(s)H(s)$	$(0, 0)$
$F(s) - 1 = G(s)H(s)$	$G(s)H(s)$	$(-1, 0)$

- The number of **clockwise encirclements** of the **origin** of the $F(s)$-plane becomes the number of **clockwise encirclements** of the $(-1, 0)$ point in the $G(s)H(S)$-plane.



The Nyquist Stability Criterion

Case 1

$\mathbf{L(S) = G(s)H(}\boldsymbol{s}\mathbf{)}$ has no **open-loop poles** in the RHP, $(\boldsymbol{P} = \mathbf{0})$

"A feedback system is stable if and only if the contour $\boldsymbol{\Gamma_L}$ in the $\mathbf{G(s)H(}\boldsymbol{s}\mathbf{)}$-plane DOES NOT encircle the $(\mathbf{-1, 0})$ point when the number of poles of $\boldsymbol{L(s)}$ in the RHP is zero $(\boldsymbol{P} = \mathbf{0})$"

$$\boldsymbol{Z = N + P = N}$$
$$\therefore \boldsymbol{Z} = \mathbf{0}$$
$$\Rightarrow \boldsymbol{N} = \mathbf{0}$$

- For no zeros of the characteristic equation on the RHP, then there should be **no encirclement** of the **point $\mathbf{-1}$ in the $\boldsymbol{L(s)}$ plane**

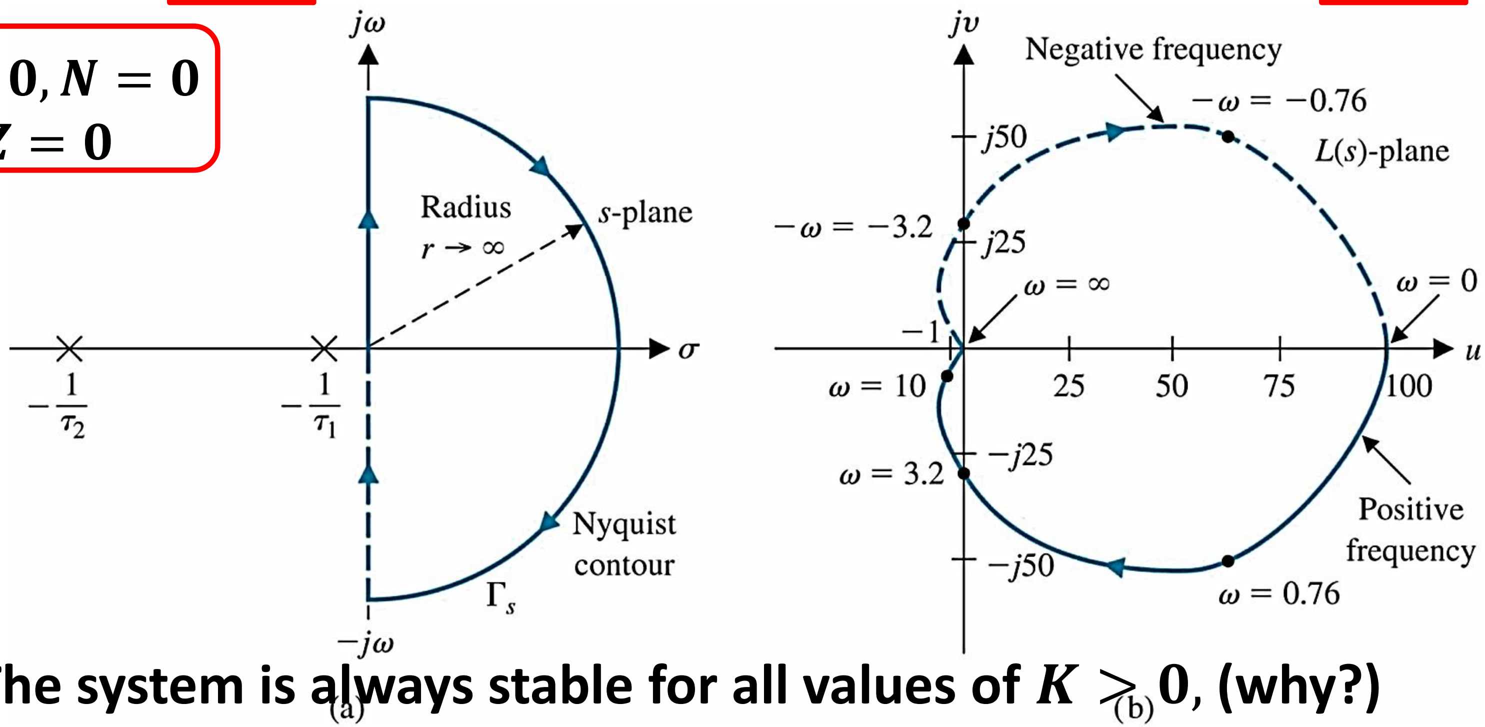
- *example*

$$L(s)=\boxed{\frac{K}{(\tau_1 s+1)(\tau_2 s+1)}}=\frac{100}{(s+1)(0.1s+1)}$$

Table 9.2 Magnitude and Phase of $L(j\omega)$

ω	0	0.1	0.76	1	2	10	20	100	∞
$\lvert L(j\omega)\rvert$	100	96	79.6	70.7	50.2	6.8	2.24	0.10	0
$\angle L(j\omega)$ (degrees)	0	−5.7	−41.5	−50.7	−74.7	−129.3	−150.5	−173.7	−180

