Chapter 5 Solutions, Susanna Epp Discrete Math 5th Edition

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Contents

1 Exercise Set 5.1

Write the first four terms of the sequences defined by the formulas in $1-6$.

1.1 Exercise 1

$a_k = \dfrac{k}{10+k}$, for every integer $k \geq 1$.

Proof. $\dfrac{1}{11}, \dfrac{2}{12}, \dfrac{3}{13}, \dfrac{4}{14}$ □

1.2 Exercise 2

$b_j = \dfrac{5-j}{5+j}$, for every integer $j \geq 1$.

Proof. $\dfrac{4}{6}, \dfrac{3}{7}, \dfrac{2}{8}, \dfrac{1}{9}$ □

1.3 Exercise 3

$c_i = \dfrac{(-1)^i}{3^i}$, for every integer $i \geq 0$.

Proof. $1, -\dfrac{1}{3}, \dfrac{1}{9}, -\dfrac{1}{27}$ □

1.4 Exercise 4

$d_m = 1 + \left(\dfrac{1}{2}\right)^m$, for every integer $m \geq 0$.

Proof. $2, \dfrac{3}{2}, \dfrac{5}{4}, \dfrac{9}{8}$ □

1.5 Exercise 5

$e_n = \left\lfloor \dfrac{n}{2} \right\rfloor \cdot 2$, for every integer $n \geq 0$.

Proof. $0, 0, 2, 2$ □

1.6 Exercise 6

$f_n = \left\lfloor \dfrac{n}{4} \right\rfloor \cdot 4$, for every integer $n \geq 1$.

Proof. $0,\ 0,\ 0,\ 4$ □

1.7 Exercise 7

Let $a_k = 2k + 1$ and $b_k = (k-1)^3 + k + 2$ for every integer $k \geq 0$. Show that the first three terms of these sequences are identical but that their fourth terms differ.

Proof. $a_0 = 2(0) + 1 = 1, a_1 = 2(1) + 1 = 3, a_2 = 2(2) + 1 = 5, a_3 = 2(3) + 1 = 7.$

$b_0 = (0-1)^3 + 0 + 2 = 1, b_1 = (1-1)^3 + 1 + 2 = 3, b_2 = (2-1)^3 + 2 + 2 = 5, b_3 = (3-1)^3 + 3 + 2 = 13.$ □

Compute the first fifteen terms of each of the sequences in 8 and 9, and describe the general behavior of these sequences in words. (a definition of logarithm is given in Section 7.1.)

1.8 Exercise 8

$g_n = \lfloor \log_2 n \rfloor$ for every integer $n \geq 1$.

Proof. $g_1 = \lfloor \log_2 1 \rfloor = 0, g_2 = \lfloor \log_2 2 \rfloor = 1, g_3 = \lfloor \log_2 3 \rfloor = 1, g_4 = \lfloor \log_2 4 \rfloor = 2,$

$g_5 = \lfloor \log_2 5 \rfloor = 2, g_6 = \lfloor \log_2 6 \rfloor = 2, g_7 = \lfloor \log_2 7 \rfloor = 2, g_8 = \lfloor \log_2 8 \rfloor = 3,$

$g_9 = \lfloor \log_2 9 \rfloor = 3, g_{10} = \lfloor \log_2 10 \rfloor = 3, g_{11} = \lfloor \log_2 11 \rfloor = 3, g_{12} = \lfloor \log_2 12 \rfloor = 3,$

$g_{13} = \lfloor \log_2 13 \rfloor = 3, g_{14} = \lfloor \log_2 14 \rfloor = 3, g_{15} = \lfloor \log_2 15 \rfloor = 3.$

When n is an integral power of 2, g_n is the exponent of that power. For instance, $8 = 2^3$ and $g_8 = 3$. More generally, if $n = 2k$, where k is an integer, then $g_n = k$. All terms of the sequence from g_{2^k} up to, but not including, $g_{2^{k+1}}$ have the same value, namely k. For instance, all terms of the sequence from g_8 through g_{15} have the value 3. □

1.9 Exercise 9

$h_n = n\lfloor \log_2 n \rfloor$ for every integer $n \geq 1$.

Proof. $h_1 = 1\lfloor \log_2 1 \rfloor = 0, h_2 = 2\lfloor \log_2 2 \rfloor = 2, h_3 = 3\lfloor \log_2 3 \rfloor = 3, h_4 = 4\lfloor \log_2 4 \rfloor = 8,$

$h_5 = 5\lfloor \log_2 5 \rfloor = 10, h_6 = 6\lfloor \log_2 6 \rfloor = 12, h_7 = 7\lfloor \log_2 7 \rfloor = 14, h_8 = 8\lfloor \log_2 8 \rfloor = 24,$

$h_9 = 9\lfloor \log_2 9 \rfloor = 27, h_{10} = 10\lfloor \log_2 10 \rfloor = 30, h_{11} = 11\lfloor \log_2 11 \rfloor = 33,$

$h_{12} = 12\lfloor \log_2 12 \rfloor = 36, h_{13} = 13\lfloor \log_2 13 \rfloor = 39, h_{14} = 14\lfloor \log_2 14 \rfloor = 42,$

$h_{15} = 15\lfloor \log_2 15 \rfloor = 45.$ □

Find explicit formulas for sequences of the form $a_1, a_2, a_3, \ldots$ with the initial terms given in $10 - 16$.

Exercises $10 - 16$ have more than one correct answer.

1.10 Exercise 10

$-1, 1, -1, 1, -1, 1$

Proof. $a_n = (-1)^n$, where n is an integer and $n \geq 1$ $\square$

1.11 Exercise 11

$0, 1, -2, 3, -4, 5$

Proof. $a_n = (n-1)(-1)^n$, where n is an integer and $n \geq 1$ $\square$

1.12 Exercise 12

$\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \frac{6}{49}$

Proof. $a_n = \frac{n}{(n+1)^2}$, where n is an integer and $n \geq 1$ $\square$

1.13 Exercise 13

$1 - \frac{1}{2}, \frac{1}{2} - \frac{1}{3}, \frac{1}{3} - \frac{1}{4}, \frac{1}{4} - \frac{1}{5}, \frac{1}{5} - \frac{1}{6}, \frac{1}{6} - \frac{1}{7}$

Proof. $a_n = \frac{1}{n} - \frac{1}{n+1}$, where n is an integer and $n \geq 1$ $\square$

1.14 Exercise 14

$\frac{1}{3}, \frac{4}{9}, \frac{9}{27}, \frac{16}{81}, \frac{25}{243}, \frac{36}{729}$

Proof. $a_n = \frac{n^2}{3^n}$, where n is an integer and $n \geq 1$ $\square$

1.15 Exercise 15

$0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \frac{6}{7}$

Proof. $a_n = \frac{n-1}{n} \cdot (-1)^{n-1}$, where n is an integer and $n \geq 1$ $\square$

1.16 Exercise 16

3, 6, 12, 24, 48, 96

Proof. $a_n = 3 \cdot 2^{n-1}$, where n is an integer and $n \geq 1$ $\square$

1.17 Exercise 17

Consider the sequence defined by $a_n = \dfrac{2n + (-1)^n - 1}{4}$ for every integer $n \geq 0$. Find an alternative explicit formula for an that uses the floor notation.

Proof. $a_0 = 0, a_1 = 0, a_2 = 1, a_3 = 1, a_4 = 2, a_5 = 2$. It seems to be following the pattern: $a_n = \left\lfloor \dfrac{n}{2} \right\rfloor$. Let's try to prove this. When n is even, $n = 2k$ for some integer k, so we have

$$a_n = a_{2k} = \frac{2(2k) + (-1)^{2k} - 1}{4} = \frac{4k + 1 - 1}{4} = \frac{4k}{4} = k = \frac{n}{2} = \left\lfloor \frac{n}{2} \right\rfloor$$

When n is odd, $n = 2k + 1$ for some integer k, so we have

$$a_n = a_{2k+1} = \frac{2(2k + 1) + (-1)^{2k+1} - 1}{4} = \frac{4k + 2 - 1 - 1}{4} = \frac{4k}{4} = k = \frac{n - 1}{2} = \left\lfloor \frac{n}{2} \right\rfloor$$

So $a_n = \left\lfloor \dfrac{n}{2} \right\rfloor$ for all $n \geq 0$. □

1.18 Exercise 18

Let $a_0 = 2, a_1 = 3, a_2 = -2, a_3 = 1, a_4 = 0, a_5 = -1$, and $a_6 = -2$. Compute each of the summations and products below.

1.18.1 (a)

$$\sum_{i=0}^{6} a_i$$

Proof. $2 + 3 + (-2) + 1 + 0 + (-1) + (-2) = 1$

□

1.18.2 (b)

$$\sum_{i=0}^{0} a_i$$

Proof. $a_0 = 2$ □

1.18.3 (c)

$$\sum_{j=1}^{3} a_{2j}$$

Proof. $a_2 + a_4 + a_6 = -2 + 0 + (-2) = -4$ □

1.18.4 (d)

$$\prod_{k=0}^{6} a_k$$

Proof. $2 \cdot 3 \cdot (-2) \cdot 1 \cdot 0 \cdot (-1) \cdot (-2) = 0$ □

1.18.5 (e)

$$\prod_{k=2}^{2} a_k$$

Proof. □

Compute the summations and products in $19 - 28$.

1.19 Exercise 19

$$\sum_{k=1}^{5} (k+1)$$

Proof. $2 + 3 + 4 + 5 + 6 = 20$ □

1.20 Exercise 20

$$\prod_{k=2}^{4} k^2$$

Proof. $2^2 \cdot 3^2 \cdot 4^2 = 576$ □

1.21 Exercise 21

$$\sum_{k=1}^{3} (k^2+1)$$

Proof. $(1^2+1) + (2^2+1) + (3^2+1) = 2 + 5 + 10 = 17$ □

1.22 Exercise 22

$$\prod_{j=0}^{4} (-1)^j$$

Proof. $(-1)^0 \cdot (-1)^1 \cdot (-1)^2 \cdot (-1)^3 \cdot (-1)^4 = 1$ □

1.23 Exercise 23

$$\sum_{i=1}^{1} i(i+1)$$

Proof. $1(1+1) = 2$ $\square$

1.24 Exercise 24

$$\sum_{j=0}^{0} (j+1) \cdot 2^j$$

Proof. $(0+1) \cdot 2^0 = 1$ $\square$

1.25 Exercise 25

$$\prod_{k=2}^{2} \left(1 - \frac{1}{k}\right)$$

Proof. $(1 - 1/2) = 1/2$ $\square$

1.26 Exercise 26

$$\sum_{k=-1}^{1} (k^2 + 3)$$

Proof. $((-1)^2 + 3) + (0^2 + 3) + (1^2 + 3) = 11$ $\square$

1.27 Exercise 27

$$\sum_{n=1}^{6} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Proof. $$\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{6}} - \frac{1}{7}\right)$$

$$= 1 - \frac{1}{7} = \frac{6}{7}$$ $\square$

1.28 Exercise 28

$$\prod_{i=2}^{5} \frac{i(i+2)}{(i-1) \cdot (i+1)}$$

Proof. $\frac{2(2+2)}{(2-1)(2+1)} \cdot \frac{3(3+2)}{(3-1)(3+1)} \cdot \frac{4(4+2)}{(4-1)(4+1)} \cdot \frac{5(5+2)}{(5-1)(5+1)}$

$= \frac{\cancel{8}}{3} \cdot \frac{\cancel{15}}{\cancel{8}} \cdot \frac{\cancel{24}}{\cancel{15}} \cdot \frac{35}{\cancel{24}} \cdot = \frac{35}{3}$ □

Write the summations in $29 - 32$ in expanded form.

