# Chapter 4 Solutions, Susanna Epp Discrete Math 5th Edition

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## Contents

# 1 Exercise Set 4.1

## 1.1 Exercise 1

Assume that $k$ is a particular integer.

### 1.1.1 (a)

Is $-17$ an odd integer?

*Proof.* Yes: $-17 = 2(-9) + 1$.

$\square$

### 1.1.2 (b)

Is 0 neither even nor odd?

*Proof.* No. 0 is even because $0 = 0 \cdot 2$. $\square$

### 1.1.3 (c)

Is $2k - 1$ odd?

*Proof.* Yes: $2k - 1 = 2(k - 1) + 1$ and $k - 1$ is an integer because it is a difference of integers. $\square$

## 1.2 Exercise 2

Assume that $c$ is a particular integer.

### 1.2.1 (a)

Is $-6c$ an even integer?

*Proof.* Yes, because $-6c = 2 \cdot (-3c) = 2k$ where $k = -3c$ is an integer. $\square$

### 1.2.2 (b)

Is $8c + 5$ an odd integer?

*Proof.* Yes, because $8c + 5 = 2(4c + 2) + 1$ and $k = 4k + 2$ is an integer. $\square$

### 1.2.3 (c)

Is $(c^2 + 1) - (c^2 - 1) - 2$ an even integer?

*Proof.* Yes, because it equals 0: $(c^2+1)-(c^2-1)-2 = c^2+1-c^2+1-2 = 2-2 = 0$. $\square$

## 1.3 Exercise 3

Assume that $m$ and $n$ are particular integers.

### 1.3.1 (a)

Is $6m + 8n$ even?

*Proof.* Yes: $6m + 8n = 2(3m + 4n)$ and $(3m + 4n)$ is an integer because $3, 4, m$, and $n$ are integers, and products and sums of integers are integers. $\square$

### 1.3.2 (b)

Is $10mn + 7$ odd?

*Proof.* Yes: $10mn + 7 = 2(5mn + 3) + 1$ and $5mn + 3$ is an integer because $3, 5, m$, and $n$ are integers, and products and sums of integers are integers. $\square$

### 1.3.3 (c)

If $m > n > 0$, is $m^2 - n^2$ composite?

*Proof.* Not necessarily. For instance, if $m = 3$ and $n = 2$, then $m^2 - n^2 = 9 - 4 = 5$, which is prime. (However, $m^2 - n^2$ is composite for many values of $m$ and $n$ because of the identity $m^2 - n^2 = (m - n)(m + n)$.) $\square$

## 1.4 Exercise 4

Assume that $r$ and $s$ are particular integers.

### 1.4.1 (a)

Is $4rs$ even?

*Proof.* Yes: $4rs = 2(2rs)$ and $2rs$ is an integer because $2, r, s$ are integers, and products of integers are integers. $\square$

### 1.4.2 (b)

Is $6r + 4s^2 + 3$ odd?

*Proof.* Yes: $6r + 4s^2 + 3 = 2(3r + 2s^2 + 1) + 1$ and $3r + 2s^2 + 1$ is an integer because $3, r, 2, s, 1$ are integers and products and sums of integers are integers. $\square$

### 1.4.3 (c)

If $r$ and $s$ are both positive, is $r^2 + 2rs + s^2$ composite?

*Proof.* Yes: $r^2 + 2rs + s^2 = (r + s)(r + s)$ and $r + s \geq 2$, therefore $r^2 + 2rs + s^2$ is a product of two integers both of which are greater than 1. $\square$

**Prove the statements in 5–11.**

## 1.5 Exercise 5

There are integers $m$ and $n$ such that $m > 1$ and $n > 1$ and $\frac{1}{m} + \frac{1}{n}$ is an integer.

*Proof.* For example, let $m = n = 2$. Then $m$ and $n$ are integers such that $m > 1$ and $n > 1$ and $\frac{1}{m} + \frac{1}{n} = \frac{1}{2} + \frac{1}{2} = 1$ which is an integer. $\square$

## 1.6 Exercise 6

There are distinct integers $m$ and $n$ such that $\frac{1}{m} + \frac{1}{n}$ is an integer.

*Proof.* For example, let $m = 1, n = -1$. Then $m$ and $n$ are integers such that $\frac{1}{m} + \frac{1}{n} = \frac{1}{1} - \frac{1}{1} = 0$ which is an integer. $\square$

## 1.7 Exercise 7

There are real numbers $a$ and $b$ such that $\sqrt{a+b} = \sqrt{a}\sqrt{b}$.

