Chapter 6 Solutions, Susanna Epp Discrete Math 5th Edition

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Contents

1 Exercise Set 6.1

1.1 Exercise 1

In each of (a)–(f), answer the following questions: Is $A \subseteq B$? Is $B \subseteq A$? Is either A or B a proper subset of the other?

1.1.1 (a)

$A = \{2, \{2\}, (\sqrt{2})^2\}, B = \{2, \{2\}, \{\{2\}\}\}$

Proof. $A = \{2, \{2\}, (\sqrt{2})^2\} = \{2, \{2\}, 2\} = \{2, \{2\}\}$, so $A \subseteq B$ because every element of A is in B, but $B \not\subseteq A$ because $\{\{2\}\} \in B$ but $\{\{2\}\} \notin A$. Thus A is a proper subset of B. □

1.1.2 (b)

$A = \{3, \sqrt{5^2 - 4^2}, 24 \mod 7\}, B = \{8 \mod 5\}$

Proof. $A = \{3, \sqrt{5^2 - 4^2}, 24 \mod 7\} = \{3, 3, 3\} = \{3\}, B = \{8 \mod 5\} = \{3\}$. So $A = B$, which means both $A \subseteq B$ and $B \subseteq A$. □

1.1.3 (c)

$A = \{\{1, 2\}, \{2, 3\}\}, B = \{1, 2, 3\}$

Proof. $A \not\subseteq B$ because $\{1, 2\} \in A$ but $\{1, 2\} \notin B$. $B \not\subseteq A$ because $1 \in B$ but $1 \notin A$. □

1.1.4 (d)

$A = \{a, b, c\}, B = \{\{a\}, \{b\}, \{c\}\}$

Proof. $A \nsubseteq B$ because $a \in A$ but $a \notin B$. $B \nsubseteq A$ because $\{a\} \in B$ but $\{a\} \notin A$. $\square$

1.1.5 (e)

$A = \{\sqrt{16}, \{4\}\}, B = \{4\}$

Proof. $A = \{\sqrt{16}, \{4\}\} = \{4, \{4\}\}$. $B \subseteq A$ because 4 is the only element of B and $4 \in A$. $A \nsubseteq B$ because $\{4\} \in A$ but $\{4\} \notin B$. $\square$

1.1.6 (f)

$A = \{x \in \mathbb{R} \,|\, \cos(x) \in \mathbb{Z}\}, B = \{x \in \mathbb{R} \,|\, \sin(x) \in \mathbb{Z}\}$

Proof. From trigonometry we know that $\cos(x)$ and $\sin(x)$ have the only integer values $-1, 0$ and 1. We also know that

$\cos(x) = -1$ if and only if $x = \pi + 2n\pi$ for some integer $n \in \mathbb{Z}$ $(\ldots, -3\pi, -\pi, \pi, 3\pi, \ldots)$,

$\cos(x) = 0$ if and only if $x = \frac{\pi}{2} + n\pi$ for some integer $n \in \mathbb{Z}$ $(\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots)$,

$\cos(x) = 1$ if and only if $x = 2n\pi$ for some integer $n \in \mathbb{Z}$ $(\ldots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots)$,

$\sin(x) = -1$ if and only if $x = -\frac{\pi}{2} + 2n\pi$ for some integer $n \in \mathbb{Z}$ $(\ldots, -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots)$,

$\sin(x) = 0$ if and only if $x = n\pi$ for some integer $n \in \mathbb{Z}$, $(\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots)$,

$\sin(x) = 1$ if and only if $x = \frac{\pi}{2} + 2n\pi$ for some integer $n \in \mathbb{Z}$ $(\ldots, -\frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \ldots)$.

So:

$$A = \{\cdots, -\frac{5\pi}{2}, -2\pi, -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}, \cdots\}$$

and

$$B = \{\cdots, -\frac{5\pi}{2}, -2\pi, -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}, \cdots\}$$

so $A = B$. $\square$

1.2 Exercise 2

Complete the proof from Example 6.1.3: Prove that $B \subseteq A$ where $A = \{m \in \mathbb{Z} \,|\, m = 2a \text{ for some integer } a\}$ and $B = \{n \in \mathbb{Z} \,|\, n = 2b - 2 \text{ for some integer } b\}$

Proof. **Proof That $B \subseteq A$:** Suppose x is a particular but arbitrarily chosen element of B.

[We must show that $x \in A$. By definition of A, this means we must show that $x = 2\cdot$(some integer).]

By definition of B, there is an integer b such that $x = 2b - 2$.

[Given that $x = 2b - 2$, can x also be expressed as $2\cdot$ (some integer)? That is, is there an integer, say, a, such that $2b - 2 = 2a$? Solve for a to obtain $a = b - 1$. Check to see if this works.]

Let $a = b - 1$.

[First check that a is an integer.]

We know that a is an integer because it is a difference of integers.

[Then check that $x = 2a$.]

By substitution, $2a = 2(b - 1) = 2b - 2 = x$. Thus, by definition of A, x is an element of A, *[as was to be shown]*. $\square$

1.3 Exercise 3

Let sets R, S, and T be defined as follows:

$$\begin{aligned} R &= \{x \in \mathbb{Z} \,|\, x \text{ is divisible by } 2\} \\ S &= \{y \in \mathbb{Z} \,|\, y \text{ is divisible by } 3\} \\ T &= \{z \in \mathbb{Z} \,|\, z \text{ is divisible by } 6\} \end{aligned}$$

Prove or disprove each of the following statements.

1.3.1 (a)

$R \subseteq T$

Proof. $R \not\subseteq T$ because there are elements in R that are not in T. For example, the number 2 is in R but 2 is not in T since 2 is not divisible by 6. $\square$

1.3.2 (b)

$T \subseteq R$

Proof. $T \subseteq R$ because every element in T is in R since every integer divisible by 6 is divisible by 2. To see why this is so, suppose n is any integer that is divisible by 6. Then $n = 6m$ for some integer m. Since $6m = 2(3m)$ and since $3m$ is an integer (being a product of integers), it follows that $n = 2\cdot$(some integer), and, hence, that n is divisible by 2. $\square$

1.3.3 (c)

$T \subseteq S$

Proof. $T \subseteq S$ because every element in T is in S since every integer divisible by 6 is divisible by 3. To see why this is so, suppose n is any integer that is divisible by 6. Then $n = 6m$ for some integer m. Since $6m = 3(2m)$ and since $2m$ is an integer (being a product of integers), it follows that $n = 3\cdot$(some integer), and, hence, that n is divisible by 3. $\square$

1.4 Exercise 4

Let $A = \{n \in \mathbb{Z} \,|\, n = 5r \text{ for some integer } r\}$ and

$B = \{m \in \mathbb{Z} \,|\, m = 20s \text{ for some integer } s\}$. Prove or disprove each of the following statements.

