Chapter 1 Solutions, Susanna Epp Discrete Math 5th Edition

https://github.com/spamegg1

April 25, 2023

Contents

1 Exercise Set 1.1

In each of 1–6, fill in the blanks using a variable or variables to rewrite the given statement.

1.1 Problem 1

Is there a real number whose square is -1?

1.1.1 (a)

Is there a real number x such that ____ ?

Proof. Is there a real number x such that $\underline{x^2 = -1}$? □

1.1.2 (b)

Does there exist ____ such that $x^2 = -1$?

Proof. Does there exist <u>a real number x</u> such that $x^2 = -1$? □

1.2 Problem 2

Is there an integer that has a remainder of 2 when it is divided by 5 and a remainder of 3 when it is divided by 6?

Note: There are integers with this property. Can you think of one?

1.2.1 (a)

Is there an integer n such that n has ____ ?

Proof. Is there an integer n such that n has <u>a remainder of 2 when it is divided by 5 and a remainder of 3 when it is divided by 6</u>? □

1.2.2 (b)

Does there exist ____ such that if n is divided by 5 the remainder is 2 and if ____ ?

Proof. Does there exist <u>an integer n</u> such that if n is divided by 5 the remainder is 2 and if <u>n is divided by 6 the remainder is 3</u>? □

1.3 Problem 3

Given any two distinct real numbers, there is a real number in between them.

1.3.1 (a)

Given any two distinct real numbers a and b, there is a real number c such that c is ____ .

Proof. Given any two distinct real numbers a and b, there is a real number c such that c is <u>between a and b</u>. □

1.3.2 (b)

For any two ____ , ____ such that c is between a and b.

Proof. For any two distinct real numbers a and b, there exists a real number c such that c is between a and b. □

1.4 Problem 4

Given any real number, there is a real number that is greater.

1.4.1 (a)

Given any real number r, there is ____ s such that s is ____ .

Proof. Given any real number r, there is a real number s such that s is greater than r. □

1.4.2 (b)

For any ____ , ____ such that $s > r$.

Proof. For any real number r, there exists a real number s such that $s > r$. □

1.5 Problem 5

The reciprocal of any positive real number is positive.

1.5.1 (a)

Given any positive real number r, the reciprocal of ____ .

Proof. Given any positive real number r, the reciprocal of r is positive. □

1.5.2 (b)

For any real number r, if r is ____ , then ____ .

Proof. For any real number r, if r is positive, then $1/r$ is positive. □

1.5.3 (c)

If a real number r ____ , then ____ .

Proof. If a real number r is positive, then $1/r$ is positive. □

1.6 Problem 6

The cube root of any negative real number is negative.

1.6.1 (a)

Given any negative real number s, the cube root of ____ .

Proof. Given any negative real number s, the cube root of $\underline{s \text{ is negative}}$. $\square$

1.6.2 (b)

For any real number s, if s is ____ , then ____ .

Proof. For any real number s, if s is $\underline{\text{negative}}$, then $\underline{\sqrt[3]{s} \text{ is negative}}$. $\square$

1.6.3 (c)

If a real number s ____ , then ____ .

Proof. If a real number s $\underline{\text{is negative}}$, then $\underline{\sqrt[3]{s} \text{ is negative}}$. $\square$

1.7 Problem 7

Rewrite the following statements less formally, without using variables. Determine, as best as you can, whether the statements are true or false.

1.7.1 (a)

There are real numbers u and v with the property that $u + v < u - v$.

Proof. Rewrite: There are real numbers such that their sum is less than their difference.

True: 0 and -1 have this property: $-1 = 0 + (-1) < 0 - (-1) = 1$ $\square$

1.7.2 (b)

There is a real number x such that $x^2 < x$.

Proof. Rewrite: there is a real number whose square is less than itself.

True: 1/2 has this property: $\frac{1}{4} = \left(\frac{1}{2}\right)^2 < \frac{1}{2}$ $\square$

1.7.3 (c)

For every positive integer n, $n^2 \geq n$.

Proof. Rewrite: The square of every positive integer is greater than or equal to itself.

True: if we look at the first few examples it holds: $1^2 = 1 \geq 1$, $2^2 = 4 \geq 2$, $3^2 = 9 \geq 3$ and so on. This is however not a proof. Later we'll learn methods to prove this for all positive integers. □

1.7.4 (d)

For all real numbers a and b, $|a + b| \leq |a| + |b|$.

Proof. Rewrite: for all two real numbers, the absolute value of their sum is less than or equal to the sum of their absolute values.

True: this is known as the Triangle Inequality and it will be proved later. □

In each of 8-13, fill in the blanks to rewrite the given statement.

1.8 Problem 8

For every object J, if J is a square then J has four sides.

1.8.1 (a)

All squares ____ .

Proof. All squares <u>have four sides</u>. □

1.8.2 (b)

Every square ____ .

Proof. Every square <u>has four sides</u>. □

1.8.3 (c)

If an object is a square, then it ____ .

Proof. If an object is a square, then it <u>has four sides</u>. □

1.8.4 (d)

If J ____ , then J ____ .

Proof. If J <u>is a square</u>, then J <u>has four sides</u>. □

1.8.5 (e)

For every square J, ____ .

Proof. For every square J, <u>J has four sides</u>. ☐

1.9 Problem 9

For every equation E, if E is quadratic then E has at most two real solutions.

1.9.1 (a)

All quadratic equations ____ .

Proof. All quadratic equations <u>have at most two real solutions</u>. ☐

1.9.2 (b)

Every quadratic equation ____ .

Proof. Every quadratic equation <u>has at most two real solutions</u>. ☐

1.9.3 (c)

If an equation is quadratic, then it ____ .

Proof. If an equation is quadratic, then it <u>has at most two real solutions</u>. ☐

1.9.4 (d)

If E ____ , then E ____ .

Proof. If E <u>is a quadratic equation</u>, then E <u>has at most two real solutions</u>. ☐

1.9.5 (e)

For every quadratic equation E, ____ .

Proof. For every quadratic equation E, <u>E has at most two real solutions</u>. ☐

1.10 Problem 10

Every nonzero real number has a reciprocal.

1.10.1 (a)

All nonzero real numbers ____ .

Proof. All nonzero real numbers <u>have reciprocals</u>. ☐

1.10.2 (b)

For every nonzero real number r, there is ____ for r.

Proof. For every nonzero real number r, there is a reciprocal for r. □

1.10.3 (c)

For every nonzero real number r, there is a real number s such that ____ .

Proof. For every nonzero real number r, there is a real number s such that $\underline{r = 1/s}$. □

1.11 Problem 11

Every positive number has a positive square root.

1.11.1 (a)

All positive numbers ____ .

Proof. All positive numbers have a positive square root. □

1.11.2 (b)

For every positive number e, there is ____ for e.

Proof. For every positive number e, there is a positive square root for e. □

1.11.3 (c)

For every positive number e, there is a positive number r such that ____ .

Proof. For every positive number e, there is a positive number r such that $\underline{r^2 = e}$. □

1.12 Problem 12

There is a real number whose product with every number leaves the number unchanged.

1.12.1 (a)

Some ____ has the property that its ____ .

Proof. Some real number has the property that its product with every number leaves the number unchanged. □

1.12.2 (b)

There is a real number r such that the product of r ____ .

Proof. There is a real number r such that the product of r <u>with every number leaves the number unchanged</u>. $\square$

1.12.3 (c)

There is a real number r with the property that for every real number s, ____ .

Proof. There is a real number r with the property that for every real number s, $\underline{r \cdot s = s}$. $\square$

1.13 Problem 13

There is a real number whose product with every real number equals zero.

1.13.1 (a)

Some ____ has the property that its ____ .

Proof. Some <u>real number</u> has the property that its <u>product with every real number is zero</u>. $\square$

1.13.2 (b)

There is a real number a such that the product of a ____ .

Proof. There is a real number a such that the product of a <u>with every real number is zero</u>. $\square$

1.13.3 (c)

There is a real number a with the property that for every real number b, ____ .

Proof. There is a real number a with the property that for every real number b, $\underline{a \cdot b = 0}$. $\square$

2 Exercise Set 1.2

2.1 Problem 1

Which of the following sets are equal?

$A = \{a, b, c, d\}$ $B = \{d, e, a, c\}$ $C = \{d, b, a, c\}$ $D = \{a, a, d, e, c, e\}$

Proof. $A = C$ because they have the same 4 elements a, b, c, d written in different orders.

$B = D$ because, after we remove the repetitions of a and e from D, they both have the same 4 elements a, c, d, e in different orders.

No other two sets are equal. □

2.2 Problem 2

Write in words how to read each of the following out loud.