1.29 Exercise 29

$$\sum_{i=1}^{n} (-2)^i$$

Proof. $(-2)^1 + (-2)^2 + (-2)^3 + \cdots + (-2)^n = -2 + 2^2 - 2^3 + \cdots + (-1)^n 2^n$ □

1.30 Exercise 30

$$\sum_{j=1}^{n} j(j+1)$$

Proof. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1)$ □

1.31 Exercise 31

$$\sum_{k=0}^{n+1} \frac{1}{k!}$$

Proof. $\sum_{k=0}^{n+1} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(n+1)!}$ □

1.32 Exercise 32

$$\sum_{i=1}^{k+1} i(i!)$$

Proof. $1(1!) + 2(2!) + 3(3!) + \cdots + (k+1)(k+1)!$ □

Evaluate the summations and products in $33 - 36$ for the indicated values of the variable.

1.33 Exercise 33

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2};\ n = 1$$

Proof. $\frac{1}{1^2} = 1$ □

1.34 Exercise 34

$1(1!) + 2(2!) + 3(3!) + \cdots + m(m!)$; $m = 2$

Proof. $1(1!) + 2(2!) = 1 + 4 = 5$ □

1.35 Exercise 35

$$\left(\frac{1}{1+1}\right)\left(\frac{2}{2+1}\right)\left(\frac{3}{3+1}\right)\cdots\left(\frac{k}{k+1}\right); \ k = 3$$

Proof. $\left(\frac{1}{1+1}\right)\left(\frac{2}{2+1}\right)\left(\frac{3}{3+1}\right) = \frac{1}{2}\frac{2}{3}\frac{3}{4} = \frac{1}{4}$ □

1.36 Exercise 36

$$\left(\frac{1\cdot 2}{3\cdot 4}\right)\left(\frac{2\cdot 3}{4\cdot 5}\right)\left(\frac{3\cdot 4}{5\cdot 6}\right)\cdots\left(\frac{m\cdot(m+1)}{(m+2)\cdot(m+3)}\right); m = 1$$

Proof. $\frac{1\cdot 2}{3\cdot 4} = \frac{3}{8}$ □

Write each of 37 – 39 as a single summation.

1.37 Exercise 37

$$\sum_{i=1}^{k} i^3 + (k+1)^3$$

Proof. $\sum_{i=1}^{k+1} i^3$ □

1.38 Exercise 38

$$\sum_{k=1}^{m} \frac{k}{k+1} + \frac{m+1}{m+2}$$

Proof. $\sum_{k=1}^{m+1} \frac{k}{k+1}$ □

1.39 Exercise 39

$$\sum_{m=0}^{n} (m+1)2^n + (n+2)2^{n+1}$$

Proof. $\sum_{m=0}^{n+1}(m+1)2^n$ $\square$

Rewrite 40 – 42 by separating off the final term.

1.40 Exercise 40

$$\sum_{i=1}^{k+1} i(i!)$$

Proof. $\sum_{i=1}^{k} i(i!) + (k+1)(k+1)!$ $\square$

1.41 Exercise 41

$$\sum_{k=1}^{m+1} k^2$$

Proof. $\sum_{k=1}^{m} k^2 + (m+1)^2$ $\square$

1.42 Exercise 42

$$\sum_{m=1}^{n+1} m(m+1)$$

Proof. $\sum_{m=1}^{n} m(m+1) + (n+1)(n+2)$ $\square$

Write each of 43 – 52 using summation or product notation.

Exercises 43 – 52 have more than one correct answer.

1.43 Exercise 43

$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + 7^2$

Proof. $\sum_{k=1}^{7}(-1)^{k+1}k^2$ or $\sum_{k=0}^{6}(-1)^k(k+1)^2$ $\square$

1.44 Exercise 44

$(1^3 - 1) - (2^3 - 1) + (3^3 - 1) - (4^3 - 1) + (5^3 - 1)$

Proof. $\displaystyle\sum_{k=1}^{5}(k^3 - 1)$ □

1.45 Exercise 45

$(2^2 - 1) \cdot (3^2 - 1) \cdot (4^2 - 1)$

Proof. $\displaystyle\prod_{k=2}^{4}(k^2 - 1)$ □

1.46 Exercise 46

$\frac{2}{3 \cdot 4} - \frac{3}{4 \cdot 5} + \frac{4}{5 \cdot 6} - \frac{5}{6 \cdot 7} + \frac{6}{7 \cdot 8}$

Proof. $\displaystyle\sum_{j=2}^{6}\frac{(-1)^j j}{(j+1)(j+2)}$ □

1.47 Exercise 47

$1 - r + r^2 - r^3 + r^4 - r^5$

Proof. $\displaystyle\sum_{i=0}^{5}(-1)^i r^i$ □

1.48 Exercise 48

$(1 - t) \cdot (1 - t^2) \cdot (1 - t^3) \cdot (1 - t^4)$

Proof. $\displaystyle\prod_{k=1}^{4}(1 - t^k)$ □

1.49 Exercise 49

$1^3 + 2^3 + 3^3 + \cdots + n^3$

Proof. $\displaystyle\sum_{k=1}^{n} k^3$ □

1.50 Exercise 50

$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots \frac{n}{(n+1)!}$

Proof. $\sum_{k=1}^{n} \frac{k}{(k+1)!}$ □

1.51 Exercise 51

$n + (n-1) + (n-2) + \cdots + 1$

Proof. $\sum_{i=0}^{n-1} (n-i)$ □

1.52 Exercise 52

$n + \frac{n-1}{2!} + \frac{n-2}{3!} + \frac{n-3}{4!} + \cdots + \frac{1}{n!}$

Proof. $\sum_{i=0}^{n-1} \frac{n-i}{(i+1)!}$ □

Transform each of 53 and 54 by making the change of variable $i = k + 1$.

1.53 Exercise 53

$\sum_{k=0}^{5} k(k-1)$

Proof. When $k = 0$, we have $i = 0 + 1 = 1$ and when $k = 5$ we have $i = 5 + 1 = 6$. Solving for k we get $k = i - 1$. So

$$\sum_{k=0}^{5} k(k-1) = \sum_{i=1}^{6} (i-1)(i-2)$$

□

1.54 Exercise 54

$\prod_{k=1}^{n} \frac{k}{k^2+4}$

Proof. When $k = 1$, we have $i = 1 + 1 = 2$ and when $k = n$ we have $i = n + 1$. Solving for k we get $k = i - 1$. So

$$\prod_{k=1}^{n} \frac{k}{k^2 + 4} = \prod_{i=2}^{n+1} \frac{i - 1}{(i - 1)^2 + 4}$$

□

Transform each of $55 - 58$ by making the change of variable $j = i - 1$.

1.55 Exercise 55

$$\sum_{i=1}^{n+1} \frac{(i-1)^2}{i \cdot n}$$

Proof. When $i = 1$, we have $j = 1 - 1 = 0$ and when $i = n+1$ we have $j = n+1-1 = n$. Solving for i we get $i = j + 1$. So

$$\sum_{i=1}^{n+1} \frac{(i-1)^2}{i \cdot n} = \sum_{j=0}^{n} \frac{(j+1-1)^2}{(j+1) \cdot n} = \sum_{j=0}^{n} \frac{j^2}{(j+1) \cdot n}$$

□

1.56 Exercise 56

$$\sum_{i=3}^{n} \frac{i}{i + n - 1}$$

Proof. When $i = 3$, we have $j = 3 - 1 = 2$ and when $i = n$ we have $j = n - 1$. Solving for i we get $i = j + 1$. So

$$\sum_{i=3}^{n} \frac{i}{i + n - 1} = \sum_{j=2}^{n-1} \frac{j + 1}{j + 1 + n - 1} = \sum_{j=2}^{n-1} \frac{j + 1}{j + n}$$

□

1.57 Exercise 57

$$\sum_{i=1}^{n-1} \frac{i}{(n-i)^2}$$

Proof. When $i = 1$, we have $j = 1 - 1 = 0$ and when $i = n - 1$ we have $j = n - 1 - 1 = n - 2$. Solving for i we get $i = j + 1$. So

$$\sum_{i=1}^{n-1} \frac{i}{(n-i)^2} = \sum_{j=0}^{n-2} \frac{j + 1}{(n - (j + 1))^2}$$

□

1.58 Exercise 58

$$\prod_{i=n}^{2n} \frac{n-i+1}{n+i}$$

Proof. When $i = n$, we have $j = n-1$ and when $i = 2n$ we have $j = 2n-1$. Solving for i we get $i = j+1$. So

$$\prod_{i=n}^{2n} \frac{n-i+1}{n+i} = \prod_{j=n-1}^{2n-1} \frac{n-(j+1)+1}{n+j+1} = \prod_{j=n-1}^{2n-1} \frac{n-j}{n+j+1}$$

$\square$

Write each of 59 – 61 as a single summation or product.

1.59 Exercise 59

$$3\sum_{k=1}^{n}(2k-3) + \sum_{k=1}^{n}(4-5k)$$

Proof. $\displaystyle\sum_{k=1}^{n}[3(2k-3)+(4-5k)] = \sum_{k=1}^{n}[6k-9+4-5k] = \sum_{k=1}^{n}[k-5]$ $\square$

1.60 Exercise 60

$$2\sum_{k=1}^{n}(3k^2+4) + 5\sum_{k=1}^{n}(2k^2-1)$$

Proof. $\displaystyle\sum_{k=1}^{n}[2(3k^2+4)+5(2k^2-1)] = \sum_{k=1}^{n}[6k^2+8+10k^2-5] = \sum_{k=1}^{n}[16k^2+3]$ $\square$

1.61 Exercise 61

$$\prod_{k=1}^{n}\frac{k}{k+1}\prod_{k=1}^{n}\frac{k+1}{k+2}$$

Proof. $\displaystyle\prod_{k=1}^{n}\frac{k}{k+1}\prod_{k=1}^{n}\frac{k+1}{k+2} = \prod_{k=1}^{n}\frac{k}{\cancel{k+1}}\frac{\cancel{k+1}}{k+2} = \prod_{k=1}^{n}\frac{k}{k+2}$ $\square$

Compute each of 62 – 76. Assume the values of the variables are restricted so that the expressions are defined.