1.4.1 (a)

$A \subseteq B$

Proof. $A \nsubseteq B$ because $5 \in A$ but $5 \notin B$. $\square$

1.4.2 (b)

$B \subseteq A$

Proof. $B \subseteq A$ is true. Suppose $m \in B$. Then $m = 20s$ for some integer s. So $m = 20s = 5(4s)$ where $4s$ is an integer, therefore $m \in A$. $\square$

1.5 Exercise 5

Let $C = \{n \in \mathbb{Z} \,|\, n = 6r - 5 \text{ for some integer } r\}$ and $D = \{m \in \mathbb{Z} \,|\, m = 3s + 1 \text{ for some integer } s\}$. Prove or disprove each of the following statements.

1.5.1 (a)

$C \subseteq D$

Proof. $C \subseteq D$ because every element in C is in D. To see why this is so, suppose n is any element of C. Then $n = 6r - 5$ for some integer r. Let $s = 2r - 2$. Then s is an integer (because products and differences of integers are integers), and $3s + 1 = 3(2r - 2) + 1 = 6r - 6 + 1 = 6r - 5$, which equals n. Thus n satisfies the condition for being in D. Hence, every element in C is in D. $\square$

1.5.2 (b)

$D \subseteq C$

Proof. $D \nsubseteq C$ because there are elements of D that are not in C. For example, 4 is in D because $4 = 3 \cdot 1 + 1$. But 4 is not in C because if it were, then $4 = 6r - 5$ for some integer r, which would imply that $9 = 6r$, or, equivalently, that $r = 3/2$, and this contradicts the fact that r is an integer. $\square$

1.6 Exercise 6

Let $A = \{x \in \mathbb{Z} \,|\, x = 5a + 2 \text{ for some integer } a\}$,

$B = \{y \in \mathbb{Z} \,|\, y = 10b - 3 \text{ for some integer } b\}$ and

$D = \{z \in \mathbb{Z} \,|\, z = 10c + 7 \text{ for some integer } c\}$. Prove or disprove each of the following statements.

1.6.1 (a)

$A \subseteq B$

Proof. $A \not\subseteq B$ because $2 \in A$ because $2 = 5 \cdot 0 + 2$, but $2 \notin B$. To see this, argue by contradiction and assume $2 \in B$. So $2 = 10b - 3$ for some integer b. So $5 = 10b$ and $b = 1/2$ is an integer, a contradiction. Therefore our supposition was false and $2 \notin B$. □

1.6.2 (b)

$B \subseteq A$

Proof. Suppose $y \in B$. Then $y = 10b - 3$ for some integer b. Then $y = 10b - 5 + 5 - 3 = 5(b-2) + 2$ where $b - 2$ is an integer. Let $a = b - 2$. Therefore $y = 5a + 2$ for some integer a, therefore $y \in A$. This proves $B \subseteq A$. □

1.6.3 (c)

$B = C$

Proof. **Sketch of proof that $B \subseteq C$:** If r is any element of B then there is an integer b such that $r = 10b - 3$. To show that r is in C, you must show that there is an integer c such that $r = 10c + 7$. In scratch work, assume that c exists and use the information that $10b - 3$ would have to equal $10c + 7$ to deduce the only possible value for c. Then show that this value is (1) an integer and (2) satisfies the equation $r = 10c + 7$, which will allow you to conclude that r is an element of C.

Sketch of proof that $C \subseteq B$: If s is any element of C then there is an integer c such that $s = 10c + 7$. To show that s is in B, you must show that there is an integer b such that $s = 10c - 3$. In scratch work, assume that b exists and use the information that $10c + 7$ would have to equal $10b - 3$ to deduce the only possible value for b. Then show that this value is (1) an integer and (2) satisfies the equation $s = 10b - 3$, which will allow you to conclude that s is an element of B. □

1.7 Exercise 7

Let $A = \{x \in \mathbb{Z} \,|\, x = 6a + 4 \text{ for some integer } a\}$,

$B = \{y \in \mathbb{Z} \,|\, y = 18b - 2 \text{ for some integer } b\}$ and

$D = \{z \in \mathbb{Z} \mid z = 18c + 16 \text{ for some integer } c\}$. Prove or disprove each of the following statements.

1.7.1 (a)

$A \subseteq B$

Proof. $A \nsubseteq B$ because $10 \in A$ but $10 \notin B$. To see this, argue by contradiction and assume $10 \in B$. Then $10 = 18b - 2$ for some integer b. Then $12 = 18b$ and $b = 12/18 = 2/3$, which is not an integer, contradiction. Therefore our supposition was false and $10 \notin B$. □

1.7.2 (b)

$B \subseteq A$

Proof. Suppose $y \in B$. Then $y = 18b - 2$ for some integer b. Then $y = 18b - 6 + 6 - 2 = 6(3b - 1) + 4$ where $3b - 1$ is an integer. Let $a = 3b - 1$. So $y = 6a + 4$ for some integer a, therefore $y \in A$. This proves $B \subseteq A$. □

1.7.3 (c)

$B = C$

Proof. Suppose $y \in B$. Then $y = 18b - 2$ for some integer b. Then $y = 18b - 18 + 18 - 2 = 18(b - 1) + 16$ where $b - 1$ is an integer. Let $c = b - 1$. So $y = 18c + 16$ for some integer c, therefore $y \in C$. This proves $B \subseteq C$.

Suppose $z \in C$. Then $z = 18c + 16$ for some integer c. Then $z = 18c + 18 - 18 + 16 = 18(c + 1) - 2$ where $c + 1$ is an integer. Let $b = c + 1$. So $z = 18b - 2$ for some integer b, therefore $z \in B$. This proves $C \subseteq B$. □

1.8 Exercise 8

Write in words how to read each of the following out loud. Then write each set using the symbols for union, intersection, set difference, or set complement.

