Automatic Control Systems

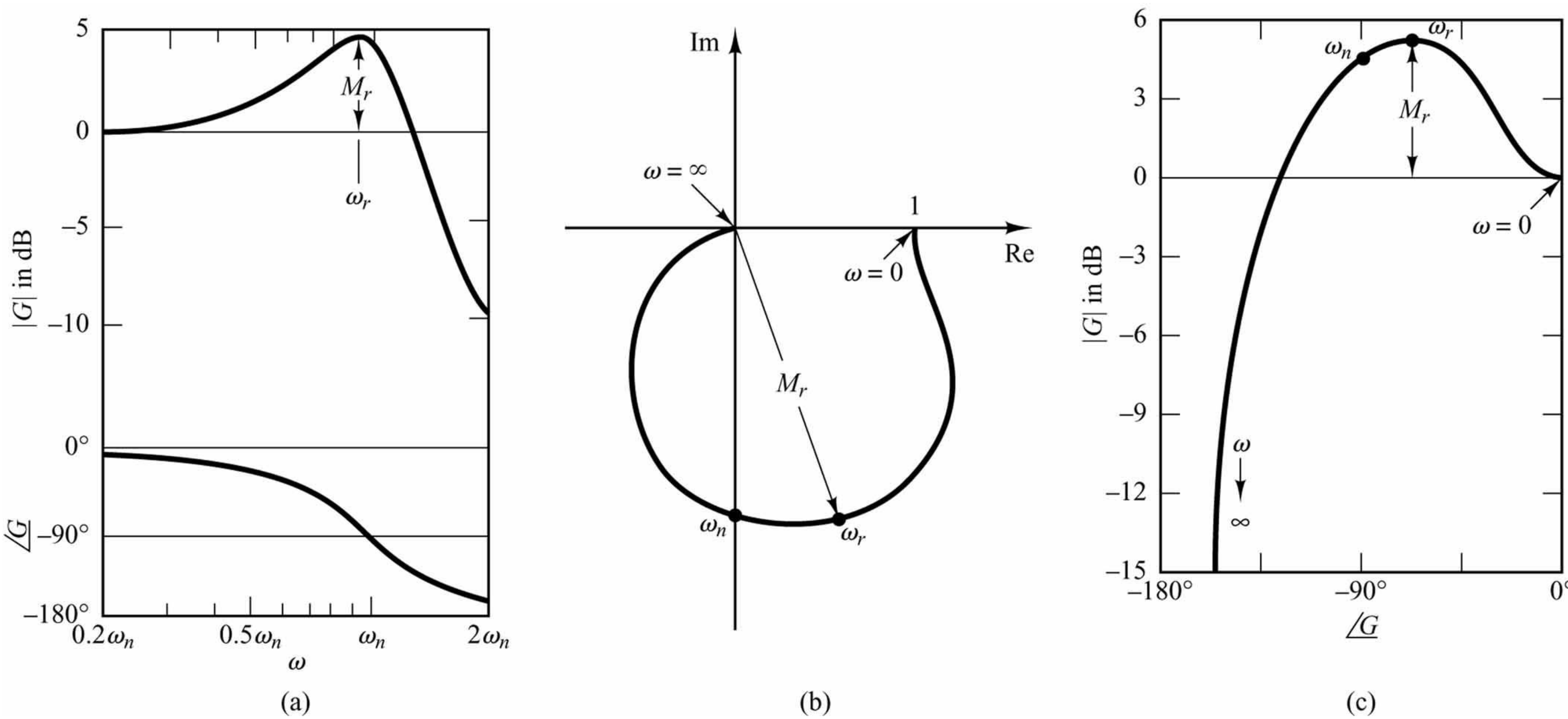
LOGARITHMC PLOTS (BODE)

alghoniemy@alexu.edu.eg

- **Reading:**
  - **chapter 8 (**Section 8.2)
  - **chapter 9 (**Section 9.5)

- **Practice problems**
  - Study table 9.6 on pages 704-711
  - Solve problems at the end of chapter 9

Frequency Domain Representations

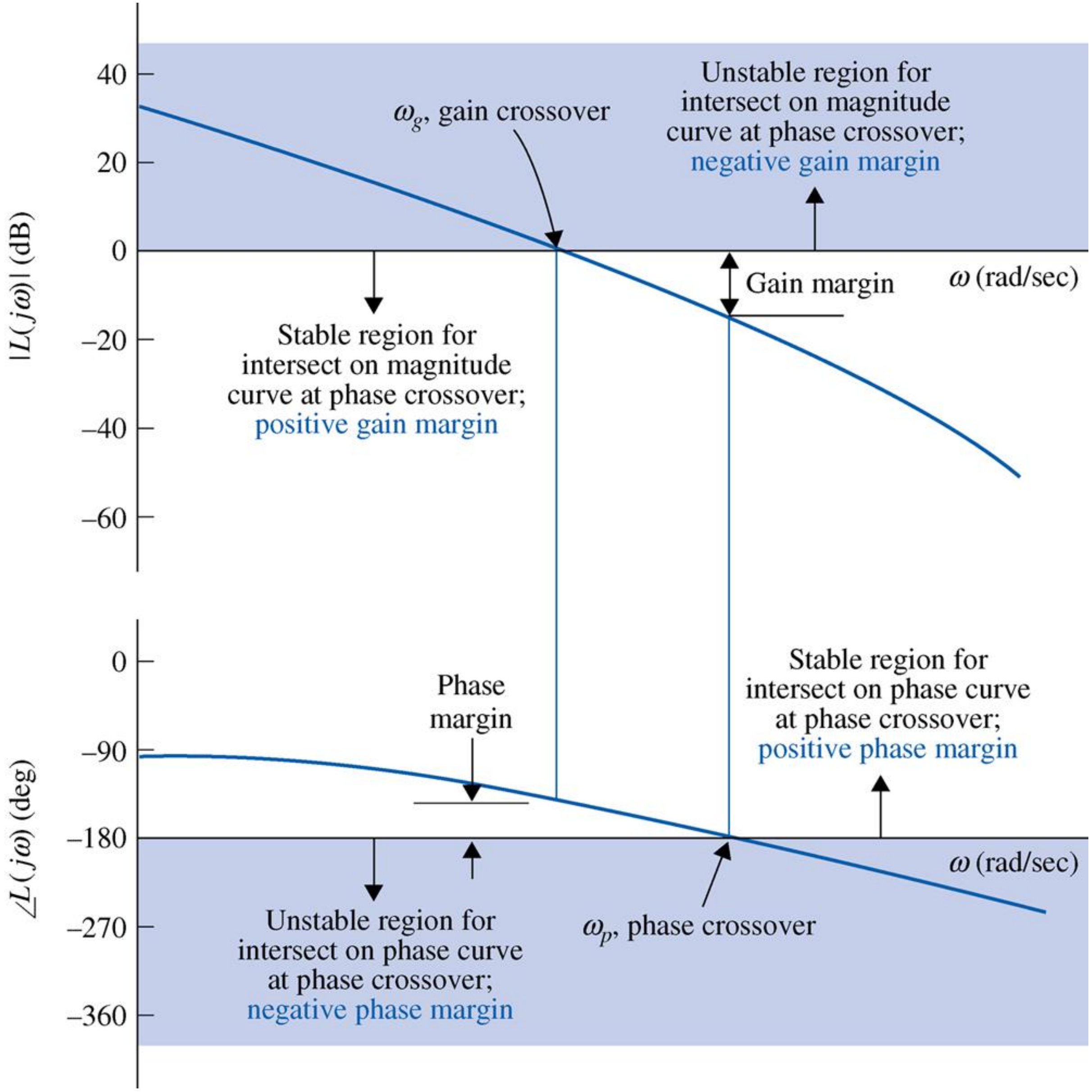


Logarithmic plots (Bode Plots)

Polar plots (Nyquist Diagrams)

Log Magnitude-Phase (Nichols Charts)

Stability analysis with the logarithmic plots (Bode Plot)

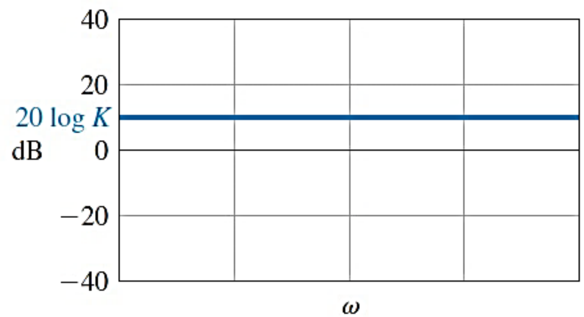


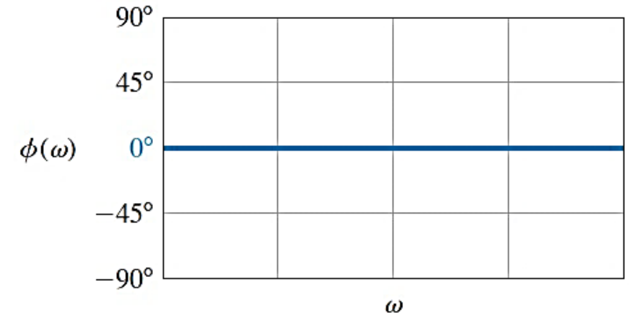
Sketching the logarithmic plots (Bode Plots)

Table 8.3 Asymptotic Curves for Basic Terms of a Transfer Function

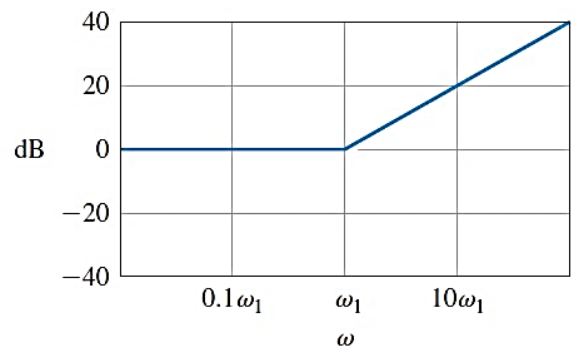
Term — Magnitude 20 log|G| — Phase $\phi(\omega)$

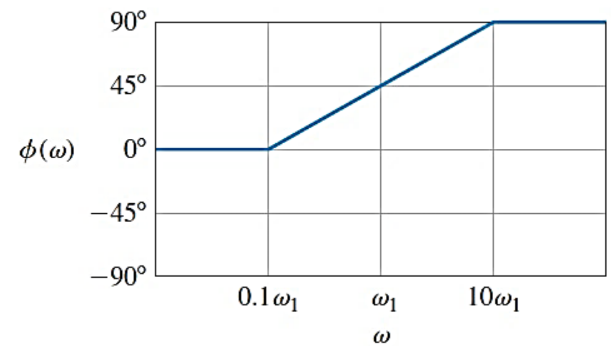
1. Gain,
   $G(j\omega) = K$



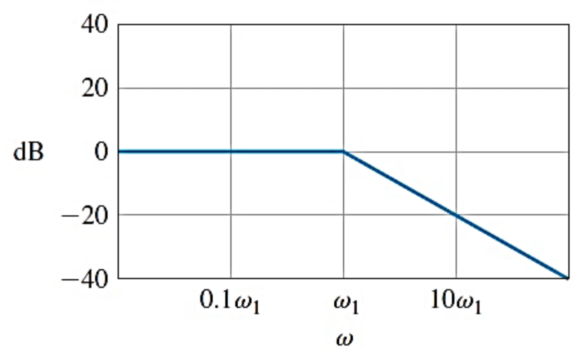


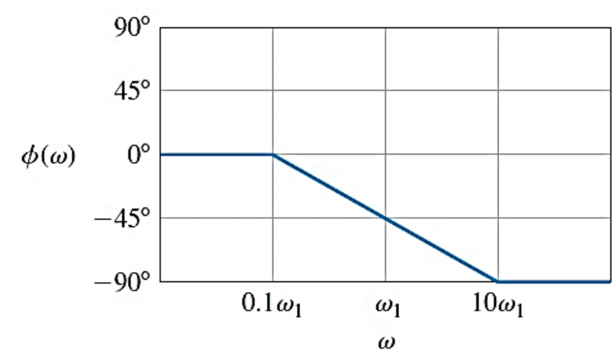
2. Zero,
   $G(j\omega) =$
   $1 + j\omega/\omega_1$





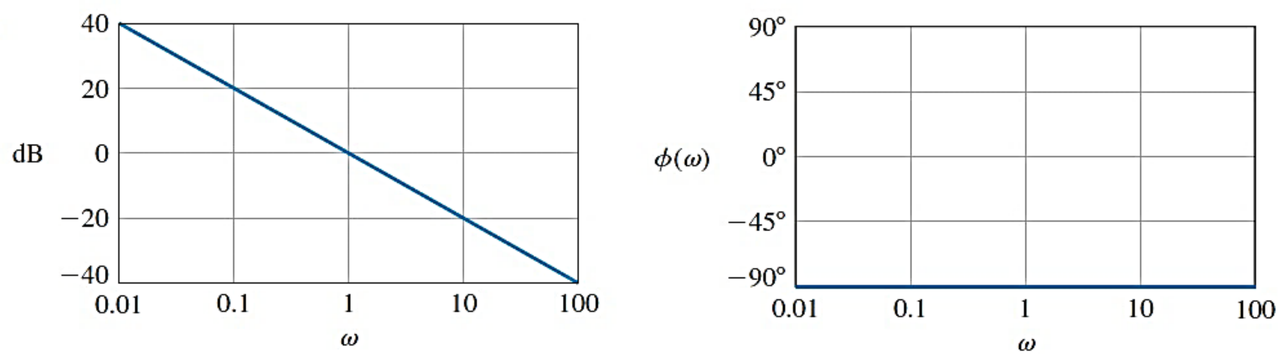
3. Pole,
   $G(j\omega) =$
   $(1 + j\omega/\omega_1)^{-1}$



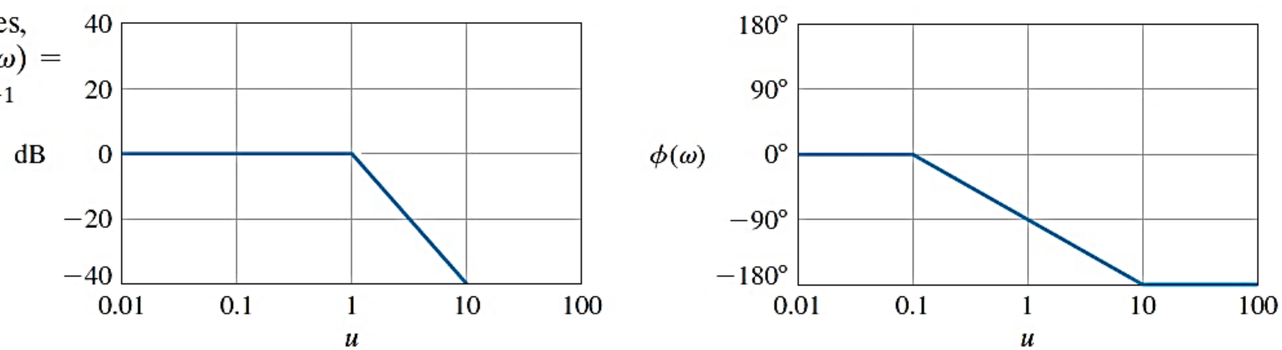


Sketching the logarithmic plots (Bode Plots)