2.2.1 (a)

$\{x \in \mathbb{R}^+ \mid 0 < x < 1\}$

Proof. The set of all positive real numbers x such that 0 is less than x and x is less than 1. □

2.2.2 (b)

$\{x \in \mathbb{R} \mid x \leq 0 \text{ or } x \geq 1\}$

Proof. The set of all reals x such that x is less than or equal to 0 or x is greater than or equal to 1. □

2.2.3 (c)

$\{n \in \mathbb{Z} \mid n \text{ is a factor of } 6\}$

Proof. The set of all integers n such that n is a factor of 6. □

2.2.4 (d)

$\{n \in \mathbb{Z}^+ \mid n \text{ is a factor of } 6\}$

Proof. The set of all positive integers n such that n is a factor of 6. □

2.3 Problem 3

2.3.1 (a)

Is $4 = \{4\}$?

Proof. No, $\{4\}$ is a set with one element, namely 4, whereas 4 is just a symbol that represents the number 4. □

2.3.2 (b)

How many elements are in the set $\{3, 4, 3, 5\}$?

Proof. Three: the elements of the set are 3, 4, and 5. $\square$

2.3.3 (c)

Proof. Three: the elements are the symbol 1, the set $\{1\}$, and the set $\{1, \{1\}\}$. $\square$

2.4 Problem 4

2.4.1 (a)

Is $2 \in \{2\}$?

Proof. Yes. $\square$

2.4.2 (b)

How many elements are in the set $\{2, 2, 2, 2\}$?

Proof. One. The only element of the set is 2. $\square$

2.4.3 (c)

How many elements are in the set $\{0, \{0\}\}$?

Proof. Two: the elements are the number 0 and the set $\{0\}$. $\square$

2.4.4 (d)

Is $\{0\} \in \{\{0\}, \{1\}\}$?

Proof. Yes. The elements of the second set are $\{0\}$ and $\{1\}$. $\square$

2.4.5 (e)

Is $0 \in \{\{0\}, \{1\}\}$?

Proof. No. The elements of the second set are $\{0\}$ and $\{1\}$, so 0 is not an element of the second set (because $0 \neq \{0\}$). $\square$

2.5 Problem 5

Which of the following sets are equal?

$$\begin{aligned} A &= \{0, 1, 2\} \\ B &= \{x \in \mathbb{R} \mid -1 \leq x < 3\} \\ C &= \{x \in \mathbb{R} \mid -1 < x < 3\} \\ D &= \{x \in \mathbb{Z} \mid -1 < x < 3\} \\ E &= \{x \in \mathbb{Z}^+ \mid -1 < x < 3\} \end{aligned}$$

Proof. B or C are not equal to any one of A, D, E because B and D contain infinitely many real numbers, whereas A, D, E contain only finitely many integers.

$B \neq C$ because $-1 \in B$ but $-1 \notin C$.

$A = D$ because D consists of integers strictly between -1 and 3, namely 0, 1, 2.

No other two sets are equal, because E does not contain 0. $\square$

2.6 Problem 6

For each integer n, let $T_n = \{n, n^2\}$. How many elements are in each of T_2, T_{-3}, T_1, and T_0? Justify your answers.

Proof. $T_2 = \{2, 2^2\} = \{2, 4\}$ has two elements.

$T_{-3} = \{-3, (-3)^2\} = \{-3, 9\}$ has two elements.

$T_1 = \{1, 1^2\} = \{1, 1\} = \{1\}$ has one element.

$T_0 = \{0, 0^2\} = \{0, 0\} = \{0\}$ has one element. $\square$

2.7 Problem 7

Use set-roster notation to indicate the elements in each of the following sets.

2.7.1 (a)

$S = \{n \in \mathbb{Z} \mid n = (-1)^k, \text{ for some integer } k\}$.

Proof. Here $S = \{-1, 1\}$ because for all integers k, $(-1)^k$ is either -1 or 1. $\square$

2.7.2 (b)

$T = \{m \in \mathbb{Z} \mid m = 1 + (-1)^i, \text{ for some integer } i\}$.

Proof. Similar to part (a), we have $T = \{0, 2\}$. $\square$

2.7.3 (c)

$U = \{r \in \mathbb{Z} \mid 2 \leq r \leq -2\}$

Proof. Here U is the empty set because there are no integers r such that $2 \leq r$ and $r \leq -2$ at the same time. □

2.7.4 (d)

$V = \{s \in \mathbb{Z} \mid s > 2 \text{ or } s < 3\}$

Proof. Here $V = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Because all integers satisfy the given property (of being either greater than 2 or less than 3). □

2.7.5 (e)

$W = \{t \in \mathbb{Z} \mid 1 < t < -3\}$

Proof. Similar to part (c), $W = \emptyset$. □

2.7.6 (f)

$X = \{u \in \mathbb{Z} \mid u \leq 4 \text{ or } u \geq 1\}$

Proof. Similar to part (d) here $X = \mathbb{Z}$. □

2.8 Problem 8

Let $A = \{c, d, f, g\}$, $B = \{f, j\}$, and $C = \{d, g\}$. Answer each of the following questions. Give reasons for your answers.

2.8.1 (a)

Is $B \subseteq A$?

Proof. No, because $j \in B$ but $j \notin A$. □

2.8.2 (b)

Is $C \subseteq A$?

Proof. Yes. Both elements of C are also elements of A. □

2.8.3 (c)

Is $C \subseteq C$?

Proof. Yes. Every element of C is an element of C. □

2.8.4 (d)

Is C a proper subset of A?

Proof. Yes. By part (b) we have $C \subseteq A$, and we also have $C \neq A$ because $c \in A$ and $c \notin C$. Therefore $C \subset A$. □

2.9 Problem 9

2.9.1 (a)

Is $3 \in \{1, 2, 3\}$?

Proof. Yes. □

2.9.2 (b)

Is $1 \subseteq \{1\}$?

Proof. No, the number 1 is not a set and so it cannot be a subset. □

2.9.3 (c)

Is $\{2\} \in \{1, 2\}$?

Proof. No. $\{2\} \subset \{1, 2\}$ and $2 \in \{1, 2\}$ but $\{2\} \notin \{1, 2\}$. □

2.9.4 (d)

Is $\{3\} \in \{1, \{2\}, \{3\}\}$?

Proof. Yes. □

2.9.5 (e)

Is $1 \in \{1\}$?

Proof. Yes. □

2.9.6 (f)

Is $\{2\} \subseteq \{1, \{2\}, \{3\}\}$?

Proof. No, the only element in $\{2\}$ is the number 2 and the number 2 is not one of the three elements in $\{1, \{2\}, \{3\}\}$. □

2.9.7 (g)

Is $\{1\} \subseteq \{1, 2\}$?

Proof. Yes. Every element of $\{1\}$, namely 1, is also an element of $\{1, 2\}$. $\square$

2.9.8 (h)

Is $1 \in \{\{1\}, 2\}$?

Proof. No. $1 \neq \{1\}$ and $1 \neq 2$. $\square$

2.9.9 (i)

Is $\{1\} \subseteq \{1, \{2\}\}$?

Proof. Yes, the only element in $\{1\}$ is the number 1, which is an element in $\{1, \{2\}\}$. $\square$

2.9.10 (j)

Is $\{1\} \subseteq \{1\}$?

Proof. Yes. Every element of $\{1\}$ is also an element of $\{1\}$. $\square$

2.10 Problem 10

2.10.1 (a)

Is $((-2)^2, -2^2) = (-2^2, (-2)^2)$?

Proof. No, because $(-2)^2 = 4$ and $-2^2 = -4$, therefore $(4, -4) \neq (-4, 4)$. $\square$

2.10.2 (b)

Is $(5, -5) = (-5, 5)$?

Proof. No, because $5 \neq -5$, and these are ordered tuples, so order matters. $\square$

2.10.3 (c)

Is $(8 - 9, \sqrt[3]{-1}) = (-1, -1)$?

Proof. Yes, because $8 - 9 = -1$ and $\sqrt[3]{-1} = -1$. So they are both $(-1, -1)$. $\square$

2.10.4 (d)

Is $(\frac{-2}{-4}, (-2)^3) = (\frac{3}{6}, -8)$?

Proof. Yes, because both $\frac{-2}{-4}$ and $\frac{3}{6}$ are equal to $1/2$, and $(-2)^3 = -8$. $\square$

2.11 Problem 11

Let $A = \{w, x, y, z\}$ and $B = \{a, b\}$. Use the set-roster notation to write each of the following sets, and indicate the number of elements that are in each set.