1.8.1 (a)

$\{x \in U \mid x \in A \text{ and } x \in B\}$

Proof. In words: The set of all x in U such that x is in A and x is in B.

In symbolic notation: $A \cap B$. □

1.8.2 (b)

$\{x \in U \mid x \in A \text{ or } x \in B\}$

Proof. *In words:* The set of all x in U such that x is in A or x is in B.

In symbolic notation: $A \cup B$. □

1.8.3 (c)

$\{x \in U \mid x \in A \text{ and } x \notin B\}$

Proof. *In words:* The set of all x in U such that x is in A and x is not in B.

In symbolic notation: $A - B$. □

1.8.4 (d)

$\{x \in U \mid x \notin A\}$

Proof. *In words:* The set of all x in U such that x is not in A.

In symbolic notation: A^c. □

1.9 Exercise 9

Complete the following sentences without using the symbols $\cup, \cap$, or $-$.

1.9.1 (a)

$x \notin A \cup B$ if, and only if, ____ .

Proof. $x \notin A$ and $x \notin B$ □

1.9.2 (b)

$x \notin A \cap B$ if, and only if, ____ .

Proof. $x \notin A$ or $x \notin B$ □

1.9.3 (c)

$x \notin A - B$ if, and only if, ____ .

Proof. $x \notin A$ or $x \in B$ □

1.10 Exercise 10

Let $A = \{1, 3, 5, 7, 9\}$, $B = \{3, 6, 9\}$, and $C = \{2, 4, 6, 8\}$. Find each of the following:

1.10.1 (a)

$A \cup B$

Proof. $A \cup B = \{1, 3, 5, 6, 7, 9\}$ □

1.10.2 (b)

$A \cap B$

Proof. $A \cap B = \{3, 9\}$ □

1.10.3 (c)

$A \cup C$

Proof. $A \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ □

1.10.4 (d)

$A \cap C$

Proof. $A \cap C = \varnothing$ □

1.10.5 (e)

$A - B$

Proof. $A - B = \{1, 5, 7\}$ □

1.10.6 (f)

$B - A$

Proof. $B - A = \{6\}$ □

1.10.7 (g)

$B \cup C$

Proof. $B \cup C = \{2, 3, 4, 6, 8, 9\}$ □

1.10.8 (h)

$B \cap C$

Proof. $B \cap C = \{6\}$ □

1.11 Exercise 11

Let the universal set be $\mathbb{R}$, the set of all real numbers, and let $A = \{x \in \mathbb{R} \,|\, 0 < x \leq 2\}$, $B = \{x \in \mathbb{R} \,|\, 1 \leq x < 4\}$, and $C = \{x \in \mathbb{R} \,|\, 3 \leq x < 9\}$. Find each of the following:

1.11.1 (a)

$A \cup B$

Proof. $A \cup B = \{x \in \mathbb{R} \,|\, 0 < x < 4\}$ □

1.11.2 (b)

$A \cap B$

Proof. $A \cap B = \{x \in \mathbb{R} \,|\, 1 \leq x \leq 2\}$ □

1.11.3 (c)

A^c

Proof. $A^c = \{x \in \mathbb{R} \,|\, x \leq 0 \text{ or } 2 < x\}$ □

1.11.4 (d)

$A \cup C$

Proof. $A \cup C = \{x \in \mathbb{R} \,|\, 0 < x \leq 2 \text{ or } 3 \leq x < 9\}$ □

1.11.5 (e)

$A \cap C$

Proof. $A \cap C = \varnothing$ □

1.11.6 (f)

B^c

Proof. $B^c = \{x \in \mathbb{R} \,|\, x < 1 \text{ or } 4 \leq x\}$ □

1.11.7 (g)

$A^c \cap B^c$

Proof. $A^c \cap B^c = \{x \in \mathbb{R} \,|\, x \leq 0 \text{ or } 4 \leq x\}$ □

1.11.8 (h)

$A^c \cup B^c$

Proof. $A^c \cup B^c = \{x \in \mathbb{R} \,|\, x < 1 \text{ or } 2 < x\}$ □

1.11.9 (i)

$(A \cap B)^c$

Proof. $(A \cap B)^c = \{x \in \mathbb{R} \,|\, x < 1 \text{ or } 2 < x\}$ □

1.11.10 (j)

$(A \cup B)^c$

Proof. $(A \cup B)^c = \{x \in \mathbb{R} \,|\, x \leq 0 \text{ or } 4 \leq x\}$ □

1.12 Exercise 12

Let the universal set be R, the set of all real numbers, and let $A = \{x \in R \,|\, -3 \leq x \leq 0\}$, $B = \{x \in R \,|\, -1 < x < 2\}$, and $C = \{x \in R \,|\, 6 < x \leq 8\}$. Find each of the following:

1.12.1 (a)

$A \cup B$

Proof. $A \cup B = \{x \in \mathbb{R} \,|\, -3 \leq x < 2\}$ □

1.12.2 (b)

$A \cap B$

Proof. $A \cap B = \{x \in \mathbb{R} \,|\, -1 < x \leq 0\}$ □

1.12.3 (c)

A^c

Proof. $A^c = \{x \in \mathbb{R} \,|\, x < -3 \text{ or } 0 < x\}$ □

1.12.4 (d)

$A \cup C$

Proof. $A \cup C = \{x \in \mathbb{R} \,|\, -3 \leq x \leq 0 \text{ or } 6 < x \leq 8\}$ □

1.12.5 (e)

$A \cap C$

Proof. $A \cap C = \varnothing$ □

1.12.6 (f)

B^c

Proof. $B^c = \{x \in \mathbb{R} \mid x \leq -1 \text{ or } 2 \leq x\}$ □

1.12.7 (g)

$A^c \cap B^c$

Proof. $A^c \cap B^c = \{x \in \mathbb{R} \mid x < -3 \text{ or } 2 \leq x\}$ □

1.12.8 (h)

$A^c \cup B^c$

Proof. $A^c \cup B^c = \{x \in \mathbb{R} \mid x \leq -1 \text{ or } 0 < x\}$ □

1.12.9 (i)