$$\boxed{\begin{aligned}&P=0,N=0\\&\Longrightarrow Z=0\end{aligned}}$$

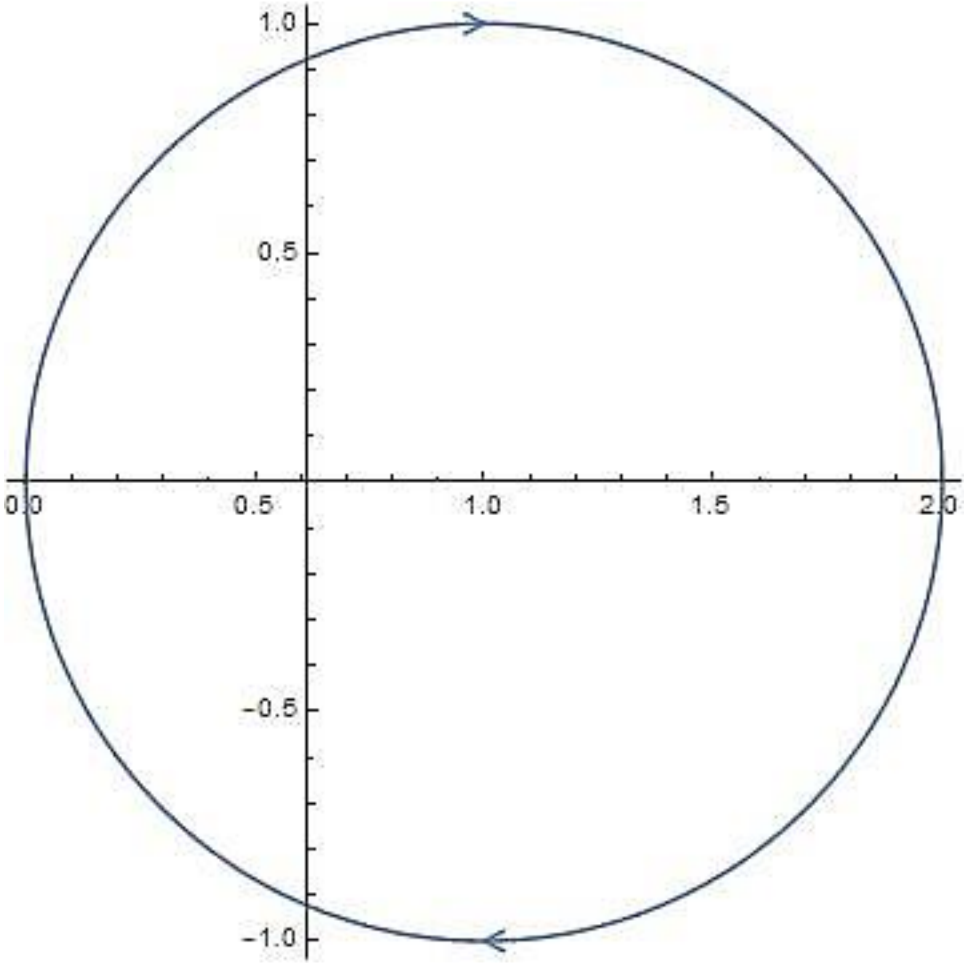


- **The system is always stable for all values of $K>0$, (why?)**

$$\boxed{G(s) = 10/(s+5)}$$

$j\omega$	$GH(j\omega)$
$\lim_{j\omega \longrightarrow 0+} GH(s)$	$+2$
$\lim_{j\omega \longrightarrow +\infty} GH(s)$	$0\angle -90^o$

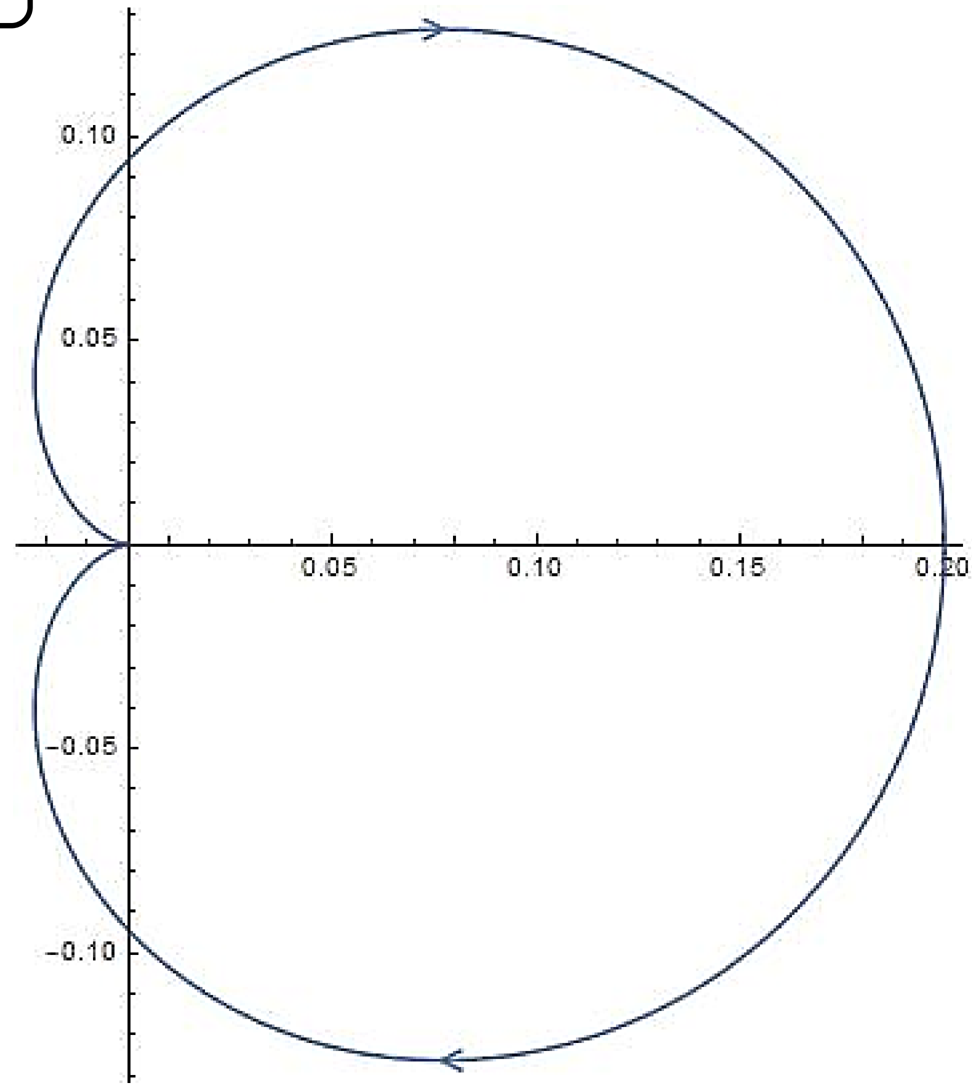
$P = 0, N = 0 \implies Z = 0\ (stable)$



$$G(s) = 10/(s+10)(s+5)$$

$j\omega$	$GH(j\omega)$
$\lim_{j\omega \to 0+} GH(s)$	$+0.2$
$\lim_{j\omega \to +\infty} GH(s)$	$0\angle -180^o$