2.11.1 (a)

$A \times B$

Proof. $A \times B = \{(w, a), (w, b), (x, a), (x, b), (y, a), (y, b), (z, a), (z, b)\}$

8 elements. $\square$

2.11.2 (b)

$B \times A$

Proof. $B \times A = \{(a, w), (b, w), (a, x), (b, x), (a, y), (b, y), (a, z), (b, z)\}$

8 elements. $\square$

2.11.3 (c)

$A \times A$

Proof. 16 elements:

$A \times A = \{(w, w), (w, x), (w.y), (w, z), (x, w), (x, x), (x, y), (x, z),$

$(y, w), (y, x), (y, y), (y, z), (z, w), (z, x), (z, y), (z, z)\}$ $\square$

2.11.4 (d)

$B \times B$

Proof. 4 elements: $B \times B = \{(a, a), (a, b), (b, a), (b, b)\}$ $\square$

2.12 Problem 12

Let $S = \{2, 4, 6\}$ and $T = \{1, 3, 5\}$. Use the set-roster notation to write each of the following sets, and indicate the number of elements that are in each set.

2.12.1 (a)

$S \times T$

Proof. $S \times T = \{(2,1),(2,3),(2,5),(4,1),(4,3),(4,5),(6,1),(6,3),(6,5)\}$
9 elements. □

2.12.2 (b)

$T \times S$

Proof. $T \times S = \{(1,2),(3,2),(5,2),(1,4),(3,4),(5,4),(1,6),(3,6),(5,6)\}$
9 elements. □

2.12.3 (c)

$S \times S$

Proof. $S \times S = \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}$
9 elements. □

2.12.4 (d)

$T \times T$

Proof. $T \times T = \{(1,1),(3,1),(5,1),(1,3),(3,3),(5,3),(1,5),(3,5),(5,5)\}$
9 elements. □

2.13 Problem 13

Let $A = \{1,2,3\}$, $B = \{u\}$, and $C = \{m,n\}$. Find each of the following sets.

2.13.1 (a)

$A \times (B \times C)$

Proof. $\{(1,(u,m)),(1,(u,n)),(2,(u,m)),(2,(u,n)),(3,(u,m)),(3,(u,n))\}$ □

2.13.2 (b)

$(A \times B) \times C$

Proof. $\{((1,u),m),((1,u),n),((2,u),m),((2,u),n),((3,u),m),((3,u),n)\}$ □

2.13.3 (c)

$A \times B \times C$

Proof. $\{(1,u,m),(1,u,n),(2,u,m),(2,u,n),(3,u,m),(3,u,n)\}$ □

2.14 Problem 14

Let $R = \{a\}$, $S = \{x, y\}$, and $T = \{p, q, r\}$. Find each of the following sets.

2.14.1 (a)

$R \times (S \times T)$

Proof. $\{(a,(x,p)),(a,(x,q)),(a,(x,r)),(a,(y,p)),(a,(y,q)),(a,(y,r))\}$ □

2.14.2 (b)

$(R \times S) \times T$

Proof. $\{((a,x),p),((a,x),q),((a,x),r),((a,y),p),((a,y),q),((a,y),r)\}$ □

2.14.3 (c)

$R \times S \times T$

Proof. $\{(a,x,p),(a,x,q),(a,x,r),(a,y,p),(a,y,q),(a,y,r)\}$ □

2.15 Problem 15

Let $S = \{0, 1\}$. List all the strings of length 4 over S that contain three or more 0's.

Proof. $0000, 0001, 0010, 0100, 1000$ □

2.16 Problem 16

Let $T = \{x, y\}$. List all the strings of length 5 over T that have exactly one y.

Proof. $xxxxy, xxxyx, xxyxx, xyxxx, yxxxx$ □

3 Exercise Set 1.3

3.1 Problem 1

Let $A = \{2, 3, 4\}$ and $B = \{6, 8, 10\}$ and define a relation R from A to B as follows: For every $(x, y) \in A \times B$,

$$(x, y) \in R \text{ means that } \frac{y}{x} \text{ is an integer.}$$

3.1.1 (a)

Is $4\ R\ 6$? Is $4\ R\ 8$? Is $(3, 8) \in R$? Is $(2, 10) \in R$?

Proof. $4 \not R\ 6$ because $6/4$ is not an integer.

$4\ R\ 8$ because $8/4 = 2$ is an integer.

$(3, 8) \notin R$ because $8/3$ is not an integer.

$(2, 10) \in R$ because $10/2 = 5$ is an integer. □

3.1.2 (b)

Write R as a set of ordered pairs.

Proof. $R = \{(2, 6), (2, 8), (2, 10), (3, 6), (4, 8)\}$ □

3.1.3 (c)

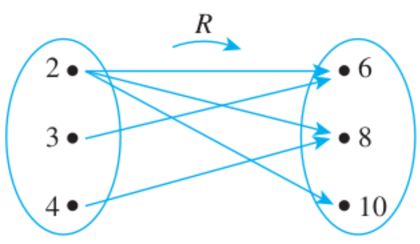
Write the domain and co-domain of R.

Proof. Domain of $R = A = \{2, 3, 4\}$

Co-domain of $R = B = \{6, 8, 10\}$ □

3.1.4 (d)

Draw an arrow diagram for R.



Proof. □

3.2 Problem 2

Let $C = D = \{-3, -2, -1, 1, 2, 3\}$ and define a relation S from C to D as follows: For every $(x, y) \in C \times D$,

$$(x, y) \in S \text{ means that } \frac{1}{x} - \frac{1}{y} \text{ is an integer.}$$

3.2.1 (a)

Is 2 S 2? Is -1 S -1? Is $(3,3) \in S$? Is $(3,-3) \in S$?

Proof. 2 S 2 is true, because $\dfrac{1}{2} - \dfrac{1}{2} = 0$ is an integer.

-1 S -1 is true, because $\dfrac{1}{-1} - \dfrac{1}{-1} = 0$ is an integer.

$(3,3) \in S$ because $\dfrac{1}{3} - \dfrac{1}{3} = 0$ is an integer.

$(3,-3) \notin S$ because $\dfrac{1}{3} - \dfrac{1}{-3} = \dfrac{2}{3}$ is not an integer. □

3.2.2 (b)

Write S as a set of ordered pairs.

Proof. $S = \{(-3,-3), (-2,-2), (-1,-1), (1,1), (2,2), (3,3), (-2,2),$
$(2,-2), (-1,1), (1,-1)\}$ □

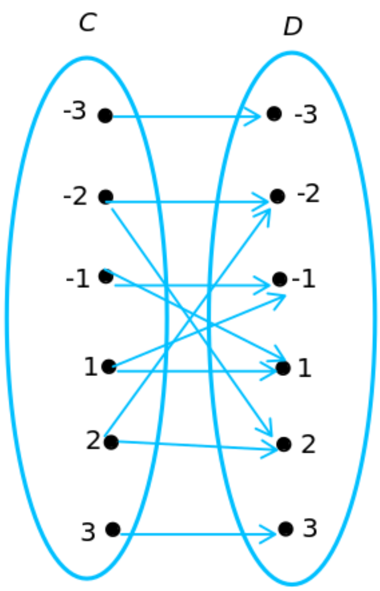
3.2.3 (c)

Write the domain and co-domain of S.

Proof. Domain of $S = C = \{-3,-2,-1,1,2,3\}$
Co-domain of $S = D = \{-3,-2,-1,1,2,3\}$ □

3.2.4 (d)

Draw an arrow diagram for S.



Proof. □

3.3 Problem 3

Let $E = \{1, 2, 3\}$ and $F = \{-2, -1, 0\}$ and define a relation T from E to F as follows: For every $(x, y) \in E \times F$,

$$(x, y) \in T \text{ means that } \frac{x-y}{3} \text{ is an integer.}$$

3.3.1 (a)

Is 3 T 0? Is 1 T (-1)? Is $(2, -1) \in T$? Is $(3, -2) \in T$?

Proof. 3 T 0 because $\dfrac{3-0}{3} = 1$ is an integer.

1 $\not{T}$ (-1) because $\dfrac{1-(-1)}{3} = 2/3$ is not an integer.

$(2, -1) \in T$ because $\dfrac{2-(-1)}{3} = 1$ is an integer.

$(3, -2) \notin T$ because $\dfrac{3-(-2)}{3} = 5/3$ is not an integer. □

3.3.2 (b)

Write T as a set of ordered pairs.

Proof. $T = \{(1, -2), (2, -1), (3, 0)\}$ □

3.3.3 (c)

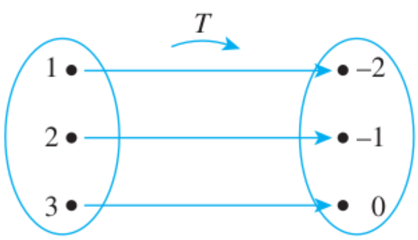
Write the domain and co-domain of T.

Proof. Domain of $T = E = \{1, 2, 3\}$

Co-domain of $T = F = \{-2, -1, 0\}$ □

3.3.4 (d)

Draw an arrow diagram for T.