$(A \cap B)^c$

Proof. $(A \cap B)^c = \{x \in \mathbb{R} \mid x \leq -1 \text{ or } 0 < x\}$ □

1.12.10 (j)

$(A \cup B)^c$

Proof. $(A \cup B)^c = \{x \in \mathbb{R} \mid x < -3 \text{ or } 2 \leq x\}$ □

1.13 Exercise 13

Let S be the set of all strings of 0's and 1's of length 4, and let A and B be the following subsets of S: $A = \{1110, 1111, 1000, 1001\}$ and $B = \{1100, 0100, 1111, 0111\}$. Find each of the following:

1.13.1 (a)

$A \cap B$

Proof. $A \cap B = \{1111\}$ □

1.13.2 (b)

$A \cup B$

Proof. $A \cup B = \{1110, 1111, 1000, 1001, 1100, 0100, 0111\}$ □

1.13.3 (c)

$A - B$

Proof. $A - B = \{1110, 1000, 1001\}$ □

1.13.4 (d)

$B - A$

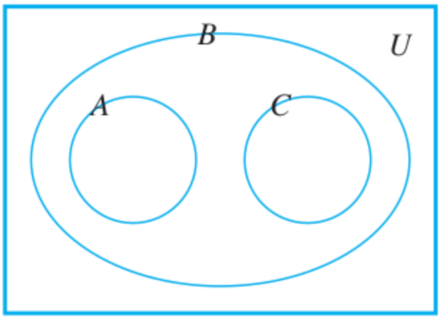
Proof. $B - A = \{1100, 0100, 0111\}$ □

1.14 Exercise 14

In each of the following, draw a Venn diagram for sets A, B, and C that satisfy the given conditions.

1.14.1 (a)

$A \subseteq B, C \subseteq B, A \cap C = \varnothing$



Proof. □

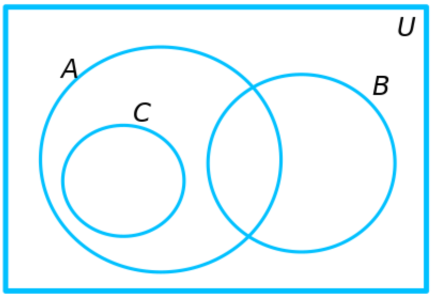
1.14.2 (b)

$C \subseteq A, B \cap C = \varnothing$

Proof. □

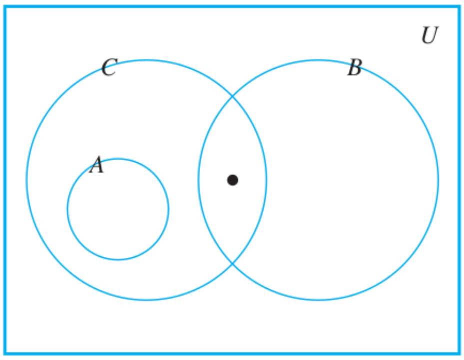
1.15 Exercise 15

In each of the following, draw a Venn diagram for sets A, B, and C that satisfy the given conditions.



1.15.1 (a)

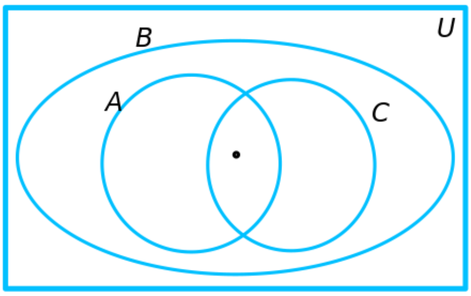
$A \cap B = \varnothing, A \subseteq C, C \cap B \neq \varnothing$



Proof. □

1.15.2 (b)

$A \subseteq B, C \subseteq B, A \cap C \neq \varnothing$



Proof. □

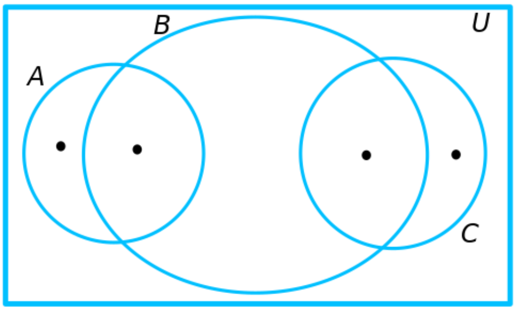
1.15.3 (c)

$A \cap B \neq \varnothing, B \cap C \neq \varnothing, A \cap C = \varnothing, A \nsubseteq B, C \nsubseteq B$

Proof. □

1.16 Exercise 16

Let $A = \{a, b, c\}, B = \{b, c, d\}, C = \{b, c, e\}$.



1.16.1 (a)

Find $A \cup (B \cap C), (A \cup B) \cap C$, and $(A \cup B) \cap (A \cup C)$. Which of these sets are equal?

Proof. $A \cup (B \cap C) = \{a, b, c\}, (A \cup B) \cap C = \{b, c\}$, and $(A \cup B) \cap (A \cup C) = \{a, b, c, d\} \cap \{a, b, c, e\} = \{a, b, c\}$. Hence $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. □

1.16.2 (b)

Find $A \cap (B \cup C), (A \cap B) \cap C$, and $(A \cap B) \cup (A \cap C)$. Which of these sets are equal?

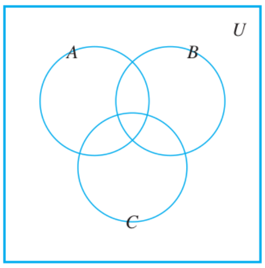
Proof. $A \cap (B \cup C) = \{b, c\}, (A \cap B) \cap C = \{b, c\}$, and $(A \cap B) \cup (A \cap C) = \{b, c\} \cup \{b, c\} = \{b, c\}$. Hence all three sets are equal. □

1.16.3 (c)

Find $(A - B) - C$ and $A - (B - C)$. Are these sets equal?