1.62 Exercise 62

$\frac{4!}{3!}$

Proof. $\frac{4 \cdot \cancel{3 \cdot 2 \cdot 1}}{\cancel{3 \cdot 2 \cdot 1}} = 4$ $\square$

1.63 Exercise 63

$\frac{6!}{8!}$

Proof. $\frac{\cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{8 \cdot 7 \cdot \cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}} = \frac{1}{56}$ $\square$

1.64 Exercise 64

$\frac{4!}{0!}$

Proof. $\frac{4!}{0!} = \frac{24}{1} = 24$ $\square$

1.65 Exercise 65

$\frac{n!}{(n-1)!}$

Proof. $\frac{n \cdot \cancel{(n-1) \cdots 2 \cdot 1}}{\cancel{(n-1) \cdots 2 \cdot 1}} = n$ $\square$

1.66 Exercise 66

$\frac{(n-1)!}{(n+1)!}$

Proof. $\frac{\cancel{(n-1) \cdots 2 \cdot 1}}{(n+1) \cdot n \cdot \cancel{(n-1) \cdots 2 \cdot 1}} = \frac{1}{(n+1)n}$ $\square$

1.67 Exercise 67

$\frac{n!}{(n-2)!}$

Proof. $\frac{n \cdot (n-1) \cdot \cancel{(n-2) \cdots 2 \cdot 1}}{\cancel{(n-2) \cdots 2 \cdot 1}} = n(n-1)$ $\square$

1.68 Exercise 68

$$\frac{((n+1)!)^2}{(n!)^2}$$

Proof. $= \left(\frac{(n+1)!}{n!}\right)^2 = \left(\frac{(n+1)\cancel{n(n-1)\cdots 2\cdot 1}}{\cancel{n(n-1)\cdots 2\cdot 1}}\right)^2 = (n+1)^2$ □

1.69 Exercise 69

$$\frac{n!}{(n-k)!}$$

Proof. $\frac{n\cdot(n-1)\cdots(n-k+1)\cdot\cancel{(n-k)(n-k-1)\cdots 2\cdot 1}}{\cancel{(n-k)(n-k-1)\cdots 2\cdot 1}} = n(n-1)\cdots(n-k+1)$

□

1.70 Exercise 70

$$\frac{n!}{(n-k+1)!}$$

Proof. $\frac{n\cdot(n-1)\cdots(n-k+2)\cdot\cancel{(n-k+1)(n-k)\cdots 2\cdot 1}}{\cancel{(n-k+1)(n-k)\cdots 2\cdot 1}} = n(n-1)\cdots(n-k+2)$

□

1.71 Exercise 71

$$\binom{5}{3}$$

Proof. $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!\cdot 2!} = \frac{5\cdot 4\cdot\cancel{3\cdot 2\cdot 1}}{(\cancel{3\cdot 2\cdot 1})\cdot(2\cdot 1)} = 10$

□

1.72 Exercise 72

$$\binom{7}{4}$$

Proof. $\binom{7}{4} = \frac{7!}{4!(7-4)!} = \frac{7!}{4!\cdot 3!} = \frac{7\cdot\cancel{6}\cdot 5\cdot\cancel{4\cdot 3\cdot 2\cdot 1}}{(\cancel{4\cdot 3\cdot 2\cdot 1})\cdot(\cancel{3\cdot 2}\cdot 1)} = 35$ □

1.73 Exercise 73

$\binom{3}{0}$

Proof. 1 $\square$

1.74 Exercise 74

$\binom{5}{5}$

Proof. 1 $\square$

1.75 Exercise 75

$\binom{n}{n-1}$

Proof. $\binom{n}{n-1} = \frac{n!}{(n-1)!(n-(n-1))!} = \frac{n!}{(n-1)! \cdot 1!} = \frac{n \cdot \cancel{(n-1)\cdots 2 \cdot 1}}{\cancel{(n-1)\cdots 2 \cdot 1}} = n$ $\square$

1.76 Exercise 76

$\binom{n+1}{n-1}$

Proof. $\binom{n+1}{n-1} = \frac{(n+1)!}{(n-1)!(n+1-(n-1))!} = \frac{(n+1)!}{(n-1)! \cdot 2!}$

$= \frac{(n+1) \cdot n \cdot \cancel{(n-1)\cdots 2 \cdot 1}}{\cancel{(n-1)\cdots 2 \cdot 1} \cdot 2} = \frac{(n+1)n}{2}$ $\square$

1.77 Exercise 77

1.77.1 (a)

Prove that $n! + 2$ is divisible by 2, for every integer $n \geq 2$.

Proof. Let n be an integer such that $n \geq 2$. By definition of factorial,

$$n! = \begin{cases} 2 \cdot 1 & \text{if } n = 2 \\ 3 \cdot 2 \cdot 1 & \text{if } n = 3 \\ n \cdot (n-1) \cdots 2 \cdot 1 & \text{if } n > 3 \end{cases}$$

In each case, $n!$ has a factor of 2, and so $n! = 2k$ for some integer k. Then $n! + 2 = 2k + 2 = 2(k+1)$. Since $k+1$ is an integer, $n! + 2$ is divisible by 2. $\square$

1.77.2 (b)

Prove that $n! + k$ is divisible by k, for every integer $n \geq 2$ and $k = 2, 3, \ldots, n$.

Proof. For every $k = 2, 3, \ldots, n$, from the definition of $n!$ in part (a), we can see that $n!$ has a factor of k, so $n! = ka$ for some integer a. Then $n! + k = ka + k = k(a+1)$ where $a + 1$ is an integer. Therefore $n! + k$ is divisible by k for every $k = 2, 3, \ldots, n$. $\square$

1.77.3 (c)

Given any integer $m \geq 2$, is it possible to find a sequence of $m - 1$ consecutive positive integers none of which is prime? Explain your answer.

Proof. Yes. By part (b), $m! + k$ is divisible by k, for all $k = 2, 3, \ldots, m$. So $m! + 2, m! + 3, \ldots, m! + m$ are $m - 1$ consecutive integers none of which is prime. $\square$

1.78 Exercise 78

Prove that for all nonnegative integers n and r with $r + 1 \leq n$, $\binom{n}{r+1} = \frac{n-r}{r+1}\binom{n}{r}$.

Proof. Suppose n and r are nonnegative integers with $r + 1 \leq n$. The right-hand side of the equation to be shown is

$$\begin{aligned}
\frac{n-r}{r+1} \cdot \binom{n}{r} &= \frac{n-r}{r+1} \cdot \frac{n!}{r!(n-r)!} \\
&= \frac{\cancel{n-r}}{r+1} \cdot \frac{n!}{r!\cancel{(n-r)}(n-r-1)!} \\
&= \frac{n!}{(r+1)!(n-r-1)!} \\
&= \frac{n!}{(r+1)!(n-(r+1))!} \\
&= \binom{n}{r+1}
\end{aligned}$$

which is the left-hand side of the equality to be shown. $\square$

1.79 Exercise 79

Prove that if p is a prime number and r is an integer with $0 < r < p$, then $\binom{p}{r}$ is divisible by p.

Proof. We know that

$$\binom{p}{r} = \frac{p!}{r!(p-r)!} = \frac{p \cdot (p-1) \cdots 2 \cdot 1}{[r \cdot (r-1) \cdots 2 \cdot 1][(p-r) \cdot (p-r-1) \cdots 2 \cdot 1]}$$

is an integer. Notice that all the factors in the denominator are less than p. So, since p is prime, p is not divisible by any of the factors in the denominator. This means that every factor in the denominator is canceled out by the factors of $(p-1)\cdots 2\cdot 1$. Thus

$$M = \frac{(p-1)\cdots 2\cdot 1}{[r\cdot(r-1)\cdots 2\cdot 1][(p-r)\cdot(p-r-1)\cdots 2\cdot 1]}$$

is also an integer (otherwise $p\cdot M$ would not be an integer, since p cannot cancel out anything in the denominator). Therefore $\binom{p}{r} = p\cdot M$ where M is an integer, so it is divisible by p. □

1.80 Exercise 80

Suppose $a[1], a[2], a[3], \ldots, a[m]$ is a one-dimensional array and consider the following algorithm segment:

$sum := 0$
for $(k := 1$ **to** $m)$
  $sum := sum + a[k]$
next k

Fill in the blanks below so that each algorithm segment performs the same job as the one shown in the exercise statement.

1.80.1 (a)

$sum := 0$
for $(i := 0$ **to** ____ $)$
  $sum :=$ ____
next i

Proof. $m-1, sum + a[i+1]$ □

1.80.2 (b)

$sum := 0$
for $(j := 2$ **to** ____ $)$
  $sum :=$ ____
next j

Proof. $m+1, sum + a[j-1]$ □

Use repeated division by 2 to convert (by hand) the integers in $81-83$ from base 10 to base 2.

1.81 Exercise 81

90

Proof.
$$\begin{aligned}
90 \ / \ 2 &= 45, && \text{remainder} = 0\\
45 \ / \ 2 &= 22, && \text{remainder} = 1\\
22 \ / \ 2 &= 11, && \text{remainder} = 0\\
11 \ / \ 2 &= 5, && \text{remainder} = 1\\
5 \ / \ 2 &= 2, && \text{remainder} = 1\\
2 \ / \ 2 &= 1, && \text{remainder} = 0\\
1 \ / \ 2 &= 0, && \text{remainder} = 1
\end{aligned}$$

So $90_{10} = 1011010_2$. □

1.82 Exercise 82

98

Proof.
$$\begin{aligned}
98 \ / \ 2 &= 49, && \text{remainder} = 0\\
49 \ / \ 2 &= 24, && \text{remainder} = 1\\
24 \ / \ 2 &= 12, && \text{remainder} = 0\\
12 \ / \ 2 &= 6, && \text{remainder} = 0\\
6 \ / \ 2 &= 3, && \text{remainder} = 0\\
3 \ / \ 2 &= 1, && \text{remainder} = 1\\
1 \ / \ 2 &= 0, && \text{remainder} = 1
\end{aligned}$$

So $98_{10} = 1100010_2$. □

1.83 Exercise 83

205

Proof.
$$\begin{aligned}
205 \ / \ 2 &= 102, && \text{remainder} = 1\\
102 \ / \ 2 &= 51, && \text{remainder} = 0\\
51 \ / \ 2 &= 25, && \text{remainder} = 1\\
25 \ / \ 2 &= 12, && \text{remainder} = 1\\
12 \ / \ 2 &= 6, && \text{remainder} = 0\\
6 \ / \ 2 &= 3, && \text{remainder} = 0\\
3 \ / \ 2 &= 1, && \text{remainder} = 1\\
1 \ / \ 2 &= 0, && \text{remainder} = 1
\end{aligned}$$

So $205_{10} = 11001101_2$. □

Make a trace table to trace the action of algorithm 5.1.1 on the input in $84 - 86$.

1.84 Exercise 84

23

Proof.

a	23					
i	0	1	2	3	4	5
q	23	11	5	2	1	0
$r[0]$		1				
$r[1]$			1			
$r[2]$				1		
$r[3]$					0	
$r[4]$						1

□

1.85 Exercise 85

28

Proof.

a	28					
i	0	1	2	3	4	5
q	28	14	7	3	1	0
$r[0]$		0				
$r[1]$			0			
$r[2]$				1		
$r[3]$					1	
$r[4]$						1

□

1.86 Exercise 86

44

Proof.

a	44						
i	0	1	2	3	4	5	6
q	44	22	11	5	2	1	0
$r[0]$		0					
$r[1]$			0				
$r[2]$				1			
$r[3]$					1		
$r[4]$						0	
$r[5]$							1

□

1.87 Exercise 87

Write an informal description of an algorithm (using repeated division by 16) to convert a nonnegative integer from decimal notation to hexadecimal notation (base 16).

Proof. Suppose a is a nonnegative integer. Divide a by 16 using the quotient-remainder theorem to obtain a quotient $q[0]$ and a remainder $r[0]$. If the quotient is nonzero, divide by 16 again to obtain a quotient $q[1]$ and a remainder $r[1]$. Continue this process until a quotient of 0 is obtained. At each stage, the remainder must be less than the divisor, which is 16. Thus each remainder is always among $0, 1, 2, \ldots, 15$. Read the divisions from the bottom up. $\square$

Use the algorithm you developed for exercise 87 to convert the integers in 88 – 90 to hexadecimal notation.