*Proof.* For example, let $a = 0, b = 0$. Then $a$ and $b$ are real numbers such that

$$\sqrt{a+b} = \sqrt{0+0} = 0 = \sqrt{0} + \sqrt{0} = \sqrt{a} + \sqrt{b}.$$

$\square$

## 1.8 Exercise 8

There is an integer $n > 5$ such that $2^n - 1$ is prime.

*Proof.* For example, let $n = 7$. Then $n$ is an integer such that $n > 5$ and $2^n - 1 = 127$, which is prime. $\square$

## 1.9 Exercise 9

There is a real number $x$ such that $x > 1$ and $2^x > x^{10}$.

*Proof.* For example, take $x = 80$. Then

$$x^{10} = 80^{10} = 8^{10} \cdot 10^{10} = (2^3)^{10} \cdot 10^{10} = 2^{30} \cdot 10^{10}.$$

We have $2^{50} \approx 1,125899907 \cdot 10^{15} > 10^{10}$. So $2^{80} = 2^{30} \cdot 2^{50} > 2^{30} \cdot 10^{10} = 80^{10}$.

Therefore $x = 80$ is a real number such that $x > 1$ and $2^x > x^{10}$. $\square$

**Definition:** An integer $n$ is called a **perfect square** if, and only if, $n = k^2$ for some integer $k$.

## 1.10 Exercise 10

There is a perfect square that can be written as the sum of two other perfect squares.

*Proof.* For example, 25, 9, and 16 are all perfect squares, because $25 = 5^2, 9 = 3^2$, and $16 = 4^2$, and $25 = 9 + 16$. Thus 25 is a perfect square that can be written as a sum of two other perfect squares. □

## 1.11 Exercise 11

There is an integer $n$ such that $2n^2 - 5n + 2$ is prime.

*Proof.* For example, take $n = 3$. Then $2n^2 - 5n + 2 = 18 - 15 + 2 = 5$ is prime. (You can find this value of $n$ by either starting at $n = 1$ and using trial and error, or noticing that $2n^2 - 5n + 2 = (2n - 1)(n - 2)$, so, for this to be prime, one of the factors has to be 1.) □

**In $12 - 13$, (a) write a negation for the given statement, and (b) use a counterexample to disprove the given statement. explain how the counterexample actually shows that the given statement is false.**

## 1.12 Exercise 12

For all real numbers $a$ and $b$, if $a < b$ then $a^2 < b^2$.

*Proof.* a. *Negation for the statement:* There exist real numbers $a$ and $b$ such that $a < b$ and $a^2 \nless b^2$.

b. *Counterexample for the statement:* Let $a = -2$ and $b = -1$. Then $a < b$ because $-2 < -1$, but $a^2 \nless b^2$ because $(-2)^2 = 4$ and $(-1)^2 = 1$ and $4 \nless 1$. *[So the hypothesis of the statement is true and its conclusion is false.]* □

## 1.13 Exercise 13

For every integer $n$, if $n$ is odd then $\frac{n-1}{2}$ is odd.

*Proof.* a. *Negation for the statement:* There exists an integer $n$ such that $n$ is odd and $\frac{n-1}{2}$ is not odd.

b. *Counterexample for the statement:* Let $n = 5$. Then $n$ is odd because $5 = 2 \cdot 2 + 1$, but $\frac{n-1}{2}$ is not odd because $\frac{5-1}{2} = 2 = 2 \cdot 1$ is even. *[So the hypothesis of the statement is true and its conclusion is false.]* □

**Disprove each of the statements in 14 – 16 by giving a counterexample. In each case explain how the counterexample actually disproves the statement.**

## 1.14 Exercise 14

For all integers $m$ and $n$, if $2m+n$ is odd then $m$ and $n$ are both odd.

*Proof.* Counterexample: Let $m=2$ and $n=1$. Then $2m+n=2\cdot 2+1=5$, which is odd. But $m$ is not odd, and so it is false that both $m$ and $n$ are odd. *[This is one counterexample among many.]* $\square$

## 1.15 Exercise 15

For every integer $p$, if $p$ is prime then $p^2-1$ is even.

*Proof.* Counterexample: Let $p=2$ which is prime, but $p^2-1=2^2-1=4-1=3$ is not even. *[This is the only counterexample! For every other prime $p$, $p^2-1$ is even.]* $\square$

## 1.16 Exercise 16

For every integer $n$, if $n$ is even then $n^2+1$ is prime.