Proof. $(A - B) - C = \{a\} - \{b, c, e\} = \{a\}$ and $A - (B - C) = \{a, b, c\} - \{d\} = \{a, b, c\}$. The sets are not equal. □

1.17 Exercise 17

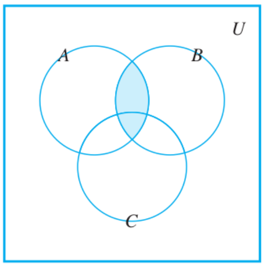


Consider the Venn diagram. For each of (a)-(f), copy the diagram and shade the region corresponding to the indicated set.

1.17.1 (a)

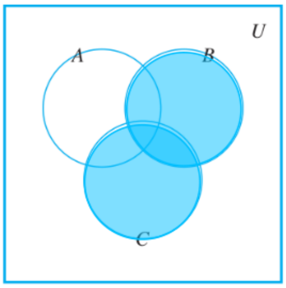
$A \cap B$

Proof. □



1.17.2 (b)

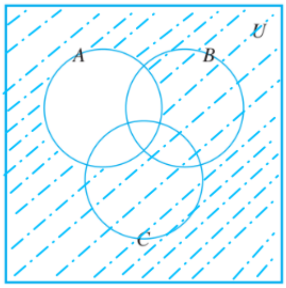
$B \cup C$



Proof. □

1.17.3 (c)

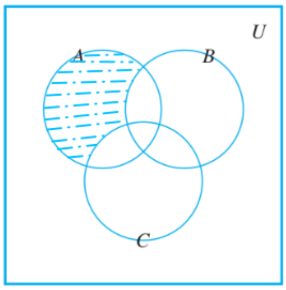
A^c



Proof. □

1.17.4 (d)

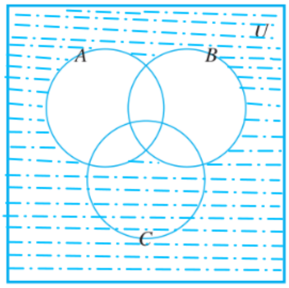
$A - (B \cup C)$



Proof. □

1.17.5 (e)

$(A \cup B)^c$



Proof. $\square$

1.17.6 (f)

$A^c \cap B^c$

Proof. Same as (e). $\square$

1.18 Exercise 18

1.18.1 (a)

Is the number 0 in $\varnothing$? Why?

Proof. The number 0 is not in $\varnothing$ because $\varnothing$ has no elements. $\square$

1.18.2 (b)

Is $\varnothing = \{\varnothing\}$? Why?

Proof. No. The left-hand set is the empty set; it does not have any elements. The right-hand set is a set with one element, namely $\varnothing$. $\square$

1.18.3 (c)

Is $\varnothing \in \{\varnothing\}$? Why?

Proof. Yes. The left-hand side is the empty set; the right-hand side is a set with one element, namely $\varnothing$. So the left- hand side is an element of the right-hand side. $\square$

1.18.4 (d)

Is $\varnothing \in \varnothing$? Why?

Proof. $\varnothing$ is not in $\varnothing$ because $\varnothing$ has no elements. $\square$

1.19 Exercise 19

Let $A_i = \{i, i^2\}$ for each integer $i = 1, 2, 3, 4$.

1.19.1 (a)

$A_1 \cup A_2 \cup A_3 \cup A_4 =?$

Proof. $A_1 = \{1, 1^2\} = \{1\}, A_2 = \{2, 2^2\} = \{2, 4\}, A_3 = \{3, 3^2\} = \{3, 9\}, A_4 = \{4, 4^2\} = \{4, 16\}$

$A_1 \cup A_2 \cup A_3 \cup A_4 = \{1\} \cup \{2, 4\} \cup \{3, 9\} \cup \{4, 16\} = \{1, 2, 3, 4, 9, 16\}$ □

1.19.2 (b)

$A_1 \cap A_2 \cap A_3 \cap A_4 =?$

Proof. $A_1 \cap A_2 \cap A_3 \cap A_4 = \{1\} \cap \{2, 4\} \cap \{3, 9\} \cap \{4, 16\} = \varnothing$ □

1.19.3 (c)

Are A_1, A_2, A_3, A_4 mutually disjoint? Explain.

Proof. A_1, A_2, A_3, A_4 are not mutually disjoint, because $A_2 \cap A_4 = \{4\} \neq \varnothing$. □

1.20 Exercise 20

Let $B_i = \{x \in \mathbb{R} \,|\, 0 \leq x \leq i\}$ for each integer $i = 1, 2, 3, 4$.

1.20.1 (a)

$B_1 \cup B_2 \cup B_3 \cup B_4 =?$

Proof. $B_1 \cup B_2 \cup B_3 \cup B_4 = B_4 = [0, 4]$ □

1.20.2 (b)

$B_1 \cap B_2 \cap B_3 \cap B_4 =?$

Proof. $B_1 \cap B_2 \cap B_3 \cap B_4 = B_1 = [0, 1]$ □

1.20.3 (c)

Are B_1, B_2, B_3, B_4 mutually disjoint? Explain.

Proof. No, because for example $B_1 \cap B_2 = B_1 \neq \varnothing$. □

1.21 Exercise 21

Let $C_i = \{-i, i\}$ for each nonnegative integer i.

1.21.1 (a)

$$\bigcup_{i=0}^{4} C_i = ?$$

Proof. $C_0 = \{0, -0\} = \{0\}, C_1 = \{1, -1\}, C_2 = \{2, -2\}, C_3 = \{3, -3\}, C_4 = \{4, -4\}$

$$\bigcup_{i=0}^{4} C_i = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$$ □

1.21.2 (b)

$$\bigcap_{i=0}^{4} C_i = ?$$

Proof. $\bigcap_{i=0}^{4} C_i = \{0\} \cap \{1, -1\} \cap \{2, -2\} \cap \{3, -3\} \cap \{4, -4\} = \varnothing$ □

1.21.3 (c)

Are $C_0, C_1, C_2, \ldots$ mutually disjoint? Explain.