1.88 Exercise 88

287

Proof.

$$\begin{array}{rcll} 287 \;/\; 16 & = & 17, & \text{remainder} = 15 = \text{F} \\ 17 \;/\; 16 & = & 1, & \text{remainder} = 1 \\ 1 \;/\; 16 & = & 0, & \text{remainder} = 1 \end{array}$$

So $287_{10} = 11F_{16}$. $\square$

1.89 Exercise 89

693

Proof.

$$\begin{array}{rcll} 693 \;/\; 16 & = & 43, & \text{remainder} = 5 \\ 43 \;/\; 16 & = & 2, & \text{remainder} = 11 = \text{B} \\ 2 \;/\; 16 & = & 0, & \text{remainder} = 2 \end{array}$$

So $693_{10} = 2B5_{16}$. $\square$

1.90 Exercise 90

2301

Proof.

$$\begin{array}{rcll} 2301 \;/\; 16 & = & 143, & \text{remainder} = 13 = \text{D} \\ 143 \;/\; 16 & = & 8, & \text{remainder} = 15 = \text{F} \\ 8 \;/\; 16 & = & 0, & \text{remainder} = 8 \end{array}$$

So $2301_{10} = 8FD_{16}$. $\square$

1.91 Exercise 91

Write a formal version of the algorithm you developed for exercise 87.
Proof:

Decimal to Hexadecimal Conversion Using Repeated Division by 16
Input: a *[a nonnegative integer]*
Algorithm Body:

$q := a, i := 0$
while $(i = 0 \text{ or } q \neq 0)$
  $r[i] := q \mod 16$
  $q := q \text{ div } 16$
  [$r[i]$ and q can be obtained by calling the division algorithm.]
end while
[After execution of this step, the values $r[0], r[1], \ldots, r[i-1]$ are all 0's and 1's, and $a = (r[i-1]r[i-2]\ldots r[1]r[0])_{16}$.]

Output: $r[0], r[1], \ldots, r[i-1]$ *[a sequence of integers]*

2 Exercise Set 5.2

2.1 Exercise 1

Use the technique illustrated at the beginning of this section to show that the statements in (a) and (b) are true.

2.1.1 (a)

If $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right) = \frac{1}{5}$ then
$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{6}\right) = \frac{1}{6}$.

Proof. The statement in part (a) is true because if

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right) = \frac{1}{5}$$

then

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{6}\right) = \frac{1}{5}\cdot\frac{5}{6} = \frac{1}{6}.$$

□

2.1.2 (b)

If $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{6}\right) = \frac{1}{6}$ then
$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{6}\right)\left(1-\frac{1}{7}\right) = \frac{1}{7}$.

Proof. The statement in part (a) is true because if

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{6}\right)=\frac{1}{6}$$

then

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{6}\right)\left(1-\frac{1}{7}\right)=\frac{1}{6}\cdot\frac{6}{7}=\frac{1}{7}.$$

□

2.2 Exercise 2

For each positive integer n, let $P(n)$ be the formula

$$1+3+5+\cdots+(2n-1)=n^2.$$

2.2.1 (a)

Write $P(1)$. Is $P(1)$ true?

Proof. $P(1)$ is the equation $1=1^2$, which is true. □

2.2.2 (b)

Write $P(k)$.

Proof. $P(k)$ is the equation $1+3+5+\cdots+(2k-1)=k^2$. □

2.2.3 (c)

Write $P(k+1)$.

Proof. $P(k+1)$ is the equation $1+3+5+\cdots+(2(k+1)-1)=(k+1)^2$. □

2.2.4 (d)

In a proof by mathematical induction that the formula holds for every integer $n \geq 1$, what must be shown in the inductive step?

Proof. In the inductive step, show that if k is any integer for which $k \geq 1$ and $1+3+5+\cdots+(2k-1)=k^2$ is true, then $1+3+5+\cdots+(2(k+1)-1)=(k+1)^2$ is also true. □

2.3 Exercise 3

For each positive integer n, let $P(n)$ be the formula

$$1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

2.3.1 (a)

Write $P(1)$. Is $P(1)$ true?

Proof. $P(1)$ is "$1^2 = \frac{1(1+1)(2\cdot 1+1)}{6}$." $P(1)$ is true because the left-hand side equals $1^2 = 1$ and the right-hand side equals $\frac{1(1+1)(2+1)}{6} = \frac{2\cdot 3}{6} = 1$ also. □

2.3.2 (b)

Write $P(k)$.

Proof. $P(k)$ is "$k^2 = \dfrac{k(k+1)(2k+1)}{6}$." □

2.3.3 (c)

Write $P(k+1)$.

Proof. $P(k+1)$ is "$(k+1)^2 = \dfrac{(k+1)(k+2)(2k+3)}{6}$." □

2.3.4 (d)

In a proof by mathematical induction that the formula holds for every integer $n \geq 1$, what must be shown in the inductive step?

Proof. In the inductive step, show that if k is any integer for which $k \geq 1$ and $k^2 = \dfrac{k(k+1)(2k+1)}{6}$ is true, then $(k+1)^2 = \dfrac{(k+1)(k+2)(2k+3)}{6}$ is also true. □

2.4 Exercise 4

For each integer n with $n \geq 2$, let $P(n)$ be the formula

$$\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}$$

2.4.1 (a)

Write $P(2)$. Is $P(2)$ true?

Proof. $P(2)$ is "$\displaystyle\sum_{i=1}^{1} i(i+1) = \frac{2(2-1)(2+1)}{3}$" It's true because the left-hand side is $1(1+1) = 2$ and the right-hand side is $\dfrac{2(1)(3)}{3} = 2$ also. □

2.4.2 (b)

Write $P(k)$.

Proof. $P(k)$ is "$\sum_{i=1}^{k-1} i(i+1) = \frac{k(k-1)(k+1)}{3}$." □

2.4.3 (c)

Write $P(k+1)$.

Proof. $P(k+1)$ is "$\sum_{i=1}^{k} i(i+1) = \frac{(k+1)k(k+2)}{3}$." □

2.4.4 (d)

In a proof by mathematical induction that the formula holds for every integer $n \geq 2$, what must be shown in the inductive step?

Proof. In the inductive step, show that if k is any integer for which $k \geq 2$ and $\sum_{i=1}^{k-1} i(i+1) = \frac{k(k-1)(k+1)}{3}$ is true, then $\sum_{i=1}^{k} i(i+1) = \frac{(k+1)k(k+2)}{3}$ is also true. □

2.5 Exercise 5

Fill in the missing pieces in the following proof that

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

for every integer $n \geq 1$.

Proof: Let the property $P(n)$ be the equation

$$1 + 3 + 5 + \cdots + (2n-1) = n^2. \leftarrow P(n)$$

Show that $P(1)$ is true: To establish $P(1)$, we must show that when 1 is substituted in place of n, the left-hand side equals the right-hand side. But when $n = 1$, the left-hand side is the sum of all the odd integers from 1 to $2 \cdot 1 - 1$, which is the sum of the odd integers from 1 to 1 and is just 1. The right-hand side is (a) ____ , which also equals 1. So $P(1)$ is true.

Show that for every integer $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 1$.

[Suppose $P(k)$ is true. That is:] Suppose

$$1 + 3 + 5 + \cdots + (2k-1) = \text{(b)}\_\_\_\_ \quad \leftarrow P(k)$$

[This is the inductive hypothesis.]

[We must show that $P(k+1)$ is true. That is:] We must show that

$$(c)\_\_\_\_ = (d)\_\_\_\_ \leftarrow P(k+1)$$

Now the left-hand side of $P(k+1)$ is $1+3+5+\cdots+(2(k+1)-1)$

$$\begin{aligned}
&= 1+3+5+\cdots+(2k+1) && \text{by algebra} \\
&= [1+3+5+\cdots+(2k-1)]+(2k+1) && \text{because (e) ____} \\
&= k^2+(2k+1) && \text{by (f) ____} \\
&= (k+1)^2 && \text{by algebra,}
\end{aligned}$$

which is the right-hand side of $P(k+1)$ *[as was to be shown.]*

[Since we have proved the basis step and the inductive step, we conclude that the given statement is true.]

Note: This proof was annotated to help make its logical flow more obvious. In standard mathematical writing, such annotation is omitted.

Proof. a. 1^2; b. k^2; c. $1+3+5+\cdots+[2(k+1)-1]$; d. $(k+1)^2$; e. the next-to-last term is $2k-1$ because the odd integer just before $2k+1$ is $2k-1$; f. inductive hypothesis □

Prove each statement in $6-9$ using mathematical induction. Do not derive them from theorem 5.2.1 or Theorem 5.2.2.

2.6 Exercise 6

For every integer $n \geq 1$,

$$2+4+6+\cdots+2n = n^2+n.$$

Proof. For the given statement, the property $P(n)$ is the equation

$$2+4+6+\cdots+2n = n^2+n. \leftarrow P(n)$$

Show that $P(1)$ is true: To prove $P(1)$, we must show that when 1 is substituted into the equation in place of n, the left-hand side equals the right-hand side. But when 1 is substituted for n, the left-hand side is the sum of all the even integers from 2 to $2 \geq 1$, which is just 2, and the right-hand side is 1^2+1, which also equals 2. Thus $P(1)$ is true.

Show that for every integer $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer with $k \geq 1$, and suppose $P(k)$ is true. That is, suppose

$$2+4+6+\cdots+2k = k^2+k. \leftarrow P(k)\text{: inductive hypothesis}$$

We must show that $P(k+1)$ is true. That is, we must show that

$$2+4+6+\cdots+2(k+1) = (k+1)^2+(k+1).$$

Because $(k+1)^2 + (k+1) = k^2 + 2k + 1 + k + 1 = k^2 + 3k + 2$, this is equivalent to showing that

$$2 + 4 + 6 + \cdots + 2(k+1) = k^2 + 3k + 2. \leftarrow P(k+1)$$

Now the left-hand side of $P(k+1)$ is $2 + 4 + 6 + \cdots + 2(k+1)$

$$\begin{aligned} &= 2 + 4 + 6 + \cdots + 2k + 2(k+1) && \text{make next-to-last term explicit} \\ &= (k^2 + k) + 2(k+1) && \text{by inductive hypothesis} \\ &= k^2 + 3k + 2 && \text{by algebra,} \end{aligned}$$

and this is the right-hand side of $P(k+1)$. Hence $P(k+1)$ is true. *[Since both the basis step and the inductive step have been proved, $P(n)$ is true for every integer $n \geq 1$.]* □

2.7 Exercise 7

For every integer $n \geq 1$,

$$1 + 6 + 11 + 16 + \cdots + (5n-4) = \frac{n(5n-3)}{2}.$$

Proof. For the given statement, the property $P(n)$ is the equation

$$1 + 6 + 11 + 16 + \cdots + (5n-4) = \frac{n(5n-3)}{2}. \leftarrow P(n)$$

Show that $P(1)$ is true: To prove $P(1)$, we must show that when 1 is substituted into the equation in place of n, the left-hand side equals the right-hand side. But when 1 is substituted for n, the left-hand side is the sum from 1 to $1 \geq 1$, which is just 1, and the right-hand side is $\frac{1(5-3)}{2}$, which also equals 1. Thus $P(1)$ is true.