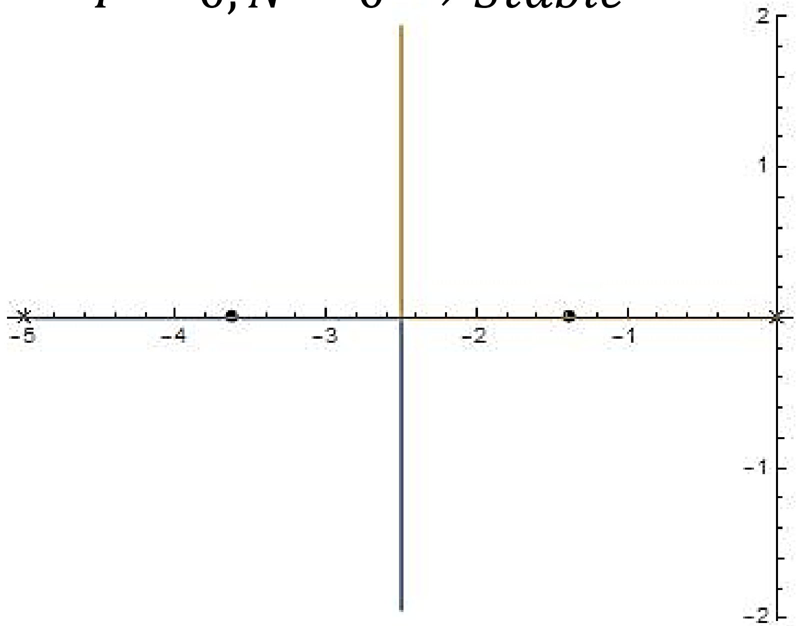
$P = 0, N = 0 \implies$
System is always stable

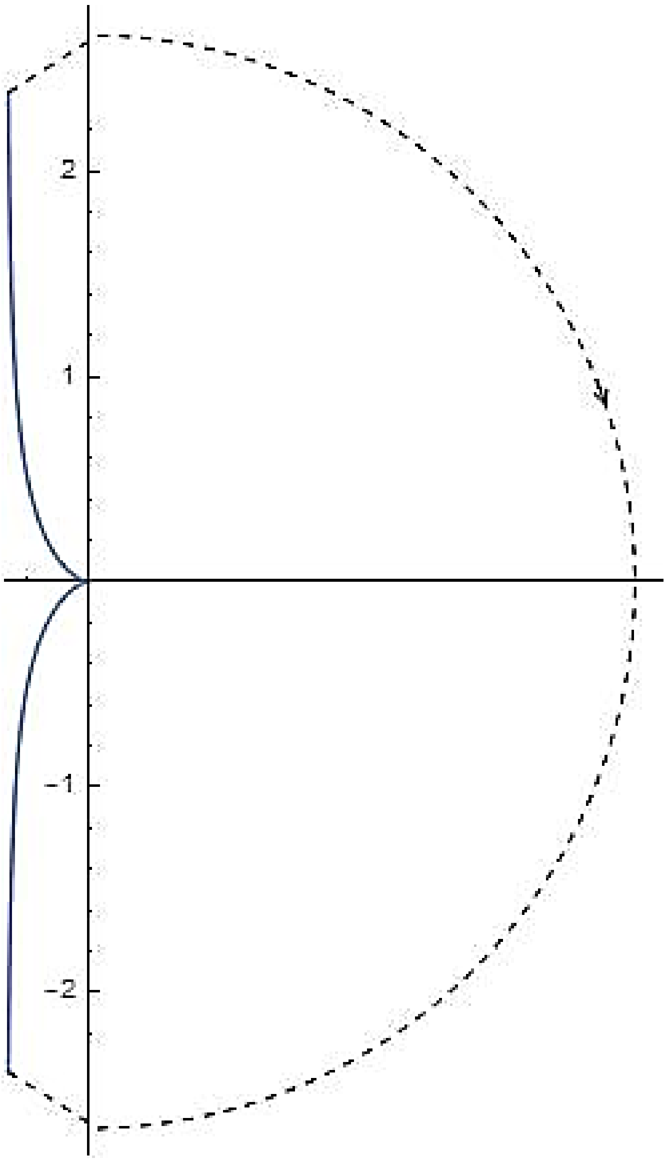


$$\boxed{G(s) = 10/s(s+5)}$$

$j\omega$	$GH(j\omega)$
$\lim_{j\omega \to 0+} GH(s)$	$\infty\angle -90^o$
$\lim_{j\omega \to +\infty} GH(s)$	$0\angle - 180^o$