*Proof.* Counterexample: Let $n=8$. Then $n$ is even. But $n^2+1=65=13\cdot 5$ is not prime. *[This is one counterexample among many.]* $\square$

**In 17 – 20, determine whether the property is true for all integers, true for no integers, or true for some integers and false for other integers. Justify your answers.**

## 1.17 Exercise 17

$(a+b)^2=a^2+b^2$

*Proof.* This property is true for some integers and false for other integers. For instance, if $a=0$ and $b=1$, the property is true because $(0+1)^2=0^2+1^2$, but if $a=1$ and $b=1$, the property is false because $(1+1)^2=4$ and $1^2+1^2=2$ and $4\neq 2$. $\square$

## 1.18 Exercise 18

$\frac{a}{b}+\frac{c}{d}=\frac{a+c}{b+d}$

*Proof.* True for some integers, false for others. For example, if $a=c=0$ and $b=d=1$ then $\frac{0}{1}+\frac{0}{1}=0=\frac{0+0}{1+1}$ is true. But if $a=1, b=2, c=3$ and $d=4$ then

$$\frac{a}{b}+\frac{c}{d}=\frac{1}{2}+\frac{3}{4}=\frac{5}{4}\neq\frac{4}{6}=\frac{1+3}{2+4}=\frac{a+c}{b+d}.$$ $\square$

## 1.19 Exercise 19

$-a^n = (-a)^n$

*Hint:* This property is true for some integers and false for other integers. To justify this answer you need to find examples of both.

*Proof.* True for some integers: let $a = 0, n = 1$. Then $-0^1 = -0 = 0$ and $(-0)^1 = 0^1 = 0$ so the equality holds. When $a = n = 2$ it is false: $-2^2 = -4$ but $(-2)^2 = 4$ and $-4 \neq 4$. □

## 1.20 Exercise 20

The average of any two odd integers is odd.

*Proof.* True for some, false for others. For example, 3 and 7 are both odd, and their average $(3+7)/2 = 5$ is odd. But 3 and 5 are both odd, and their average is $(3+5)/2 = 4$ is even. □

**Prove the statement in 21 and 22 by the method of exhaustion.**

## 1.21 Exercise 21

Every positive even integer less than 26 can be expressed as a sum of three of fewer perfect squares. (For instance, $10 = 1^2 + 3^2$ and $16 = 4^2$.)

*Proof.* $2 = 1^2 + 1^2$

$4 = 2^2$

$6 = 2^2 + 1^2 + 1^2$

$8 = 2^2 + 2^2$

$10 = 1^2 + 3^2$

$12 = 2^2 + 2^2 + 2^2$

$16 = 4^2$

$18 = 4^2 + 1^2 + 1^2$

$20 = 4^2 + 2^2$

$22 = 2^2 + 2^2 + 3^2$

$24 = 4^2 + 2^2 + 2^2$ □

## 1.22 Exercise 22

For each integer $n$ with $1 \leq n \leq 10$, $n^2 - n + 11$ is a prime number.

*Proof.* $1^2 - 1 + 11 = 11$ is prime.

$2^2 - 2 + 11 = 13$ is prime.

$3^2 - 3 + 11 = 17$ is prime.

$4^2 - 4 + 11 = 23$ is prime.

$5^2 - 5 + 11 = 31$ is prime.

$6^2 - 6 + 11 = 41$ is prime.

$7^2 - 7 + 11 = 53$ is prime.

$8^2 - 8 + 11 = 67$ is prime.

$9^2 - 9 + 11 = 83$ is prime.

$10^2 - 10 + 11 = 101$ is prime. □

**Each of the statements in $23 - 26$ is true. For each, (a) rewrite the statement with the quantification implicit as If ____ , then ____ , and (b) write the first sentence of a proof (the "starting point") and the last sentence of a proof (the "conclusion to be shown"). (Note that you do not need to understand the statements in order to be able to do these exercises.)**

## 1.23 Exercise 23

For every integer $m$, if $m > 1$ then $0 < \frac{1}{m} < 1$.

*Proof.* a. If an integer is greater than 1, then its reciprocal is between 0 and 1.

b. *Start of proof:* Suppose $m$ is any integer such that $m > 1$. *Conclusion to be shown:* $0 < 1/m < 1$. □

## 1.24 Exercise 24

For every real number $x$, if $x > 1$ then $x^2 > x$.