Proof. □

3.4 Problem 4

Let $G = \{-2, 0, 2\}$ and $H = \{4, 6, 8\}$ and define a relation V from G to H as follows: For every $(x, y) \in G \times H$,

$$(x, y) \in V \text{ means that } \frac{x-y}{4} \text{ is an integer.}$$

3.4.1 (a)

Is $2\ V\ 6$? Is $(-2)\ V\ 8$? Is $(0, 6) \in V$? Is $(2, 4) \in V$?

Proof. $2\ V\ 6$ because $\dfrac{2-6}{4} = -1$ is an integer.

$(-2) \not V\ 8$ because $\dfrac{-2-8}{4} = -5/2$ is not an integer.

$(0, 6) \notin V$ because $\dfrac{0-6}{4} = -3/2$ is an not integer.

$(2, 4) \notin V$ because $\dfrac{2-4}{4} = -1/2$ is not an integer.

□

3.4.2 (b)

Write V as a set of ordered pairs.

Proof. $V = \{(-2, 6), (0, 4), (0, 8), (2, 6)\}$ □

3.4.3 (c)

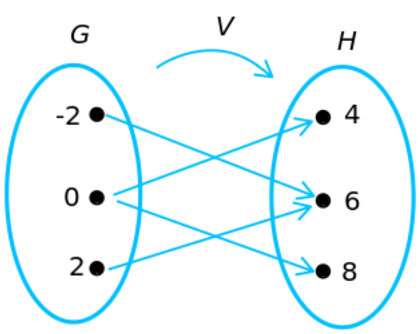
Write the domain and co-domain of V.

Proof. Domain of $V = G = \{-2, 0, 2\}$

Co-domain of $V = H = \{4, 6, 8\}$ □

3.4.4 (d)

Draw an arrow diagram for V.



Proof. $\square$

3.5 Problem 5

Define a relation S from $\mathbb{R}$ to $\mathbb{R}$ as follows: For every $(x, y) \in \mathbb{R} \times \mathbb{R}$,

$$(x, y) \in S \text{ means that } x \geq y.$$

3.5.1 (a)

Is $(2, 1) \in S$? Is $(2, 2) \in S$? Is $2\ S\ 3$? Is $(-1)\ S\ (-2)$?

Proof. $(2, 1) \in S$ because $2 \geq 1$.

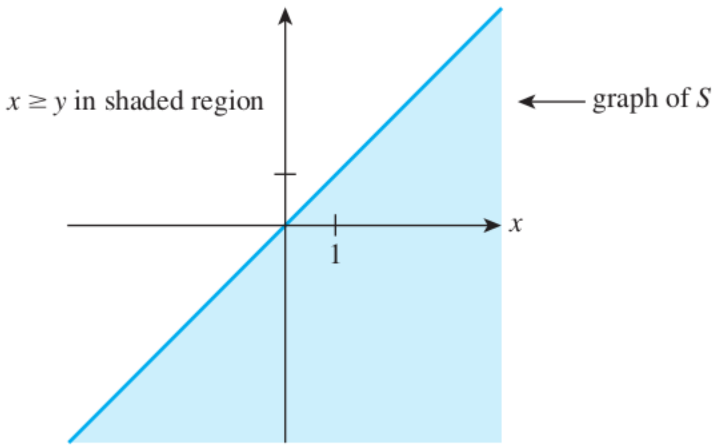
$(2, 2) \in S$ because $2 \geq 2$.

$2 \not{V}\ 3$ because $2 \not\geq 3$.

$(-1)\ V\ (-2)$ because $-1 \geq -2$. $\square$

3.5.2 (b)

Draw the graph of S in the Cartesian plane.



Proof. $\square$

3.6 Problem 6

Define a relation R from $\mathbb{R}$ to $\mathbb{R}$ as follows: For every $(x, y) \in \mathbb{R} \times \mathbb{R}$,

$$(x, y) \in R \text{ means that } y = x^2.$$

3.6.1 (a)

Is $(2, 4) \in R$? Is $(4, 2) \in R$? Is $(-3)\ R\ 9$? Is $9\ R\ (-3)$?

Proof. $(2,4) \in R$ because $4 = 2^2$.

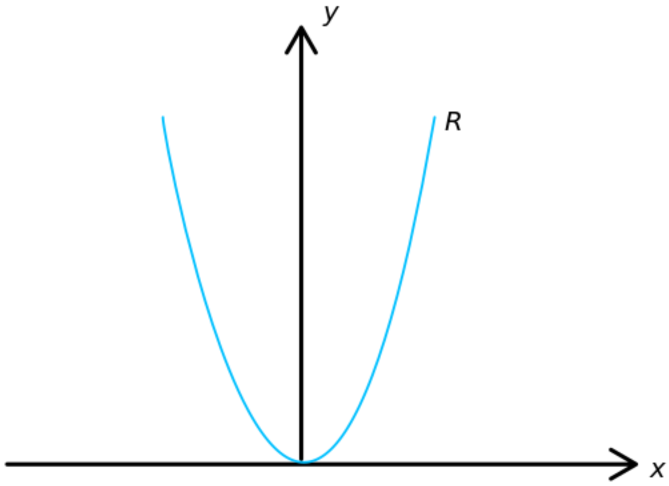
$(4,2) \notin R$ because $2 \neq 4^2$.

$(-3)\ R\ 9$ because $9 = (-3)^2$.

$9 \not R\ (-3)$ because $-3 \neq 9^2$. □

3.6.2 (b)

Draw the graph of R in the Cartesian plane.



Proof. □

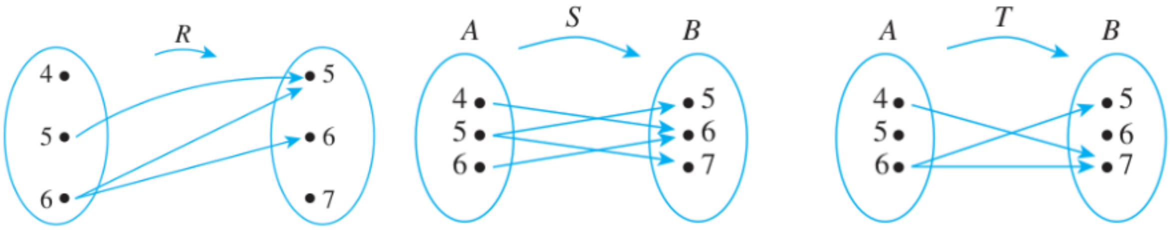
3.7 Problem 7

Let $A = \{4,5,6\}$ and $B = \{5,6,7\}$ and define relations R, S, and T from A to B as follows: For every $(x,y) \in A \times B$:

$$(x,y) \in R \text{ means that } x \geq y.$$
$$(x,y) \in S \text{ means that } \frac{x-y}{2} \text{ is an integer.}$$
$$T = \{(4,7),(6,5),(6,7)\}$$

3.7.1 (a)

Draw arrow diagrams for R, S, and T.



Proof. □

3.7.2 (b)

Indicate whether any of the relations R, S, and T are functions.

Proof. R is not a function because it satisfies neither property (1) nor property (2) of the definition. It fails property (1) because $(4, y) \notin R$, for any y in B. It fails property (2) because (6, 5) [R and (6, 6) [R and 5 ? 6.

S is not a function because $(5, 5) \in S$ and $(5, 7) \in S$ and $5 \neq 7$. So S does not satisfy property (2) of the definition of function.

T is not a function both because $(5, x) \notin T$ for any x in B and because $(6, 5) \in T$ and $(6, 7) \in T$ and $5 \neq 7$. So T does not satisfy either property (1) or property (2) of the definition of function. □

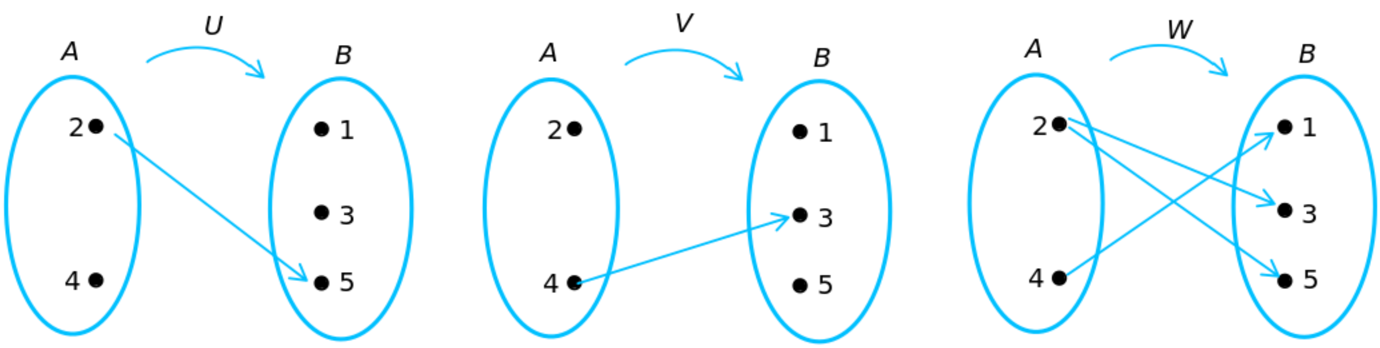
3.8 Problem 8

Let $A = \{2, 4\}$ and $B = \{1, 3, 5\}$ and define relations U, V, and W from A to B as follows: For every $(x, y) \in A \times B$:

$$(x, y) \in U \text{ means that } y - x > 2.$$
$$(x, y) \in V \text{ means that } y - 1 = \tfrac{x}{2}.$$
$$W = \{(2, 5), (4, 1), (2, 3)\}.$$

3.8.1 (a)

Draw arrow diagrams for U, V, and W.



