Chapter 5 Solutions, Susanna Epp Discrete Math 5th Edition

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Contents

1 Exercise Set 5.1

Write the first four terms of the sequences defined by the formulas in $1-6$.

1.1 Exercise 1

$a_k = \dfrac{k}{10+k}$, for every integer $k \geq 1$.

Proof. $\dfrac{1}{11}, \dfrac{2}{12}, \dfrac{3}{13}, \dfrac{4}{14}$ □

1.2 Exercise 2

$b_j = \dfrac{5-j}{5+j}$, for every integer $j \geq 1$.

Proof. $\dfrac{4}{6}, \dfrac{3}{7}, \dfrac{2}{8}, \dfrac{1}{9}$ □

1.3 Exercise 3

$c_i = \frac{(-1)^i}{3^i}$, for every integer $i \geq 0$.

Proof. $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}$ □

1.4 Exercise 4

$d_m = 1 + \left(\frac{1}{2}\right)^m$, for every integer $m \geq 0$.

Proof. $2, \frac{3}{2}, \frac{5}{4}, \frac{9}{8}$ □

1.5 Exercise 5

$e_n = \left\lfloor \frac{n}{2} \right\rfloor \cdot 2$, for every integer $n \geq 0$.

Proof. $0, 0, 2, 2$ □

1.6 Exercise 6

$f_n = \left\lfloor \frac{n}{4} \right\rfloor \cdot 4$, for every integer $n \geq 1$.

Proof. 0, 0, 0, 4 □

1.7 Exercise 7

Let $a_k = 2k + 1$ and $b_k = (k-1)^3 + k + 2$ for every integer $k \geq 0$. Show that the first three terms of these sequences are identical but that their fourth terms differ.

Proof. $a_0 = 2(0) + 1 = 1, a_1 = 2(1) + 1 = 3, a_2 = 2(2) + 1 = 5, a_3 = 2(3) + 1 = 7$.

$b_0 = (0-1)^3 + 0 + 2 = 1, b_1 = (1-1)^3 + 1 + 2 = 3, b_2 = (2-1)^3 + 2 + 2 = 5, b_3 = (3-1)^3 + 3 + 2 = 13$. □

Compute the first fifteen terms of each of the sequences in 8 and 9, and describe the general behavior of these sequences in words. (a definition of logarithm is given in Section 7.1.)

1.8 Exercise 8

$g_n = \lfloor \log_2 n \rfloor$ for every integer $n \geq 1$.

Proof. $g_1 = \lfloor \log_2 1 \rfloor = 0, g_2 = \lfloor \log_2 2 \rfloor = 1, g_3 = \lfloor \log_2 3 \rfloor = 1, g_4 = \lfloor \log_2 4 \rfloor = 2,$

$g_5 = \lfloor \log_2 5 \rfloor = 2, g_6 = \lfloor \log_2 6 \rfloor = 2, g_7 = \lfloor \log_2 7 \rfloor = 2, g_8 = \lfloor \log_2 8 \rfloor = 3,$

$g_9 = \lfloor \log_2 9 \rfloor = 3, g_{10} = \lfloor \log_2 10 \rfloor = 3, g_{11} = \lfloor \log_2 11 \rfloor = 3, g_{12} = \lfloor \log_2 12 \rfloor = 3,$

$g_{13} = \lfloor \log_2 13 \rfloor = 3, g_{14} = \lfloor \log_2 14 \rfloor = 3, g_{15} = \lfloor \log_2 15 \rfloor = 3.$

When n is an integral power of 2, g_n is the exponent of that power. For instance, $8 = 2^3$ and $g_8 = 3$. More generally, if $n = 2k$, where k is an integer, then $g_n = k$. All terms of the sequence from g_{2^k} up to, but not including, $g_{2^{k+1}}$ have the same value, namely k. For instance, all terms of the sequence from g_8 through g_{15} have the value 3. □

1.9 Exercise 9

$h_n = n\lfloor \log_2 n \rfloor$ for every integer $n \geq 1$.

Proof. $h_1 = 1\lfloor \log_2 1 \rfloor = 0, h_2 = 2\lfloor \log_2 2 \rfloor = 2, h_3 = 3\lfloor \log_2 3 \rfloor = 3, h_4 = 4\lfloor \log_2 4 \rfloor = 8,$

$h_5 = 5\lfloor \log_2 5 \rfloor = 10, h_6 = 6\lfloor \log_2 6 \rfloor = 12, h_7 = 7\lfloor \log_2 7 \rfloor = 14, h_8 = 8\lfloor \log_2 8 \rfloor = 24,$

$h_9 = 9\lfloor \log_2 9 \rfloor = 27, h_{10} = 10\lfloor \log_2 10 \rfloor = 30, h_{11} = 11\lfloor \log_2 11 \rfloor = 33,$

$h_{12} = 12\lfloor \log_2 12 \rfloor = 36, h_{13} = 13\lfloor \log_2 13 \rfloor = 39, h_{14} = 14\lfloor \log_2 14 \rfloor = 42,$

$h_{15} = 15\lfloor \log_2 15 \rfloor = 45.$ □

Find explicit formulas for sequences of the form $a_1, a_2, a_3, \ldots$ with the initial terms given in $10 - 16$.