Show that for every integer $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer with $k \geq 1$, and suppose $P(k)$ is true. That is, suppose

$$1 + 6 + 11 + 16 + \cdots + (5k-4) = \frac{k(5k-3)}{2}. \leftarrow P(k)\text{: inductive hypothesis}$$

We must show that $P(k+1)$ is true. That is, we must show that

$$1 + 6 + 11 + 16 + \cdots + (5(k+1)-4) = \frac{(k+1)(5(k+1)-3)}{2}.$$

Because $5(k+1) - 4 = 5k+1$ and $5(k+1) - 3 = 5k+2$, this is equivalent to showing that

$$1 + 6 + 11 + 16 + \cdots + (5k+1) = \frac{(k+1)(5k+2)}{2}. \leftarrow P(k+1)$$

Now the left-hand side of $P(k+1)$ is $1 + 6 + 11 + 16 + \cdots + (5k+1)$

$$
\begin{aligned}
&= 1+6+11+16+\cdots+(5k-4)+(5k+1) && \text{make next-to-last term explicit} \\
&= \frac{k(5k-3)}{2}+(5k+1) && \text{by inductive hypothesis} \\
&= \frac{5k^2-3k}{2}+\frac{10k+2}{2} && \text{by algebra} \\
&= \frac{5k^2-3k+10k+2}{2} && \text{by algebra} \\
&= \frac{5k^2+7k+2}{2} && \text{by algebra} \\
&= \frac{(k+1)(5k+2)}{2} && \text{by factoring,}
\end{aligned}
$$

and this is the right-hand side of $P(k+1)$. Hence $P(k+1)$ is true. *[Since both the basis step and the inductive step have been proved, $P(n)$ is true for every integer $n \geq 1$.]* $\square$

2.8 Exercise 8

For every integer $n \geq 0$,

$$1+2+2^2+\cdots+2^n = 2^{n+1}-1.$$

Proof. For the given statement, the property $P(n)$ is the equation

$$1+2+2^2+\cdots+2^n = 2^{n+1}-1. \leftarrow P(n)$$

Show that $P(0)$ is true: The left-hand side of $P(0)$ is 1, and the right-hand side is $2^{0+1}-1 = 2-1 = 1$ also. Thus $P(0)$ is true.

Show that for every integer $k \geq 0$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 0$, and suppose $P(k)$ is true. That is, suppose

$$1+2+2^2+\cdots+2^k = 2^{k+1}-1. \leftarrow P(k)\text{: inductive hypothesis}$$

We must show that $P(k+1)$ is true. That is, we must show that

$$1+2+2^2+\cdots+2^{k+1} = 2^{(k+1)+1}-1,$$

or, equivalently,

$$1+2+2^2+\cdots+2^{k+1} = 2^{k+2}-1. \leftarrow P(k+1)$$

Now the left-hand side of $P(k+1)$ is $1+2+2^2+\cdots+2^{k+1}$

$$
\begin{aligned}
&= 1+2+2^2+\cdots+2^k+2^{k+1} && \text{make next-to-last term explicit} \\
&= (2^{k+1}-1)+2^{k+1} && \text{by inductive hypothesis} \\
&= 2\cdot 2^{k+1}-1 && \text{by combining like terms} \\
&= 2^{k+2}-1 && \text{by the laws of exponents}
\end{aligned}
$$

and this is the right-hand side of $P(k+1)$. Hence $P(k+1)$ is true. *[Since both the basis step and the inductive step have been proved, $P(n)$ is true for every integer $n \geq 0$.]* $\square$

2.9 Exercise 9

For every integer $n \geq 3$,

$$4^3 + 4^4 + 4^5 + \cdots + 4^n = \frac{4(4^n - 16)}{3}.$$

Proof. For the given statement, the property $P(n)$ is the equation

$$4^3 + 4^4 + 4^5 + \cdots + 4^n = \frac{4(4^n - 16)}{3}. \leftarrow P(n)$$

Show that $P(3)$ is true: The left-hand side of $P(3)$ is $4^3 = 64$, and the right-hand side is $\frac{4(4^3-16)}{3} = 4 \cdot 48/3 = 64$ also. Thus $P(3)$ is true.

Show that for every integer $k \geq 3$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer with $k \geq 3$, and suppose $P(k)$ is true. That is, suppose

$$4^3 + 4^4 + 4^5 + \cdots + 4^k = \frac{4(4^k - 16)}{3}. \leftarrow P(k)\text{: inductive hypothesis}$$

We must show that $P(k+1)$ is true. That is, we must show that

$$4^3 + 4^4 + 4^5 + \cdots + 4^{k+1} = \frac{4(4^{k+1} - 16)}{3}. \leftarrow P(k+1)$$

Now the left-hand side of $P(k+1)$ is $4^3 + 4^4 + 4^5 + \cdots + 4^{k+1}$

$$\begin{aligned}
&= 4^3 + 4^4 + 4^5 + \cdots + 4^k + 4^{k+1} && \text{make next-to-last term explicit}\\
&= \frac{4(4^k - 16)}{3} + 4^{k+1} && \text{by inductive hypothesis}\\
&= \frac{4(4^k - 16)}{3} + 4 \cdot 4^k && \text{by the laws of exponents}\\
&= 4\left(\frac{4^k - 16}{3} + 4^k\right) && \text{by factoring}\\
&= 4\left(\frac{4^k - 16}{3} + \frac{3 \cdot 4^k}{3}\right) && \text{by algebra}\\
&= 4\left(\frac{4^k - 16 + 3 \cdot 4^k}{3}\right) && \text{by algebra}\\
&= 4\left(\frac{4 \cdot 4^k - 16}{3}\right) && \text{by combining like terms}\\
&= 4\left(\frac{4^{k+1} - 16}{3}\right) && \text{by the laws of exponents}
\end{aligned}$$

and this is the right-hand side of $P(k+1)$. Hence $P(k+1)$ is true. *[Since both the basis step and the inductive step have been proved, $P(n)$ is true for every integer $n \geq 3$.]* □

Prove each of the statements in $10 - 18$ by mathematical induction.

2.10 Exercise 10

$1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$, for every integer $n \geq 1$.

Proof. For the given statement, the property $P(n)$ is the equation

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \leftarrow P(n)$$

Show that $P(1)$ is true: The left-hand side of $P(1)$ is $1^2 = 1$, and the right-hand side is $\dfrac{1(1+1)(2+1)}{6} = \dfrac{2 \cdot 3}{6} = 1$ also. Thus $P(1)$ is true.

Show that for every integer $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer with $k \geq 1$, and suppose $P(k)$ is true. That is, suppose

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}. \leftarrow P(k)\text{: inductive hypothesis}$$

We must show that $P(k+1)$ is true. That is, we must show that

$$1^2 + 2^2 + \cdots + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6},$$

or, equivalently,

$$1^2 + 2^2 + \cdots + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}. \leftarrow P(k+1)$$

Now the left-hand side of $P(k+1)$ is $1^2 + 2^2 + \cdots + (k+1)^2$

$$\begin{aligned}
&= 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 && \text{make next-to-last term explicit}\\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{by inductive hypothesis}\\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} && \text{because } \tfrac{6}{6} = 1\\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} && \text{by adding fractions}\\
&= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} && \text{by factoring out } (k+1)\\
&= \frac{(k+1)[2k^2 + k + 6k + 6]}{6} && \text{by multiplying out}\\
&= \frac{(k+1)[2k^2 + 7k + 6]}{6} && \text{by combining like terms}\\
&= \frac{(k+1)(k+2)(2k+3)}{6} && \text{by factoring}
\end{aligned}$$

and this is the right-hand side of $P(k+1)$. Hence $P(k+1)$ is true. *[Since both the basis step and the inductive step have been proved, $P(n)$ is true for every integer $n \geq 1$.]* □

2.11 Exercise 11

$1^3 + 2^3 + \cdots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$, for every integer $n \geq 1$.

Proof. For the given statement, the property $P(n)$ is the equation

$$1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2. \leftarrow P(n)$$

Show that $P(1)$ is true: The left-hand side of $P(1)$ is $1^3 = 1$, and the right-hand side is $\left[\dfrac{1(1+1)}{2}\right]^2 = 1^2 = 1$ also. Thus $P(1)$ is true.

Show that for every integer $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 1$, and suppose $P(k)$ is true. That is, suppose

$$1^3 + 2^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2}\right]^2. \leftarrow P(k)\text{: inductive hypothesis}$$

We must show that $P(k+1)$ is true. That is, we must show that

$$1^3 + 2^3 + \cdots + (k+1)^3 = \left[\frac{(k+1)(k+2)}{2}\right]^2. \leftarrow P(k+1)$$

Now the left-hand side of $P(k+1)$ is $1^3 + 2^3 + \cdots + (k+1)^3$

$$\begin{aligned}
&= 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 && \text{make next-to-last term explicit}\\
&= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 && \text{by inductive hypothesis}\\
&= \frac{k^2(k+1)^2}{4} + (k+1)(k+1)^2 && \text{by algebra}\\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)(k+1)^2}{4} && \text{because } \tfrac{4}{4} = 1\\
&= \frac{k^2(k+1)^2 + 4(k+1)(k+1)^2}{4} && \text{by adding fractions}\\
&= \frac{(k+1^2)[k^2 + 4(k+1)]}{4} && \text{by factoring out } (k+1)^2\\
&= \frac{(k+1)^2[k^2 + 4k + 4]}{4} && \text{by multiplying out}\\
&= \frac{(k+1)^2(k+2)^2}{4} && \text{by factoring}\\
&= \left[\frac{(k+1)(k+2)}{2}\right]^2 && \text{by factoring}
\end{aligned}$$

and this is the right-hand side of $P(k+1)$. Hence $P(k+1)$ is true. *[Since both the basis step and the inductive step have been proved, $P(n)$ is true for every integer $n \geq 1$.]* □

2.12 Exercise 12

$\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$, for every integer $n\geq 1$.

Proof. For the given statement, the property $P(n)$ is the equation

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}. \quad \leftarrow P(n)$$

Show that $P(1)$ is true: The left-hand side of $P(1)$ is $\frac{1}{1\cdot(1+1)}=1/2$, and the right-hand side is $\frac{1}{1+1}=1/2$ also. Thus $P(1)$ is true.

Show that for every integer $k\geq 1$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer with $k\geq 1$, and suppose $P(k)$ is true. That is, suppose

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1}. \quad \leftarrow P(k)\text{: inductive hypothesis}$$

We must show that $P(k+1)$ is true. That is, we must show that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)((k+1)+1)}=\frac{k+1}{(k+1)+1}. \quad \leftarrow P(k+1)$$

Now the left-hand side of $P(k+1)$ is $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{(k+1)(k+2)}$

$$\begin{aligned}
&= \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)} && \text{make next-to-last term explicit}\\
&= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)} && \text{by inductive hypothesis}\\
&= \frac{k(k+2)}{(k+1)(k+2)}+\frac{1}{(k+1)(k+2)} && \text{because } \tfrac{k+2}{k+2}=1\\
&= \frac{k^2+2k}{(k+1)(k+2)}+\frac{1}{(k+1)(k+2)} && \text{by algebra}\\
&= \frac{k^2+2k+1}{(k+1)(k+2)} && \text{by algebra}\\
&= \frac{(k+1)^2}{(k+1)(k+2)} && \text{by algebra}\\
&= \frac{k+1}{k+2} && \text{by canceling } k+1
\end{aligned}$$

and this is the right-hand side of $P(k+1)$. Hence $P(k+1)$ is true. *[Since both the basis step and the inductive step have been proved, $P(n)$ is true for every integer $n\geq 1$.]* □

2.13 Exercise 13

$\displaystyle\sum_{i=1}^{n-1} i(i+1)=\frac{n(n-1)(n+1)}{3}$, for every integer $n\geq 2$.

Proof. For the given statement, the property $P(n)$ is the equation

$$\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}. \quad \leftarrow P(n)$$

Show that $P(2)$ is true: The left-hand side of $P(1)$ is $\sum_{i=1}^{2-1} i(i+1) = 1(1+1) = 2$, and the right-hand side is $\frac{2(2-1)(2+1)}{3} = \frac{6}{3} = 2$ also. Thus $P(2)$ is true.