- $P = 0, N = 0 \Rightarrow Stable$

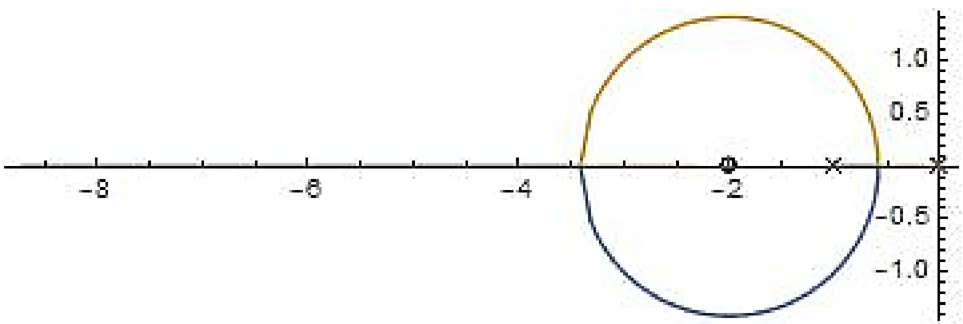


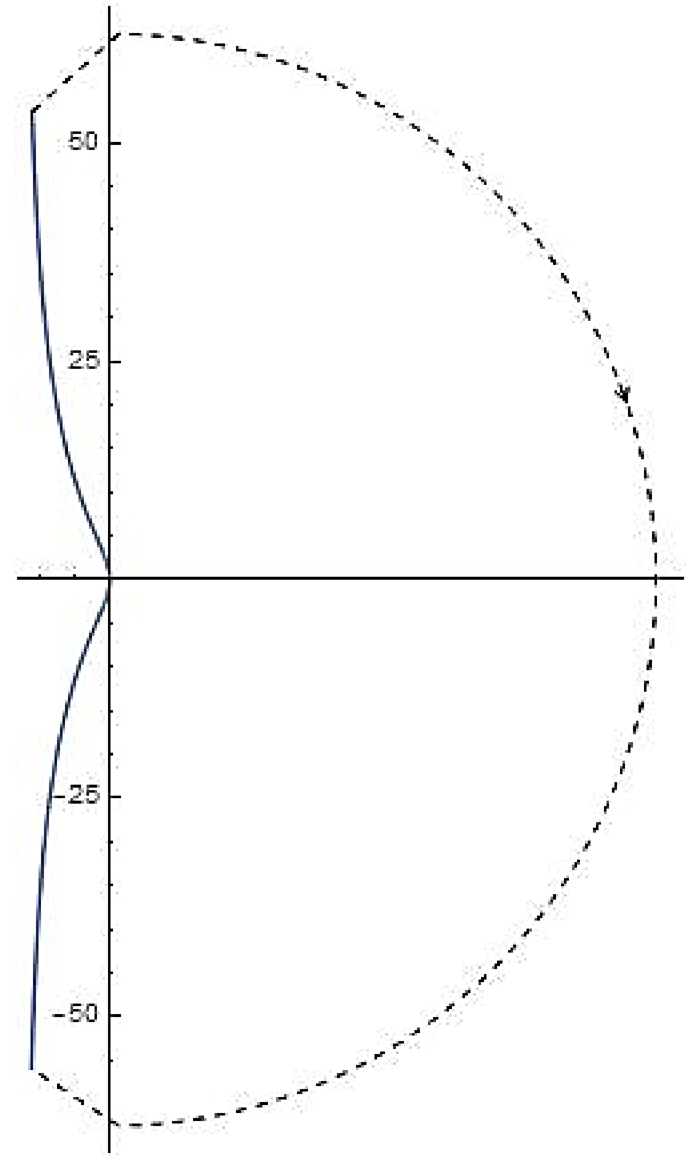


$$\boxed{G(s) = 10(s+2)/s(s+1)}$$

$j\omega$	$GH(j\omega)$
$\lim_{j\omega \to 0+} GH(s)$	$\infty\angle -90^o$
$\lim_{j\omega \to +\infty} GH(s)$	$0\angle -90^o$

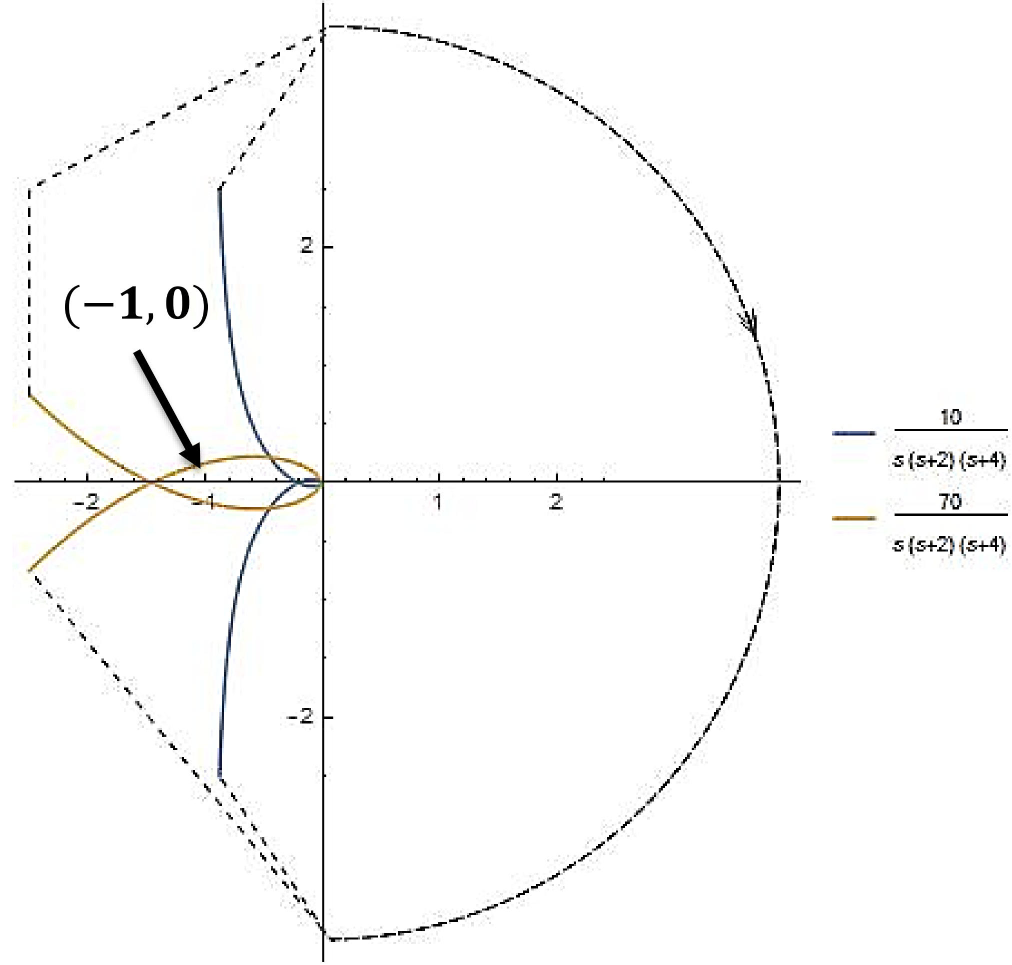
- $N = 0, P = 0 \Rightarrow Stable$

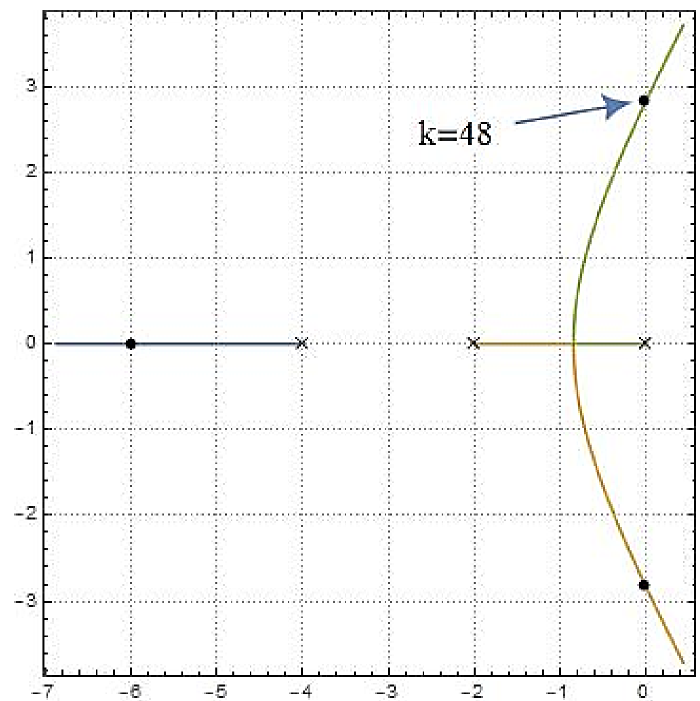




$$\boxed{G(s) = K/s(s+2)(s+4)}$$

$j\omega$	$GH(j\omega)$
$\lim\limits_{j\omega \to 0+} GH(s)$	$\infty\angle -90^o$
$\lim\limits_{j\omega \to +\infty} GH(s)$	$0\angle -270^o$