Exercises $10 - 16$ have more than one correct answer.

1.10 Exercise 10

$-1, 1, -1, 1, -1, 1$

Proof. $a_n = (-1)^n$, where n is an integer and $n \geq 1$ □

1.11 Exercise 11

$0, 1, -2, 3, -4, 5$

Proof. $a_n = (n-1)(-1)^n$, where n is an integer and $n \geq 1$ □

1.12 Exercise 12

$\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \frac{6}{49}$

Proof. $a_n = \frac{n}{(n+1)^2}$, where n is an integer and $n \geq 1$ □

1.13 Exercise 13

$1-\frac{1}{2}, \frac{1}{2}-\frac{1}{3}, \frac{1}{3}-\frac{1}{4}, \frac{1}{4}-\frac{1}{5}, \frac{1}{5}-\frac{1}{6}, \frac{1}{6}-\frac{1}{7}$

Proof. $a_n = \frac{1}{n} - \frac{1}{n+1}$, where n is an integer and $n \geq 1$ □

1.14 Exercise 14

$\frac{1}{3}, \frac{4}{9}, \frac{9}{27}, \frac{16}{81}, \frac{25}{243}, \frac{36}{729}$

Proof. $a_n = \frac{n^2}{3^n}$, where n is an integer and $n \geq 1$ □

1.15 Exercise 15

$0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \frac{6}{7}$

Proof. $a_n = \frac{n-1}{n} \cdot (-1)^{n-1}$, where n is an integer and $n \geq 1$ □

1.16 Exercise 16

3, 6, 12, 24, 48, 96

Proof. $a_n = 3 \cdot 2^{n-1}$, where n is an integer and $n \geq 1$ □

1.17 Exercise 17

Consider the sequence defined by $a_n = \frac{2n + (-1)^n - 1}{4}$ for every integer $n \geq 0$. Find an alternative explicit formula for an that uses the floor notation.

Proof. $a_0 = 0, a_1 = 0, a_2 = 1, a_3 = 1, a_4 = 2, a_5 = 2$. It seems to be following the pattern: $a_n = \left\lfloor \frac{n}{2} \right\rfloor$. Let's try to prove this. When n is even, $n = 2k$ for some integer k, so we have

$$a_n = a_{2k} = \frac{2(2k) + (-1)^{2k} - 1}{4} = \frac{4k + 1 - 1}{4} = \frac{4k}{4} = k = \frac{n}{2} = \left\lfloor \frac{n}{2} \right\rfloor$$

When n is odd, $n = 2k + 1$ for some integer k, so we have

$$a_n = a_{2k+1} = \frac{2(2k+1) + (-1)^{2k+1} - 1}{4} = \frac{4k + 2 - 1 - 1}{4} = \frac{4k}{4} = k = \frac{n-1}{2} = \left\lfloor \frac{n}{2} \right\rfloor$$

So $a_n = \left\lfloor \frac{n}{2} \right\rfloor$ for all $n \geq 0$. □

1.18 Exercise 18

Let $a_0 = 2, a_1 = 3, a_2 = -2, a_3 = 1, a_4 = 0, a_5 = -1$, and $a_6 = -2$. Compute each of the summations and products below.

1.18.1 (a)

$$\sum_{i=0}^{6} a_i$$

Proof. $2 + 3 + (-2) + 1 + 0 + (-1) + (-2) = 1$ $\square$

1.18.2 (b)

$$\sum_{i=0}^{0} a_i$$

Proof. $a_0 = 2$ $\square$

1.18.3 (c)

$$\sum_{j=1}^{3} a_{2j}$$

Proof. $a_2 + a_4 + a_6 = -2 + 0 + (-2) = -4$ $\square$

1.18.4 (d)

$$\prod_{k=0}^{6} a_k$$

Proof. $2 \cdot 3 \cdot (-2) \cdot 1 \cdot 0 \cdot (-1) \cdot (-2) = 0$ $\square$

1.18.5 (e)

$$\prod_{k=2}^{2} a_k$$

Proof. $\square$

Compute the summations and products in 19 – 28.