*Proof.* a. If a real number is greater than 1, then its square is greater than itself.

b. *Start of proof:* Suppose $x$ is any real number such that $x > 1$. *Conclusion to be shown:* $x^2 > x$. □

## 1.25 Exercise 25

For all integers $m$ and $n$, if $mn = 1$ then $m = n = 1$ or $m = n = -1$.

*Proof.* a. If the product of two integers is 1, then either both are 1 or both are $-1$.

b. *Start of proof:* Suppose $m$ and $n$ are any integers with $mn = 1$. *Conclusion to be shown:* $m = n = 1$ or $m = n = -1$. □

## 1.26 Exercise 26

For every real number $x$, if $0 < x < 1$ then $x^2 < x$.

*Proof.* a. If a real number is strictly between 0 and 1, then its square is less than itself.

b. *Start of proof:* Suppose $x$ is any real number such that $0 < x < 1$. *Conclusion to be shown:* $x^2 < x$. □

## 1.27 Exercise 27

Fill in the blanks in the following proof.

**Theorem:** For every odd integer $n$, $n^2$ is odd.

**Proof:** Suppose $n$ is any (a) ____ . By definition of odd, $n = 2k + 1$ for some integer $k$. Then

$$\begin{aligned} n^2 &= \text{(b) } (\_\_\_\_)^2 && \text{by substitution} \\ &= 4k^2 + 4k + 1 && \text{by multiplying} \\ &= 2(2k^2 + 2k) + 1 && \text{by factoring out a 2} \end{aligned}$$

Now $2k^2 + 2k$ is an integer because it is a sum of products of integers. Therefore, $n^2$ equals $2\cdot$ (an integer) $+1$, and so (c) ____ is odd by definition of odd.

Because we have not assumed anything about $n$ except that it is an odd integer, it follows from the principle of (d) ____ that for every odd integer $n$, $n^2$ is odd.

*Proof.* (a) particular but arbitrarily chosen odd integer (b) $2k + 1$ (c) $n^2$ (d) universal generalization □

**In each of 28 – 31:**
**a. Rewrite the theorem in three different ways: as $\forall$ ____ , if ____ , then ____ ; as $\forall$ ____ , ____ (without using the words if or then), and as If ____ , then ____ (without using an explicit universal quantifier).**
**b. Fill in the blanks in the proof of the theorem.**

## 1.28 Exercise 28

**Theorem:** The sum of any two odd integers is even.

**Proof:** Suppose $m$ and $n$ are any *[particular but arbitrarily chosen]* odd integers.

*[We must show that $m + n$ is even.]*

By (a) ____ , $m = 2r + 1$ and $n = 2s + 1$ for some integers $r$ and $s$. Then

$$\begin{aligned} m + n &= (2r + 1) + (2s + 1) && \text{by (b) } \_\_\_\_ \\ &= 2r + 2s + 2 \\ &= 2(r + s + 1) && \text{by algebra} \end{aligned}$$

Let $u = r + s + 1$. Then $u$ is an integer because $r, s$ and 1 are integers and because (c) ____ .

Hence $m + n = 2u$, where $u$ is an integer, and so, by (d) ____ , $m + n$ is even *[as was to be shown]*.

*Proof.* a. ∀ integers $m$ and $n$, if $m$ and $n$ are odd then $m + n$ is even.

∀ odd integers $m$ and $n$, $m + n$ is even.

If $m$ and $n$ are any odd integers, then $m + n$ is even.

b. (a) definition of odd, (b) substitution, (c) any sum of integers is an integer, (d) definition of even □

## 1.29 Exercise 29

**Theorem:** The negative of any even integer is even.

**Proof:** Suppose $n$ is any *[particular but arbitrarily chosen]* even integer.

*[We must show that $-n$ is even.]*

By (a) ____ , $n = 2k$ for some integer $k$.

Then

$$\begin{aligned} -n &= -(2k) && \text{by (b) \_\_\_\_} \\ &= 2(-k) && \text{by algebra} \end{aligned}$$

Let $r = -k$. Then $r$ is an integer because $-1$ and $k$ are integers and (c) ____ .

Hence $-n = 2r$, where $r$ is an integer, and so $-n$ is even by (d) ____ *[as was to be shown]*.

*Proof.* a. ∀ integer $n$, if $n$ is even then $-n$ is even.

∀ even integer $n$, $-n$ is even.