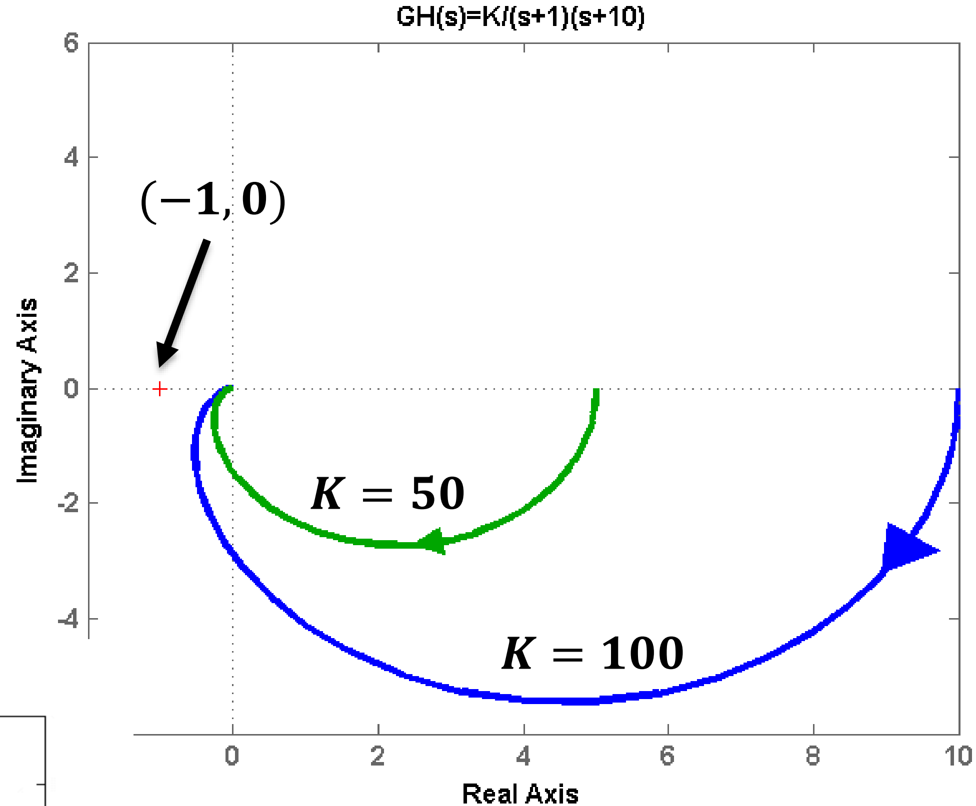
$$K = 10 \Rightarrow \; N = 0, P = 0 \; \Rightarrow Z = 0 (Stable)$$

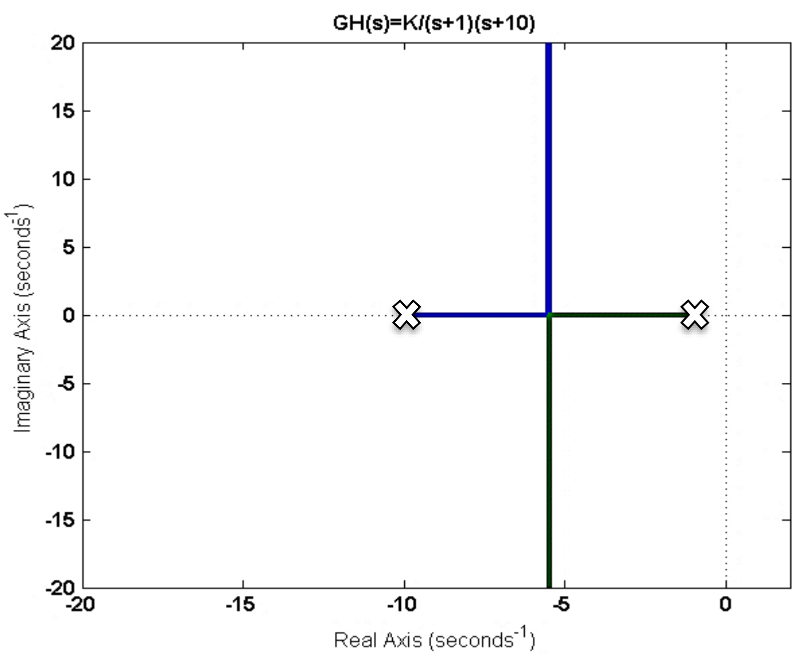
$$K = 70 \; \Rightarrow N = 2, P = 0 \; \Rightarrow Z = 2 \; (unstable)$$

- ***example***

$$\boxed{GH(s) = K/(s+1)(s+10)}$$

$j\omega$	$GH(j\omega)$
$\lim_{j\omega \to 0+} GH(s)$	$K/10$
$\lim_{j\omega \to +\infty} GH(s)$	$0\angle -180^o$