1.19 Exercise 19

$$\sum_{k=1}^{5}(k+1)$$

Proof. $2+3+4+5+6=20$ $\square$

1.20 Exercise 20

$$\prod_{k=2}^{4}k^2$$

Proof. $2^2\cdot 3^2\cdot 4^2=576$ $\square$

1.21 Exercise 21

$$\sum_{k=1}^{3}(k^2+1)$$

Proof. $(1^2+1)+(2^2+1)+(3^2+1)=2+5+10=17$ $\square$

1.22 Exercise 22

$$\prod_{j=0}^{4}(-1)^j$$

Proof. $(-1)^0\cdot(-1)^1\cdot(-1)^2\cdot(-1)^3\cdot(-1)^4=1$ $\square$

1.23 Exercise 23

$$\sum_{i=1}^{1}i(i+1)$$

Proof. $1(1{+}1)=2$ $\square$

1.24 Exercise 24

$$\sum_{j=0}^{0}(j+1)\cdot 2^j$$

Proof. $(0+1)\cdot 2^0=1$ $\square$

1.25 Exercise 25

$$\prod_{k=2}^{2}\left(1-\frac{1}{k}\right)$$

Proof. $(1-1/2)=1/2$ $\square$

1.26 Exercise 26

$$\sum_{k=-1}^{1}(k^2+3)$$

Proof. $((-1)^2+3)+(0^2+3)+(1^2+3)=11$ $\square$

1.27 Exercise 27

$$\sum_{n=1}^{6}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

Proof. $\left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{5}}-\cancel{\frac{1}{6}}\right)+\left(\cancel{\frac{1}{6}}-\frac{1}{7}\right)$

$=1-\frac{1}{7}=\frac{6}{7}$ $\square$

1.28 Exercise 28

$$\prod_{i=2}^{5}\frac{i(i+2)}{(i-1)\cdot(i+1)}$$

Proof. $\frac{2(2+2)}{(2-1)(2+1)}\cdot\frac{3(3+2)}{(3-1)(3+1)}\cdot\frac{4(4+2)}{(4-1)(4+1)}\cdot\frac{5(5+2)}{(5-1)(5+1)}$

$=\frac{\cancel{8}}{3}\cdot\frac{\cancel{15}}{\cancel{8}}\cdot\frac{\cancel{24}}{\cancel{15}}\cdot\frac{35}{\cancel{24}}\cdot=\frac{35}{3}$ $\square$

Write the summations in $29-32$ in expanded form.

1.29 Exercise 29

$$\sum_{i=1}^{n}(-2)^i$$

Proof. $(-2)^1+(-2)^2+(-2)^3+\cdots+(-2)^n=-2+2^2-2^3+\cdots+(-1)^n2^n$ $\square$

1.30 Exercise 30

$$\sum_{j=1}^{n} j(j+1)$$

Proof. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1)$ □

1.31 Exercise 31

$$\sum_{k=0}^{n+1} \frac{1}{k!}$$

Proof. $\displaystyle\sum_{k=0}^{n+1} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(n+1)!}$ □

1.32 Exercise 32

$$\sum_{i=1}^{k+1} i(i!)$$

Proof. $1(1!) + 2(2!) + 3(3!) + \cdots + (k+1)(k+1)!$ □

Evaluate the summations and products in $33-36$ for the indicated values of the variable.

1.33 Exercise 33

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2};\ n = 1$$

Proof. $\dfrac{1}{1^2} = 1$ □

1.34 Exercise 34

$1(1!) + 2(2!) + 3(3!) + \cdots + m(m!);\ m = 2$

Proof. $1(1!) + 2(2!) = 1 + 4 = 5$ □

1.35 Exercise 35

$$\left(\frac{1}{1+1}\right)\left(\frac{2}{2+1}\right)\left(\frac{3}{3+1}\right)\cdots\left(\frac{k}{k+1}\right);\ k = 3$$

Proof. $\left(\dfrac{1}{1+1}\right)\left(\dfrac{2}{2+1}\right)\left(\dfrac{3}{3+1}\right) = \dfrac{1}{2}\dfrac{2}{3}\dfrac{3}{4} = \dfrac{1}{4}$ □

1.36 Exercise 36

$$\left(\frac{1\cdot 2}{3\cdot 4}\right)\left(\frac{2\cdot 3}{4\cdot 5}\right)\left(\frac{3\cdot 4}{5\cdot 6}\right)\cdots\left(\frac{m\cdot(m+1)}{(m+2)\cdot(m+3)}\right); m=1$$

Proof. $\dfrac{1\cdot 2}{3\cdot 4}=\dfrac{3}{8}$ □

Write each of 37 – 39 as a single summation.