4. Pole at the origin, $G(j\omega) = 1/j\omega$



5. Two complex poles, $0.1 < \zeta < 1$, $G(j\omega) = (1 + j2\zeta u - u^2)^{-1}$
$u = \omega/\omega_n$



- ***example***

$$L(s) = \frac{2500}{s(s+50)(s+5)}$$

$$L(s) = \frac{10}{s(0.02s+1)(0.2s+1)}$$

- **Phase crossover**

$$\omega_p = 15.88 \quad \text{rad/s}$$

- **Gain Margin**

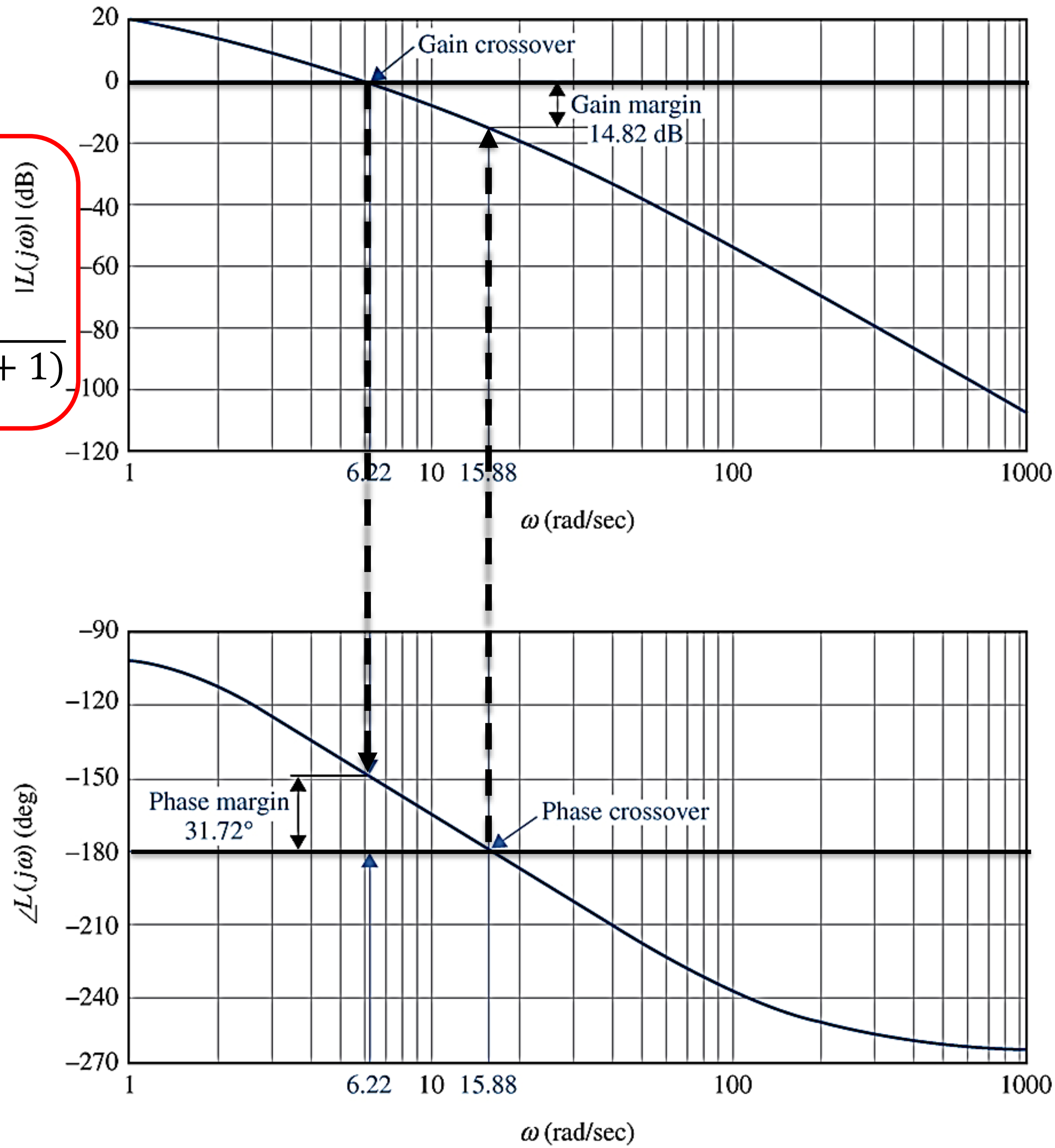
$$G.M. = 14.82 \ \text{dB}$$

- **Gain crossover**

$$\omega_g = 6.22 \quad \text{rad/s}$$

- **Phase Margin**

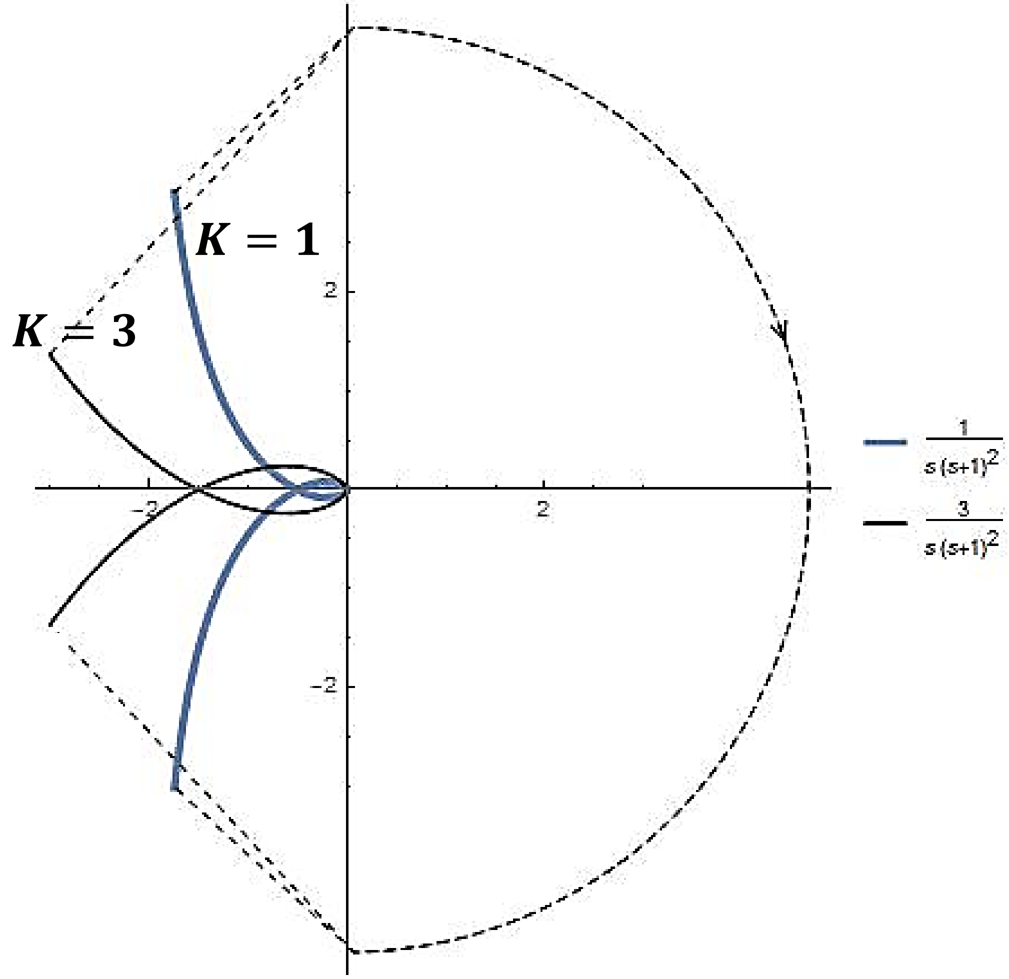
$$P.M. = 31.72^o$$

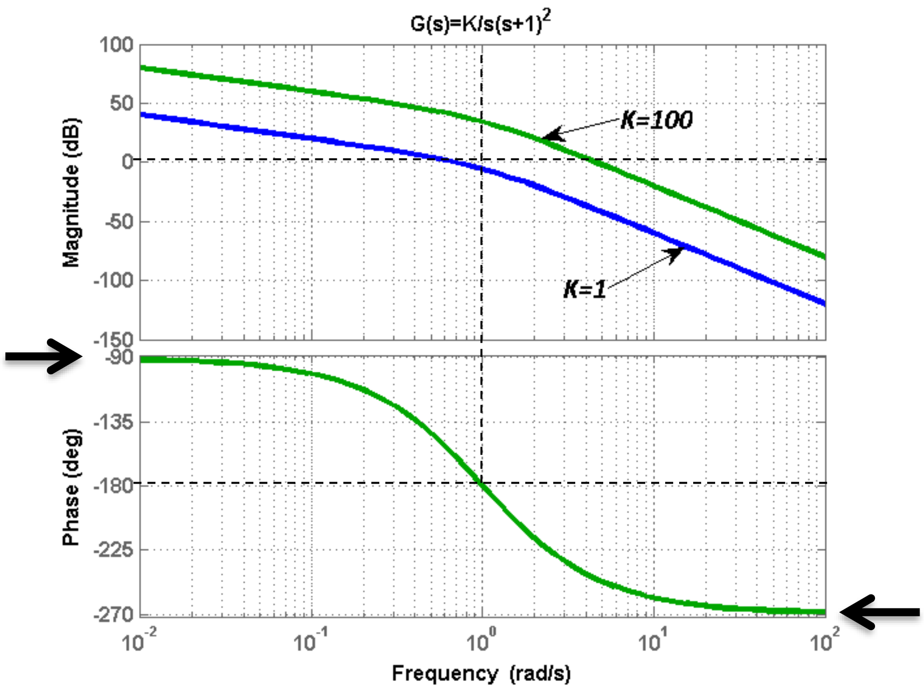


- ***example***:

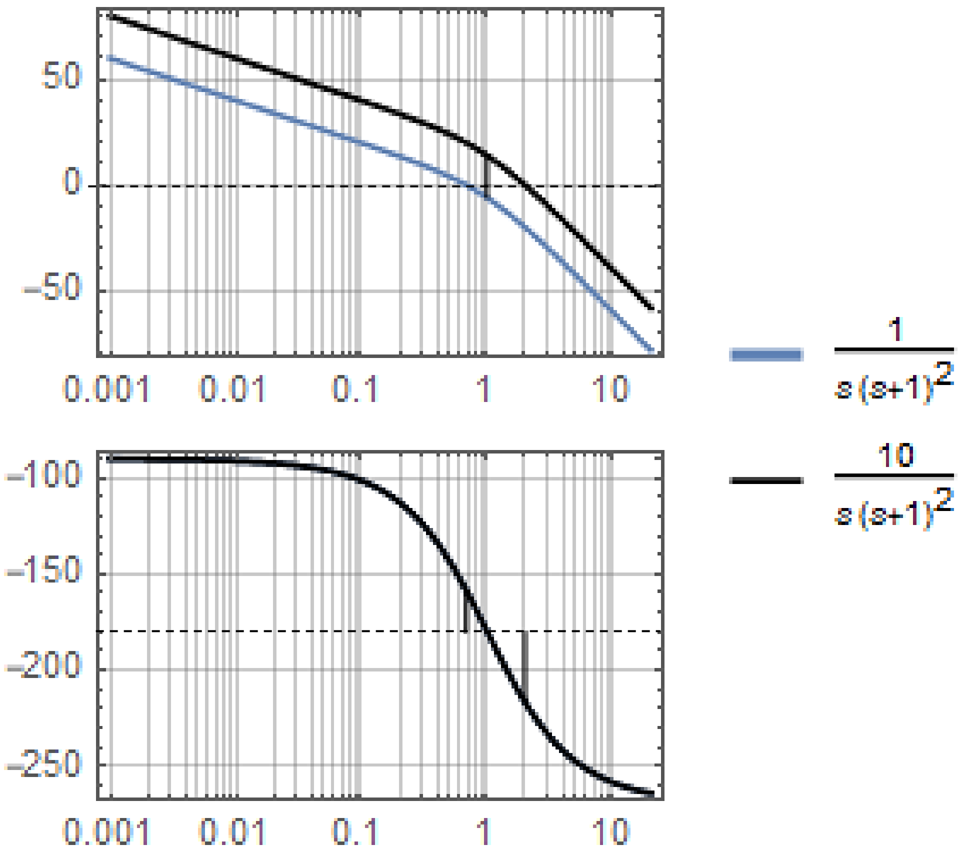
$$\boxed{G(s) = K/s(s+1)^2}$$

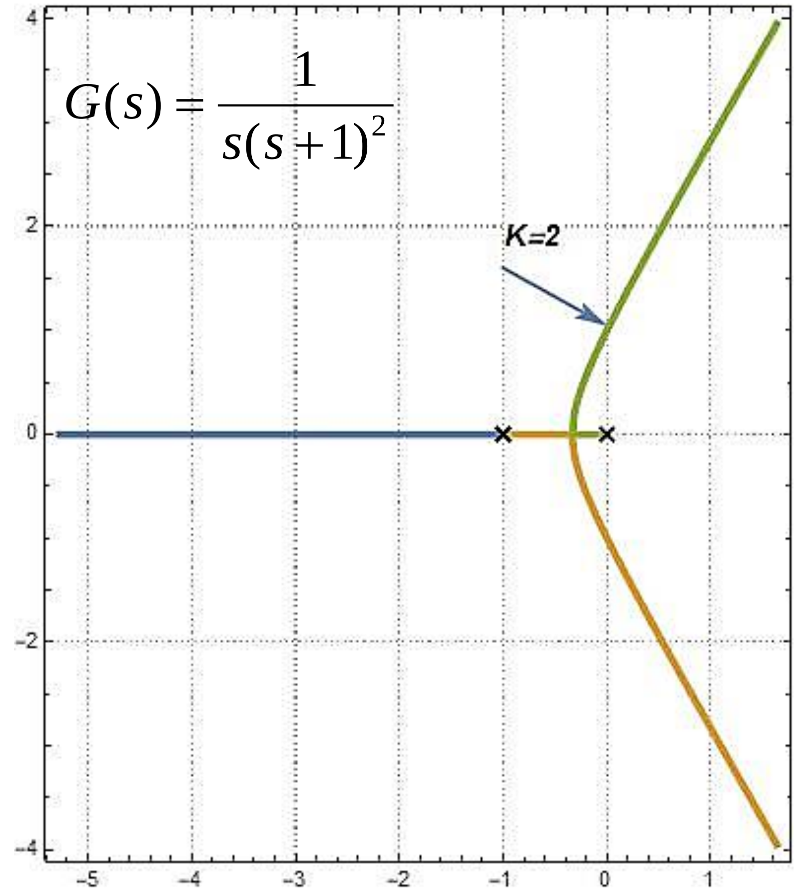
s	$G(s)$
$j\omega \longrightarrow 0+$	$\infty\angle -90$
$j\omega \longrightarrow \infty+$	$0\angle -270$