Proof. $C_0, C_1, C_2, \ldots$ are mutually disjoint because no two of the sets have any elements in common. □

1.21.4 (d)

$$\bigcup_{i=0}^{n} C_i = ?$$

Proof. $\bigcup_{i=0}^{n} C_i = \{-n, -(n-1), \ldots, -2, -1, 0, 1, 2, \ldots, n-1, n\}$ □

1.21.5 (e)

$$\bigcap_{i=0}^{n} C_i = ?$$

Proof. $\bigcap_{i=0}^{n} C_i = \varnothing$ □

1.21.6 (f)

$$\bigcup_{i=0}^{\infty} C_i = ?$$

Proof. $\bigcup_{i=0}^{\infty} C_i = \mathbb{Z}$, the set of all integers $\square$

1.21.7 (g)

$$\bigcap_{i=0}^{\infty} C_i = ?$$

Proof. $\bigcap_{i=0}^{\infty} C_i = \varnothing$ $\square$

1.22 Exercise 22

Let $D_i = \{x \in \mathbb{R} \mid -i \leq x \leq i\} = [-i, i]$ for each nonnegative integer i.

1.22.1 (a)

$$\bigcup_{i=0}^{4} D_i = ?$$

Proof. $D_0 = [-0, 0] = \{0\}, D_1 = [-1, 1], D_2 = [-2, 2], D_3 = [-3, 3], D_4 = [-4, 4]$

$$\bigcup_{i=0}^{4} D_i = \{0\} \cup [-1, 1] \cup [-2, 2] \cup [-3, 3] \cup [-4, 4] = [-4, 4]$$ $\square$

1.22.2 (b)

$$\bigcap_{i=0}^{4} D_i = ?$$

Proof. $\bigcap_{i=0}^{4} D_i = \{0\} \cap [-1, 1] \cap [-2, 2] \cap [-3, 3] \cap [-4, 4] = \{0\}$ $\square$

1.22.3 (c)

Are $D_0, D_1, D_2, \ldots$ mutually disjoint? Explain.

Proof. $D_0, D_1, D_2, \ldots,$ are not mutually disjoint. In fact, each $D_k \subseteq D_{k+1}$. $\square$

1.22.4 (d)

$$\bigcup_{i=0}^{n} D_i =?$$

Proof. $\bigcup_{i=0}^{n} D_i = [-n, n]$ □

1.22.5 (e)

$$\bigcap_{i=0}^{n} D_i =?$$

Proof. $\bigcap_{i=0}^{n} D_i = \{0\}$ □

1.22.6 (f)

$$\bigcup_{i=0}^{\infty} D_i =?$$

Proof. $\bigcup_{i=0}^{\infty} D_i = \mathbb{R}$, the set of all real numbers □

1.22.7 (g)

$$\bigcap_{i=0}^{\infty} D_i =?$$

Proof. $\bigcap_{i=0}^{\infty} D_i = \{0\}$ □

1.23 Exercise 23

Let $V_i = \{x \in \mathbb{R} \mid -\frac{1}{i} \leq x \leq \frac{1}{i}\} = [-\frac{1}{i}, \frac{1}{i}]$ for each positive integer i.

1.23.1 (a)

$$\bigcup_{i=1}^{4} V_i =?$$

Proof. $\bigcup_{i=1}^{4} V_i = V_1 = [-1, 1]$ □

1.23.2 (b)

$$\bigcap_{i=1}^{4} V_i = ?$$

Proof. $\displaystyle\bigcap_{i=1}^{4} V_i = V_4 = \left[-\frac{1}{4}, \frac{1}{4}\right]$ $\square$

1.23.3 (c)

Are $V_1, V_2, V_3, \ldots$ mutually disjoint? Explain.

Proof. $V_1, V_2, V_3, \ldots,$ are not mutually disjoint. In fact, each $V_{k+1} \subseteq V_k$. $\square$

1.23.4 (d)

$$\bigcup_{i=1}^{n} V_i = ?$$

Proof. $\displaystyle\bigcup_{i=1}^{n} V_i = V_1 = [-1, 1]$ $\square$

1.23.5 (e)

$$\bigcap_{i=1}^{n} V_i = ?$$

Proof. $\displaystyle\bigcap_{i=1}^{n} V_i = V_n = \left[-\frac{1}{n}, \frac{1}{n}\right]$ $\square$

1.23.6 (f)

$$\bigcup_{i=1}^{\infty} V_i = ?$$

Proof. $\displaystyle\bigcup_{i=1}^{\infty} V_i = V_1 = [-1, 1]$ $\square$

1.23.7 (g)

$$\bigcap_{i=1}^{\infty} V_i = ?$$

Proof. $\bigcap_{i=1}^{\infty} V_i = \{0\}$ □

1.24 Exercise 24

Let $W_i = \{x \in \mathbb{R} \,|\, i < x\} = (i, \infty)$ for each nonnegative integer i.

1.24.1 (a)

$$\bigcup_{i=0}^{4} W_i = ?$$

Proof. $\bigcup_{i=0}^{4} W_i = W_0 = (0, \infty)$ □

1.24.2 (b)

$$\bigcap_{i=0}^{4} W_i = ?$$

Proof. $\bigcap_{i=0}^{4} W_i = W_4 = (4, \infty)$ □

1.24.3 (c)

Are $W_0, W_1, W_2, \ldots$ mutually disjoint? Explain.

Proof. $W_0, W_1, W_2, \ldots,$ are not mutually disjoint. In fact, each $W_{k+1} \subseteq W_k$. □

1.24.4 (d)

$$\bigcup_{i=0}^{n} W_i = ?$$

Proof. $\bigcup_{i=0}^{n} W_i = W_0 = (0, \infty)$ □

1.24.5 (e)

$$\bigcap_{i=0}^{n} W_i = ?$$

Proof. $\bigcap_{i=0}^{n} W_i = W_n = (n, \infty)$ □

1.24.6 (f)

$$\bigcup_{i=0}^{\infty} W_i = ?$$

Proof. $\displaystyle\bigcup_{i=0}^{\infty} W_i = W_0 = (0, \infty)$ □

1.24.7 (g)

$$\bigcap_{i=0}^{\infty} W_i = ?$$

Proof. $\displaystyle\bigcap_{i=0}^{\infty} W_i = \varnothing$ □

1.25 Exercise 25

Let $R_i = \{x \in \mathbb{R} \,|\, 1 \leq x \leq 1 + \frac{1}{i}\} = \left[1, 1 + \frac{1}{i}\right]$ for each positive integer i.