Show that for every integer $k \geq 2$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer with $k \geq 2$, and suppose $P(k)$ is true. That is, suppose

$$\sum_{i=1}^{k-1} i(i+1) = \frac{k(k-1)(k+1)}{3}. \quad \leftarrow P(k)\text{: inductive hypothesis}$$

We must show that $P(k+1)$ is true. That is, we must show that

$$\sum_{i=1}^{k+1-1} i(i+1) = \frac{(k+1)(k+1-1)(k+1+1)}{3},$$

or, equivalently,

$$\sum_{i=1}^{k} i(i+1) = \frac{(k+1)k(k+2)}{3}. \quad \leftarrow P(k+1)$$

Now the left-hand side of $P(k+1)$ is $\sum_{i=1}^{k} i(i+1)$

$$\begin{aligned}
&= \sum_{i=1}^{k-1} i(i+1) + k(k+1) && \text{make next-to-last term explicit} \\
&= \frac{k(k-1)(k+1)}{3} + k(k+1) && \text{by inductive hypothesis} \\
&= \frac{k(k-1)(k+1)}{3} + \frac{3k(k+1)}{3} && \text{because } \tfrac{3}{3} = 1 \\
&= \frac{k(k-1)(k+1) + 3k(k+1)}{3} && \text{by adding fractions} \\
&= \frac{k(k+1)[(k-1)+3]}{3} && \text{by factoring out } k(k+1) \\
&= \frac{k(k+1)(k+2)}{3} && \text{by algebra}
\end{aligned}$$

and this is the right-hand side of $P(k+1)$. Hence $P(k+1)$ is true. *[Since both the basis step and the inductive step have been proved, $P(n)$ is true for every integer $n \geq 2$.]* $\square$

2.14 Exercise 14

$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$, for every integer $n \geq 0$.

Proof. For the given statement, the property $P(n)$ is the equation

$$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2. \leftarrow P(n)$$

Show that $P(0)$ is true: The left-hand side of $P(0)$ is $\sum_{i=1}^{0+1} i \cdot 2^i = 1 \cdot 2^i = 2$, and the right-hand side is $0 \cdot 2^{0+2} + 2 = 0 + 2 = 2$ also. Thus $P(0)$ is true.

Show that for every integer $k \geq 0$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer with $k \geq 0$, and suppose $P(k)$ is true. That is, suppose

$$\sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{k+2} + 2. \leftarrow P(k)\text{: inductive hypothesis}$$

We must show that $P(k+1)$ is true. That is, we must show that

$$\sum_{i=1}^{(k+1)+1} i \cdot 2^i = (k+1) \cdot 2^{(k+1)+2} + 2,$$

or, equivalently,

$$\sum_{i=1}^{k+2} i \cdot 2^i = (k+1) \cdot 2^{k+3} + 2. \leftarrow P(k+1).$$

Now the left-hand side of $P(k+1)$ is $\sum_{i=1}^{k+2} i \cdot 2^i$

$$\begin{aligned}
&= \sum_{i=1}^{k+1} i \cdot 2^i + (k+2) \cdot 2^{k+2} && \text{make next-to-last term explicit} \\
&= k \cdot 2^{k+2} + 2 + (k+2) \cdot 2^{k+2} && \text{by inductive hypothesis} \\
&= (2k+2) \cdot 2^{k+2} + 2 && \text{by combining like terms} \\
&= 2(k+1) \cdot 2^{k+2} + 2 && \text{by factoring out 2} \\
&= (k+1) \cdot 2^{k+3} + 2 && \text{by laws of exponents}
\end{aligned}$$

and this is the right-hand side of $P(k+1)$. Hence $P(k+1)$ is true. *[Since both the basis step and the inductive step have been proved, $P(n)$ is true for every integer $n \geq 0$.]* □

2.15 Exercise 15

$\sum_{i=1}^{n} i(i!) = (n+1)! - 1$, for every integer $n \geq 1$.

Proof. For the given statement, the property $P(n)$ is the equation

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1. \leftarrow P(n)$$

Show that $P(1)$ is true: The left-hand side of $P(1)$ is $\displaystyle\sum_{i=1}^{1} i(i!) = 1(1!) = 1$, and the right-hand side is $(1+1)! - 1 = 2 - 1 = 1$ also. Thus $P(1)$ is true.

Show that for every integer $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer with $k \geq 1$, and suppose $P(k)$ is true. That is, suppose

$$\sum_{i=1}^{k} i(i!) = (k+1)! - 1. \leftarrow P(k)\text{: inductive hypothesis}$$

We must show that $P(k+1)$ is true. That is, we must show that

$$\sum_{i=1}^{k+1} i(i!) = (k+2)! - 1. \leftarrow P(k+1)$$

Now the left-hand side of $P(k+1)$ is $\sum_{i=1}^{k+1} i(i!)$

$$\begin{aligned}
&= \sum_{i=1}^{k} i(i!) + (k+1)(k+1)! && \text{make next-to-last term explicit} \\
&= (k+1)! - 1 + (k+1)(k+1)! && \text{by inductive hypothesis} \\
&= (k+1)![1 + (k+1)] - 1 && \text{by factoring out } (k+1)! \\
&= (k+1)!(k+2) - 1 && \text{by algebra} \\
&= (k+2)! - 1 && \text{by definition of } !
\end{aligned}$$

and this is the right-hand side of $P(k+1)$. Hence $P(k+1)$ is true. *[Since both the basis step and the inductive step have been proved, $P(n)$ is true for every integer $n \geq 1$.]* □

2.16 Exercise 16

$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$, for every integer $n \geq 2$.

Proof. For the given statement, the property $P(n)$ is the equation

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}. \leftarrow P(n)$$

Show that $P(2)$ is true: The left-hand side of $P(2)$ is $1 - \dfrac{1}{2^2} = 1 - \dfrac{1}{4} = \dfrac{3}{4}$, and the right-hand side is $\dfrac{2+1}{2 \cdot 2} = \dfrac{3}{4}$ also. Thus $P(2)$ is true.

Show that for every integer $k \geq 2$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer with $k \geq 2$, and suppose $P(k)$ is true. That is, suppose

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}. \leftarrow P(k)\text{: inductive hypothesis}$$

We must show that $P(k+1)$ is true. That is, we must show that

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)=\frac{(k+1)+1}{2(k+1)}. \leftarrow P(k+1)$$

Now the left-hand side of $P(k+1)$ is $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)$

$$\begin{aligned}
&= \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right) && \text{show next-to-last term}\\
&= \frac{k+1}{2k}\cdot\left(1-\frac{1}{(k+1)^2}\right) && \text{by inductive hypothesis}\\
&= \frac{k+1}{2k}\cdot\left(\frac{(k+1)^2}{(k+1)^2}-\frac{1}{(k+1)^2}\right) && \text{because } \tfrac{(k+1)^2}{(k+1)^2}=1\\
&= \frac{k+1}{2k}\cdot\frac{(k+1)^2-1}{(k+1)^2} && \text{by adding fractions}\\
&= \frac{k+1}{2k}\cdot\frac{k^2+2k+1-1}{(k+1)^2} && \text{by algebra}\\
&= \frac{k+1}{2k}\cdot\frac{k^2+2k}{(k+1)^2} && \text{by algebra}\\
&= \frac{k+1}{2k}\cdot\frac{k(k+2)}{(k+1)^2} && \text{by factoring out } k\\
&= \frac{k+1}{2}\cdot\frac{k+2}{(k+1)^2} && \text{by canceling out } k\\
&= \frac{1}{2}\cdot\frac{k+2}{k+1} && \text{by canceling out } k+1\\
&= \frac{k+2}{2(k+1)} && \text{by multiplying fractions}
\end{aligned}$$

and this is the right-hand side of $P(k+1)$. Hence $P(k+1)$ is true. *[Since both the basis step and the inductive step have been proved, $P(n)$ is true for every integer $n \geq 2$.]* □

2.17 Exercise 17

$$\prod_{i=0}^{n}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)=\frac{1}{(2n+2)!}, \text{ for every integer } n \geq 0.$$

Proof. For the given statement, the property $P(n)$ is the equation

$$\prod_{i=0}^{n}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)=\frac{1}{(2n+2)!}. \leftarrow P(n)$$

Show that $P(0)$ is true: The left-hand side of $P(0)$ is $\prod_{i=0}^{0}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right) = \frac{1}{2\cdot 0+1}\cdot\frac{1}{2\cdot 0+2} = \frac{1}{1}\cdot\frac{1}{2} = \frac{1}{2}$, and the right-hand side is $\frac{1}{(2\cdot 0+2)!} = \frac{1}{2}$ also. Thus $P(0)$ is true.

Show that for every integer $k \geq 0$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer with $k \geq 0$, and suppose $P(k)$ is true. That is, suppose

$$\prod_{i=0}^{k}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right) = \frac{1}{(2k+2)!}. \quad \leftarrow P(k)\text{: inductive hypothesis}$$

We must show that $P(k+1)$ is true. That is, we must show that

$$\prod_{i=0}^{k+1}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right) = \frac{1}{(2(k+1)+2)!}. \quad \leftarrow P(k+1)$$

Now the left-hand side of $P(k+1)$ is $\prod_{i=0}^{k+1}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)$

$$\begin{aligned}
&= \prod_{i=0}^{k}\left(\frac{1}{2i+1}\frac{1}{2i+2}\right)\left(\frac{1}{2(k+1)+1}\frac{1}{2(k+1)+2}\right) && \text{make next-to-last term explicit} \\
&= \frac{1}{(2k+2)!}\cdot\left(\frac{1}{2(k+1)+1}\cdot\frac{1}{2(k+1)+2}\right) && \text{by inductive hypothesis} \\
&= \frac{1}{(2k+2)!}\cdot\frac{1}{2k+3}\cdot\frac{1}{2k+4} && \text{by algebra} \\
&= \frac{1}{(2k+2)!\cdot(2k+3)\cdot(2k+4)} && \text{by multiplying fractions} \\
&= \frac{1}{(2k+4)!} && \text{by definition of factorial}
\end{aligned}$$

and this is the right-hand side of $P(k+1)$. Hence $P(k+1)$ is true. *[Since both the basis step and the inductive step have been proved, $P(n)$ is true for every integer $n \geq 0$.]* □

2.18 Exercise 18

$\prod_{i=2}^{n}\left(1-\frac{1}{i}\right) = \frac{1}{n}$, for every integer $n \geq 2$.

Hint: See the discussion at the beginning of this section.

Proof. For the given statement, the property $P(n)$ is the equation

$$\prod_{i=2}^{n}\left(1-\frac{1}{i}\right) = \frac{1}{n}. \quad \leftarrow P(n)$$

Show that $P(2)$ is true: The left-hand side of $P(2)$ is $\prod_{i=2}^{2}\left(1-\frac{1}{i}\right)=1-\frac{1}{2}=\frac{1}{2}$, and the right-hand side is $\frac{1}{2}$ also. Thus $P(2)$ is true.