Proof. □

3.8.2 (b)

Indicate whether any of the relations U, V, and W are functions.

Proof. U and V are not functions because they are undefined for $4 \in A$ and $2 \in A$ respectively. So they fail property (1) of a function.

W is not a function because both $(2, 3) \in W$ and $(2, 5) \in W$ and $3 \neq 5$. So it fails property (2) of a function. □

3.9 Problem 9

3.9.1 (a)

Find all functions from $\{0, 1\}$ to $\{1\}$.

Proof. There is only 1 function from $\{0, 1\}$ to $\{1\}$.

Any function must be defined on both inputs $0, 1$. There is only one option for the output, namely 1. So the only function is: $\{(0, 1), (1, 1)\}$. □

3.9.2 (b)

Find two relations from $\{0, 1\}$ to $\{1\}$ that are not functions.

Proof. $R = \{(0, 1)\}$ is a relation that is not a function, because it's undefined on input 1, so it fails property (1) of a function.

$S = \{(1, 1)\}$ is a relation but not a function, for similar reasons. □

3.10 Problem 10

Find four relations from $\{a, b\}$ to $\{x, y\}$ that are not functions from $\{a, b\}$ to $\{x, y\}$.

Proof. $R = \{(a, x)\}$ (undefined for input b)

$S = \{(a, y)\}$ (undefined for input b)

$T = \{(b, x)\}$ (undefined for input a)

$U = \{(b, y)\}$ (undefined for input a) □

3.11 Problem 11

Let $A = \{0, 1, 2\}$ and let S be the set of all strings over A. Define a relation L from S to $\mathbb{Z}^{\text{nonneg}}$ as follows: For every string s in S and every nonnegative integer n,

$$(s, n) \in L \text{ means that the length of } s \text{ is } n.$$

Then L is a function because every string in S has one and only one length. Find $L(0201)$ and $L(12)$.

Proof. $L(0201) = 4$ and $L(12) = 2$, because those are the lengths of the strings. □

3.12 Problem 12

Let $A = \{x, y\}$ and let S be the set of all strings over A. Define a relation C from S to S as follows: For all strings s and t in S,

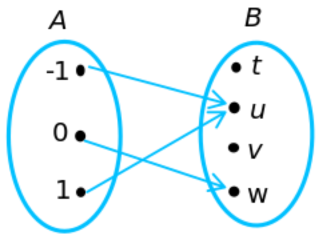
$$(s, t) \in C \text{ means that } t = ys.$$

Then C is a function because every string in S consists entirely of x's and y's and adding an additional y on the left creates a single new string that consists of x's and y's and is, therefore, also in S. Find $C(x)$ and $C(yyxyx)$.

Proof. The function C adds an additional y on the left of its input. Therefore: $C(x) = yx$ and $C(yyxyx) = yyyxyx$. □

3.13 Problem 13

Let $A = \{-1, 0, 1\}$ and $B = \{t, u, v, w\}$. Define a function $F : A \to B$ by the following arrow diagram:



3.13.1 (a)

Write the domain and co-domain of F.

Proof. Domain $= A = \{-1, 0, 1\}$

Co-domain $= B = \{t, u, v, w\}$ □

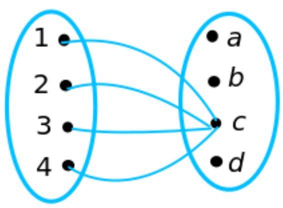
3.13.2 (b)

Find $F(-1)$, $F(0)$, and $F(1)$.

Proof. $F(-1) = u$, $F(0) = w$, and $F(1) = u$. □

3.14 Problem 14

Let $C = \{1, 2, 3, 4\}$ and $D = \{a, b, c, d\}$. Define a function $G : C \to D$ by the following arrow diagram:



3.14.1 (a)

Write the domain and co-domain of G.

Proof. Domain $= C = \{1, 2, 3, 4\}$

Co-domain $= D = \{a, b, c, d\}$ □

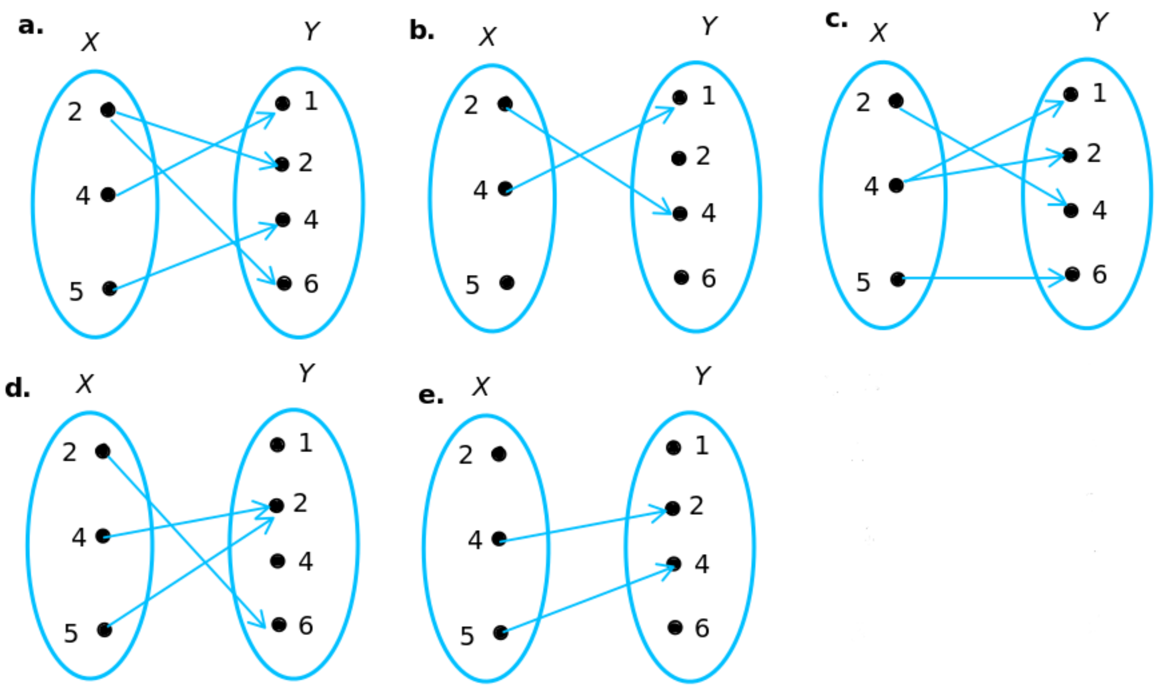
3.14.2 (b)

Find $G(1), G(2), G(3)$, and $G(4)$.

Proof. $G(1) = c, G(2) = c, G(3) = c$, and $G(4) = c$. □

3.15 Problem 15

Let $X = \{2, 4, 5\}$ and $Y = \{1, 2, 4, 6\}$. Which of the following arrow diagrams determine functions from X to Y?



3.15.1 (a)

Proof. Not a function. Fails property (2) because both $(2, 2)$ and $(2, 6)$ are included in the relation. □

3.15.2 (b)

Proof. Not a function. Fails property (1) because it's undefined on input 5. □

3.15.3 (c)

Proof. Not a function. Fails property (2) because both $(4, 1)$ and $(4, 2)$ are included in the relation. □

3.15.4 (d)

Proof. Is a function. Satisfies both properties. □

3.15.5 (e)

Proof. Not a function. Fails property (1) because it's undefined on input 2. □

3.16 Problem 16

Let f be the squaring function defined in Example 1.3.6. Find $f(-1)$, $f(0)$, and $f(1/2)$.

Proof. $f(-1) = (-1)^2 = 1$, $f(0) = 0^2 = 0$, and $f(1/2) = (1/2)^2 = 1/4$. □

3.17 Problem 17

Let g be the successor function defined in Example 1.3.6. Find $g(-1000)$, $g(0)$, and $g(999)$.