1.25.1 (a)

$$\bigcup_{i=1}^{4} R_i = ?$$

Proof. $\displaystyle\bigcup_{i=1}^{4} R_i = R_1 = [1, 2]$ □

1.25.2 (b)

$$\bigcap_{i=1}^{4} R_i = ?$$

Proof. $\displaystyle\bigcap_{i=1}^{4} R_i = R_4 = \left[1, \frac{5}{4}\right]$ □

1.25.3 (c)

Are $R_1, R_2, R_3, \ldots$ mutually disjoint? Explain.

Proof. No, in fact each $R_{k+1} \subseteq R_k$. □

1.25.4 (d)

$$\bigcup_{i=1}^{n} R_i = ?$$

Proof. $\displaystyle\bigcup_{i=1}^{n} R_i = R_1 = [1, 2]$ □

1.25.5 (e)

$$\bigcap_{i=1}^{n} R_i = ?$$

Proof. $\displaystyle\bigcap_{i=1}^{n} R_i = R_n = \left[1, 1 + \frac{1}{n}\right]$ □

1.25.6 (f)

$$\bigcup_{i=1}^{\infty} R_i = ?$$

Proof. $\displaystyle\bigcup_{i=1}^{\infty} R_i = R_1 = [1, 2]$ □

1.25.7 (g)

$$\bigcap_{i=1}^{\infty} R_i = ?$$

Proof. $\displaystyle\bigcap_{i=1}^{\infty} R_i = \{1\}$ □

1.26 Exercise 26

Let $S_i = \{x \in \mathbb{R} \,|\, 1 < x < 1 + \frac{1}{i}\} = \left(1, 1 + \frac{1}{i}\right)$ for each positive integer i.

1.26.1 (a)

$$\bigcup_{i=1}^{4} S_i = ?$$

Proof. $\displaystyle\bigcup_{i=1}^{4} S_i = S_1 = (1, 2)$ □

1.26.2 (b)

$$\bigcap_{i=1}^{4} S_i =?$$

Proof. $\bigcap_{i=1}^{4} S_i = S_4 = (1, 5/4)$ □

1.26.3 (c)

Are $S_0, S_1, S_2, \ldots$ mutually disjoint? Explain.

Proof. No, in fact each $S_{k+1} \subseteq S_k$. □

1.26.4 (d)

$$\bigcup_{i=1}^{n} S_i =?$$

Proof. $\bigcup_{i=1}^{n} S_i = S_1 = (1, 2)$ □

1.26.5 (e)

$$\bigcap_{i=1}^{n} S_i =?$$

Proof. $\bigcap_{i=1}^{n} S_i = S_n = (1, \frac{n+1}{n})$ □

1.26.6 (f)

$$\bigcup_{i=1}^{\infty} S_i =?$$

Proof. $\bigcup_{i=1}^{\infty} S_i = S_1 = (1, 2)$ □

1.26.7 (g)

$$\bigcap_{i=1}^{\infty} S_i =?$$

Proof. $\bigcap_{i=1}^{\infty} S_i = (1,1) = \varnothing$ □

1.27 Exercise 27

1.27.1 (a)

Is $\{\{a,d,e\},\{b,c\},\{d,f\}\}$ a partition of $\{a,b,c,d,e,f\}$?

Proof. No. The element d is in two of the sets. □

1.27.2 (b)

Is $\{\{w,x,v\},\{u,y,q\},\{p,z\}\}$ a partition of $\{p,q,u,v,w,x,y,z\}$?

Proof. Yes. □

1.27.3 (c)

Is $\{\{5,4\},\{7,2\},\{1,3,4\},\{6,8\}\}$ a partition of $\{1,2,3,4,5,6,7,8\}$?

Proof. No. The element 4 is in two of the sets. □

1.27.4 (d)

Is $\{\{3,7,8\},\{2,9\},\{1,4,5\}\}$ a partition of $\{1,2,3,4,5,6,7,8,9\}$?

Proof. No. None of the sets contains 6. □

1.27.5 (e)

Is $\{\{1,5\},\{4,7\},\{2,8,6,3\}\}$ a partition of $\{1,2,3,4,5,6,7,8\}$?

Proof. Yes. □

1.28 Exercise 28

Let E be the set of all even integers and O the set of all odd integers. Is $\{E,O\}$ a partition of $\mathbb{Z}$, the set of all integers? Explain your answer.

Proof. Yes. Every integer is either even or odd, and no integer is both even and odd. □

1.29 Exercise 29

Let $\mathbb{R}$ be the set of all real numbers. Is $\{R^+, R^-, \{0\}\}$ a partition of $\mathbb{R}$? Explain your answer.

Proof. Yes. Every real number is either positive or negative or zero, and no real number is both positive and negative, and zero is neither negative nor positive (so the three sets are mutually disjoint). $\square$

1.30 Exercise 30

Let Z be the set of all integers and let

$$\begin{aligned}
A_1 &= \{n \in \mathbb{Z} \,|\, n = 4k \text{ for some integer } k\} \\
A_2 &= \{n \in \mathbb{Z} \,|\, n = 4k + 1 \text{ for some integer } k\} \\
A_3 &= \{n \in \mathbb{Z} \,|\, n = 4k + 2 \text{ for some integer } k\} \\
A_4 &= \{n \in \mathbb{Z} \,|\, n = 4k + 3 \text{ for some integer } k\}
\end{aligned}$$

Is $\{A_0, A_1, A_2, A_3\}$ a partition of $\mathbb{Z}$? Explain your answer.

Proof. Yes. These sets are mutually disjoint, and by the quotient-remainder theorem, every integer has exactly one of the forms $n = 4k$ or $n = 4k + 1$ or $n = 4k + 2$ or $n = 4k + 3$. $\square$

1.31 Exercise 31

Suppose $A = \{1, 2\}, B = \{2, 3\}$. Find each of the following:

1.31.1 (a)

$\mathscr{P}(A \cap B)$

Proof. $A \cap B = \{2\}$, so $\mathscr{P}(A \cap B) = \{\varnothing, \{2\}\}$. $\square$

1.31.2 (b)

$\mathscr{P}(A)$

Proof. $A = \{1, 2\}$, so $\mathscr{P}(A) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$. $\square$

1.31.3 (c)

$\mathscr{P}(A \cup B)$

Proof. $A \cup B = \{1, 2, 3\}$, so $\mathscr{P}(A \cup B) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. $\square$

1.31.4 (d)