If $n$ is any even integer, then $-n$ is even.

b. (a) definition of even, (b) substitution, (c) any product of integers is an integer, (d) definition of even □

## 1.30 Exercise 30

**Theorem:** The sum of any even integer and any odd integer is odd.

**Proof:** Suppose $m$ is any even integer and $n$ is any (a) ____ . By definition of even, $m = 2r$ for some (b) ____ , and by definition of odd, $n = 2s + 1$ for some integer $s$. By substitution and algebra,

$$m + n = \text{(c) \_\_\_\_} = 2(r + s) + 1$$

Since $r$ and $s$ are integers, so is their sum $r + s$. Hence $m + n$ has the form twice some integer plus one, and so, by (d) ____ by definition of odd.

*Proof.* a. $\forall$ integers $m$ and $n$, if $m$ is even and $n$ is odd, then $m + n$ is odd.

$\forall$ even integers $m$ and odd integers $n$, $m + n$ is odd.

If $m$ is any even integer and $n$ is any odd integer, then $m + n$ is odd.

b. (a) any odd integer (b) integer $r$ (c) $2r + (2s + 1)$ (d) $m + n$ is odd □

### 1.31 Exercise 31

**Theorem:** Whenever $n$ is an odd integer, $5n^2 + 7$ is even.

**Proof:** Suppose $n$ is any *[particular but arbitrarily chosen]* odd integer.

*[We must show that $5n^2 + 7$ is even.]*

By definition of odd, $n =$ (a) ____ , for some integer $k$.

Then

$$\begin{aligned} 5n^2 + 7 &= \text{(b) \_\_\_\_} && \text{substitution} \\ &= 5(4k^2 + 4k + 1) + 7 \\ &= 20k^2 + 20k + 12 \\ &= 2(10k^2 + 10k + 6) && \text{by algebra} \end{aligned}$$

Let $t =$ (c) ____ . Then $t$ is an integer because products and sums of integers are integers.

Hence $5n^2 + 7 = 2t$, where $t$ is an integer, and thus (d) ____ by definition of even *[as was to be shown]*.

*Proof.* a. $\forall$ integer $n$, if $n$ is odd, then $5n^2 + 7$ is even.

$\forall$ odd integer $n$, $5n^2 + 7$ is even.

If $n$ is any odd integer, then $5n^2 + 7$ is even.

b. (a) $2k + 1$ (b) $5(2k + 1)^2 + 7$ (c) $10k^2 + 10k + 6$ (d) $5n^2 + 7$ is even □

## 2 Exercise Set 4.2

**Prove the statements in $1 - 11$. In each case use only the definitions of the terms and the assumptions listed on page 161, not any previously established properties of odd and even integers. Follow the directions given in this section for writing proofs of universal statements.**

### 2.1 Exercise 1

For every integer $n$, if $n$ is odd then $3n + 5$ is even.

*Proof.* Suppose $n$ is any *[particular but arbitrarily chosen]* odd integer.

*[We must show that $3n+5$ is even. By definition of even, this means we must show that $3n+5 = 2\cdot$(some integer).]*

By definition of odd, $n = 2r+1$, for some integer $r$.

Then

$$\begin{aligned} 3n+5 &= 3(2r+1)+5 && \text{by substitution} \\ &= 6r+3+5 \\ &= 6r+8 \\ &= 2(3r+4) && \text{by algebra} \end{aligned}$$

*[Idea for the rest of the proof: We want to show that $3n+5 = 2\cdot$(some integer). At this point we know that $3n+5 = 2(3r+4)$. So is $3r+4$ an integer? Yes, because products and sums of integers are integers.]*

Let $k = 3r+4$.

Then $k$ is an integer because products and sums of integers are integers.

Hence $3n+5 = 2(3r+4) = 2k$ where $k$ is an integer. Hence by definition of even $3n+5$ is even *[as was to be shown]*. □

## 2.2 Exercise 2

For every integer $m$, if $m$ is even then $3m+5$ is odd.

*Proof.* Suppose $m$ is any *[particular but arbitrarily chosen]* even integer.

*[We must show that $3m+5$ is odd. By definition of odd, this means we must show that $3m+5 = 2\cdot$(some integer)$+1$.]*

By definition of even, $m = 2r$, for some integer $r$.