Proof. $g(-1000) = -1000 + 1 = -999$, $g(0) = 0 + 1 = 1$, and $g(999) = 999 + 1 = 1000$. □

3.18 Problem 18

Let h be the constant function defined in Example 1.3.6. Find $h(-12/5)$, $h(0/1)$, and $h(9/17)$.

Proof. $h(-12/5) = 2$, $h(0/1) = 2$, and $h(9/17) = 2$. □

3.19 Problem 19

Define functions f and g from $\mathbb{R}$ to $\mathbb{R}$ by the following formulas: For every $x \in \mathbb{R}$,

$$f(x) = 2x \text{ and } g(x) = \frac{2x^3 + 2x}{x^2 + 1}.$$

Does $f = g$? Explain.

Proof. Yes. Because by algebra, for all $x \in \mathbb{R}$

$$g(x) = \frac{2x^3 + 2x}{x^2 + 1} = \frac{2x(x^2 + 1)}{x^2 + 1} = 2x.$$

Therefore by definition of equality of functions, $f = g$. □

3.20 Problem 20

Define functions H and K from $\mathbb{R}$ to $\mathbb{R}$ by the following formulas: For every $x \in \mathbb{R}$,

$$H(x) = (x-2)^2 \text{ and } K(x) = (x-1)(x-3)+1.$$

Does $H = K$? Explain.

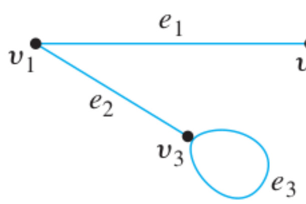
Proof. Yes. Because by algebra, for all $x \in \mathbb{R}$ we have $H(x) = (x-2)^2 = x^2 - 4x + 4$ and $K(x) = (x-1)(x-3)+1 = x^2 - 4x + 3 + 1 = x^2 - 4x + 4$.

So, for all $x \in \mathbb{R}$ we have $H(x) = K(x)$. Therefore by definition of equality of functions, $H = K$. $\square$

4 Exercises 1.4

In 1 and 2, graphs are represented by drawings. Define each graph formally by specifying its vertex set, its edge set, and a table giving the edge-endpoint function.

4.1 Problem 1

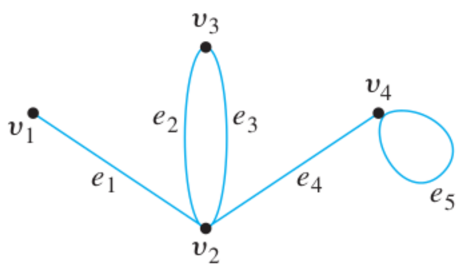


Proof. The graph has vertex set $\{v_1, v_2, v_3, v_4\}$ and edge set $\{e_1, e_2, e_3\}$ with edge-endpoint function as follows:

Edge	Endpoints
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_3\}$
e_3	$\{v_3\}$

$\square$

4.2 Problem 2



Proof. The graph has vertex set $\{v_1, v_2, v_3, v_4\}$ and edge set $\{e_1, e_2, e_3, e_4, e_5\}$ with edge-endpoint function as follows:

Edge	Endpoints
e_1	$\{v_1, v_2\}$
e_2	$\{v_2, v_3\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_2, v_4\}$
e_5	$\{v_4\}$

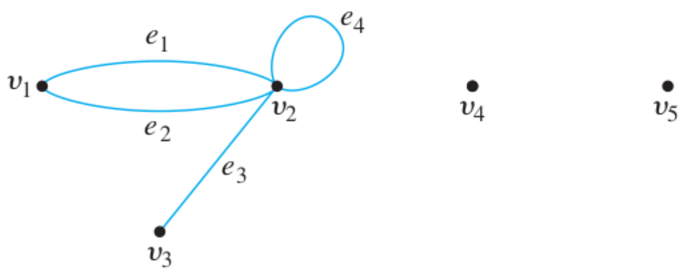
$\square$

In 3 and 4, draw pictures of the specified graphs.

4.3 Problem 3

Graph G has vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{e_1, e_2, e_3, e_4\}$ with edge-endpoint function as follows:

Edge	Endpoints
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_2\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_2\}$

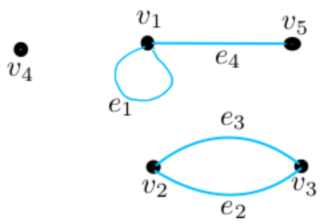


Proof. $\square$

4.4 Problem 4

Graph H has vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{e_1, e_2, e_3, e_4\}$ with edge-endpoint function as follows:

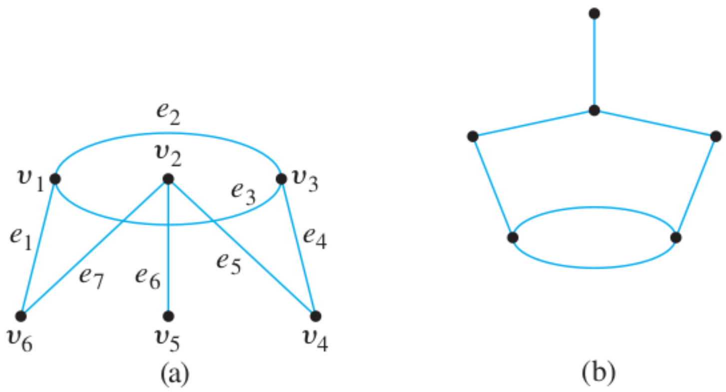
Edge	Endpoints
e_1	$\{v_1\}$
e_2	$\{v_2, v_3\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_1, v_5\}$



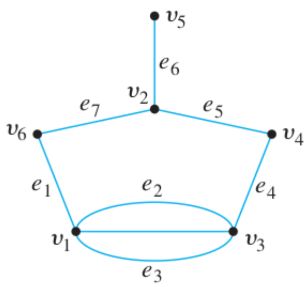
Proof. □

In 5-7, show that the two drawings represent the same graph by labeling the vertices and edges of the right-hand drawing to correspond to those of the left-hand drawing.

4.5 Problem 5

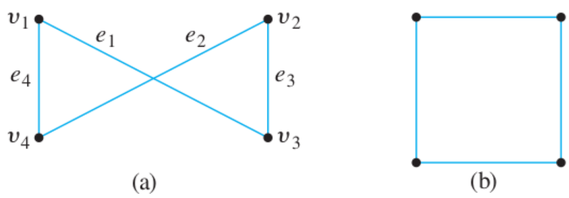


Proof. Imagine that the edges are strings and the vertices are knots. You can pick up the left-hand figure and lay it down again to form the right-hand figure as shown below.

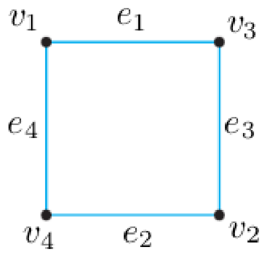


□

4.6 Problem 6

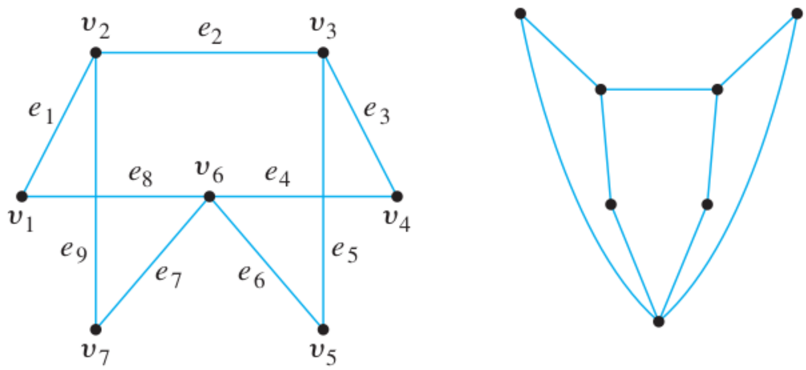


Proof. We can hold the edges e_3 and e_4 in our hands, and "un-twist" the crossing edges e_1 and e_2 by rotating one of the edges, say e_3, upside down (which switches v_2 and v_3), to get the shape below:

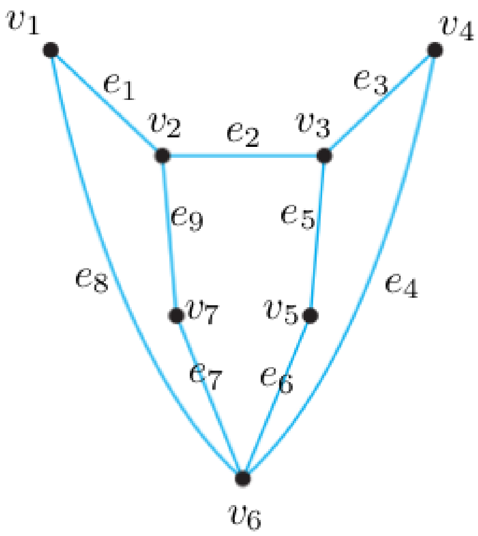


□

4.7 Problem 7



(a) (b)