- ***What is the Gain Margin?***





- **example "with time delay"**

$$L(s) = \frac{Ke^{-T_d s}}{s(s+1)(s+2)}$$

- $\boldsymbol{T_d = 0, K = 1}$

$\omega_g = 0.446 \;\; \text{rad/s}$

$P.M. = 53.4^o$

$\omega_p \approx 1.5 \;\; \text{rad/s}$

$G.M. \approx 16 \;\; \text{dB}$

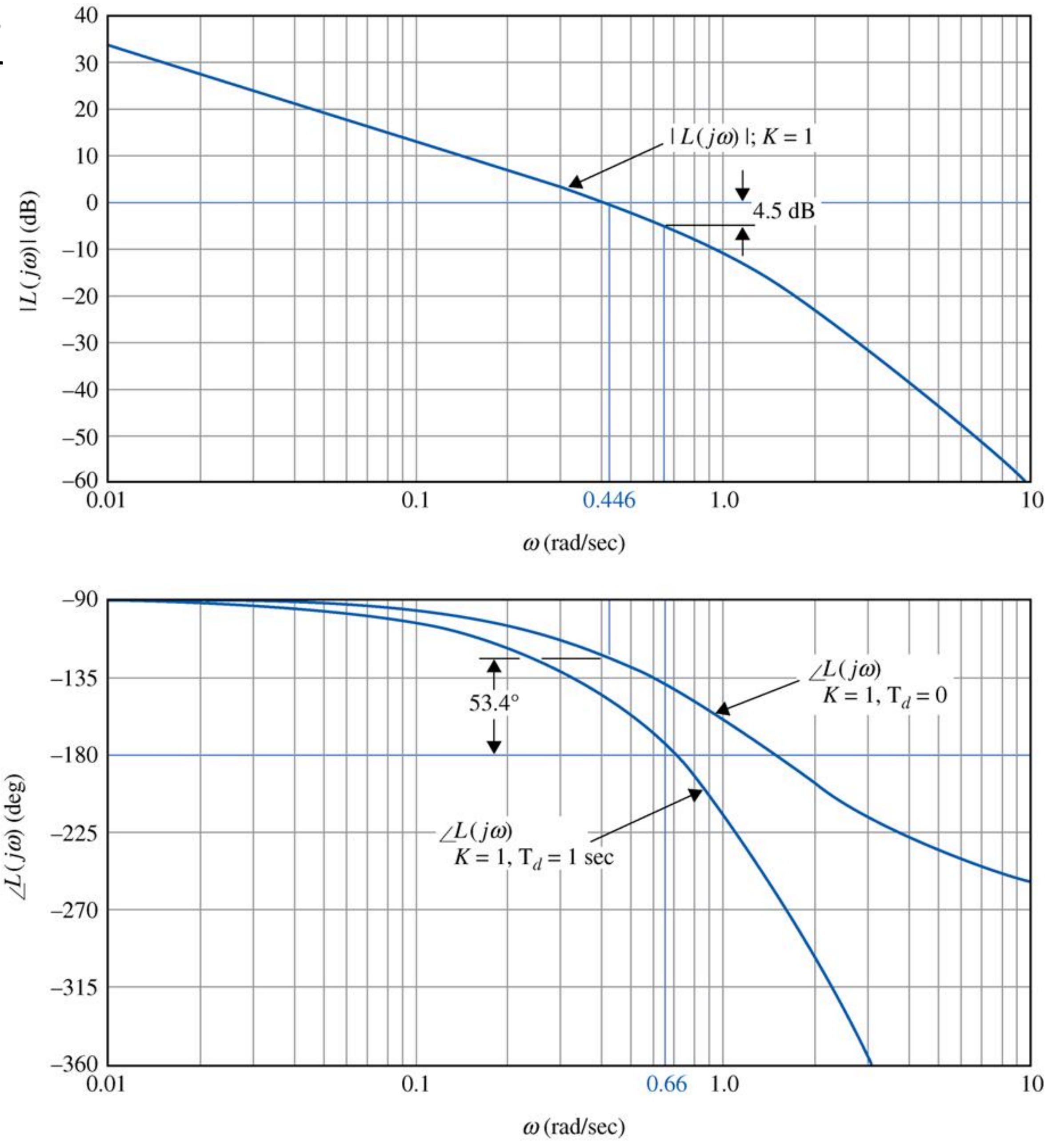
- $\boldsymbol{T_d = 1, K = 1}$

$\omega_g = 0.446 \;\; \text{rad/s}$

$P.M. \approx 40^o$

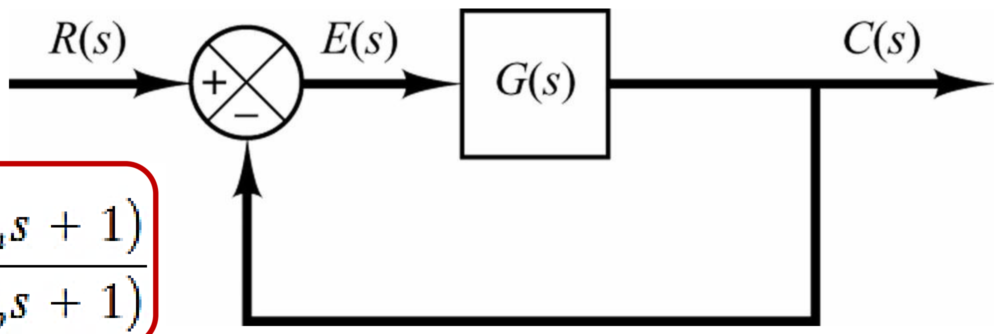
$\omega_p = 0.66 \;\; \text{rad/s}$

$G.M. = 4.5 \;\; \text{dB}$



find the critical value of T for stability?

Relationship between the steady state error and the Logarithmic plots



$$G(s) = \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{s^N(T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)}$$

System type

Determination of steady state error

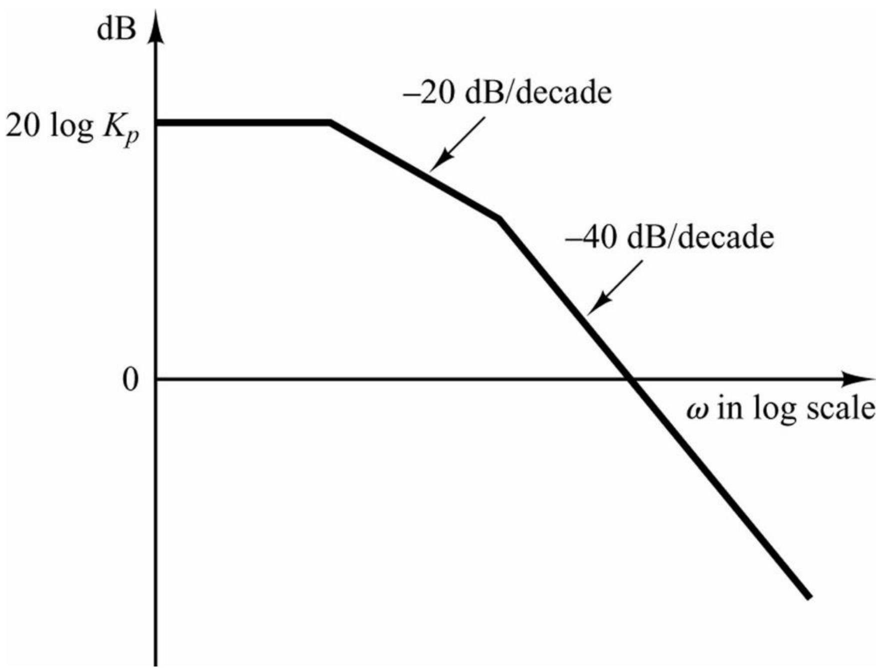
- **Type-zero system $(N = 0)$**

The static position error constant K_P

$$\boxed{K_P = \lim_{j\omega \to 0} G(j\omega) = K}$$

- **Steady state error due to step-input**

$$E_{ss} = \frac{1}{1 + K_P}$$



• Type-one system ($N = 1$)

the static velocity error constant K_v

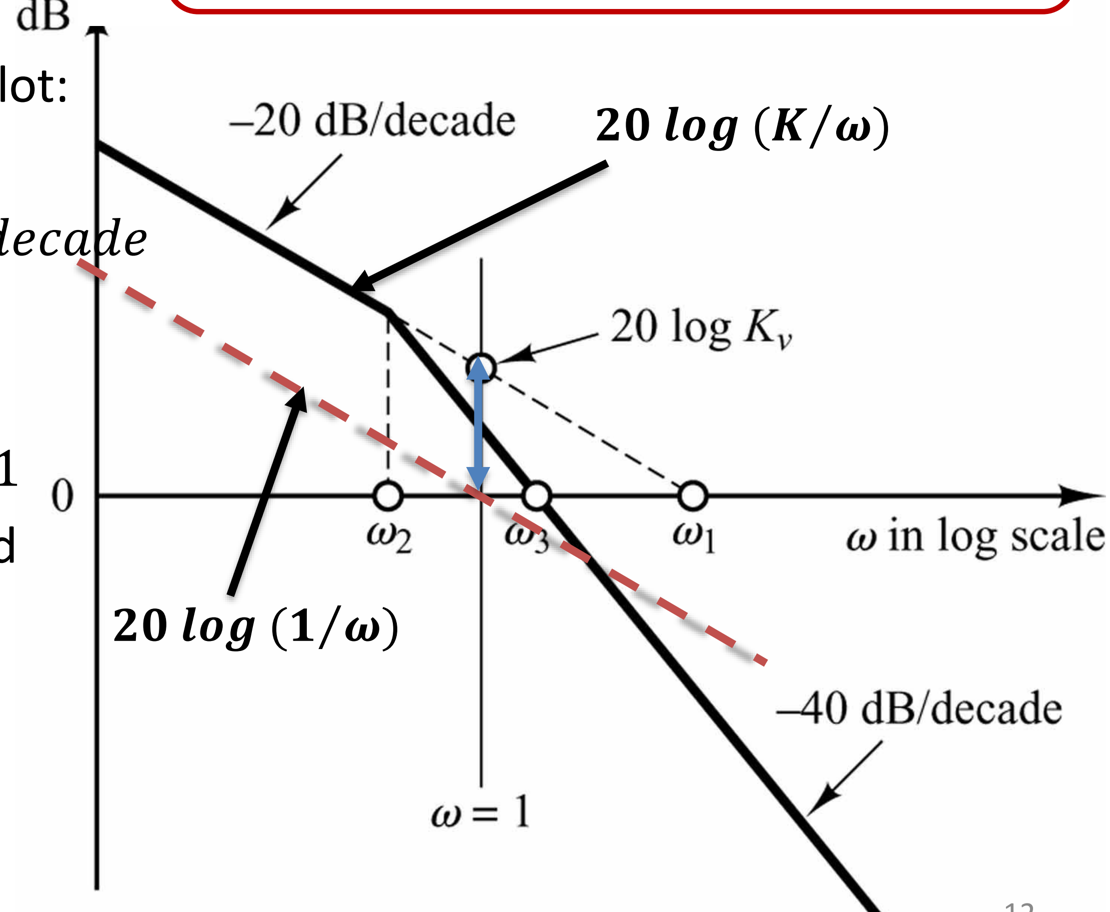
$$\boxed{K_v = \lim_{s\to 0} s\,G(s) = K}$$

$$\boxed{G(s) = \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{s\ (T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)}}$$

- In order to get K from Bode Plot:

- The line with slope $-20\ dB/decade$ represents K/s.

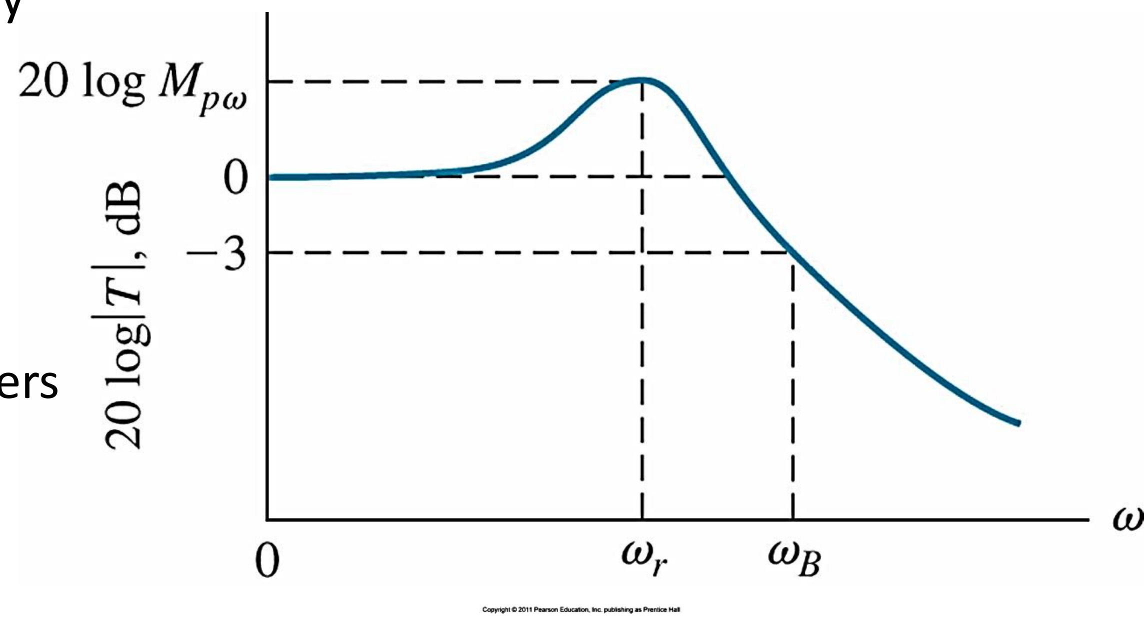
- The value of $K = \mathrm{K/s@}\ \omega = 1$ i.e., the intersection of $\omega = 1$ and The extension of K/s as shown in the figure



Time domain performance criteria and the frequency domain

The relation between the closed-loop frequency response and the transient response

- The transient performance of a feedback system can be estimated form the **closed-loop** frequency



closed-loop frequency response

- We need to relate

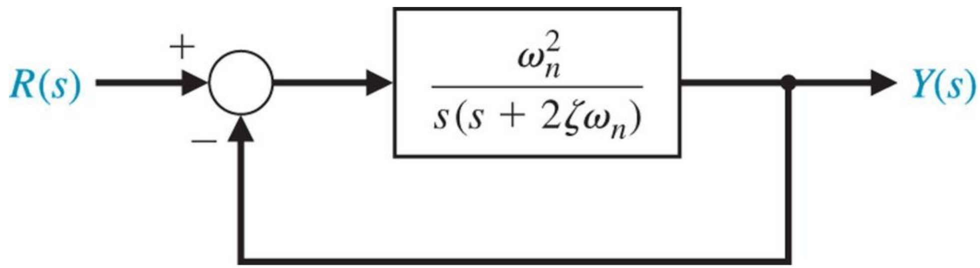
$\boldsymbol{M_r, \omega_r,}$ & $\boldsymbol{BW}$

to the time domain parameters

$\boldsymbol{\zeta}$ **and** $\boldsymbol{\omega_n}$

Frequency domain parameters	
M_r	Peak resonance
ω_r	Resonant frequency
ω_B	System BW

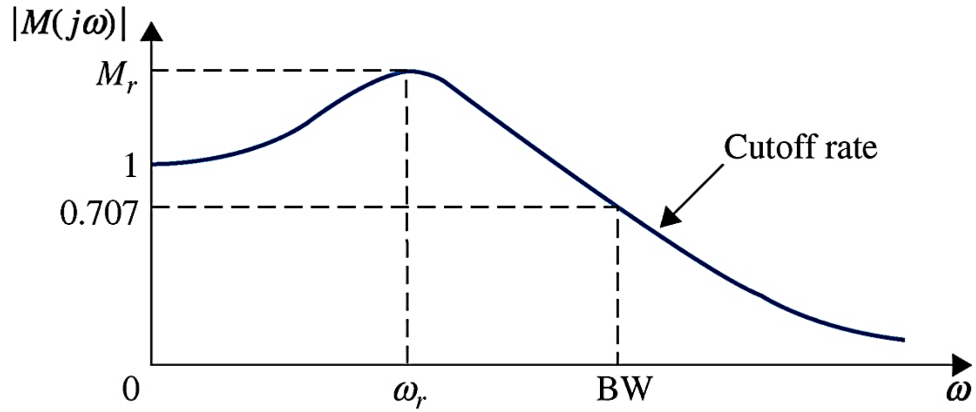
Second-order system



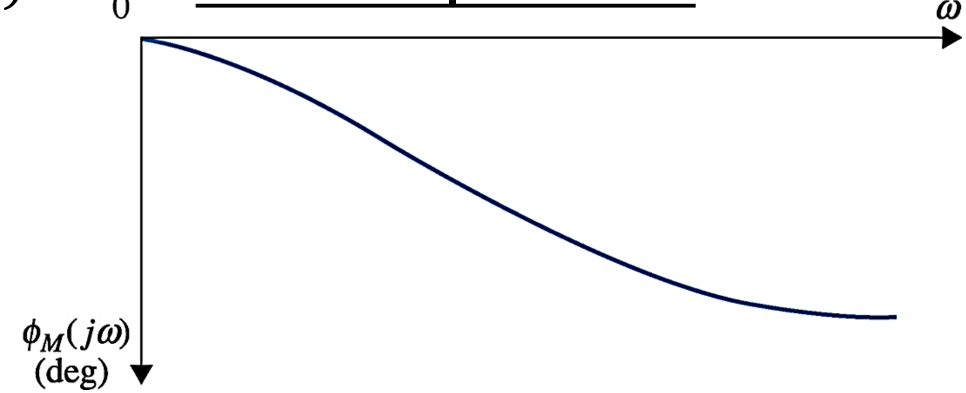
Closed-loop transfer function

$$T(s)=\frac{Y(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$T(\omega)=\frac{1}{1+(2\zeta/\omega_n)j\omega+(j\omega/\omega_n)^2}$$