1.37 Exercise 37

$$\sum_{i=1}^{k} i^3+(k+1)^3$$

Proof. $\displaystyle\sum_{i=1}^{k+1} i^3$ □

1.38 Exercise 38

$$\sum_{k=1}^{m}\frac{k}{k+1}+\frac{m+1}{m+2}$$

Proof. $\displaystyle\sum_{k=1}^{m+1}\frac{k}{k+1}$ □

1.39 Exercise 39

$$\sum_{m=0}^{n}(m+1)2^n+(n+2)2^{n+1}$$

Proof. $\displaystyle\sum_{m=0}^{n+1}(m+1)2^n$ □

Rewrite 40 – 42 by separating off the final term.

1.40 Exercise 40

$$\sum_{i=1}^{k+1} i(i!)$$

Proof. $\displaystyle\sum_{i=1}^{k} i(i!)+(k+1)(k+1)!$ □

1.41 Exercise 41

$$\sum_{k=1}^{m+1} k^2$$

Proof. $\sum_{k=1}^{m} k^2 + (m+1)^2$ □

1.42 Exercise 42

$$\sum_{m=1}^{n+1} m(m+1)$$

Proof. $\sum_{m=1}^{n} m(m+1) + (n+1)(n+2)$ □

Write each of $43 - 52$ using summation or product notation.

Exercises $43 - 52$ have more than one correct answer.

1.43 Exercise 43

$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + 7^2$

Proof. $\sum_{k=1}^{7} (-1)^{k+1} k^2$ or $\sum_{k=0}^{6} (-1)^k (k+1)^2$ □

1.44 Exercise 44

$(1^3 - 1) - (2^3 - 1) + (3^3 - 1) - (4^3 - 1) + (5^3 - 1)$

Proof. $\sum_{k=1}^{5} (k^3 - 1)$ □

1.45 Exercise 45

$(2^2 - 1) \cdot (3^2 - 1) \cdot (4^2 - 1)$

Proof. $\prod_{k=2}^{4} (k^2 - 1)$ □

1.46 Exercise 46

$$\frac{2}{3\cdot 4}-\frac{3}{4\cdot 5}+\frac{4}{5\cdot 6}-\frac{5}{6\cdot 7}+\frac{6}{7\cdot 8}$$

Proof. $\displaystyle\sum_{j=2}^{6}\frac{(-1)^j j}{(j+1)(j+2)}$ $\square$

1.47 Exercise 47

$1-r+r^2-r^3+r^4-r^5$

Proof. $\displaystyle\sum_{i=0}^{5}(-1)^i r^i$ $\square$

1.48 Exercise 48

$(1-t)\cdot(1-t^2)\cdot(1-t^3)\cdot(1-t^4)$

Proof. $\displaystyle\prod_{k=1}^{4}(1-t^k)$ $\square$

1.49 Exercise 49

$1^3+2^3+3^3+\cdots+n^3$

Proof. $\displaystyle\sum_{k=1}^{n}k^3$ $\square$

1.50 Exercise 50

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots\frac{n}{(n+1)!}$$

Proof. $\displaystyle\sum_{k=1}^{n}\frac{k}{(k+1)!}$ $\square$

1.51 Exercise 51

$n+(n-1)+(n-2)+\cdots+1$

Proof. $\displaystyle\sum_{i=0}^{n-1}(n-i)$ $\square$

1.52 Exercise 52

$n + \frac{n-1}{2!} + \frac{n-2}{3!} + \frac{n-3}{4!} + \cdots + \frac{1}{n!}$

Proof. $\sum_{i=0}^{n-1} \frac{n-i}{(i+1)!}$ □

Transform each of 53 and 54 by making the change of variable $i = k + 1$.

1.53 Exercise 53

$\sum_{k=0}^{5} k(k-1)$

Proof. When $k = 0$, we have $i = 0 + 1 = 1$ and when $k = 5$ we have $i = 5 + 1 = 6$. Solving for k we get $k = i - 1$. So

$$\sum_{k=0}^{5} k(k-1) = \sum_{i=1}^{6} (i-1)(i-2)$$

□

1.54 Exercise 54

$\prod_{k=1}^{n} \frac{k}{k^2 + 4}$

Proof. When $k = 1$, we have $i = 1 + 1 = 2$ and when $k = n$ we have $i = n + 1$. Solving for k we get $k = i - 1$. So

$$\prod_{k=1}^{n} \frac{k}{k^2 + 4} = \prod_{i=2}^{n+1} \frac{i-1}{(i-1)^2 + 4}$$

□

Transform each of $55 - 58$ by making the change of variable $j = i - 1$.