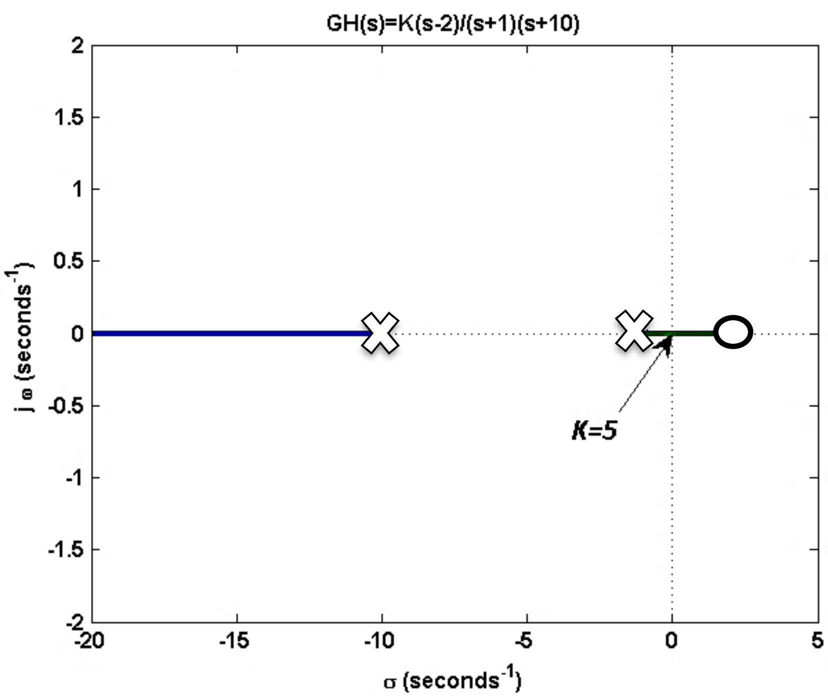


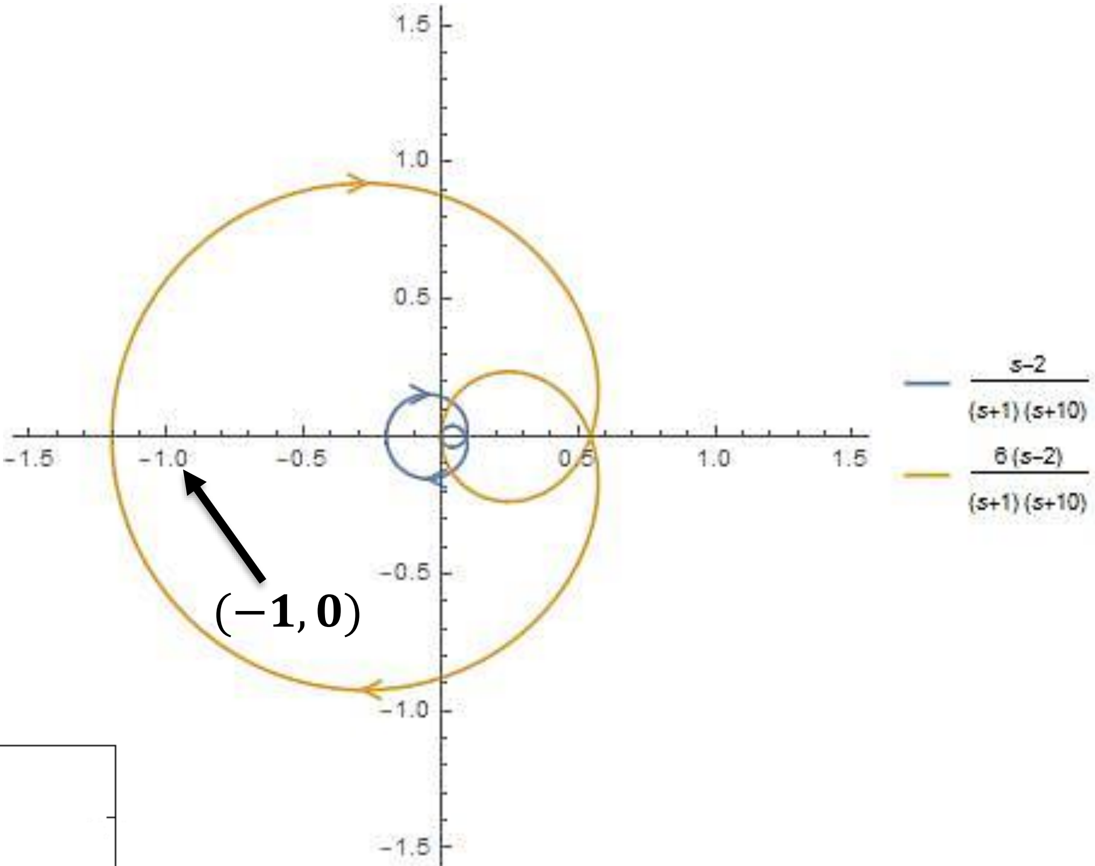
System is always stable for all values of K

- ***example***

$$GH(s) = \frac{K(s-2)}{(s+1)(s+10)}$$

$j\omega$	$GH(j\omega)$
$\lim_{j\omega \to 0+} GH(s)$	$-0.2K$
$\lim_{j\omega \to +\infty} GH(s)$	$0\angle -90^o$