The magnitude $\boldsymbol{M(\omega)}$ and phase $\boldsymbol{\phi(\omega)}$ of the **<u>closed-loop transfer</u>** function



$$M(\omega)=|T(\omega)|=\frac{1}{\sqrt{[1-u^2]^2+(2\zeta u)^2}}$$

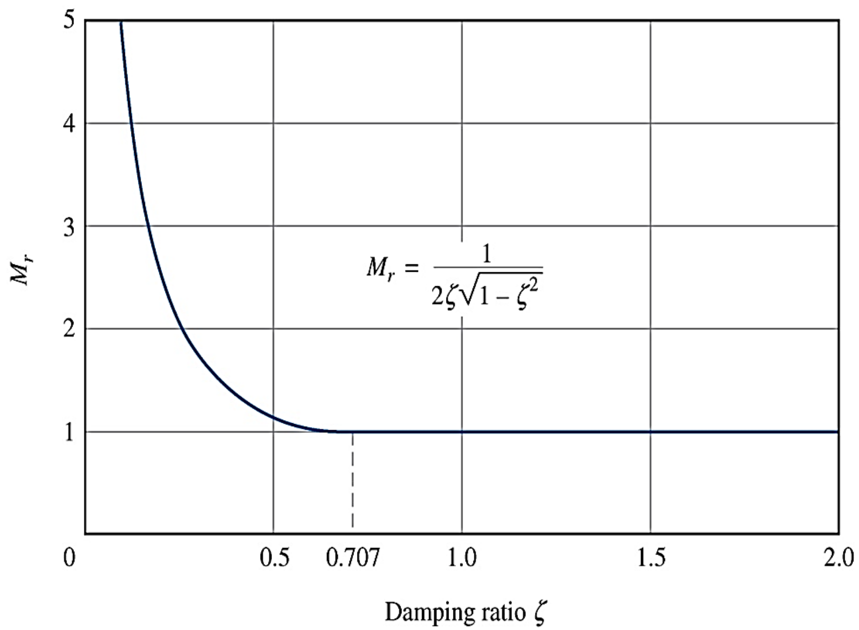
$$\phi(\omega)=\angle T(\omega)=-\tan^{-1}\frac{2\zeta u}{1-u^2}$$

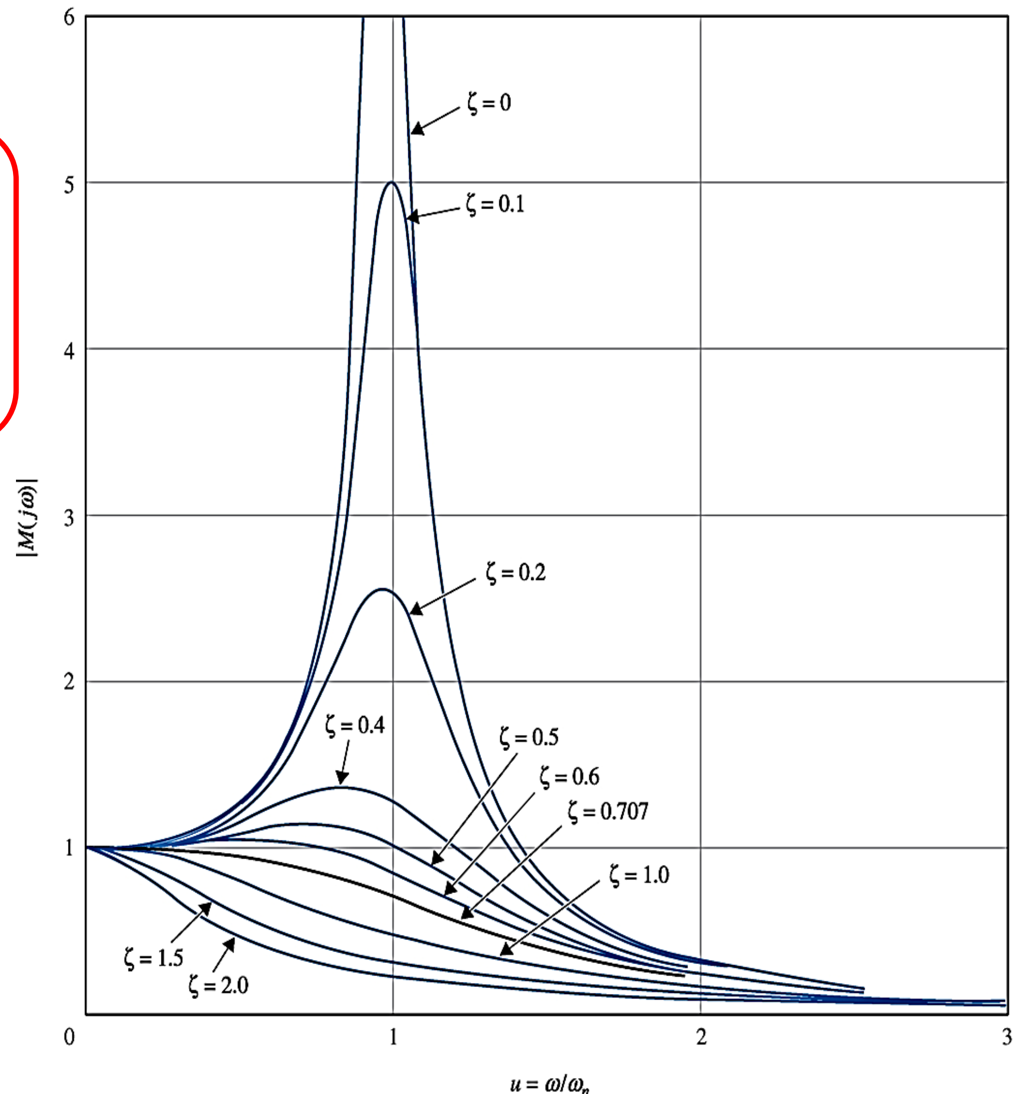
$u=\omega/\omega_n$

- To get the **resonant frequency ω_r** and the **<u>closed-loop peak resonant magnitude M_r</u>** for the **second order system**

$$\partial M / \partial u = 0$$

$$\boxed{\begin{aligned} \omega_r &= \omega_n \sqrt{1 - 2\zeta^2} \\ M_r &= \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad \zeta \le 0.707 \end{aligned}}$$





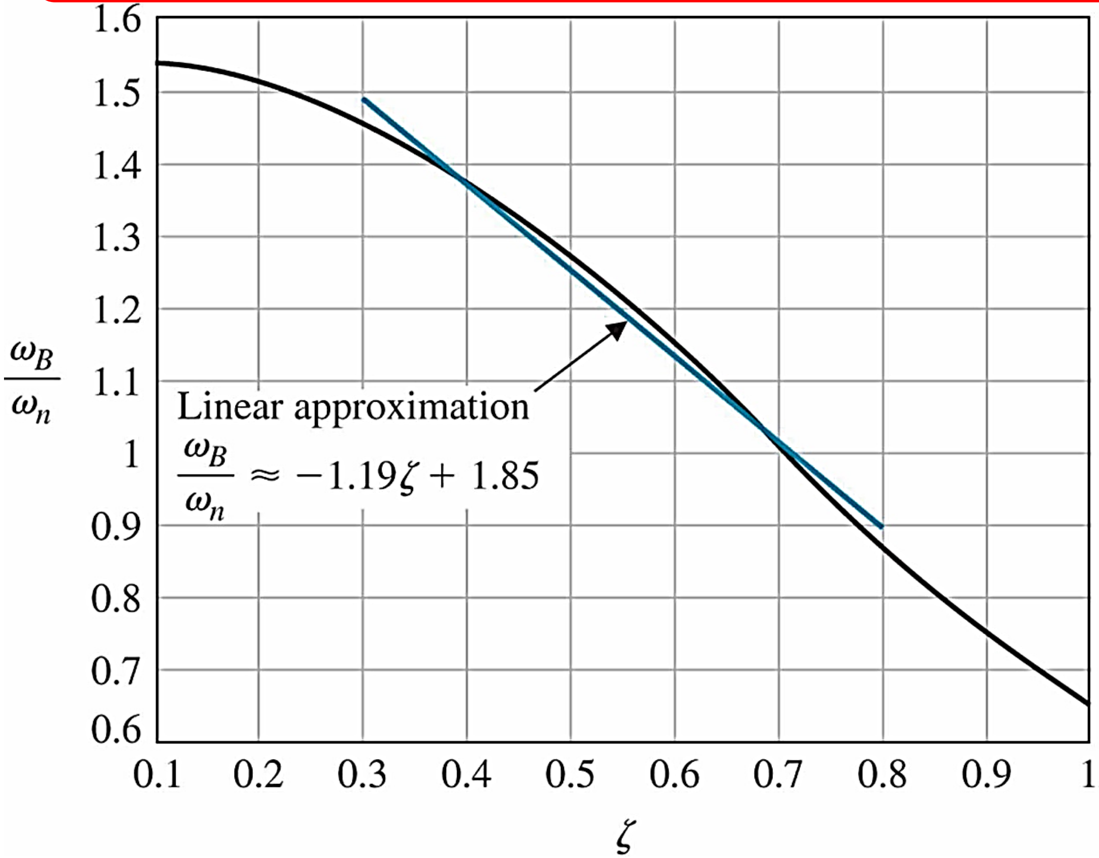
*The **closed-loop peak magnitude M_r** indicates <u>the **relative stability**</u> of the system*

- to get the bandwidth of the **second order system**

$$M(\omega) = 0.707$$

$$BW = \omega_n \left[(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2} \right]^{1/2}$$

$$\boxed{BW \cong \omega_n[-1.19\zeta + 1.85], \qquad 0.3 \leq \zeta \leq 0.8}$$



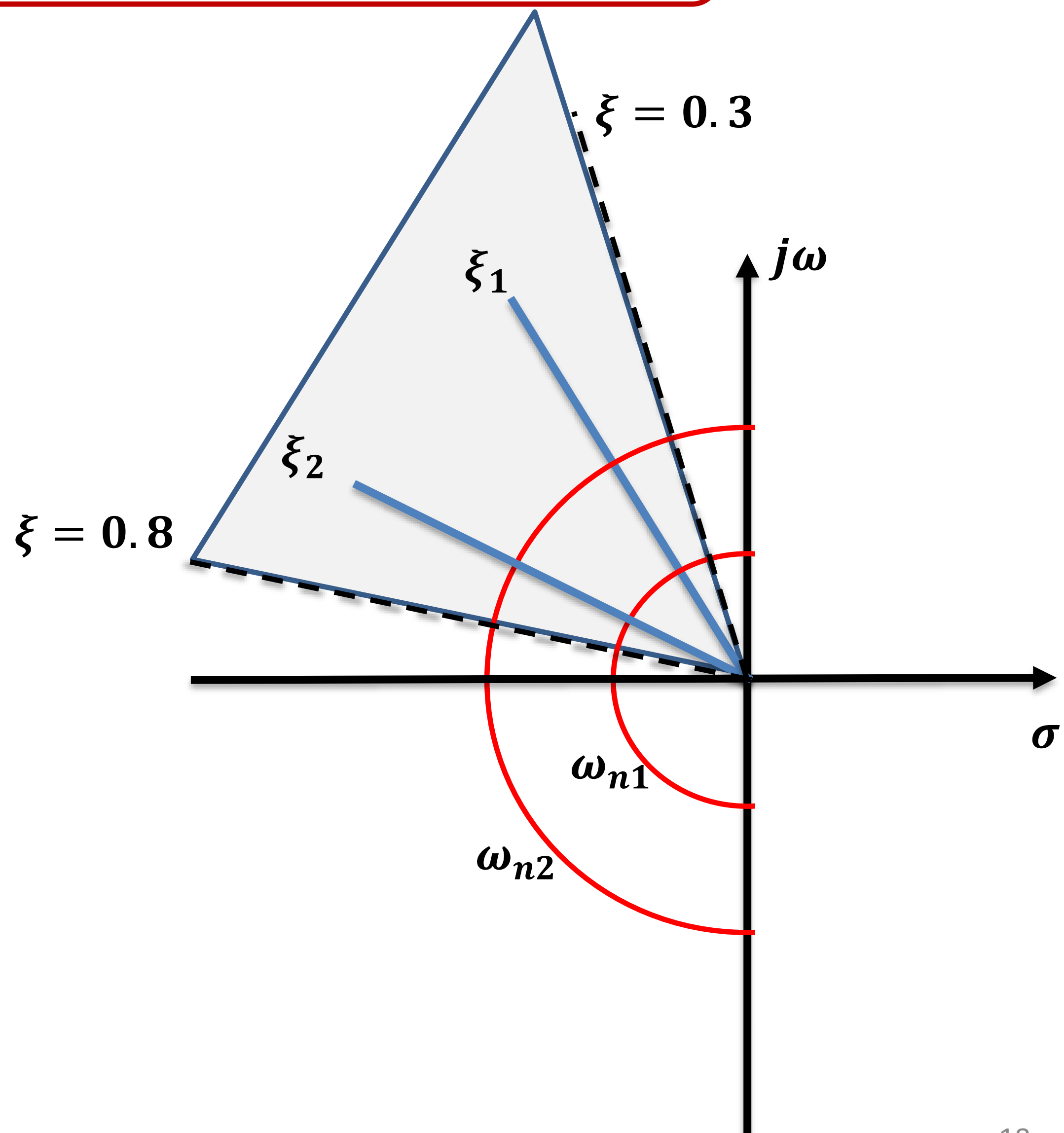
$$BW \cong \omega_n[-1.19\xi + 1.85], \qquad 0.3 \le \xi \le 0.8$$