1.55 Exercise 55

$\sum_{i=1}^{n+1} \frac{(i-1)^2}{i \cdot n}$

Proof. When $i = 1$, we have $j = 1-1 = 0$ and when $i = n+1$ we have $j = n+1-1 = n$. Solving for i we get $i = j + 1$. So

$$\sum_{i=1}^{n+1} \frac{(i-1)^2}{i \cdot n} = \sum_{j=0}^{n} \frac{(j+1-1)^2}{(j+1) \cdot n} = \sum_{j=0}^{n} \frac{j^2}{(j+1) \cdot n}$$

□

1.56 Exercise 56

$$\sum_{i=3}^{n} \frac{i}{i+n-1}$$

Proof. When $i = 3$, we have $j = 3 - 1 = 2$ and when $i = n$ we have $j = n - 1$. Solving for i we get $i = j + 1$. So

$$\sum_{i=3}^{n} \frac{i}{i+n-1} = \sum_{j=2}^{n-1} \frac{j+1}{j+1+n-1} = \sum_{j=2}^{n-1} \frac{j+1}{j+n}$$

□

1.57 Exercise 57

$$\sum_{i=1}^{n-1} \frac{i}{(n-i)^2}$$

Proof. When $i = 1$, we have $j = 1 - 1 = 0$ and when $i = n - 1$ we have $j = n - 1 - 1 = n - 2$. Solving for i we get $i = j + 1$. So

$$\sum_{i=1}^{n-1} \frac{i}{(n-i)^2} = \sum_{j=0}^{n-2} \frac{j+1}{(n-(j+1))^2}$$

□

1.58 Exercise 58

$$\prod_{i=n}^{2n} \frac{n-i+1}{n+i}$$

Proof. When $i = n$, we have $j = n - 1$ and when $i = 2n$ we have $j = 2n - 1$. Solving for i we get $i = j + 1$. So

$$\prod_{i=n}^{2n} \frac{n-i+1}{n+i} = \prod_{j=n-1}^{2n-1} \frac{n-(j+1)+1}{n+j+1} = \prod_{j=n-1}^{2n-1} \frac{n-j}{n+j+1}$$

□

Write each of 59 – 61 as a single summation or product.

1.59 Exercise 59

$$3\sum_{k=1}^{n}(2k-3)+\sum_{k=1}^{n}(4-5k)$$

Proof. $\displaystyle\sum_{k=1}^{n}[3(2k-3)+(4-5k)]=\sum_{k=1}^{n}[6k-9+4-5k]=\sum_{k=1}^{n}[k-5]$ □

1.60 Exercise 60

$$2\sum_{k=1}^{n}(3k^2+4)+5\sum_{k=1}^{n}(2k^2-1)$$

Proof. $\displaystyle\sum_{k=1}^{n}[2(3k^2+4)+5(2k^2-1)]=\sum_{k=1}^{n}[6k^2+8+10k^2-5]=\sum_{k=1}^{n}[16k^2+3]$ □

1.61 Exercise 61

$$\prod_{k=1}^{n}\frac{k}{k+1}\prod_{k=1}^{n}\frac{k+1}{k+2}$$

Proof. $\displaystyle\prod_{k=1}^{n}\frac{k}{k+1}\prod_{k=1}^{n}\frac{k+1}{k+2}=\prod_{k=1}^{n}\frac{k}{\cancel{k+1}}\frac{\cancel{k+1}}{k+2}=\prod_{k=1}^{n}\frac{k}{k+2}$ □

Compute each of $62-76$. Assume the values of the variables are restricted so that the expressions are defined.

1.62 Exercise 62

$$\frac{4!}{3!}$$

Proof. $\displaystyle\frac{4\cdot\cancel{3\cdot 2\cdot 1}}{\cancel{3\cdot 2\cdot 1}}=4$ □

1.63 Exercise 63

$$\frac{6!}{8!}$$

Proof. $\displaystyle\frac{\cancel{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}}{8\cdot 7\cdot\cancel{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}}=\frac{1}{56}$ □

1.64 Exercise 64

$$\frac{4!}{0!}$$

Proof. $\dfrac{4!}{0!} = \dfrac{24}{1} = 24$ □

1.65 Exercise 65

$$\frac{n!}{(n-1)!}$$

Proof. $\dfrac{n \cdot \cancel{(n-1)\cdots 2 \cdot 1}}{\cancel{(n-1)\cdots 2 \cdot 1}} = n$ □