Then

$$\begin{aligned} 3m+5 &= 3(2r)+5 && \text{by substitution} \\ &= 6r+5 \\ &= 6r+4+1 \\ &= 2(3r+2)+1 && \text{by algebra} \end{aligned}$$

*[Idea for the rest of the proof: We want to show that $3m+5 = 2\cdot$(some integer)$+1$. At this point we know that $3m+5 = 2(3r+2)+1$. So is $3r+2$ an integer? Yes, because products and sums of integers are integers.]*

Let $k = 3r+2$.

Then $k$ is an integer because products and sums of integers are integers.

Hence $3m+5 = 2(3r+2)+1 = 2k+1$ where $k$ is an integer. Hence by definition of odd $3n+5$ is odd *[as was to be shown]*. □

## 2.3 Exercise 3

For every integer $n$, $2n-1$ is odd.

*Proof.* Suppose $n$ is any *[particular but arbitrarily chosen]* integer.

*[We must show that* $2n-1$ *is odd. By definition of odd, this means we must show that* $2n-1 = 2\cdot$ *(some integer)* $+1$.*]*

Then

$$\begin{aligned} 2n-1 &= 2n-2+2-1 && \text{because } -2+2=0 \\ &= 2(n-1)+2-1 \\ &= 2(n-1)+1 && \text{by algebra} \end{aligned}$$

Let $k = n-1$.

Then $k$ is an integer because the difference of two integers ($n$ and $1$) is an integer.

Hence $2n-1 = 2(n-1)+1 = 2k+1$ where $k$ is an integer, and thus by definition of odd $2n-1$ is odd *[as was to be shown]*. □

## 2.4 Exercise 4

The difference of any even integer minus any odd integer is odd.

*Proof.* Suppose $a$ is any even integer and $b$ is any odd integer. *[We must show that* $a-b$ *is odd.]* By definition of even and odd, $a = 2r$ and $b = 2s+1$, for some integers $r, s$. By substitution and algebra,

$$a-b = 2r-(2s+1) = 2r-2s-1 = 2r-2s-2+2-1 = 2(r-s-1)+1$$

Let $t = r-s-1$. Then $t$ is an integer because differences of integers are integers.

Thus $a-b = 2t+1$ where $t$ is an integer, and so by definition of odd $a-b$ is odd *[as was to be shown]*. □

## 2.5 Exercise 5

If $a$ and $b$ are any odd integers, then $a^2+b^2$ is even.

*Proof.* Suppose $a, b$ are any *[particular but arbitrarily chosen]* odd integers.

*[We must show that* $a^2+b^2$ *is even.]*

By definition of odd, $a = 2r+1$ and $b = 2s+1$, for some integers $r, s$.

Then

$$\begin{aligned}
a^2+b^2 &= (2r+1)^2+(2s+1)^2 && \text{by substitution}\\
&= (4r^2+4r+1)+(4s^2+4s+1) && \text{by multiplying}\\
&= 4r^2+4r+4s^2+4s+2 && \text{by adding}\\
&= 2(2r^2+2r+2s^2+2s+1) && \text{by factoring out}
\end{aligned}$$

Let $k = 2r^2+2r+2s^2+2s+1$.

Then $k$ is an integer because squares, products and sums of integers are integers.

Hence $a^2+b^2 = 2k$ where $k$ is an integer, and thus by definition of even $a^2+b^2$ is even *[as was to be shown]*. □

## 2.6 Exercise 6

If $k$ is any odd integer and $m$ is any even integer, then $k^2+m^2$ is odd.

*Proof.* Suppose $k$ is any odd integer and $m$ is any even integer.

*[We must show that $k^2+m^2$ is odd.]*

By definition of odd and even, $k=2a+1$ and $m=2b$, for some integers $a, b$. Then

$$\begin{aligned}
k^2+m^2 &= (2a+1)^2+(2b)^2 && \text{by substitution}\\
&= 4a^2+4a+1+4b^2\\
&= 4(a^2+a+b^2)+1\\
&= 2(2a^2+2a+2b^2)+1 && \text{by algebra}
\end{aligned}$$

But $2a^2+2a+2b^2$ is an integer because it is a sum of products of integers. Thus $k^2+m^2$ is twice an integer plus 1, and so $k^2+m^2$ is odd *[as was to be shown]*. □

## 2.7 Exercise 7

The difference between the squares of any two consecutive integers is odd.

*Proof.* Suppose $m$ and $n$ are any *[particular but arbitrarily chosen]* two consecutive integers.

*[We must show that $m^2-n^2$ (or $n^2-m^2$) is odd.]*

By definition of consecutive, $m=k$ and $n=k+1$, for some integer $k$.