Show that for every integer $k \geq 2$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 2$, and suppose $P(k)$ is true. That is, suppose

$$\prod_{i=2}^{k}\left(1-\frac{1}{i}\right)=\frac{1}{k}. \leftarrow P(k)\text{: inductive hypothesis}$$

We must show that $P(k+1)$ is true. That is, we must show that

$$\prod_{i=2}^{k+1}\left(1-\frac{1}{i}\right)=\frac{1}{k+1}. \leftarrow P(k+1)$$

Now the left-hand side of $P(k+1)$ is $\prod_{i=2}^{k+1}\left(1-\frac{1}{i}\right)$

$$\begin{aligned}
&= \prod_{i=2}^{k}\left(1-\frac{1}{i}\right)\cdot\left(1-\frac{1}{k+1}\right) && \text{make next-to-last term explicit}\\
&= \frac{1}{k}\cdot\left(1-\frac{1}{k+1}\right) && \text{by inductive hypothesis}\\
&= \frac{1}{k}\cdot\left(\frac{k+1}{k+1}-\frac{1}{k+1}\right) && \text{because } \tfrac{k+1}{k+1}=1\\
&= \frac{1}{k}\cdot\frac{k}{k+1} && \text{by adding fractions}\\
&= \frac{1}{k+1} && \text{by canceling } k
\end{aligned}$$

and this is the right-hand side of $P(k+1)$. Hence $P(k+1)$ is true. *[Since both the basis step and the inductive step have been proved, $P(n)$ is true for every integer $n \geq 2$.]* □

2.19 Exercise 19

(For students who have studied calculus) Use mathematical induction, the product rule from calculus, and the facts that $\frac{d(x)}{dx}=1$ and that $x^{k+1}=x\cdot x^k$ to prove that for every integer $n \geq 1$, $\frac{d(x^n)}{dx}=nx^{n-1}$.

Proof. For the given statement, the property $P(n)$ is the equation

$$\frac{d(x^n)}{dx}=nx^{n-1}. \leftarrow P(n)$$

Show that $P(1)$ is true: The left-hand side of $P(1)$ is $\dfrac{d(x^1)}{dx} = \dfrac{d(x)}{dx} = 1$, and the right-hand side is $1 \cdot x^{1-1} = 1$ also. Thus $P(1)$ is true.

Show that for every integer $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer with $k \geq 1$, and suppose $P(k)$ is true. That is, suppose

$$\frac{d(x^k)}{dx} = kx^{k-1}. \leftarrow P(k)\text{: inductive hypothesis}$$

We must show that $P(k+1)$ is true. That is, we must show that

$$\frac{d(x^{k+1})}{dx} = (k+1)x^{k+1-1}. \leftarrow P(k+1)$$

Now the left-hand side of $P(k+1)$ is $\dfrac{d(x^{k+1})}{dx}$

$$\begin{aligned}
&= \frac{d(x \cdot x^k)}{dx} && \text{because } x^{k+1} = x \cdot x^k \\
&= \frac{d(x)}{dx} \cdot x^k + x \cdot \frac{d(x^k)}{dx} && \text{by the product rule} \\
&= 1 \cdot x^k + x \cdot \frac{d(x^k)}{dx} && \text{because } \tfrac{d(x)}{dx} = 1 \\
&= x^k + x \cdot (kx^{k-1}) && \text{by inductive hypothesis} \\
&= x^k + kx^k && \text{by algebra} \\
&= (k+1)x^k && \text{by combining like terms}
\end{aligned}$$

and this is the right-hand side of $P(k+1)$. Hence $P(k+1)$ is true. *[Since both the basis step and the inductive step have been proved, $P(n)$ is true for every integer $n \geq 1$.]* □

Use the formula for the sum of the first n integers and/or the formula for the sum of a geometric sequence to evaluate the sums in $20 - 29$ or to write them in closed form.

2.20 Exercise 20

$4 + 8 + 12 + 16 + \cdots + 200$

Proof. $4 + 8 + 12 + 16 + \cdots + 200 = 4(1 + 2 + \cdots + 50) = 4 \cdot \dfrac{50 \cdot 51}{2} = 5100$ □

2.21 Exercise 21

$5 + 10 + 15 + 20 + \cdots + 300$

Proof. $5 + 10 + 15 + 20 + \cdots + 300 = 5(1 + 2 + \cdots + 60) = 5 \cdot \dfrac{60 \cdot 61}{2} = 9150$ □

2.22 Exercise 22

2.22.1 (a)

$3 + 4 + 5 + 6 + \cdots + 1000$

Proof. $3 + 4 + 5 + 6 + \cdots + 1000 = (1 + 2 + \cdots + 1000) - (1 + 2) = \dfrac{1000 \cdot 1001}{2} - 3 = 500500 - 3 = 500497$ □

2.22.2 (b)

$3 + 4 + 5 + 6 + \cdots + m$

Proof. $3 + 4 + 5 + 6 + \cdots + m = (1 + 2 + \cdots + m) - (1 + 2) = \dfrac{m(m+1)}{2} - 3$ □

2.23 Exercise 23

2.23.1 (a)

$7 + 8 + 9 + 10 + \cdots + 600$

Proof. $7 + 8 + 9 + 10 + \cdots + 600 = 1 + 2 + \cdots + 600 - (1 + 2 + 3 + 4 + 5 + 6) = \dfrac{600 \cdot 601}{2} - \dfrac{6 \cdot 7}{2} = 18300 - 21 = 18279$ □

2.23.2 (b)

$7 + 8 + 9 + 10 + \cdots + k$

Proof. $7+8+9+10+\cdots+k = 1+2+\cdots+k-(1+2+3+4+5+6) = \dfrac{k(k+1)}{2} - 21$ □

2.24 Exercise 24

$1 + 2 + 3 + \cdots + (k - 1)$, where k is any integer with $k \geq 2$.

Proof. $1 + 2 + 3 + \cdots + (k-1) = \dfrac{(k-1)(k-1+1)}{2} = \dfrac{(k-1)k}{2}$ □

2.25 Exercise 25

$1 + 2 + 2^2 + \cdots + 2^{25}$

2.25.1 (a)

Proof. $1 + 2 + 2^2 + \cdots + 2^{25} = \dfrac{2^{26} - 1}{2 - 1} = 2^{26} - 1$ □

2.25.2 (b)

$2 + 2^2 + 2^3 + \cdots + 2^{26}$

Proof. $2 + 2^2 + 2^3 + \cdots + 2^{26} = 2(1 + 2 + 2^2 + \cdots + 2^{25}) = 2(2^{26} - 1) = 2^{27} - 2$ □

2.25.3 (c)

$2 + 2^2 + 2^3 + \cdots + 2^n$

Proof. $2 + 2^2 + 2^3 + \cdots + 2^n = 2(1 + 2 + 2^2 + \cdots + 2^{n-1}) = 2(2^n - 1) = 2^{n+1} - 2$ □

2.26 Exercise 26

$3 + 3^2 + 3^3 + \cdots + 3^n$, where n is any integer with $n \geq 1$.

Proof. $3+3^2+3^3+\cdots+3^n = 1+3+3^2+3^3+\cdots+3^n-1 = \dfrac{3^{n+1}-1}{3-1}-1 = \dfrac{3^{n+1}-1}{2}-1$ □

2.27 Exercise 27

$5^3 + 5^4 + 5^5 + \cdots + 5^k$, where k is any integer with $k \geq 3$.

Proof. $5^3 + 5^4 + 5^5 + \cdots + 5^k = 1 + 5 + 5^2 + 5^3 + 5^4 + 5^5 + \cdots + 5^k - (1 + 5 + 5^2) =$
$(5^{k+1} - 1) - 31 = 5^{k+1} - 32$ □

2.28 Exercise 28

$1 + \dfrac{1}{2} + \dfrac{1}{2^2} + \cdots + \dfrac{1}{2^n}$, where n is any positive integer.

Proof. $1 + \dfrac{1}{2} + \dfrac{1}{2^2} + \cdots + \dfrac{1}{2^n} = \dfrac{\left(\frac{1}{2}\right)^{n+1} - 1}{\frac{1}{2} - 1} = -2\left[\left(\dfrac{1}{2}\right)^{n+1} - 1\right] = 2 - \dfrac{1}{2^n}$ □

2.29 Exercise 29

$1 - 2 + 2^2 - 2^3 + \cdots + (-1)^n 2^n$, where n is any positive integer.

Proof. $1-2+2^2-2^3+\cdots+(-1)^n 2^n = \dfrac{(-2)^{n+1}-1}{-2-1} = \dfrac{(-2)^{n+1}-1}{-3} = \dfrac{1-(-2)^{n+1}}{3}$ □

2.30 Exercise 30

Observe that $\dfrac{1}{1\cdot 3} = \dfrac{1}{3}$, $\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} = \dfrac{2}{5}$, $\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} = \dfrac{3}{7}$,
$\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} + \dfrac{1}{7\cdot 9} = \dfrac{4}{9}$
Guess a general formula and prove it by induction.

Proof. General formula: For every integer $n \geq 1$,

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

Proof by mathematical induction: For the given statement, the property $P(n)$ is the equation

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}. \quad \leftarrow P(n)$$

Show that $P(1)$ is true: The left-hand side of $P(1)$ is $\frac{1}{1 \cdot 3} = \frac{1}{3}$, and the right-hand side is $\frac{1}{2 \cdot 1 + 1} = \frac{1}{3}$ also. Thus $P(1)$ is true.

Show that for every integer $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer with $k \geq 1$, and suppose $P(k)$ is true. That is, suppose

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}. \quad \leftarrow P(k)\text{: inductive hypothesis}$$

We must show that $P(k+1)$ is true. That is, we must show that

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2(k+1)-1)(2(k+1)+1)} = \frac{k+1}{2(k+1)+1},$$

or, equivalently,

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}. \quad \leftarrow P(k+1)$$

Now the left-hand side of $P(k+1)$ is $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2k+1)(2k+3)}$

$$\begin{aligned}
&= \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} && \text{show next-to-last term} \\
&= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} && \text{by inductive hypothesis} \\
&= \frac{k(2k+3)}{(2k+1)(2k+3)} + \frac{1}{(2k+1)(2k+3)} && \text{because } \tfrac{2k+3}{2k+3} = 1 \\
&= \frac{2k^2+3k+1}{(2k+1)(2k+3)} && \text{by adding fractions} \\
&= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} && \text{by factoring} \\
&= \frac{k+1}{2k+3} && \text{by canceling } 2k+1
\end{aligned}$$

and this is the right-hand side of $P(k+1)$. Hence $P(k+1)$ is true. *[Since both the basis step and the inductive step have been proved, $P(n)$ is true for every integer $n \geq 1$.]* □

2.31 Exercise 31

Compute values of the product

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{n}\right)$$

for small values of n in order to conjecture a general formula for the product. Prove your conjecture by mathematical induction.

Proof. $\left(1+\frac{1}{1}\right)=2$, $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)=3$, $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)=4$.

General formula: For every integer $n \geq 1$,

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{n}\right)=n+1.$$

Proof by mathematical induction: For the given statement, the property $P(n)$ is the equation

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{n}\right)=n+1. \leftarrow P(n)$$

Show that $P(1)$ is true: The left-hand side of $P(1)$ is $1+\frac{1}{1}=2$, and the right-hand side is $1+1=2$ also. Thus $P(1)$ is true.