- **for fixed ξ**:

as ω_n gets larger, BW **increases** and the system **responds faster**

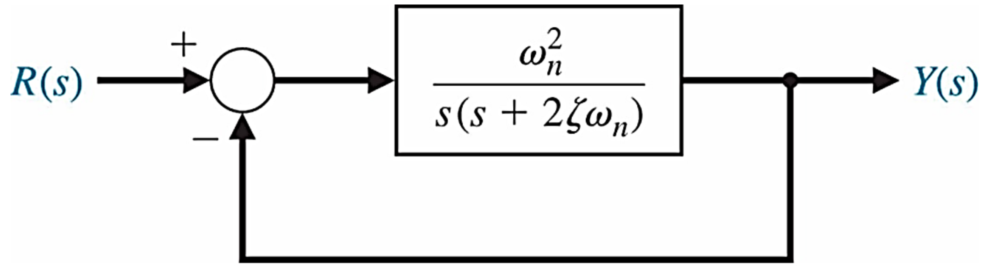
- **for fixed ω_n**:

as ξ gets larger, BW **decreases** and the system **Responds slower**

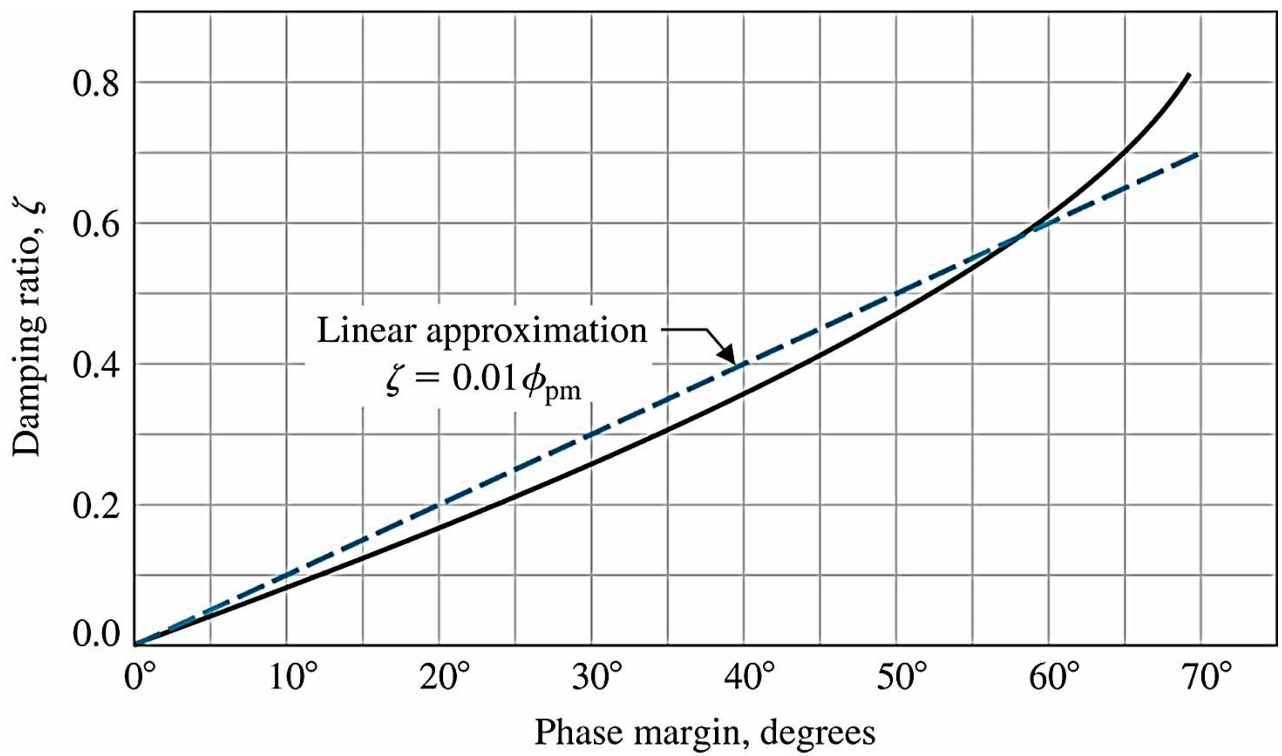


The relation between the phase margin and the damping ratio of a second order system

$$\phi_{pm} = 180° - 90° - \tan^{-1}\frac{\omega_c}{2\zeta\omega_n}$$
$$= 90° - \tan^{-1}\left(\frac{1}{2\zeta}[(4\zeta^4 + 1)^{1/2} - 2\zeta^2]^{1/2}\right)$$
$$= \tan^{-1}\frac{2}{[(4 + 1/\zeta^4)^{1/2} - 2]^{1/2}}.$$

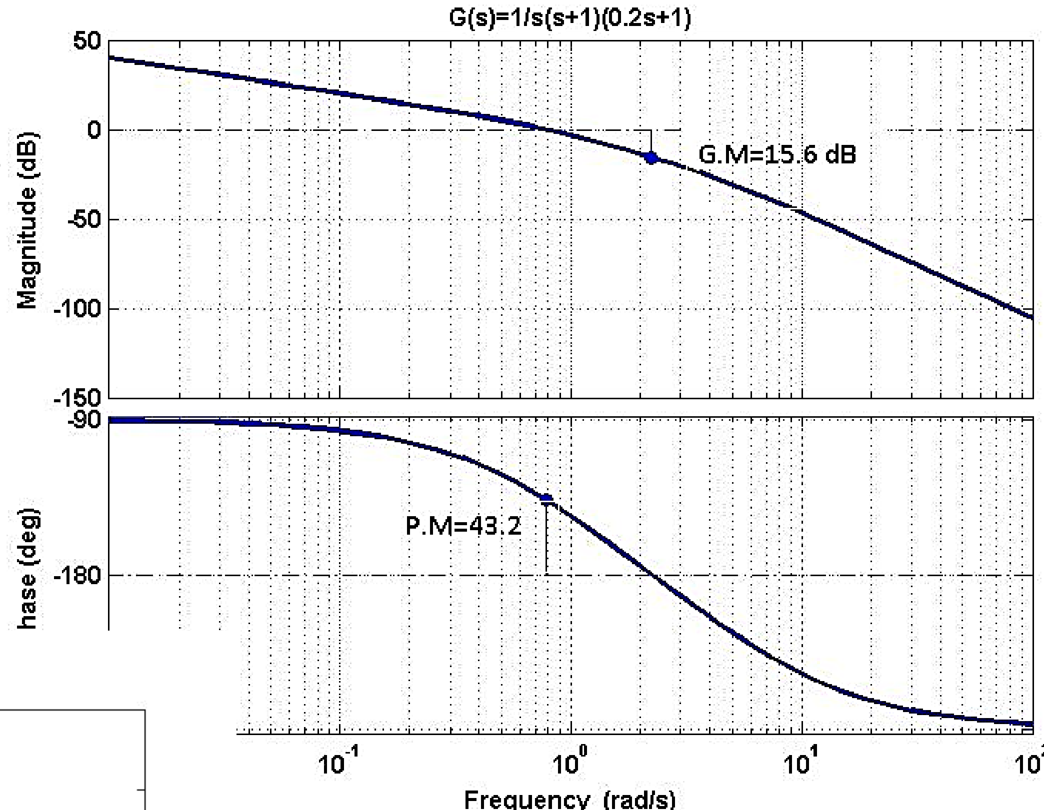


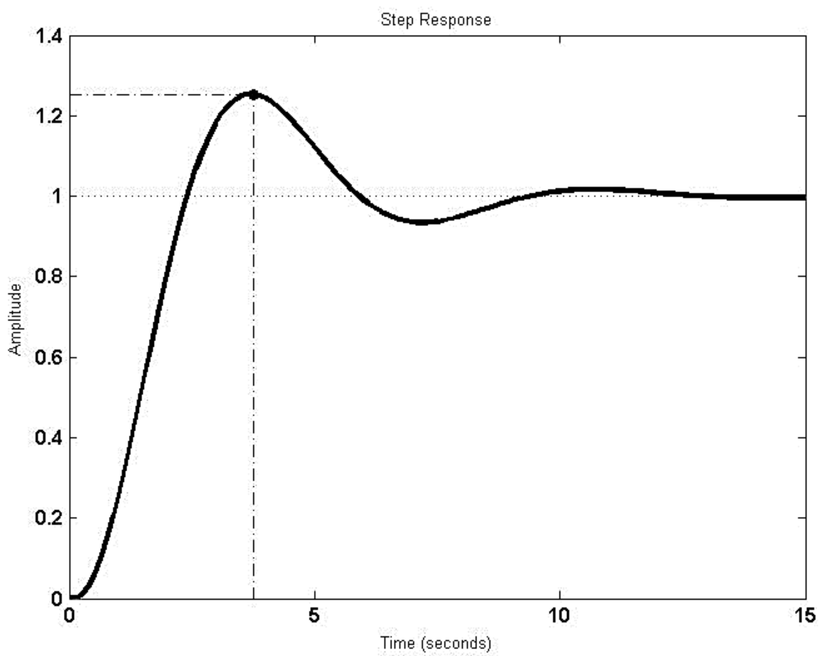
$$\xi \cong 0.01\phi_{pm}$$
$$\text{for } \xi \leq 0.707$$



- ***example***

$$G(j\omega) = \frac{1}{j\omega(j\omega+1)(0.2j\omega+1)}$$

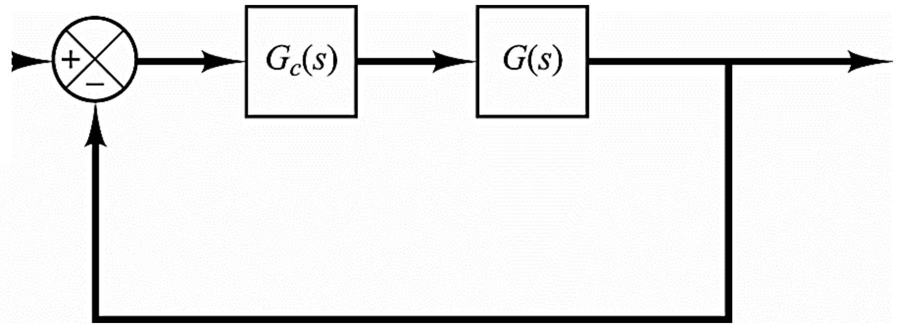




$\boldsymbol{\phi_{pm}}$	$\mathbf{43^o}$
$\boldsymbol{\xi \cong 0.01\phi_{pm}}$	$\mathbf{0.43 < 0.707}$
$\boldsymbol{P.O.}$	**25%**

- for **higher-order systems**, analytical solution is ***tedious***

- We need **graphical methods** for determining $\boldsymbol{M_r}, \boldsymbol{\omega_r}$, & $\boldsymbol{BW}$



$$T(j\omega) = \frac{G_c(j\omega)G(j\omega)}{1+G_c(j\omega)G(j\omega)}$$

$$T(j\omega) = M(\omega)e^{j\phi(\omega)}$$

- $\boldsymbol{M(\omega)}$: is the **magnitude** of the **<u>closed-loop</u>** transfer function
- $\boldsymbol{\phi(\omega)}$: is the **phase** of the **<u>closed-loop</u>** transfer function

Circles of constant magnitude of a closed-loop system (M-circles)

The **closed-loop** transfer function

$$\boxed{T(j\omega) = \frac{G_c(j\omega)G(j\omega)}{1 + G_c(j\omega)G(j\omega)}}$$

$$T(j\omega) = M(\omega)e^{j\phi(\omega)}$$

The coordinates of $G_c G(j\omega)$ $are\ u\ \&\ v$ in the **Nyquist plane**

$$G_c(j\omega)G(j\omega) = u + jv$$

- The **magnitude** of the **closed-loop** transfer function

$$M(\omega) = \left|\frac{G_c(j\omega)G(j\omega)}{1 + G_c(j\omega)G(j\omega)}\right| = \left|\frac{u + jv}{1 + u + jv}\right| = \frac{\left(u^2 + v^2\right)^{1/2}}{\left[\left(1 + u^2\right) + v^2\right]^{1/2}}$$

Squaring Equation (9.66) and rearranging, we obtain

$$(1 - M^2)u^2 + (1 - M^2)v^2 - 2M^2u = M^2. \tag{9.67}$$

Dividing Equation (9.67) by $1 - M^2$ and adding the term $[M^2/(1 - M^2)]^2$ to both sides, we have

$$u^2 + v^2 - \frac{2M^2u}{1 - M^2} + \left(\frac{M^2}{1 - M^2}\right)^2 = \left(\frac{M^2}{1 - M^2}\right) + \left(\frac{M^2}{1 - M^2}\right)^2. \tag{9.68}$$

Rearranging, we obtain

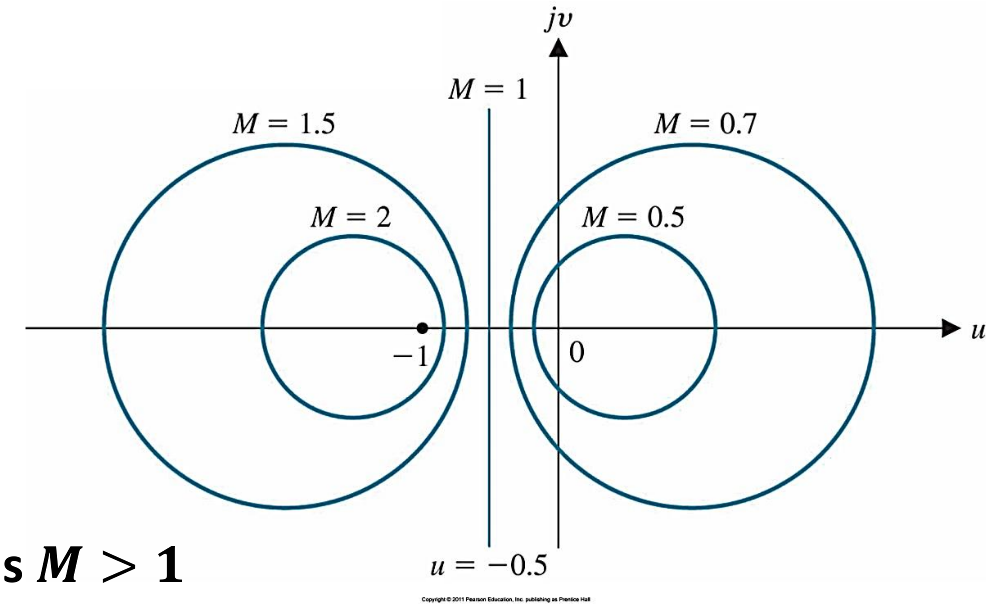
$$\boxed{\left(u - \frac{M^2}{1 - M^2}\right)^2 + v^2 = \left(\frac{M}{1 - M^2}\right)^2,} \tag{9.69}$$

which is the equation of a circle on the (u, v)-plane with the center at

$$\boxed{u = \frac{M^2}{1 - M^2}, \qquad v = 0.}$$

- The **radius** of the M circle

$$\boxed{r = \left|\frac{M}{1 - M^2}\right|}$$



- **circles to the left of $u = -0.5$ has $M > 1$**
- **circles to the right of $u = -0.5$ has $M < 1$**

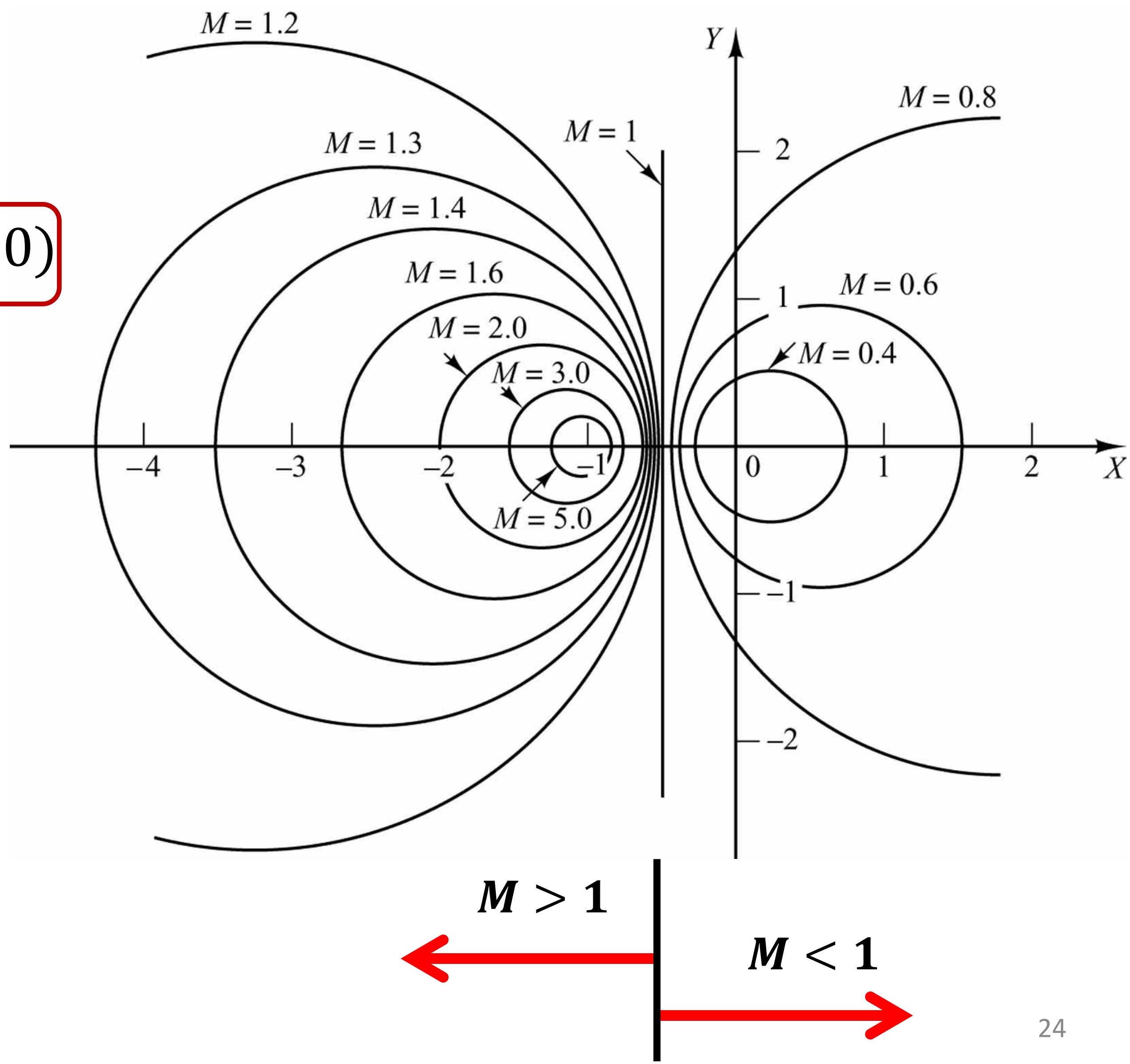
Circles of constant magnitude of a closed-loop system ($\boldsymbol{M}$-circles)