Then

$$\begin{aligned}
m^2-n^2 &= k^2-(k+1)^2 && \text{by substitution}\\
&= k^2-(k^2+2k+1)\\
&= k^2-k^2-2k-1\\
&= -2k-1\\
&= -2k-2+2-1\\
&= 2(-k-1)+1 && \text{by algebra}
\end{aligned}$$

Let $r = -k - 1$. Then $r$ is an integer because it is a difference of integers.

Hence $m^2 - n^2 = 2r + 1$ where $r$ is an integer, and thus by definition of odd $m^2 - n^2$ is odd *[as was to be shown]*. □

## 2.8 Exercise 8

For any integers $m$ and $n$, if $m$ is even and $n$ is odd then $5m + 3n$ is odd.

*Proof.* Suppose $m$ is any even integer and $n$ is any odd integer.

*[We must show that $5m + 3n$ is odd.]*

By definition of even and odd, $m = 2r$ and $n = 2s + 1$, for some integers $r, s$.

Then

$$\begin{aligned} 5m + 3n &= 5(2r) + 3(2s + 1) && \text{by substitution} \\ &= 10r + 6s + 3 \\ &= 10r + 6s + 2 + 1 \\ &= 2(5r + 3s + 1) + 1 && \text{by algebra} \end{aligned}$$

Let $k = 5r + 3s + 1$.

Then $k$ is an integer because it is a sum of products of integers.

Hence $5m + 3n = 2k + 1$ where $k$ is an integer, and thus by definition of odd $5m + 3n$ is odd *[as was to be shown]*. □

## 2.9 Exercise 9

If an integer greater than 4 is a perfect square, then the immediately preceding integer is not prime.

*Proof.* Suppose $n$ is any integer greater than 4 that is a perfect square.

*[We must show that $n - 1$ is not prime, in other words, $n - 1$ is composite.]*

By definition of perfect square, $n = k^2$, for some integer $k$.

Without loss of generality, we may assume $k > 0$, because $n = k^2 = (-k)^2$, and if $k$ is negative, we can replace it with $-k$ which is positive.

Since $n > 4$ we have $n - 4 > 0$. So $k^2 - 4 > 0$. So $(k - 2)(k + 2) > 0$. So either $k - 2$ and $k + 2$ are both negative, or they are both positive. Since $k > 0$, $k + 2 > 2 > 0$, so they have to be both positive. Therefore, $k - 2 > 0$ so $k > 2$.

Then

$$\begin{aligned} n - 1 &= k^2 - 1 && \text{by substitution} \\ &= (k - 1)(k + 1) && \text{by algebra} \end{aligned}$$

Since $k > 2$ we have both $k - 1 > 1$ and $k + 1 > 3 > 1$.

Hence $n - 1$ is a product of two positive integers both greater than 1, and thus by definition of composite $n - 1$ is composite *[as was to be shown]*. □

## 2.10 Exercise 10

If $n$ is any even integer, then $(-1)^n = 1$.

*Proof.* Suppose $n$ is any even integer. *[We must show that $(-1)^n = 1$.]*

By definition of even, $n = 2k$, for some integer $k$.

Then by the laws of exponents from algebra $(-1)^n = (-1)^{2k} = ((-1)^2)^k = 1^k = 1$, *[as was to be shown]*. □

## 2.11 Exercise 11

If $n$ is any odd integer, then $(-1)^n = -1$.

*Proof.* Suppose $n$ is any odd integer. *[We must show that $(-1)^n = -1$.]*

By definition of even, $n = 2k + 1$, for some integer $k$.

Then by the laws of exponents from algebra

$$(-1)^n = (-1)^{2k+1} = (-1)^{2k} \cdot (-1) = ((-1)^2)^k \cdot (-1) = 1^k \cdot (-1) = 1 \cdot (-1) = -1$$

*[as was to be shown]*. □

**Prove that the statements in $12 - 14$ are false.**

## 2.12 Exercise 12

There exists an integer $m \geq 3$ such that $m^2 - 1$ is prime.

*Proof.* To prove the given statement is false, we prove that its negation is true.

The negation of the statement is "For every integer $m \geq 3$, $m^2 - 1$ is not prime."

*Proof of the negation:* Suppose $m$ is any integer with $m \geq 3$.

By basic algebra, $m^2 - 1 = (m - 1)(m + 1)$.