1.66 Exercise 66

$$\frac{(n-1)!}{(n+1)!}$$

Proof. $\dfrac{\cancel{(n-1)\cdots 2 \cdot 1}}{(n+1) \cdot n \cdot \cancel{(n-1)\cdots 2 \cdot 1}} = \dfrac{1}{(n+1)n}$ □

1.67 Exercise 67

$$\frac{n!}{(n-2)!}$$

Proof. $\dfrac{n \cdot (n-1) \cdot \cancel{(n-2)\cdots 2 \cdot 1}}{\cancel{(n-2)\cdots 2 \cdot 1}} = n(n-1)$ □

1.68 Exercise 68

$$\frac{((n+1)!)^2}{(n!)^2}$$

Proof. $= \left(\dfrac{(n+1)!}{n!}\right)^2 = \left(\dfrac{(n+1)\cancel{n(n-1)\cdots 2 \cdot 1}}{\cancel{n(n-1)\cdots 2 \cdot 1}}\right)^2 = (n+1)^2$ □

1.69 Exercise 69

$$\frac{n!}{(n-k)!}$$

Proof. $\dfrac{n \cdot (n-1)\cdots(n-k+1) \cdot \cancel{(n-k)(n-k-1)\cdots 2 \cdot 1}}{\cancel{(n-k)(n-k-1)\cdots 2 \cdot 1}} = n(n-1)\cdots(n-k+1)$

□

1.70 Exercise 70

$$\frac{n!}{(n-k+1)!}$$

Proof. $$\frac{n \cdot (n-1) \cdots (n-k+2) \cdot \cancel{(n-k+1)(n-k) \cdots 2 \cdot 1}}{\cancel{(n-k+1)(n-k) \cdots 2 \cdot 1}} = n(n-1) \cdots (n-k+2)$$ □

1.71 Exercise 71

$$\binom{5}{3}$$

Proof. $$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3! \cdot 2!} = \frac{5 \cdot 4 \cdot \cancel{3 \cdot 2 \cdot 1}}{(\cancel{3 \cdot 2 \cdot 1}) \cdot (2 \cdot 1)} = 10$$ □

1.72 Exercise 72

$$\binom{7}{4}$$

Proof. $$\binom{7}{4} = \frac{7!}{4!(7-4)!} = \frac{7!}{4! \cdot 3!} = \frac{7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4 \cdot 3 \cdot 2 \cdot 1}}{(\cancel{4 \cdot 3 \cdot 2 \cdot 1}) \cdot (\cancel{3 \cdot 2} \cdot 1)} = 35$$ □

1.73 Exercise 73

$$\binom{3}{0}$$

Proof. 1 □

1.74 Exercise 74

$$\binom{5}{5}$$

Proof. 1 □

1.75 Exercise 75

$$\binom{n}{n-1}$$

Proof. $$\binom{n}{n-1} = \frac{n!}{(n-1)!(n-(n-1))!} = \frac{n!}{(n-1)! \cdot 1!} = \frac{n \cdot \cancel{(n-1) \cdots 2 \cdot 1}}{\cancel{(n-1) \cdots 2 \cdot 1}} = n$$ □

1.76 Exercise 76

$$\binom{n+1}{n-1}$$

Proof. $\displaystyle\binom{n+1}{n-1} = \frac{(n+1)!}{(n-1)!(n+1-(n-1))!} = \frac{(n+1)!}{(n-1)! \cdot 2!}$

$\displaystyle = \frac{(n+1) \cdot n \cdot \cancel{(n-1) \cdots 2 \cdot 1}}{\cancel{(n-1) \cdots 2 \cdot 1} \cdot 2} = \frac{(n+1)n}{2}$ □

1.77 Exercise 77

1.77.1 (a)

Prove that $n! + 2$ is divisible by 2, for every integer $n \geq 2$.

Proof. Let n be an integer such that $n \geq 2$. By definition of factorial,

$$n! = \begin{cases} 2 \cdot 1 & \text{if } n = 2 \\ 3 \cdot 2 \cdot 1 & \text{if } n = 3 \\ n \cdot (n-1) \cdots 2 \cdot 1 & \text{if } n > 3 \end{cases}$$

In each case, $n!$ has a factor of 2, and so $n! = 2k$ for some integer k. Then $n! + 2 = 2k + 2 = 2(k+1)$. Since $k+1$ is an integer, $n! + 2$ is divisible by 2. □

1.77.2 (b)

Prove that $n! + k$ is divisible by k, for every integer $n \geq 2$ and $k = 2, 3, \ldots, n$.