- center:

$$\boxed{C = (M^2/(1-M^2), 0)}$$

- radius:

$$\boxed{r = |M/(1-M^2)|}$$



Determining the M value of the M-circle

The value M of the M-circle that passes through the point $(\alpha, 0)$ is

$$\boxed{M = \frac{\alpha}{1+\alpha}}$$

Proof:

$$\alpha = center + radius$$

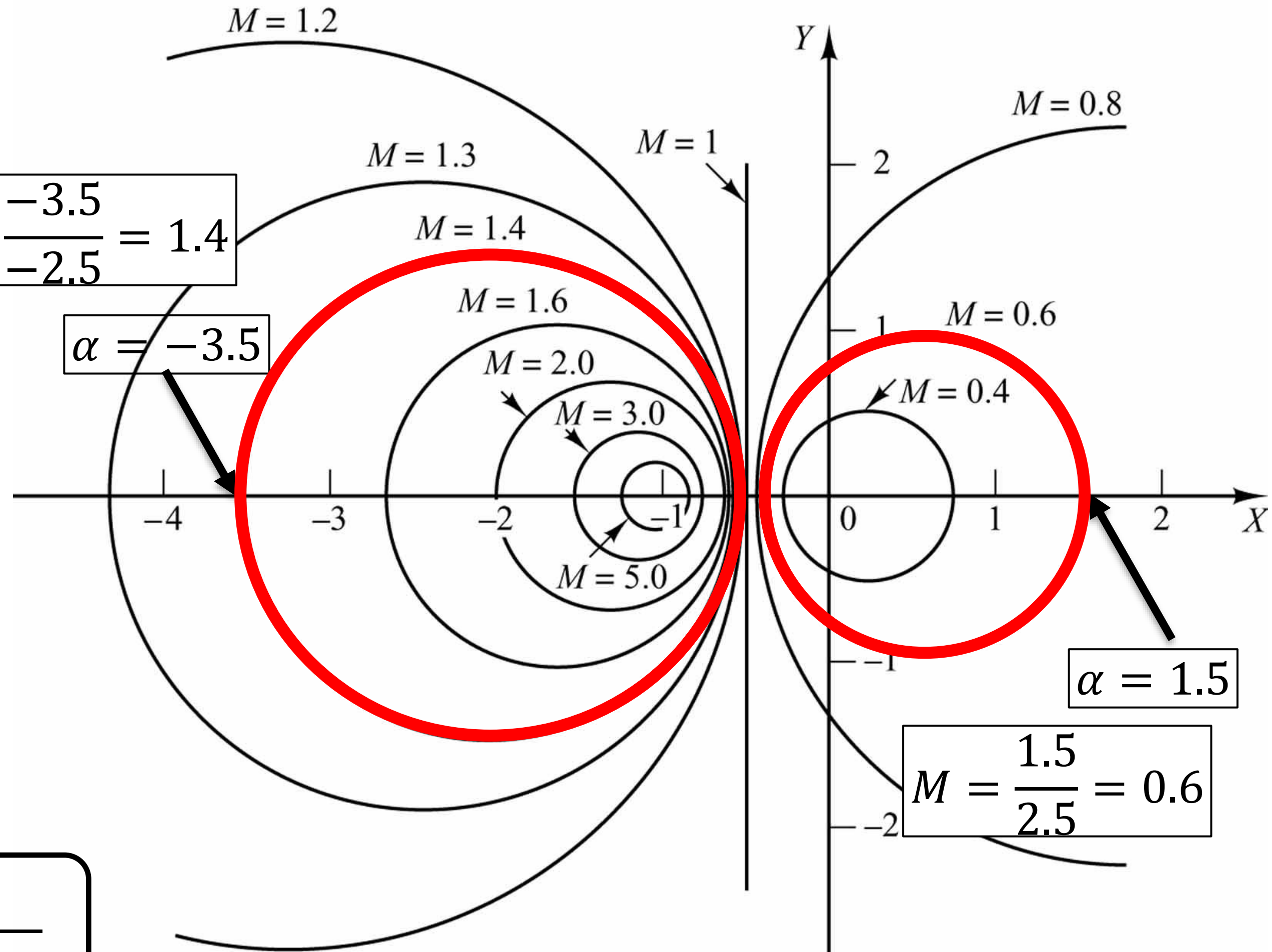
$$\alpha = \frac{M^2}{1-M^2} + \frac{M}{1-M^2}$$

$$= \frac{M(M+1)}{(1-M)(1+M)}$$

$$\alpha = \frac{M}{1-M} \qquad \Rightarrow \boxed{M = \frac{\alpha}{1+\alpha}}$$

$$M = \frac{-3.5}{-2.5} = 1.4$$

$$\alpha = -3.5$$

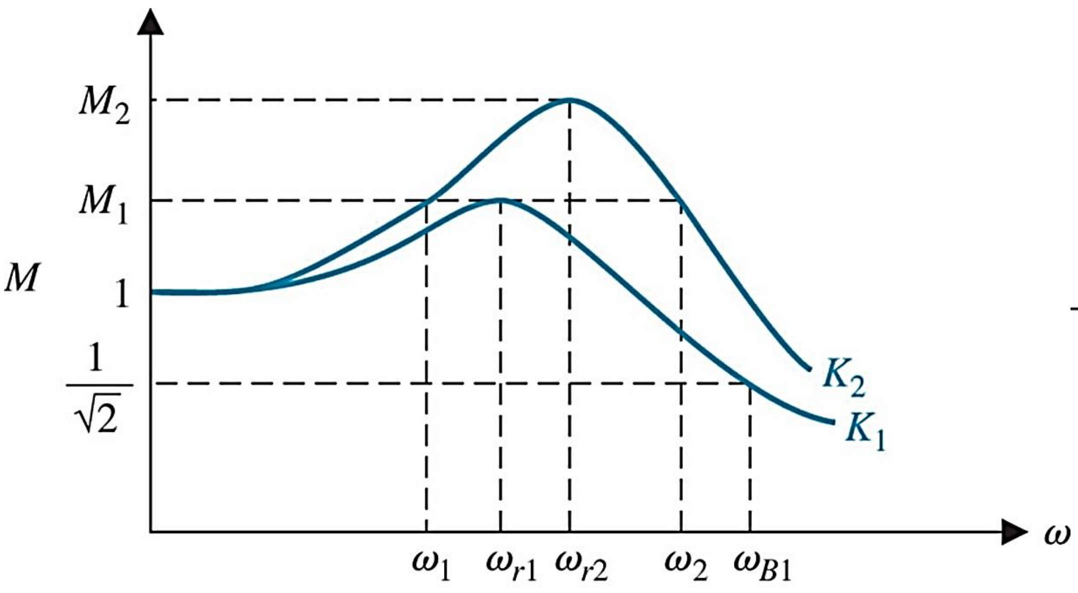


$$\alpha = 1.5$$

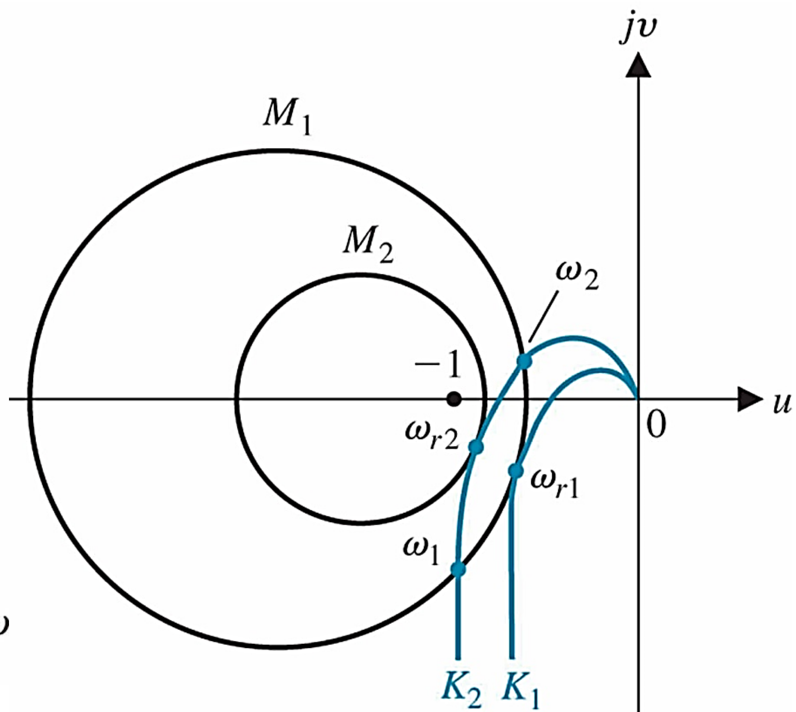
$$M = \frac{1.5}{2.5} = 0.6$$

Nyquist plot and the constant M-circles

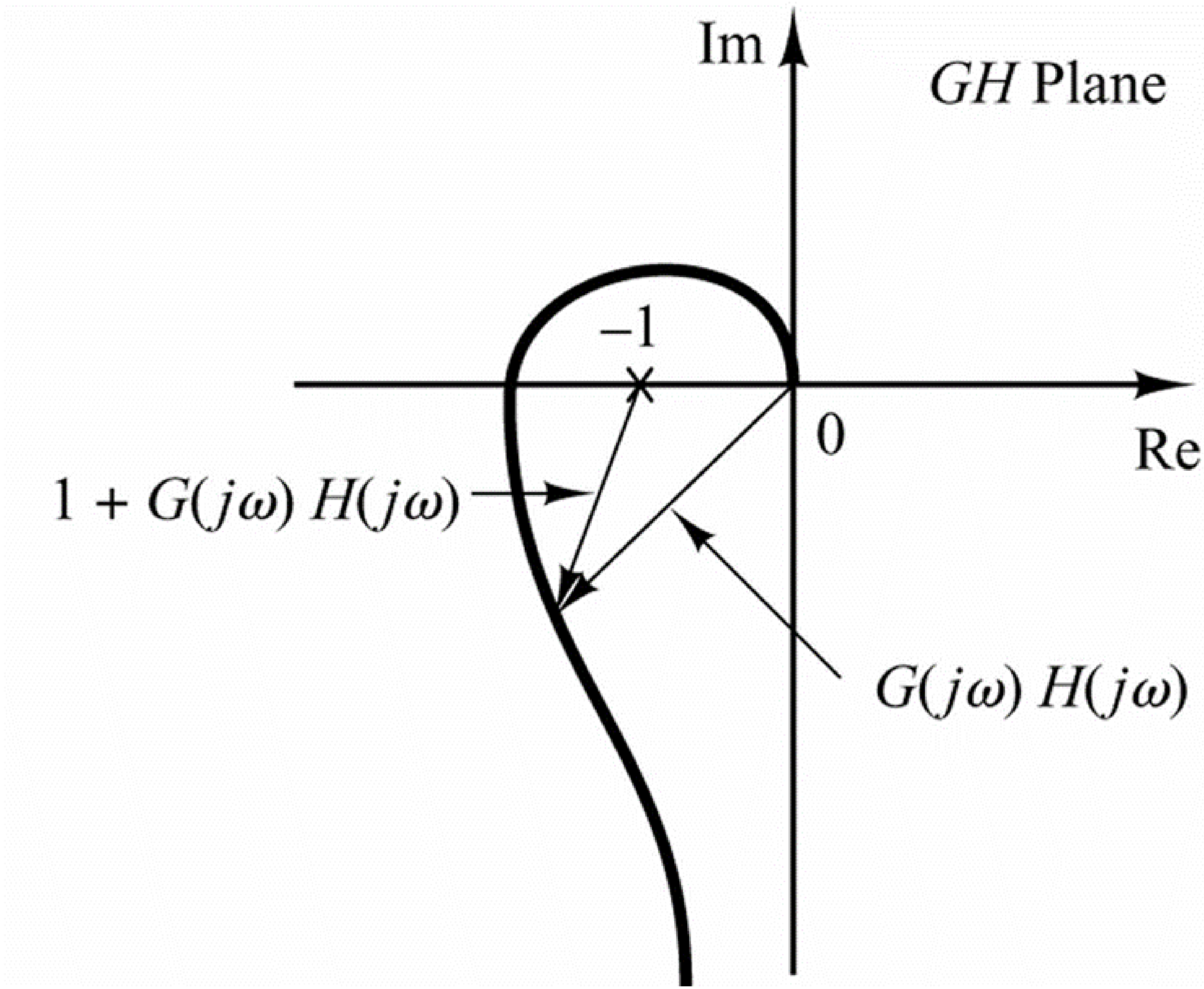
- The peak resonance $\boldsymbol{M_r}$is found by locating the **smallest circle** that is **tangent to** $G(j\omega)$ plot
- The **resonant frequency** $\boldsymbol{\omega_r}$ is found at the point of tangency
- The **bandwidth** of the closed-loop system is found at the intersect of the $G(j\omega)$ curve and the circle $\boldsymbol{M = 0.707}$



<u>Closed-loop frequency response</u>



Determining the closed loop gain from Nyquist plot



<u>*Modern control engineering – PEARSON EDUCATION, Ogata 5th ed*</u>

Circles of constant phase of a closed-loop system (N-circles)

- The angle relation

$$\phi = \angle T(j\omega) = \angle (u + jv)/(1 + u + jv)$$

$$= \tan^{-1}\left(\frac{v}{u}\right) - \tan^{-1}\left(\frac{v}{1+u}\right). \tag{9.70}$$

Taking the tangent of both sides and rearranging, we have

$$u^2 + v^2 + u - \frac{v}{N} = 0, \tag{9.71}$$

where $N = \tan\phi$. Adding the term $1/4[1 + 1/N^2]$ to both sides of the equation and simplifying, we obtain

$$\boxed{\left(u + \frac{1}{2}\right)^2 + \left(v - \frac{1}{2N}\right)^2 = \frac{1}{4}\left(1 + \frac{1}{N^2}\right),} \tag{9.72}$$

- equation of a circle with the **center** $(-\mathbf{1}/\mathbf{2}\,, \mathbf{1}/(\mathbf{2}\boldsymbol{N}))$