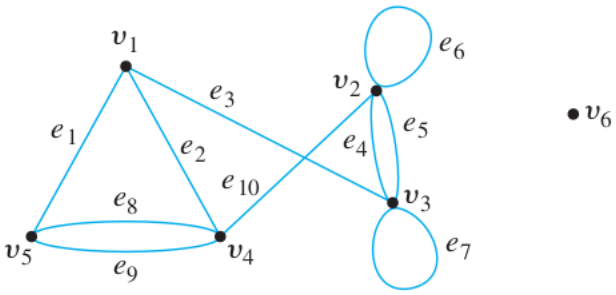
Proof. □

For each of the graphs in 8 and 9:

1. Find all edges that are incident on v_1.
2. Find all vertices that are adjacent to v_3.

3. Find all edges that are adjacent to e_1.
4. Find all loops.
5. Find all parallel edges.
6. Find all isolated vertices.
7. Find the degree of v_3.

4.8 Problem 8

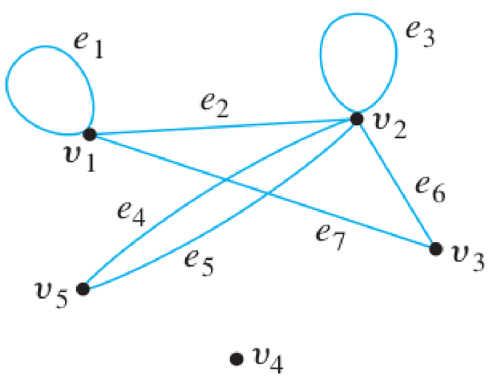


Proof. 1. e_1, e_2, and e_3 are incident on v_1.

2. v_1, v_2, and v_3 are adjacent to v_3.
3. e_2, e_8, e_9, and e_3 are adjacent to e_1.
4. Loops are e_6 and e_7.
5. e_8 and e_9 are parallel; e_4 and e_5 are parallel.
6. v_6 is an isolated vertex.
7. degree of $v_3 = 5$.

□

4.9 Problem 9



Proof. 1. e_1, e_2, e_7 are incident on v_1.

2. v_1, v_2 are adjacent to v_3.

3. e_2, e_7 are adjacent to e_1.
4. Loops are e_1 and e_3.
5. e_4 and e_5 are parallel.
6. v_4 is an isolated vertex.
7. degree of $v_3 = 2$.

□

4.10 Problem 10

Use the graph of Example 1.4.6 to determine

4.10.1 (a)

whether Sports Illustrated contains printed writing;

Proof. Yes. According to the graph, Sports Illustrated is an instance of a sports magazine, a sports magazine is a periodical, and a periodical contains printed writing. □

4.10.2 (b)

whether Poetry Magazine contains long words.

Proof. Yes. Poetry Magazine is an instance of a literary journal, a literary journal is a scholarly journal, and a scholarly journal contains long words. □

4.11 Problem 11

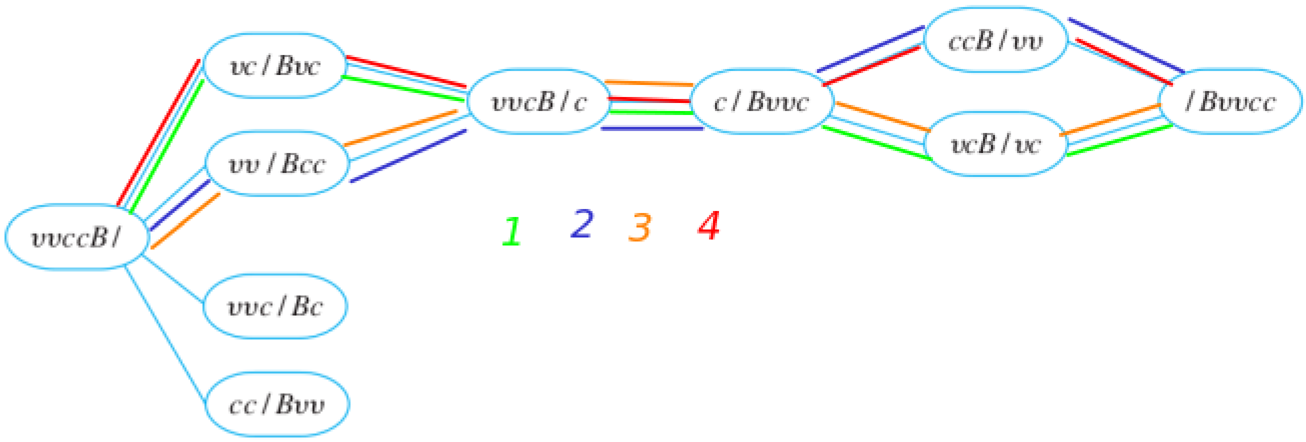
Find three other winning sequences of moves for the vegetarians and the cannibals in Example 1.4.7.

Proof. The three other winning sequences of moves are apparent from the picture. There are 2 places where the graph branches into 2, giving us $2 \times 2 = 4$ possible solutions. In the example we've seen 1 out of 4 solutions. The other 3 are:

1. $vvccB/ \to vc/Bvc \to vvcB/c \to c/Bvvc \to vcB/vc \to /Bvvcc$
2. $vvccB/ \to vv/Bcc \to vvcB/c \to c/Bvvc \to ccB/vv \to /Bvvcc$
3. $vvccB/ \to vv/Bcc \to vvcB/c \to c/Bvvc \to vcB/vc \to /Bvvcc$

Including the original solution as (4), these solutions correspond to the following paths in the solution graph:

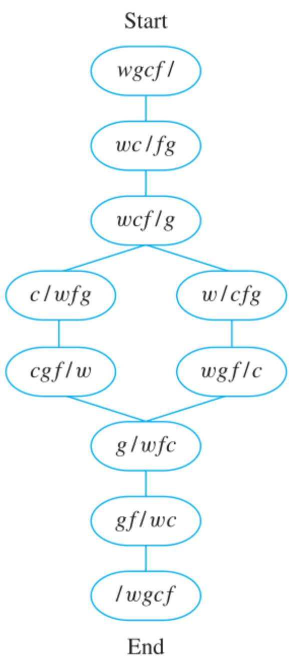
□



4.12 Problem 12

Another famous puzzle used as an example in the study of artificial intelligence seems first to have appeared in a collection of problems, *Problems for the Quickening of the Mind*, which was compiled about A.D. 775. It involves a wolf, a goat, a bag of cabbage, and a ferryman. From an initial position on the left bank of a river, the ferryman is to transport the wolf, the goat, and the cabbage to the right bank. The difficulty is that the ferryman's boat is only big enough for him to transport one object at a time, other than himself. Yet, for obvious reasons, the wolf cannot be left alone with the goat, and the goat cannot be left alone with the cabbage. How should the ferryman proceed?

Proof. To solve this puzzle using a graph, introduce a notation in which, for example, wc/fg means that the wolf and the cabbage are on the left bank of the river and the ferryman and the goat are on the right bank. Then draw those arrangements of wolf, cabbage, goat, and ferryman that can be reached from the initial arrangement ($wgcf/$) and that are not arrangements to be avoided (such as (wg/fc)). At each stage ask yourself, "Where can I go from here?" and draw lines or arrows pointing to those arrangements. This method gives the graph shown below.



$\square$

4.13 Problem 13

Solve the vegetarians-and-cannibals puzzle for the case where there are three vegetarians and three cannibals to be transported from one side of a river to the other.