Proof. For every $k = 2, 3, \ldots, n$, from the definition of $n!$ in part (a), we can see that $n!$ has a factor of k, so $n! = ka$ for some integer a. Then $n! + k = ka + k = k(a+1)$ where $a + 1$ is an integer. Therefore $n! + k$ is divisible by k for every $k = 2, 3, \ldots, n$. □

1.77.3 (c)

Given any integer $m \geq 2$, is it possible to find a sequence of $m - 1$ consecutive positive integers none of which is prime? Explain your answer.

Proof. Yes. By part (b), $m! + k$ is divisible by k, for all $k = 2, 3, \ldots, m$. So $m! + 2, m! + 3, \ldots, m! + m$ are $m - 1$ consecutive integers none of which is prime. □

1.78 Exercise 78

Prove that for all nonnegative integers n and r with $r + 1 \leq n$, $\displaystyle\binom{n}{r+1} = \frac{n-r}{r+1}\binom{n}{r}$.

Proof. Suppose n and r are nonnegative integers with $r+1 \le n$. The right-hand side of the equation to be shown is

$$
\begin{aligned}
\frac{n-r}{r+1} \cdot \binom{n}{r} &= \frac{n-r}{r+1} \cdot \frac{n!}{r!(n-r)!} \\
&= \frac{\cancel{n-r}}{r+1} \cdot \frac{n!}{r!\cancel{(n-r)}(n-r-1)!} \\
&= \frac{n!}{(r+1)!(n-r-1)!} \\
&= \frac{n!}{(r+1)!(n-(r+1))!} \\
&= \binom{n}{r+1}
\end{aligned}
$$

which is the left-hand side of the equality to be shown. □

1.79 Exercise 79

Prove that if p is a prime number and r is an integer with $0 < r < p$, then $\binom{p}{r}$ is divisible by p.

Proof. We know that

$$\binom{p}{r} = \frac{p!}{r!(p-r)!} = \frac{p \cdot (p-1) \cdots 2 \cdot 1}{[r \cdot (r-1) \cdots 2 \cdot 1][(p-r) \cdot (p-r-1) \cdots 2 \cdot 1]}$$

is an integer. Notice that all the factors in the denominator are less than p. So, since p is prime, p is not divisible by any of the factors in the denominator. This means that every factor in the denominator is canceled out by the factors of $(p-1) \cdots 2 \cdot 1$. Thus

$$M = \frac{(p-1) \cdots 2 \cdot 1}{[r \cdot (r-1) \cdots 2 \cdot 1][(p-r) \cdot (p-r-1) \cdots 2 \cdot 1]}$$

is also an integer (otherwise $p \cdot M$ would not be an integer, since p cannot cancel out anything in the denominator). Therefore $\binom{p}{r} = p \cdot M$ where M is an integer, so it is divisible by p. □

1.80 Exercise 80

Suppose $a[1], a[2], a[3], \ldots, a[m]$ is a one-dimensional array and consider the following algorithm segment:

$$
\begin{aligned}
&\mathit{sum} := 0 \\
&\textbf{for } (k := 1 \textbf{ to } m) \\
&\qquad \mathit{sum} := \mathit{sum} + a[k] \\
&\textbf{next } k
\end{aligned}
$$

Fill in the blanks below so that each algorithm segment performs the same job as the one shown in the exercise statement.

1.80.1 (a)

$sum := 0$
for $(i := 0$ **to** ____ $)$
    $sum :=$ ____
next i

Proof. $m - 1, sum + a[i + 1]$ □

1.80.2 (b)

$sum := 0$
for $(j := 2$ **to** ____ $)$
    $sum :=$ ____
next j

Proof. $m + 1, sum + a[j - 1]$ □

Use repeated division by 2 to convert (by hand) the integers in $81 - 83$ from base 10 to base 2.

1.81 Exercise 81

90

Proof.

$$\begin{aligned} 90 \ / \ 2 &= 45, && \text{remainder} = 0 \\ 45 \ / \ 2 &= 22, && \text{remainder} = 1 \\ 22 \ / \ 2 &= 11, && \text{remainder} = 0 \\ 11 \ / \ 2 &= 5, && \text{remainder} = 1 \\ 5 \ / \ 2 &= 2, && \text{remainder} = 1 \\ 2 \ / \ 2 &= 1, && \text{remainder} = 0 \\ 1 \ / \ 2 &= 0, && \text{remainder} = 1 \end{aligned}$$

So $90_{10} = 1011010_2$. □

1.82 Exercise 82

98

Proof.

$$\begin{aligned} 98 \ / \ 2 &= 49, && \text{remainder} = 0 \\ 49 \ / \ 2 &= 24, && \text{remainder} = 1 \\ 24 \ / \ 2 &= 12, && \text{remainder} = 0 \\ 12 \ / \ 2 &= 6, && \text{remainder} = 0 \\ 6 \ / \ 2 &= 3, && \text{remainder} = 0 \\ 3 \ / \ 2 &= 1, && \text{remainder} = 1 \\ 1 \ / \ 2 &= 0, && \text{remainder} = 1 \end{aligned}$$