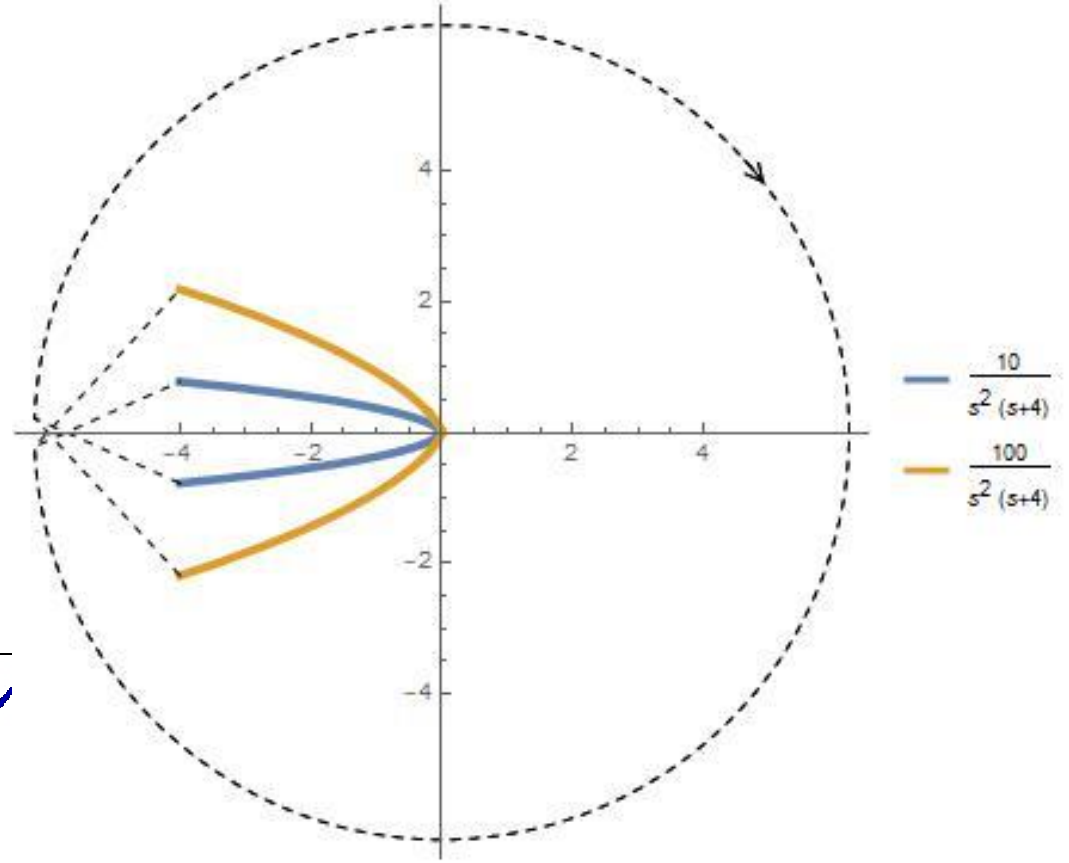


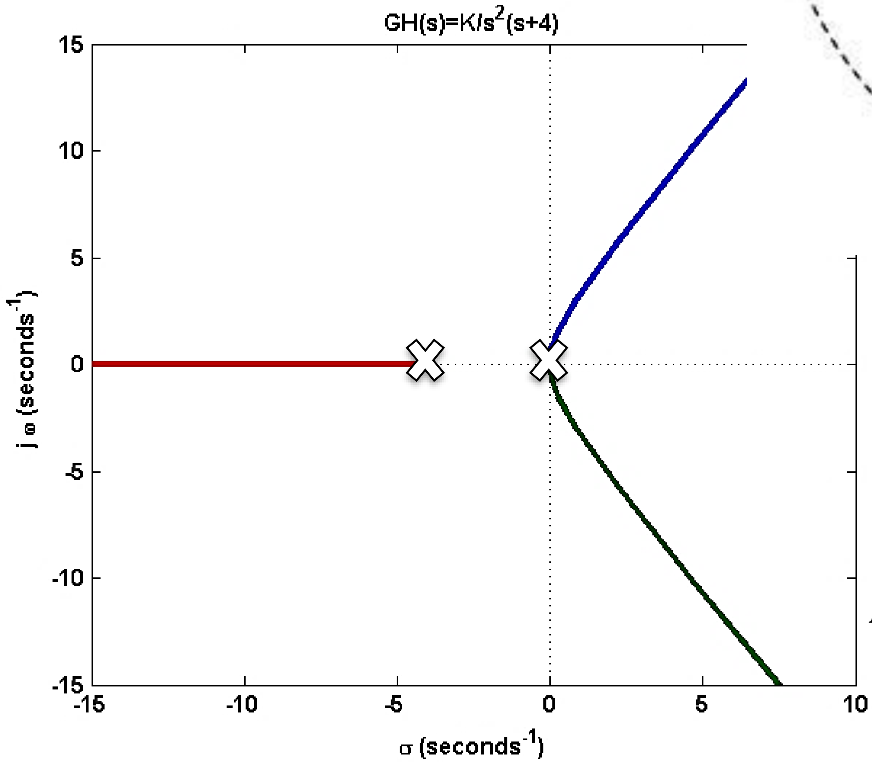
$$K = 1 \Rightarrow \ N = 0, P = 0 \ \Rightarrow Z = 0(stable)$$

$$K = 6 \Rightarrow \ N = 1, P = 0 \ \Rightarrow Z = 1(ustable)$$

- $\boxed{GH(s) = K/s^2(s+4)}$

ω	$GH(j\omega)$
$j\omega \to 0_+$	$\infty \angle -180^o$
$j\omega \to +\infty$	$0 \angle -270^o$





The system is **always unstable**

$$K = 10 \Rightarrow\ N = 2, P = 0\ \Rightarrow Z = 2(unstable)$$

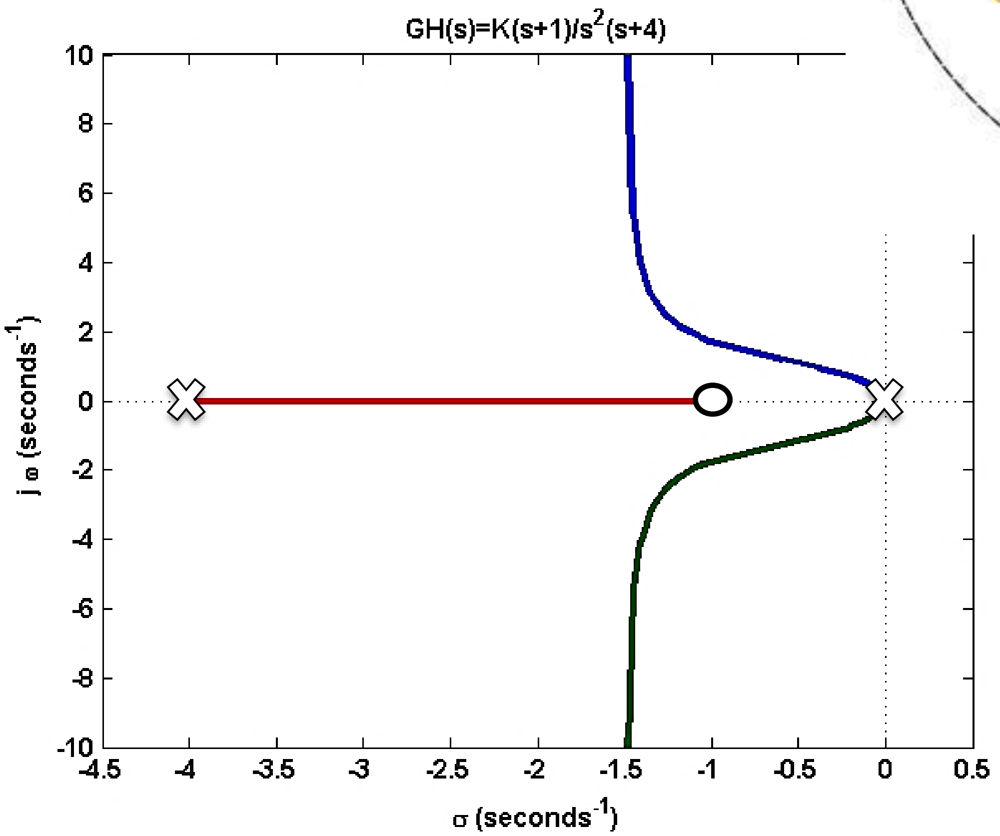
$$K = 100 \Rightarrow\ N = 2, P = 0\ \Rightarrow Z = 2(unstable)$$