Proof. This one is quite difficult: draw the graph and go through all the possibilities. Almost all the possible moves are illegal, so you have to rule them out. Showing the full graph with all the illegal moves ruled out gets too big; therefore I will show the only two possible winning sequences:

The two winning sequences only differ in the first two moves:

(1) $vvvcccB/ \overset{vc}{\rightarrow} vvcc/Bvc \overset{v}{\leftarrow} vvvccB/c$

(2) $vvvcccB/ \overset{cc}{\rightarrow} vvvc/Bcc \overset{c}{\leftarrow} vvvccB/c$

After that, they are the same, because there is always only one legal move:

$vvvccB/c \overset{cc}{\rightarrow} vvv/cccB \overset{c}{\leftarrow} vvvcB/cc \overset{vv}{\rightarrow} vc/vvccB \overset{vc}{\leftarrow} vvccB/vc \overset{vv}{\rightarrow} cc/vvvcB$

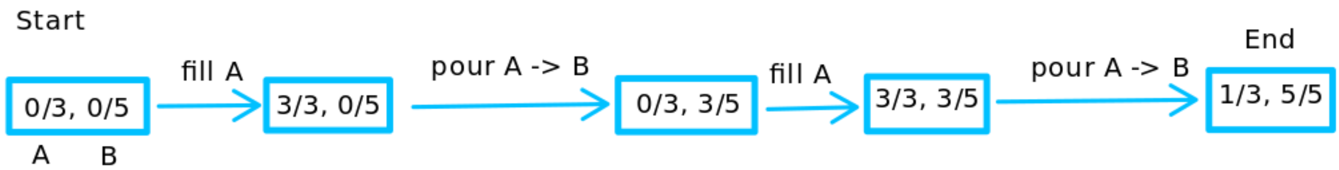
$\overset{c}{\leftarrow} cccB/vvv \overset{cc}{\rightarrow} c/vvvccB \overset{c}{\leftarrow} ccB/vvvc \overset{cc}{\rightarrow} /vvvcccB$

$\square$

4.14 Problem 14

Two jugs A and B have capacities of 3 quarts and 5 quarts, respectively. Can you use the jugs to measure out exactly 1 quart of water, while obeying the following restrictions? You may fill either jug to capacity from a water tap; you may empty the contents of either jug into a drain; and you may pour water from either jug into the other.

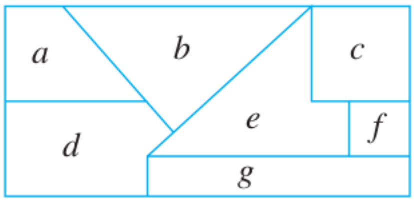
Proof. Represent possible amounts of water in jugs A and B by ordered pairs. For instance, the ordered pair $(1, 3)$ would indicate that there is one quart of water in jug A and three quarts in jug B. Starting with $(0, 0)$, draw arrows from one ordered pair to another if it is possible to go from the situation represented by one pair to that represented by the other by either filling a jug, emptying a jug, or transferring water from one jug to another. You need only draw arrows from states that have arrows pointing to them; the other states cannot be reached. Then find a directed path (sequence of directed edges) from the initial state $(0, 0)$ to a final state $(1, 0)$ or $(0, 1)$. Here is one of the solutions (there are infinitely many):



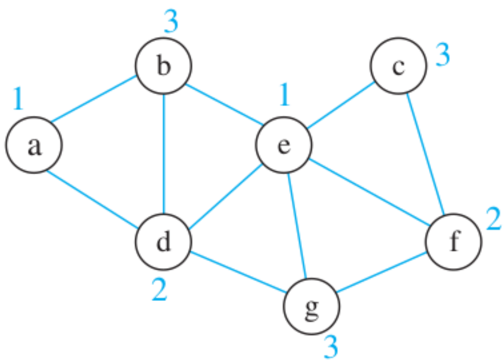
$\square$

4.15 Problem 15

Imagine that the diagram shown below is a map with countries labeled $a - g$. Is it possible to color the map with only three colors so that no two adjacent countries have the same color? To answer this question, follow the procedure suggested by Example 1.4.9. Draw and analyze a graph in which each country is represented by a vertex and two vertices are connected by an edge if, and only if, the countries share a common border.



Proof. Vertex e has maximal degree, so color it with color #1. Vertex a does not share an edge with e, and so color #1 may also be used for it. From the remaining uncolored vertices, all of d, g, and f have maximal degree. Choose any one of them, say, d, and use color #2 for it. Observe that vertices c and f do not share an edge with d, but they do share an edge with each other, which means that color #2 may be used for one but not the other. Choose to color f with color #2 because the degree of f is greater than the degree of c. The remaining uncolored vertices, b, c, and g, are unconnected, and so color #3 may be used for all three.



□

4.16 Problem 16

In this exercise a graph is used to help solve a scheduling problem. Twelve faculty members in a mathematics department serve on the following committees:

Undergraduate Education: Tenner, Peterson, Kashina, Degras

Graduate Education: Hu, Ramsey, Degras, Bergen

Colloquium: Carroll, Drupieski, Au-Yeung

Library: Ugarcovici, Tenner, Carroll

Hiring: Hu, Drupieski, Ramsey, Peterson

Personnel: Ramsey, Wang, Ugarcovici

The committees must all meet during the first week of classes, but there are only three time slots available. Find a schedule that will allow all faculty members to attend the meetings of all committees on which they serve. To do this, represent each committee as the vertex of a graph, and draw an edge between two vertices if the two committees have a common member. Find a way to color the vertices using only three colors so that no two committees have the same color, and explain how to use the result to schedule the meetings.

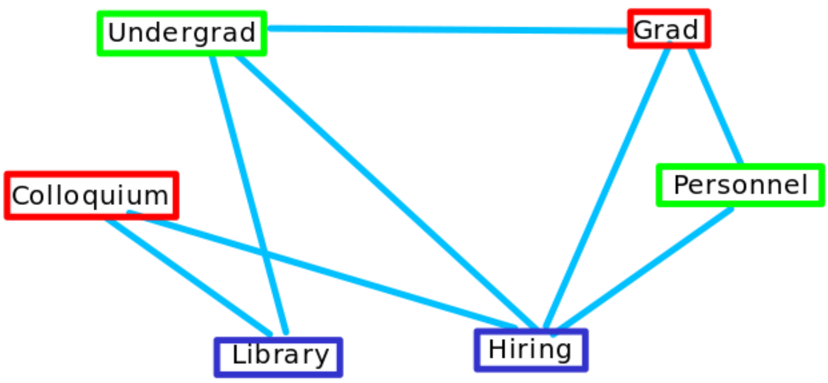
Proof. Undergraduate is connected to Library (because of Tenner), Hiring (because of Peterson), Graduate (because of Degras).

Graduate is connected to Hiring (because of Hu and Ramsey), Personnel (because of Ramsey).

Colloquium is connected to Library (because of Carroll), Hiring (because of Drupieski).

Hiring is connected to Personnel (because of Ramsey).

Based on this info we have one possible solution:



So we can do the following:

Time 1: BLUE: hiring, library

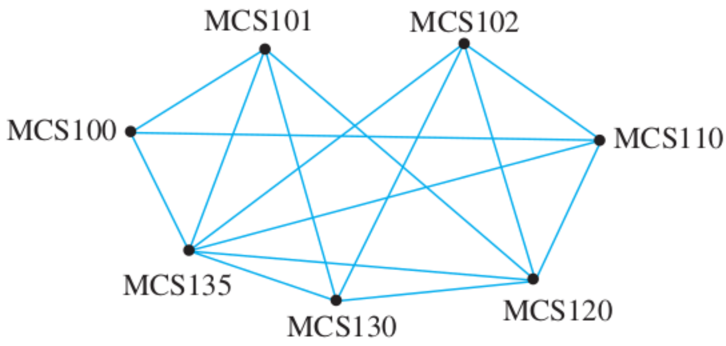
Time 2: GREEN: personnel, undergraduate education

Time 3: RED: graduate education, colloquium

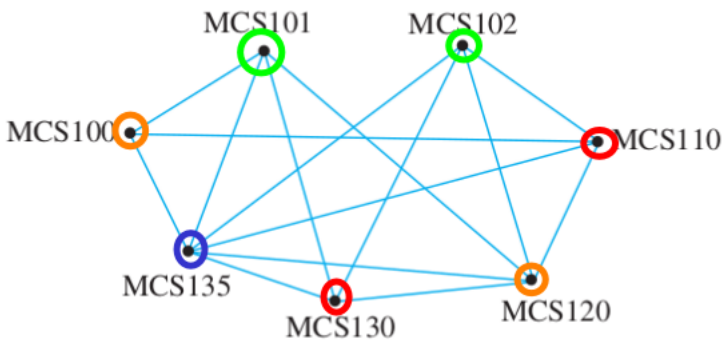
(We could have also colored Colloquium with GREEN instead. That's another valid solution.) □

4.17 Problem 17

A department wants to schedule final exams so that no student has more than one exam on any given day. The vertices of the graph below show the courses that are being taken by more than one student, with an edge connecting two vertices if there is a student in both courses. Find a way to color the vertices of the graph with only four colors so that no two adjacent vertices have the same color and explain how to use the result to schedule the final exams.



Proof. One possible solution:



We can schedule the exams of the:

green courses (MCS101 and MCS102) on Monday,

blue course (MCS135) on Tuesday,

red courses (MCS130 and MCS110) on Wednesday,

orange courses (MCS100 and MCS120) on Thursday.

Since there is no edge connecting any two courses of the same color, there are no students taking two courses of the same color. □