Show that for every integer $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer with $k \geq 1$, and suppose $P(k)$ is true. That is, suppose

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{k}\right)=k+1. \leftarrow P(k)\text{: inductive hypothesis}$$

We must show that $P(k+1)$ is true. That is, we must show that

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{k+1}\right)=k+2. \leftarrow P(k+1)$$

Now the left-hand side of $P(k+1)$ is $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{k+1}\right)$

$$\begin{aligned}
&= \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{k}\right)\left(1+\frac{1}{k+1}\right) && \text{show next-to-last term}\\
&= (k+1)\cdot\left(1+\frac{1}{k+1}\right) && \text{by inductive hypothesis}\\
&= (k+1)\cdot\left(\frac{k+1}{k+1}+\frac{1}{k+1}\right) && \text{because } \tfrac{k+1}{k+1}=1\\
&= (k+1)\cdot\frac{k+2}{k+1} && \text{by adding fractions}\\
&= k+2 && \text{by canceling } k+1
\end{aligned}$$

and this is the right-hand side of $P(k+1)$. Hence $P(k+1)$ is true. *[Since both the basis step and the inductive step have been proved, $P(n)$ is true for every integer $n \geq 1$.]* □

2.32 Exercise 32

Observe that

$$\begin{aligned} 1 &= 1 \\ 1-4 &= -(1+2) \\ 1-4+9 &= 1+2+3 \\ 1-4+9-16 &= -(1+2+3+4) \\ 1-4+9-16+25 &= 1+2+3+4+5 \end{aligned}$$

Guess a general formula and prove it by mathematical induction.

Proof. General formula: For every integer $n \geq 1$,

$$\sum_{i=1}^{n}(-1)^{i+1}i^2 = (-1)^{n+1}\sum_{j=1}^{n} j = (-1)^{n+1}\frac{n(n+1)}{2}.$$

Proof by mathematical induction: For the given statement, the property $P(n)$ is the equation

$$\sum_{i=1}^{n}(-1)^{i+1}i^2 = (-1)^{n+1}\frac{n(n+1)}{2}. \quad \leftarrow P(n)$$

Show that $P(1)$ is true: The left-hand side of $P(1)$ is $\sum_{i=1}^{1}(-1)^{i+1}i^2 = (-1)^2 1^2 = 1$, and the right-hand side is $(-1)^{1+1}\frac{1(1+1)}{2} = 1$ also. Thus $P(1)$ is true.

Show that for every integer $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true:

Let k be any integer with $k \geq 1$, and suppose $P(k)$ is true. That is, suppose

$$\sum_{i=1}^{k}(-1)^{i+1}i^2 = (-1)^{k+1}\frac{k(k+1)}{2}. \quad \leftarrow P(k)\text{: inductive hypothesis}$$

We must show that $P(k+1)$ is true. That is, we must show that

$$\sum_{i=1}^{k+1}(-1)^{i+1}i^2 = (-1)^{k+2}\frac{(k+1)(k+2)}{2}. \quad \leftarrow P(k+1)$$

Now the left-hand side of $P(k+1)$ is $\sum_{i=1}^{k+1}(-1)^{i+1}i^2$

$$\begin{aligned} &= \sum_{i=1}^{k}(-1)^{i+1}i^2 + (-1)^{k+2}(k+1)^2 && \text{make next-to-last term explicit} \\ &= (-1)^{k+1}\frac{k(k+1)}{2} + (-1)^{k+2}(k+1)^2 && \text{by inductive hypothesis} \\ &= (-1)^{k+1}\frac{k(k+1)}{2} + (-1)^{k+1}(-1)(k+1)^2 && \text{by factoring a power of } -1 \\ &= (-1)^{k+1}\left[\frac{k(k+1)}{2} - (k+1)^2\right] && \text{by factoring out } (-1)^{k+1} \end{aligned}$$

$$
\begin{aligned}
&= (-1)^{k+1}\left[\frac{k(k+1)}{2} - \frac{2(k+1)^2}{2}\right] && \text{because } \tfrac{2}{2} = 1 \\
&= (-1)^{k+1}\frac{k(k+1) - 2(k+1)^2}{2} && \text{by adding fractions} \\
&= (-1)^{k+1}\frac{k^2 + k - 2(k^2 + 2k + 1)}{2} && \text{by algebra} \\
&= (-1)^{k+1}\frac{k^2 + k - 2k^2 - 4k - 2}{2} && \text{by algebra} \\
&= (-1)^{k+1}\frac{-k^2 - 3k - 2}{2} && \text{by algebra} \\
&= (-1)^{k+2}\frac{k^2 + 3k + 2}{2} && \text{by factoring out } (-1) \\
&= (-1)^{k+2}\frac{(k+2)(k+1)}{2} && \text{by factoring}
\end{aligned}
$$

and this is the right-hand side of $P(k+1)$. Hence $P(k+1)$ is true. *[Since both the basis step and the inductive step have been proved, $P(n)$ is true for every integer $n \geq 1$.]* □

2.33 Exercise 33

Find a formula in n, a, m, and d for the sum $(a + md) + (a + (m+1)d) + (a + (m+2)d) + \cdots + (a + (m+n)d)$, where m and n are integers, $n \geq 0$, and a and d are real numbers. Justify your answer.

Proof. $(a + md) + (a + (m+1)d) + (a + (m+2)d) + \cdots + (a + (m+n)d)$

$$
\begin{aligned}
&= \sum_{i=1}^{n}(a + (m+i)d) && \text{by summation notation} \\
&= \sum_{i=1}^{n}(a + md + id) && \text{by multiplying} \\
&= \sum_{i=1}^{n}(a + md) + \sum_{i=1}^{n} id && \text{by splitting the sum} \\
&= (a + md)\sum_{i=1}^{n} 1 + d\sum_{i=1}^{n} i && \text{by moving out constants} \\
&= (a + md)n + d\sum_{i=1}^{n} i && \text{because } \textstyle\sum_{i=1}^{n} 1 = n \\
&= (a + md)n + d \cdot \frac{n(n+1)}{2} && \text{because } \textstyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}
\end{aligned}
$$

□

2.34 Exercise 34

Find a formula in a, r, m, and n for the sum $ar^m + ar^{m+1} + ar^{m+2} + \cdots + ar^{m+n}$ where m and n are integers, $n \geq 0$, and a and r are real numbers. Justify your answer.

Proof. $ar^m + ar^{m+1} + ar^{m+2} + \cdots + ar^{m+n} = ar^m(1 + r + r^2 + \cdots + r^n) = ar^m \cdot \dfrac{r^{n+1} - 1}{r - 1}$ □

2.35 Exercise 35

You have two parents, four grandparents, eight great-grandparents, and so forth.

2.35.1 (a)

If all your ancestors were distinct, what would be the total number of your ancestors for the past 40 generations (counting your parents' generation as number one)? (Hint: Use the formula for the sum of a geometric sequence.)

Proof. $2^1 + 2^2 + \cdots + 2^{40} = 2(1 + 2^1 + \cdots 2^{39}) = 2 \cdot \dfrac{2^{40} - 1}{2 - 1} = 2^{41} - 2$ □

2.35.2 (b)

Assuming that each generation represents 25 years, how long is 40 generations?

Proof. $25 \cdot 40 = 1000$ □

2.35.3 (c)

The total number of people who have ever lived is approximately 10 billion, which equals 10^{10} people. Compare this fact with the answer to part (a). What can you deduce?

Proof. $2^{41} - 2 = 2,199,023,255,550 > 10,000,000,000 = 10^{10}$ so many of my ancestors are not distinct. □

Find the mistakes in the proof fragments in $36 - 38$.

2.36 Exercise 36

Theorem: For any integer $n \geq 1$, $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

"Proof (by mathematical induction): Certainly the theorem is true for $n = 1$ because $1^2 = 1$ and $\dfrac{1(1+1)(2+1)}{6} = 1$. So the basis step is true. For the inductive step, suppose that k is any integer with $k \geq 1, k^2 = \dfrac{k(k+1)(2k+1)}{6}$. We must show that $(k+1)^2 = \dfrac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$."

Proof. In the inductive step, both the inductive hypothesis and what is to be shown are wrong. The inductive hypothesis should be:

Suppose that for some integer $k \geq 1$,

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

And what is to be shown should be:

$$1^2 + 2^2 + \cdots + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.$$

□

2.37 Exercise 37

Theorem: For any integer $n \geq 0$,

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1.$$

"Proof (by mathematical induction): Let the property $P(n)$ be

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1.$$

Show that $P(0)$ is true:
The left-hand side of $P(0)$ is $1 + 2 + 2^2 + \cdots + 2^0 = 1$ and the right-hand side is $2^{0+1} - 1 = 2 - 1 = 1$ also. So $P(0)$ is true."

Hint: See the Caution note in Section 5.1, page 262.

Proof. The left-hand side of $P(0)$ is wrong; it should be simply 1 instead of $1+2+2^2+\cdots+2^0$. □

2.38 Exercise 38

Theorem: For any integer $n \geq 1$,

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1.$$

"Proof (by mathematical induction): Let the property $P(n)$ be

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1.$$

Show that $P(1)$ is true: When $n = 1$,

$$\sum_{i=1}^{1} i(i!) = (1+1)! - 1.$$

So $1(1!) = 2! - 1$, and $1 = 1$. Thus $P(1)$ is true.

Hint: See the subsection Proving an Equality on page 284 in Section 5.2.

Proof. For $P(1)$ this proof is assuming what is to be shown. The equality $\sum_{i=1}^{1} i(i!) = (1+1)! - 1$ is $P(1)$ which is what we need to show, but the proof assumes that this is true, then does operations on both sides to reach a true conclusion $1 = 1$. This proves that if $P(1)$ is true, then $1 = 1$ is true, but that does not prove $P(1)$ is true. □

2.39 Exercise 39

Use Theorem 5.2.1 to prove that if m and n are any positive integers and m is odd, then $\sum_{k=0}^{m-1}(n+k)$ is divisible by m. Does the conclusion hold if m is even? Justify your answer.

Proof. $\sum_{k=0}^{m-1}(n+k) = \sum_{k=0}^{m-1} n + \sum_{k=0}^{m-1} k = n\sum_{k=0}^{m-1} 1 + (0 + \sum_{k=1}^{m-1} k) = nm + \frac{(m-1)(m-1+1)}{2}$ $= nm + \frac{(m-1)m}{2} = m(n+(m-1)/2)$ which is divisible by m if and only if $n+(m-1)/2$ is an integer, if and only if $m-1$ is even, if and only if m is odd. So the statement is true when m is odd and false when m is even. □

2.40 Exercise 40

Use Theorem 5.2.1 and the result of Exercise 10 to prove that if p is any prime number with $p \geq 5$, then the sum of the squares of any p consecutive integers is divisible by p.

Proof. Assume p is any prime number with $p \geq 5$. Assume n is any integer and consider the p consecutive integers $n, n+1, \ldots, n+p-1$. *[We want to show $n^2 + (n+1)^2 + \cdots + (n+p-1)^2$ is divisible by p.]* Then

$n^2 + (n+1)^2 + (n+2)^2 + \cdots + (n+p-1)^2$

$$\begin{aligned}
&= n^2 + (n^2 + 2n + 1) + (n^2 + 2n \cdot 2 + 2^2) + \cdots + (n^2 + 2n(p-1) + (p-1)^2) \\
&= pn^2 + 2n(1 + 2 + \cdots + (p-1)) + (1^2 + 2^2 + \cdots + (p-1)^2) \\
&= pn^2 + 2n \cdot \frac{(p-1)(p-1+1)}{2} + \frac{(p-1)(p-1+1)(2(p-1)+1)}{6} \\
&= pn^2 + n(p-1)p + \frac{(p-1)p(2p-1)}{6}
\end{aligned}$$

We know that the sum of squares formula gives us an integer, so $\frac{(p-1)p(2p-1)}{6}$ is an integer. Since $p \geq 5$ is a prime, it is not divisible by 6, therefore $\frac{(p-1)(2p-1)}{6}$ is an integer. So

$$\begin{aligned}
&= pn^2 + n(p-1)p + p\frac{(p-1)(2p-1)}{6} \\
&= p\left[n^2 + n(p-1) + \frac{(p-1)(2p-1)}{6}\right]
\end{aligned}$$

is divisible by p. $\square$