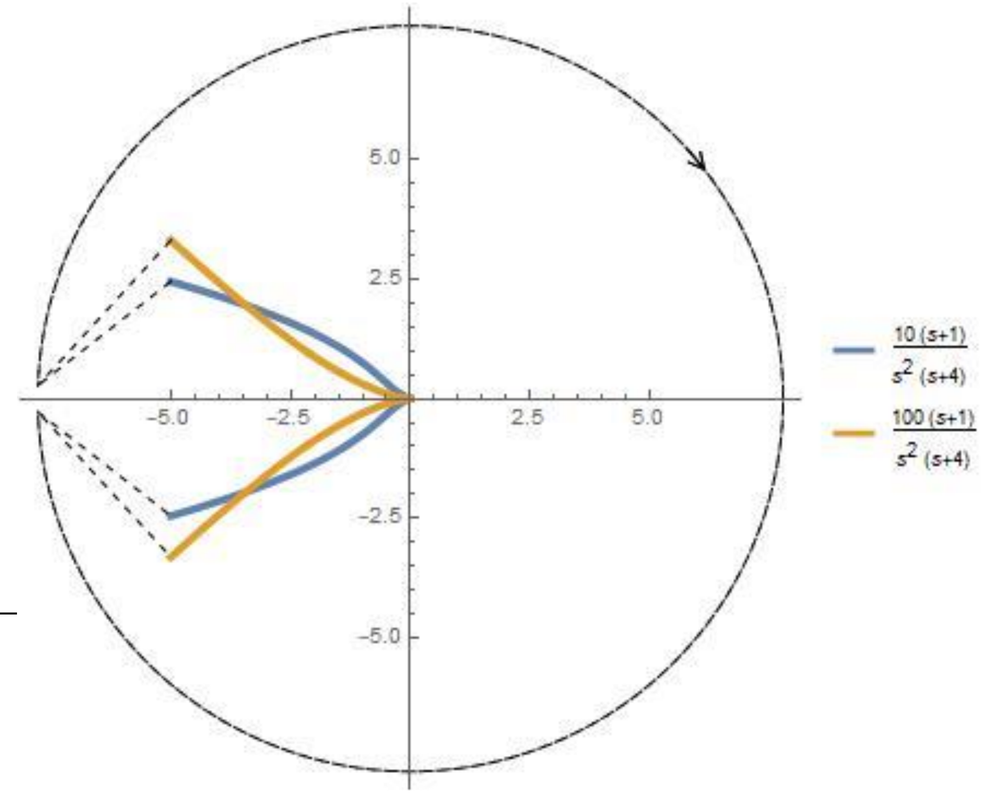
- adding a **differentiator**

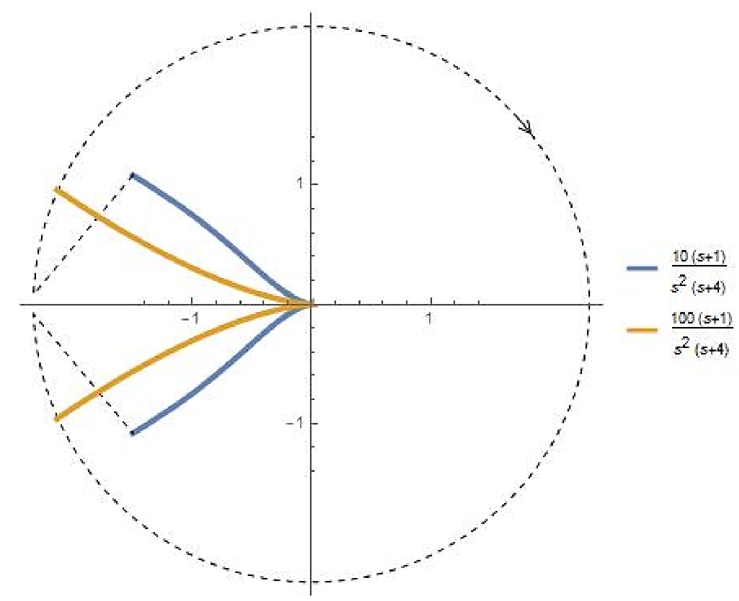
$$\boxed{GH(s) = K(s+1)/s^2(s+4)}$$

ω	$GH(j\omega)$
$j\omega \to 0_+$	$\infty \angle -\pi$
$j\omega \to +\infty$	$0 \angle -\pi$

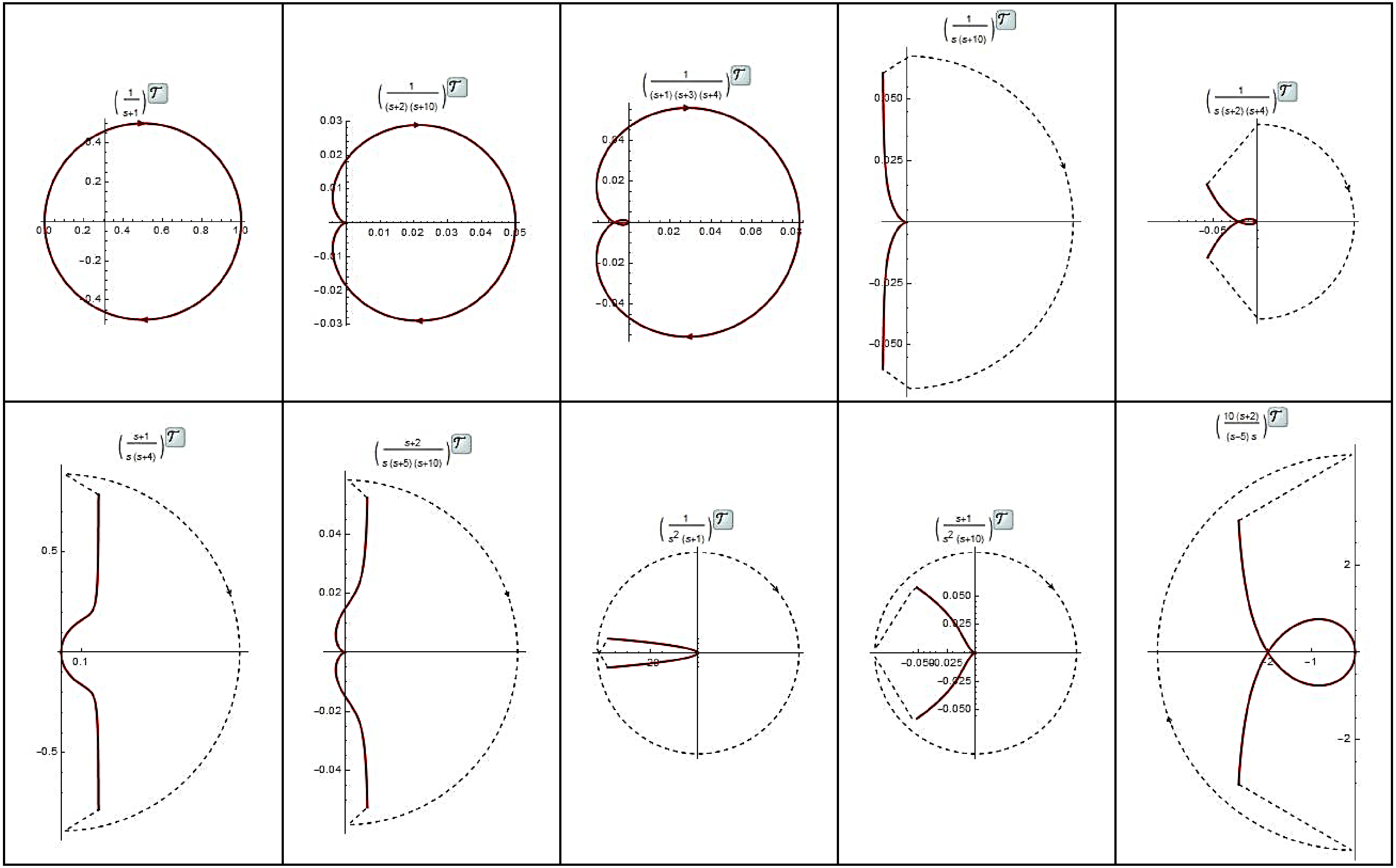
- The system is **always stable**







Nyquist plot examples



Equivalent root locus plots