$\mathscr{P}(A \times B)$

Proof. $A \times B = \{(1,2),(1,3),(2,2),(2,3)\}$, so $\mathscr{P}(A \times B) = \{\varnothing, \{(1,2)\}, \{(1,3)\}, \{(2,2)\}, \{(2,3)\}, \{(1,2),(1,3)\}, \{(1,2),(2,2)\}, \{(1,2),(2,3)\}, \{(1,3),(2,2)\}, \{(1,3),(2,3)\}, \{(2,2),(2,3)\}, \{(1,2),(1,3),(2,2)\}, \{(1,2),(1,3),(2,3)\}, \{(1,2),(2,2),(2,3)\}, \{(1,3),(2,2),(2,3)\}, \{(1,2),(1,3),(2,2),(2,3)\}\}$. □

1.32 Exercise 32

1.32.1 (a)

Suppose $A = \{1\}, B = \{u, v\}$. Find $\mathscr{P}(A \times B)$.

Proof. $\mathscr{P}(A \times B) = \{\varnothing, \{(1,u)\}, \{(1,v)\}, \{(1,u),(1,v)\}\}$ □

1.32.2 (b)

Suppose $X = \{a, b\}, Y = \{x, y\}$. Find $\mathscr{P}(X \times Y)$.

Proof. $X \times Y = \{(a,x),(a,y),(b,x),(b,y)\}$
$\mathscr{P}(X \times Y) = \{\varnothing, \{(a,x)\}, \{(a,y)\}, \{(b,x)\}, \{(b,y)\}, \{(a,x),(a,y)\}, \{(a,x),(b,x)\}, \{(a,x),(b,y)\}, \{(a,y),(b,x)\}, \{(a,y),(b,y)\}, \{(b,x),(b,y)\}, \{(a,x),(a,y),(b,x)\}, \{(a,x),(a,y),(b,y)\}, \{(a,x),(b,x),(b,y)\}, \{(a,y),(b,x),(b,y)\}, \{(a,x),(a,y),(b,x),(b,y)\}\}$ □

1.33 Exercise 33

1.33.1 (a)

Find $\mathscr{P}(\varnothing)$.

Proof. $\mathscr{P}(\varnothing) = \{\varnothing\}$ □

1.33.2 (b)

Find $\mathscr{P}(\mathscr{P}(\varnothing))$.

Proof. $\mathscr{P}(\mathscr{P}(\varnothing)) = \mathscr{P}(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$ □

1.33.3 (c)

Find $\mathscr{P}(\mathscr{P}(\mathscr{P}(\varnothing)))$.

Proof. $\mathscr{P}(\mathscr{P}(\mathscr{P}(\varnothing))) = \mathscr{P}(\{\varnothing, \{\varnothing\}\}) = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$ □

1.34 Exercise 34

Let $A_1 = \{1\}, A_2 = \{u, v\}, A_3 = \{m, n\}$. Find each of the following sets:

1.34.1 (a)

$A_1 \cup (A_2 \times A_3)$

Proof. $A_1 \cup (A_2 \times A_3) = \{1\} \cup \{(u, m), (u, n), (v, m), (v, n)\}$
$= \{1, (u, m), (u, n), (v, m), (v, n)\}$ □

1.34.2 (b)

$(A_1 \cup A_2) \times A_3$

Proof. $(A_1 \cup A_2) \times A_3 = \{1, u, v\} \times \{m, n\} = \{(1, m), (1, n), (u, m), (u, n), (v, m), (v, n)\}$
□

1.35 Exercise 35

Let $A = \{a, b\}, B = \{1, 2\}, C = \{2, 3\}$. Find each of the following sets.

1.35.1 (a)

$A \times (B \cup C)$

Proof. $A \times (B \cup C) = \{a, b\} \times \{1, 2, 3\} \ = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$ □

1.35.2 (b)

$(A \times B) \cup (A \times C)$

Proof. $(A \times B) \cup (A \times C) = \{(a, 1), (a, 2), (b, 1), (b, 2), (a, 2), (a, 3), (b, 2), (b, 3)\}$
$= \{(a, 1), (a, 2), (b, 1), (b, 2), (a, 3), (b, 3)\}$ □

1.35.3 (c)

$A \times (B \cap C)$

Proof. $A \times (B \cap C) = \{a, b\} \times \{2\} \ = \{(a, 2), (b, 2)\}$ □

1.35.4 (d)

$(A \times B) \cap (A \times C)$

Proof. $(A \times B) \cap (A \times C) = \{(a, 1), (a, 2), (b, 1), (b, 2)\} \cap \{(a, 2), (a, 3), (b, 2), (b, 3)\}$
$= \{(a, 2), (b, 2)\}$ □

1.36 Exercise 36

Trace the action of Algorithm 6.1.1 on the variables $i, j, found$, and $answer$ for $m = 3, n = 3$, and sets A and B represented as the arrays $a[1] = u, a[2] = v, a[3] = w, b[1] = w, b[2] = u, b[3] = v$.

i	1				2				3			4
j		1	2	3	1	2	3	4	1		2	
found		no	yes		no		yes		no	yes		
answer	$A \subseteq B$											

Proof. □

1.37 Exercise 37

Trace the action of Algorithm 6.1.1 on the variables $i, j, found$, and $answer$ for $m = 4, n = 4$, and sets A and B represented as the arrays $a[1] = u, a[2] = v, a[3] = w, a[4] = x, b[1] = r, b[2] = u, b[3] = y, b[4] = z$.

Proof.

i	1				2			
j	1	2	3	4	1	2	3	4
found	no	yes			no			
answer	$A \subseteq B$							$A \not\subseteq B$

□

1.38 Exercise 38

Write an algorithm to determine whether a given element x belongs to a given set that is represented as the array $a[1], a[2], \ldots, a[n]$.

Algorithm: Testing whether $x \in A$
Input: x (an element), n (a positive integer), $a[1], \ldots, a[n]$ (a one-dimensional array representing the set A).
Algorithm Body:
$i := 1, answer :=$ "$x \notin A$"
while ($i \leq n$ and $answer =$ "$x \notin A$")
  if $x = a[i]$ **then** $answer =$ "$x \in A$"
  $i := i + 1$
end while
Output: *answer [a string]*