So $98_{10} = 1100010_2$. □

1.83 Exercise 83

205

Proof.

$$\begin{array}{rcll} 205 \ / \ 2 &=& 102, & \text{remainder} = 1 \\ 102 \ / \ 2 &=& 51, & \text{remainder} = 0 \\ 51 \ / \ 2 &=& 25, & \text{remainder} = 1 \\ 25 \ / \ 2 &=& 12, & \text{remainder} = 1 \\ 12 \ / \ 2 &=& 6, & \text{remainder} = 0 \\ 6 \ / \ 2 &=& 3, & \text{remainder} = 0 \\ 3 \ / \ 2 &=& 1, & \text{remainder} = 1 \\ 1 \ / \ 2 &=& 0, & \text{remainder} = 1 \end{array}$$

So $205_{10} = 11001101_2$. □

Make a trace table to trace the action of algorithm 5.1.1 on the input in 84 – 86.

1.84 Exercise 84

23

Proof.

a	23					
i	0	1	2	3	4	5
q	23	11	5	2	1	0
$r[0]$		1				
$r[1]$			1			
$r[2]$				1		
$r[3]$					0	
$r[4]$						1

□

1.85 Exercise 85

28

Proof.

a	28					
i	0	1	2	3	4	5
q	28	14	7	3	1	0
$r[0]$		0				
$r[1]$			0			
$r[2]$				1		
$r[3]$					1	
$r[4]$						1

□

1.86 Exercise 86

44

Proof.

a	44						
i	0	1	2	3	4	5	6
q	44	22	11	5	2	1	0
$r[0]$		0					
$r[1]$			0				
$r[2]$				1			
$r[3]$					1		
$r[4]$						0	
$r[5]$							1

□

1.87 Exercise 87

Write an informal description of an algorithm (using repeated division by 16) to convert a nonnegative integer from decimal notation to hexadecimal notation (base 16).

Proof. Suppose a is a nonnegative integer. Divide a by 16 using the quotient-remainder theorem to obtain a quotient $q[0]$ and a remainder $r[0]$. If the quotient is nonzero, divide by 16 again to obtain a quotient $q[1]$ and a remainder $r[1]$. Continue this process until a quotient of 0 is obtained. At each stage, the remainder must be less than the divisor, which is 16. Thus each remainder is always among $0, 1, 2, \ldots, 15$. Read the divisions from the bottom up. □

Use the algorithm you developed for exercise 87 to convert the integers in $88 - 90$ to hexadecimal notation.

1.88 Exercise 88

287

Proof.

$$\begin{aligned} 287\ /\ 16 &= 17, & \text{remainder} &= 15 = \text{F} \\ 17\ /\ 16 &= 1, & \text{remainder} &= 1 \\ 1\ /\ 16 &= 0, & \text{remainder} &= 1 \end{aligned}$$

So $287_{10} = 11F_{16}$. □

1.89 Exercise 89

693

Proof.

$$\begin{aligned} 693\ /\ 16 &= 43, & \text{remainder} &= 5 \\ 43\ /\ 16 &= 2, & \text{remainder} &= 11 = \text{B} \\ 2\ /\ 16 &= 0, & \text{remainder} &= 2 \end{aligned}$$

So $693_{10} = 2B5_{16}$. □

1.90 Exercise 90

2301

Proof.
$$\begin{array}{rcll} 2301 \ / \ 16 & = & 143, & \text{remainder} = 13 = \text{D} \\ 143 \ / \ 16 & = & 8, & \text{remainder} = 15 = \text{F} \\ 8 \ / \ 16 & = & 0, & \text{remainder} = 8 \end{array}$$

So $2301_{10} = 8FD_{16}$. □

1.91 Exercise 91

Write a formal version of the algorithm you developed for exercise 87.
Proof:

Decimal to Hexadecimal Conversion Using Repeated Division by 16
Input: a *[a nonnegative integer]*
Algorithm Body:

$q := a, i := 0$
while $(i = 0 \text{ or } q \neq 0)$
$r[i] := q \mod 16$
$q := q \text{ div } 16$
[$r[i]$ and q can be obtained by calling the division algorithm.]
end while
[After execution of this step, the values $r[0], r[1], \ldots, r[i-1]$ are all 0's and 1's, and $a = (r[i-1]r[i-2]\ldots r[1]r[0])_{16}$].

Output: $r[0], r[1], \ldots, r[i-1]$ *[a sequence of integers]*