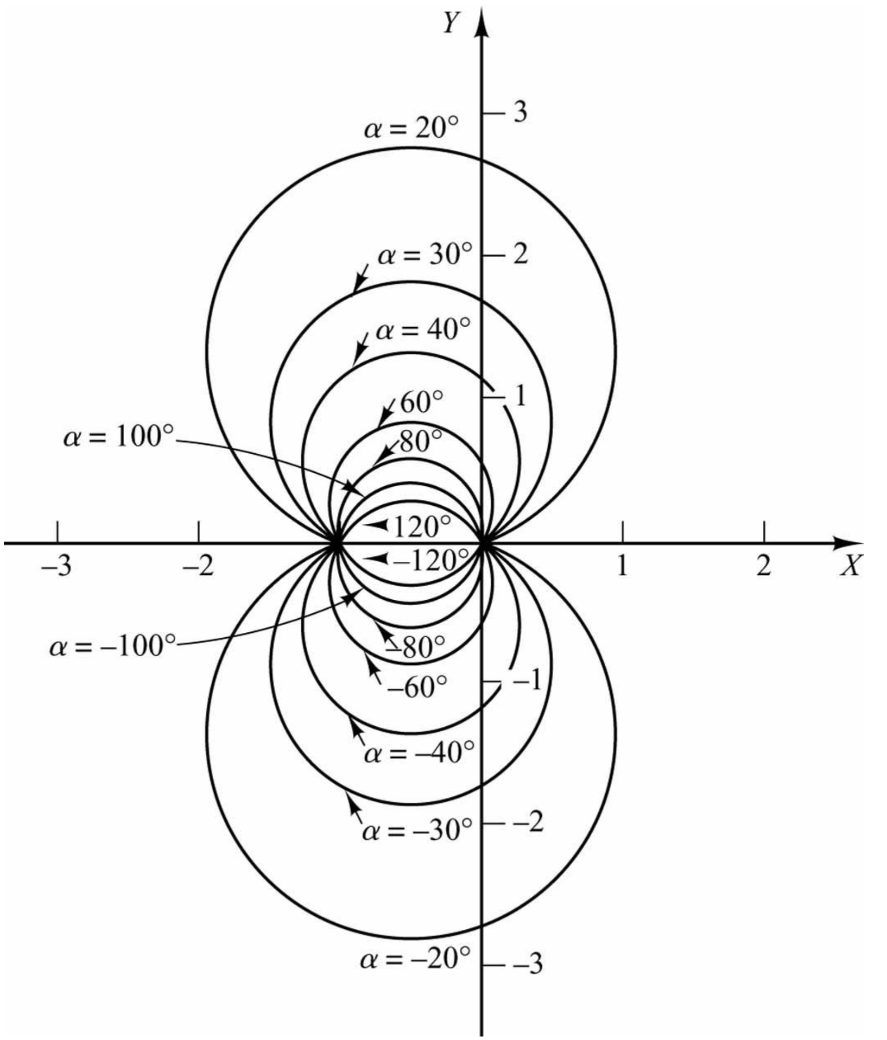
Circles of constant phase of a closed-loop system (N-circles)

- center:

$$\boxed{C = (-1/2\,,1/2N)}$$

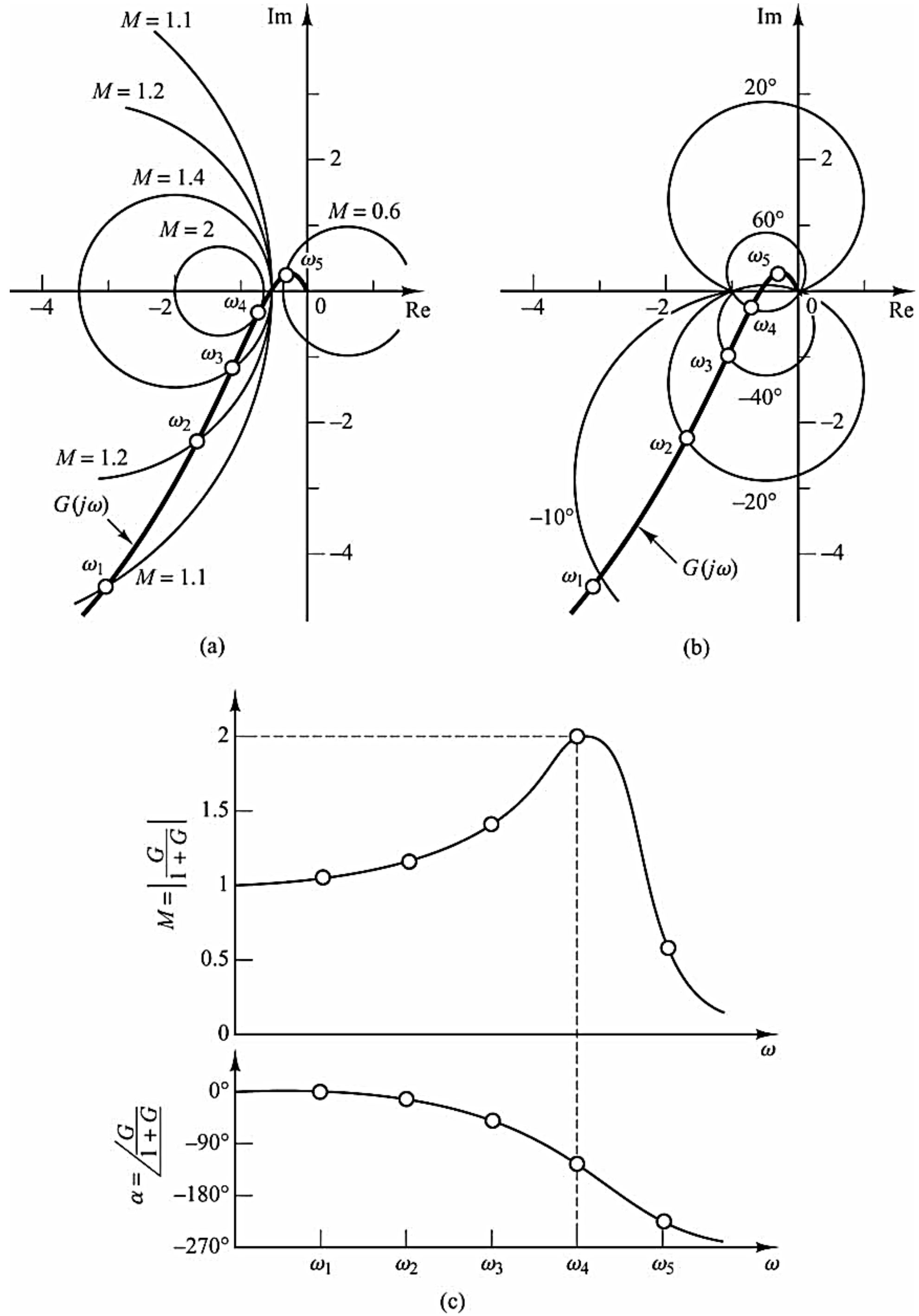
- radius:

$$\boxed{r = \frac{1}{2}\left(1 + \frac{1}{N^2}\right)^{0.5}}$$



- **example**

closed-loop response

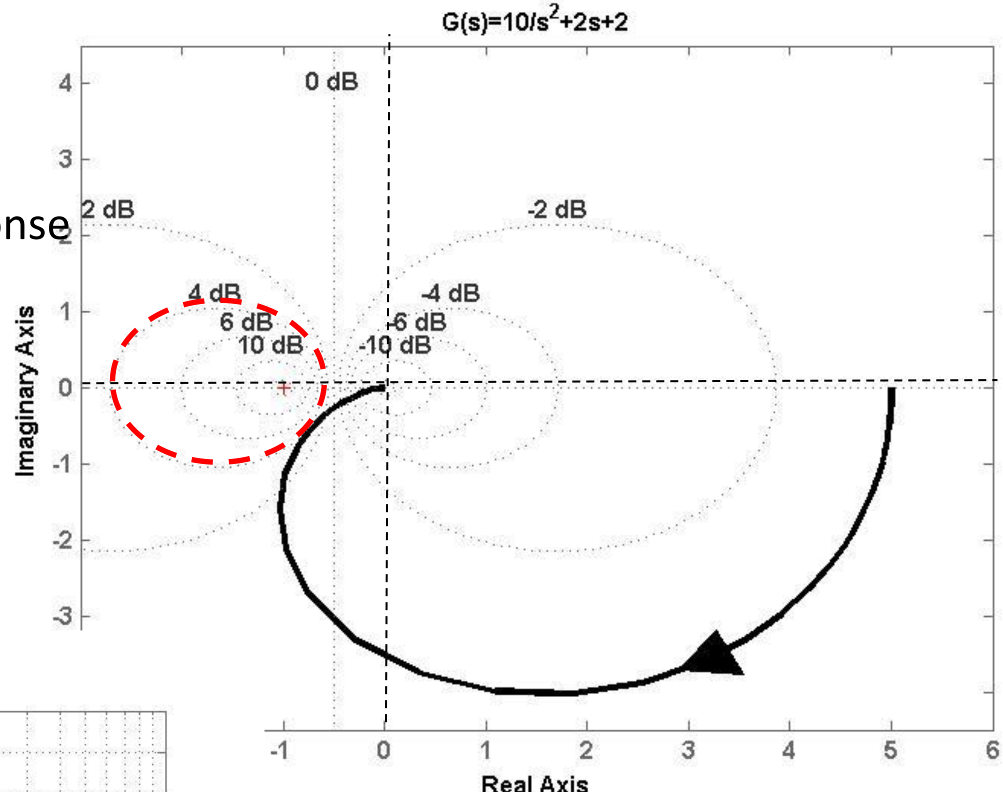


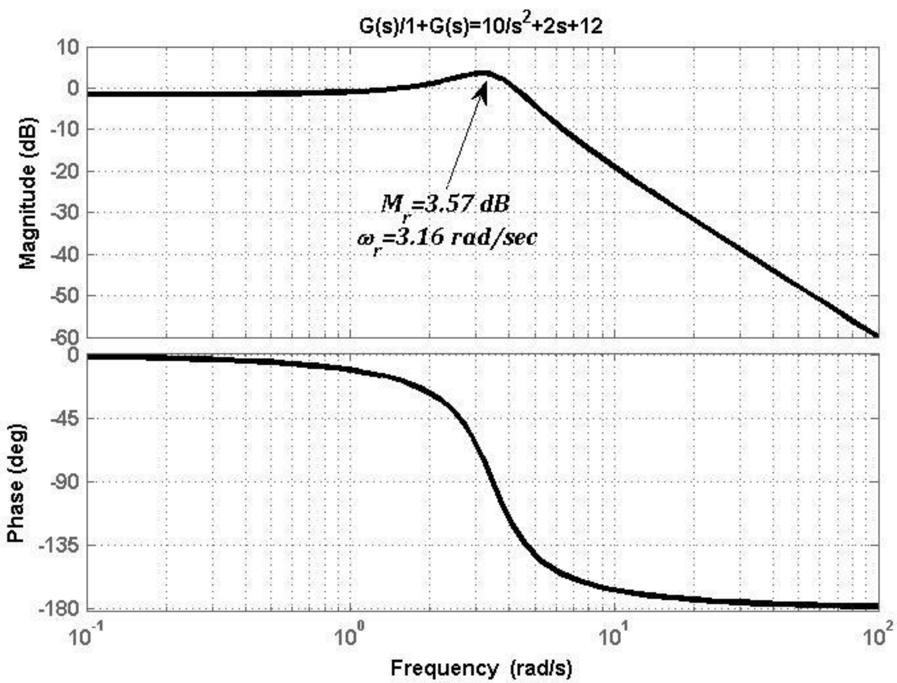
- ***example*** $\boxed{G(s) = \dfrac{10}{s^2 + 2s + 2}}$

- from Nyquist plot:

The **closed-loop** peak frequency response

$$M_r \cong 4dB \ @ \ \omega_r = 3.22 \, rad/sec$$



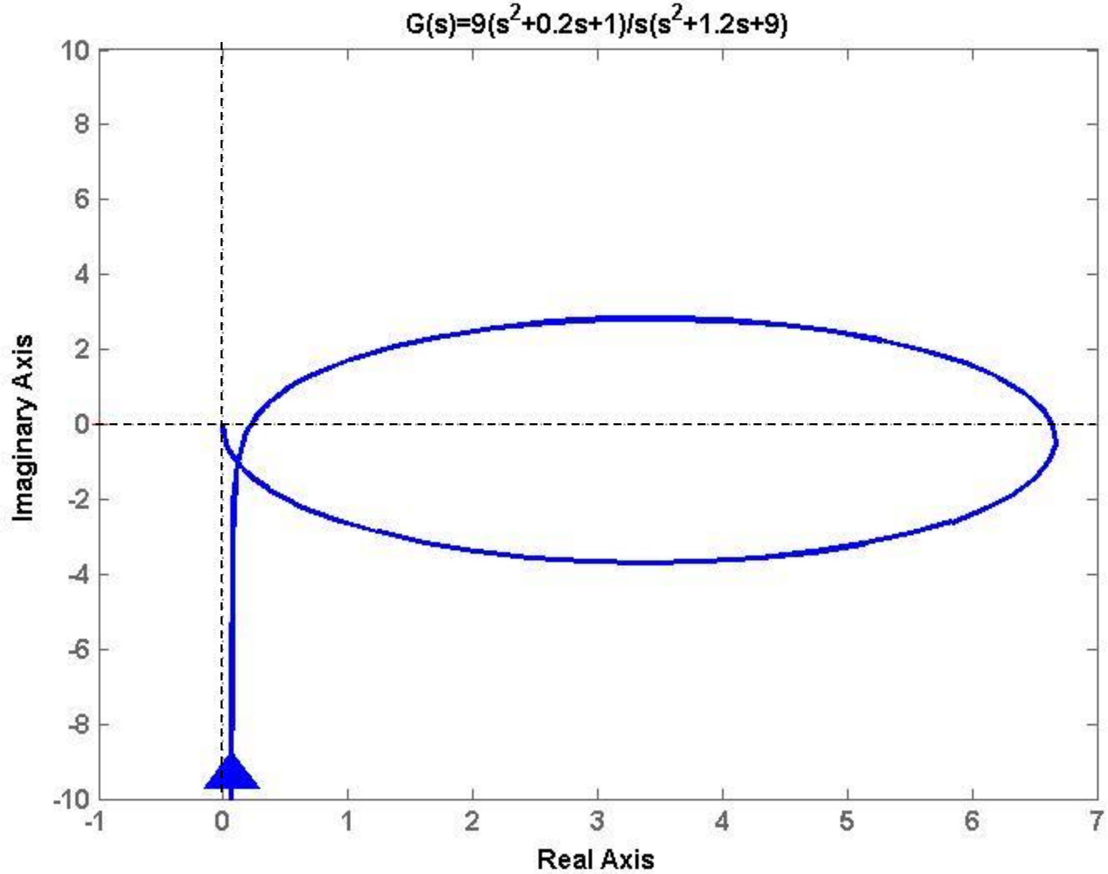


Closed-loop frequency response

$$\boxed{\frac{G(s)}{1+G(s)} = \frac{10}{s^2 + 2s + 12}}$$

- **example**

$$G(s) = \frac{9(s^2 + 0.2s + 1)}{s(s^2 + 1.2s + 9)}$$



- The system is stable
- $G.M = \infty$

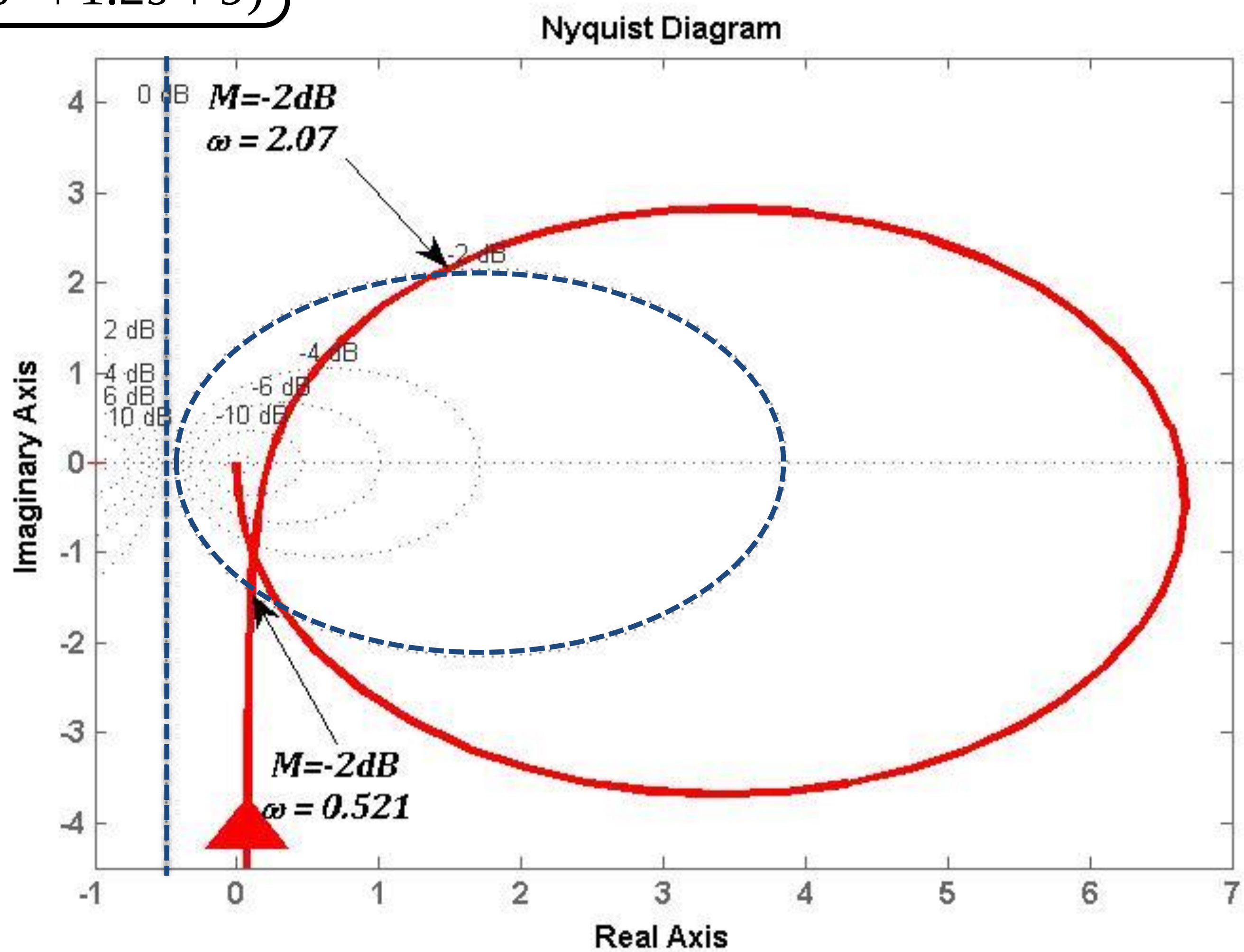
- The system behaves as a **first order system**

s	$G(s)$
$j\omega \longrightarrow 0+$	$\infty\angle -90$
$j\omega \rightarrow \infty +$	$0\angle -90$

b=9*[1 0.2 1]; a=[1 1.2 9 0]; sys=tf(b, a); nyquist(sys)

Closed-loop frequency response and the Nyquist plot

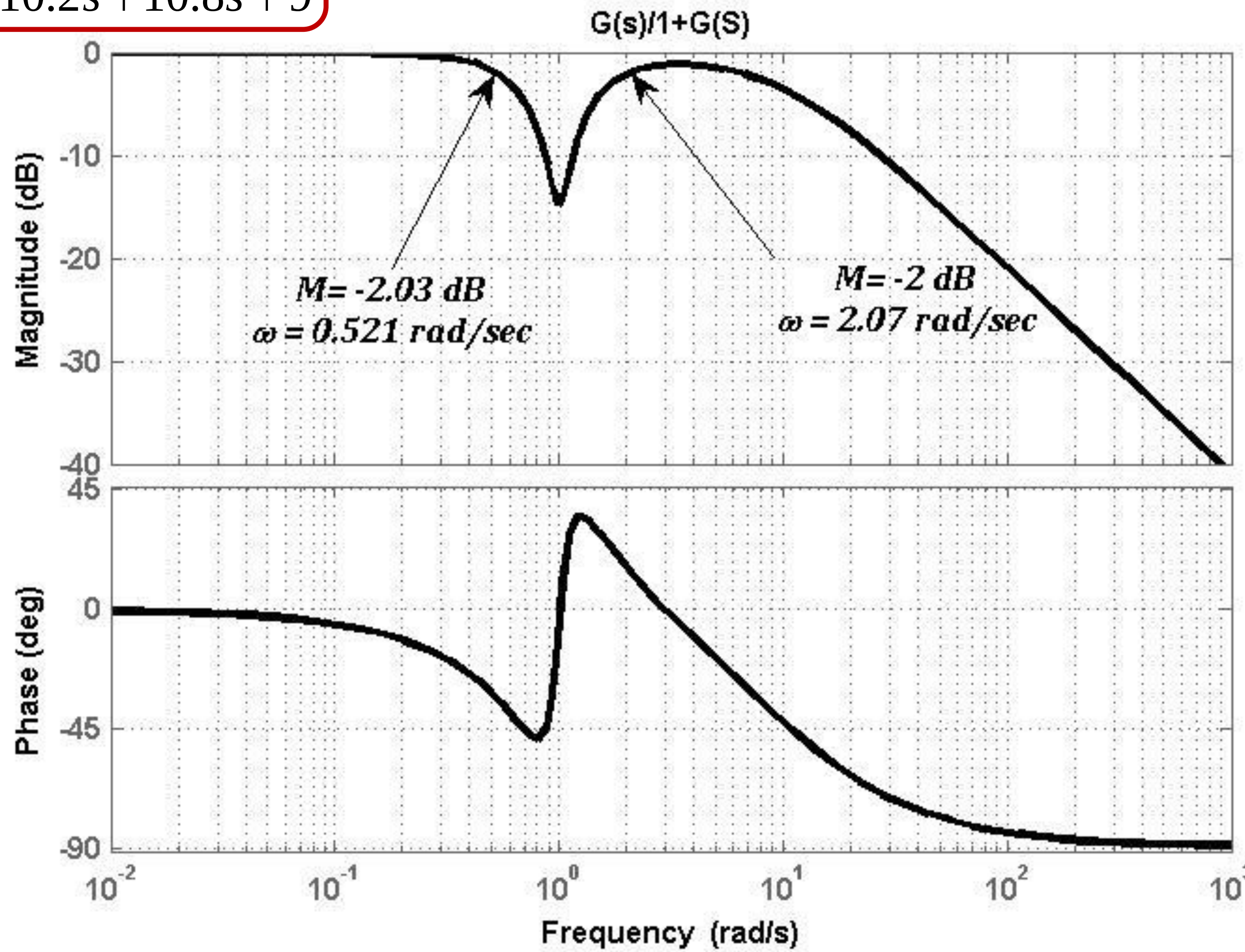
$$G(s) = \frac{9(s^2 + 0.2s + 1)}{s(s^2 + 1.2s + 9)}$$



Closed-loop frequency response

$$\frac{G(s)}{1+G(s)} = \frac{9(s^2+0.2s+1)}{s^3+10.2s+10.8s+9}$$

```
bc=9*[1 0.2 1]; ac=[1 10.2 10.8 9]; sysc=tf(bc,ac); bode(sysc)
```



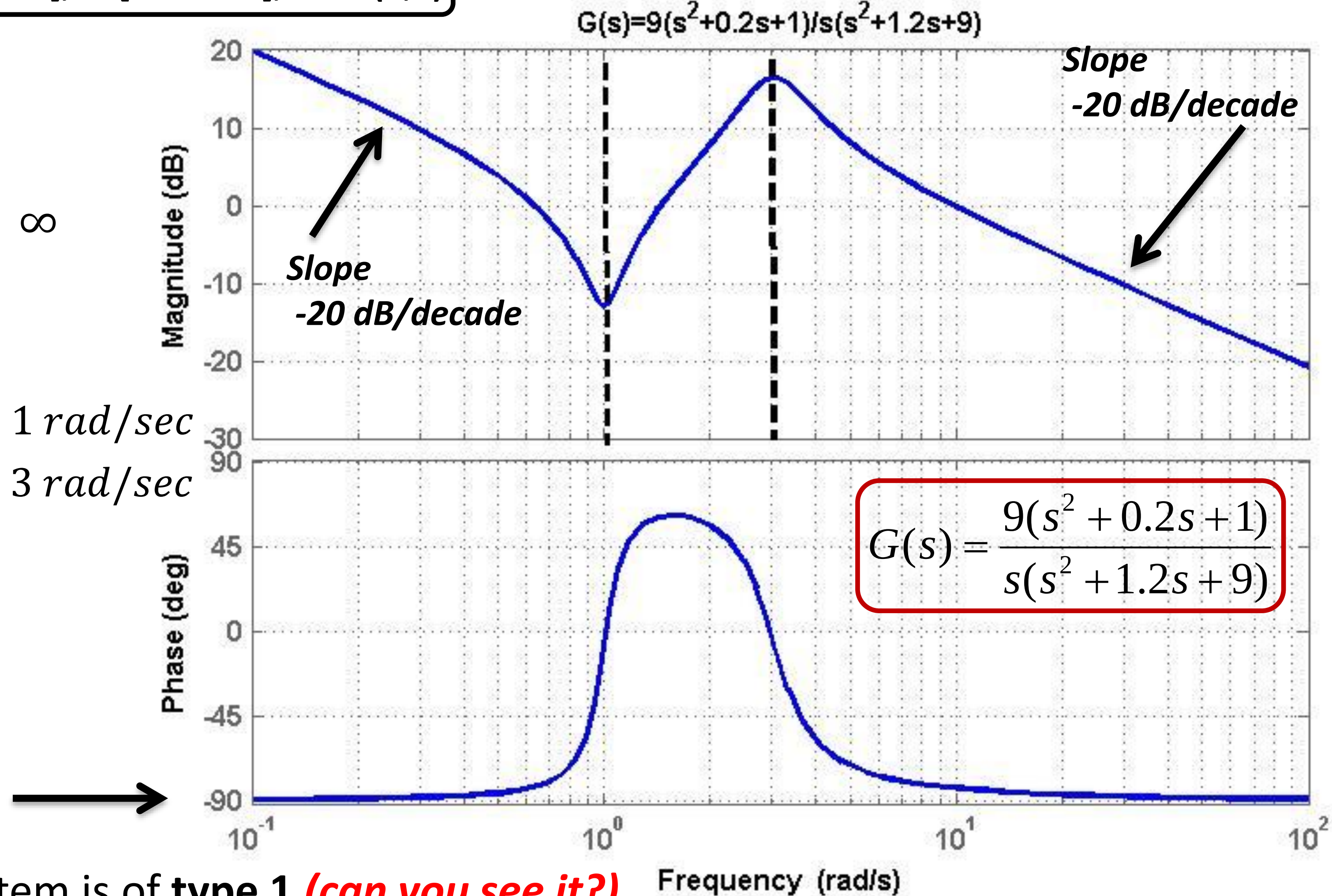
- Closed-loop Peak frequency response $M_r = 0dB\ @\ \omega \rightarrow 0$

Open-loop frequency response

```
b=9*[1 0.2 1]; a=[1 1.2 9 0]; bode(b,a)
```

- $G.M = \infty$

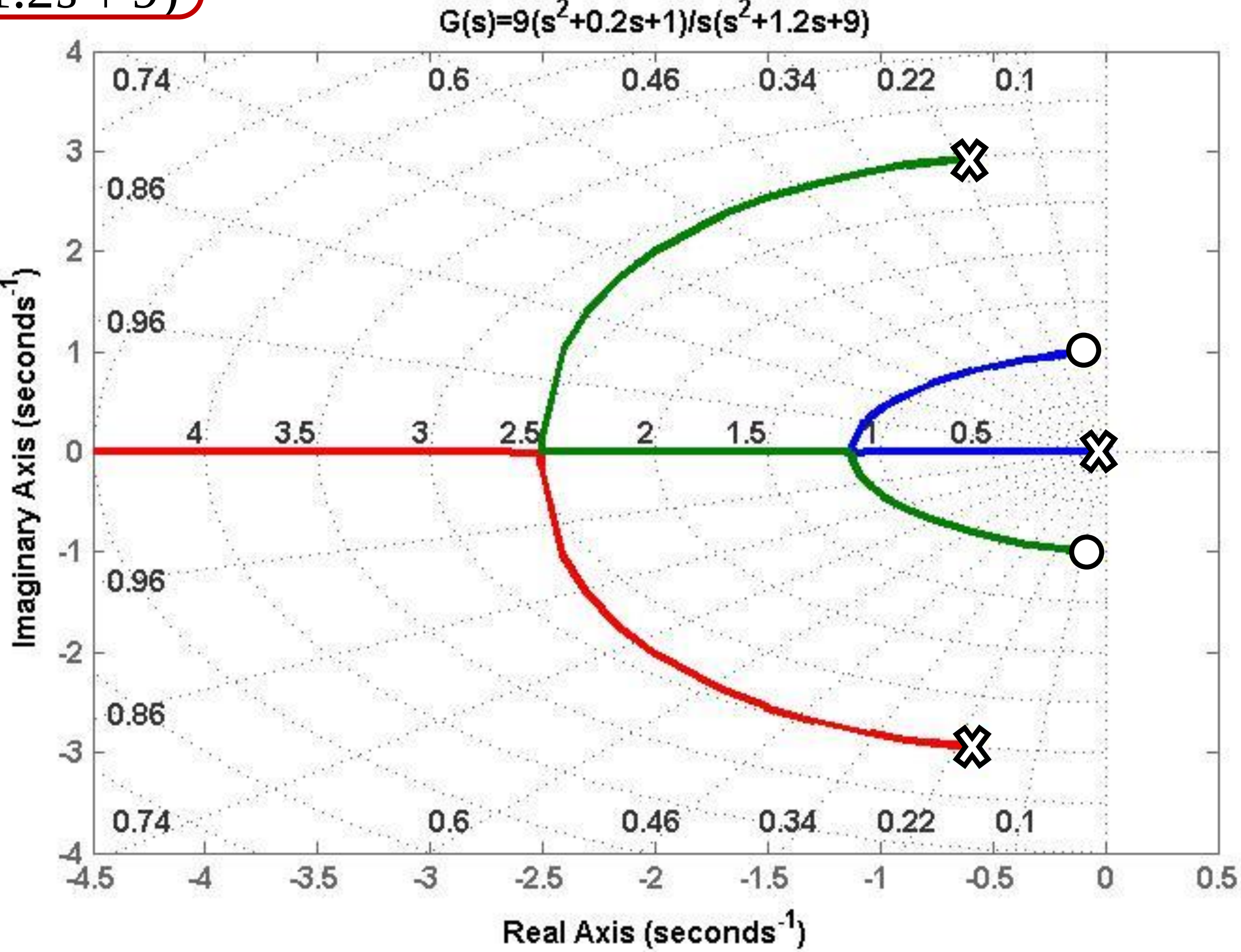
- $\omega_{zero} = 1\ rad/sec$
- $\omega_{pole} = 3\ rad/sec$



- The system is of **type 1** ***(can you see it?)***
- Can you determine the **frequency location** of the poles and zeros?
- What is the **damping factor** at $K = 0, K = \infty$? ***(check from root locus)***

$$G(s) = \frac{9(s^2 + 0.2s + 1)}{s(s^2 + 1.2s + 9)}$$

b=[1 0.2 1]; a=[1 1.2 9 0]; rlocus(b,a)



- The effect of zeros is clear

- The system is always stable $\Longrightarrow G.M = \infty$

- *Can you determine the **velocity** of each pole?*