Because $m \geq 3$, both $m - 1$ and $m + 1$ are positive integers greater than 1, and each is smaller than $m^2 - 1$.

So $m^2 - 1$ is a product of two smaller positive integers, each greater than 1, and hence $m^2 - 1$ is not prime. □

## 2.13 Exercise 13

There exists an integer $n$ such that $6n + 27$ is prime.

*Proof.* To prove the given statement is false, we prove that its negation is true.

The negation of the statement is "For every integer $n$, $6n + 27$ is not prime. In other words, $6n + 27$ is composite."

*Proof of the negation:* Suppose $n$ is any integer. By basic algebra, $6n+27 = 3(2n+9)$.

Hence $6n + 27$ is the product of two integers greater than 1. Therefore by definition of composite, $6n + 27$ is composite. □

## 2.14 Exercise 14

There exists an integer $k \geq 4$ such that $2k^2 - 5k + 2$ is prime.

*Proof.* To prove the given statement is false, we prove that its negation is true.

The negation of the statement is "For every integer $k \geq 4$, $2k^2 - 5k + 2$ is composite."

*Proof of the negation:* Suppose $k$ is any integer.

By basic algebra, $2k^2 - 5k + 2 = (2k - 1)(k - 2)$.

Because $k \geq 4$, $2k - 1 \geq 7$ and $k - 2 \geq 2$. So both $2k - 1$ and $k - 2$ are integers greater than 1.

Hence $2k^2 - 5k + 2$ is the product of two integers greater than 1. Therefore by definition of composite, $2k^2 - 5k + 2$ is composite. □

**Find the mistakes in the "proofs" shown in 15 − 19.**

## 2.15 Exercise 15

**Theorem:** For every integer $k$, if $k > 0$ then $k^2 + 2k + 1$ is composite.

"**Proof:** For $k = 2$, $k > 0$ and $k^2 + 2k + 1 = 2^2 + 2 \cdot 2 + 1 = 9$. And since $9 = 3 \cdot 3$, then 9 is composite. Hence the theorem is true."

*Proof.* The incorrect proof just shows the theorem to be true in the one case where $k = 2$. A real proof must show that it is true for *every* integer $k > 0$. □

## 2.16 Exercise 16

**Theorem:** The difference between any odd integer and any even integer is odd.

"**Proof:** Suppose $n$ is any odd integer, and $m$ is any even integer. By definition of odd, $n = 2k+1$ where $k$ is an integer, and by definition of even, $m = 2k$ where $k$ is an integer. Then $n - m = (2k + 1) - 2k = 1$, and 1 is odd. Therefore, the difference between any odd integer and any even integer is odd."

*Proof.* The mistake in the "proof" is that the same symbol, $k$, is used to represent two different quantities. By setting $m = 2k$ and $n = 2k + 1$, the proof implies that $n = m + 1$, and thus it deduces the conclusion only for this one situation. When $m = 4$ and $n = 17$, for instance, the computations in the proof indicate that $n - m = 1$, but actually $n - m = 13$. In other words, the proof does not deduce the conclusion for an arbitrarily chosen even integer $m$ and odd integer $n$, and hence it is invalid. □

## 2.17 Exercise 17

**Theorem:** For every integer $k$, if $k > 0$ then $k^2 + 2k + 1$ is composite.

**Proof:** Suppose $k$ is any integer such that $k > 0$. If $k^2 + 2k + 1$ is composite, then $k^2 + 2k + 1 = rs$ for some integers $r$ and $s$ such that

$1 < r < k^2 + 2k + 1$ and $1 < s < k^2 + 2k + 1$.

Since $k^2 + 2k + 1 = rs$ and both $r$ and $s$ are strictly between 1 and $k^2 + 2k + 1$, then $k^2 + 2k + 1$ is not prime. So $k^2 + 2k + 1$ is composite as was to be shown.

*Proof.* This incorrect proof assumes what is to be proved. The word since in the third sentence is completely unjustified. The second sentence tells only what happens if $k^2 + 2k = 1$ is composite. But at that point in the proof, it has not been established that $k^2 + 2k + 1$ is composite. In fact, that is exactly what is to be proved. □

## 2.18 Exercise 18

**Theorem:** The product of any even integer and any odd integer is even.

"**Proof:** Suppose $m$ is any even integer and $n$ is any odd integer. If $m \cdot n$ is even, then by definition of even there exists an integer $r$ such that $m \cdot n = 2r$.

Also since $m$ is even, there exists an integer $p$ such that $m = 2p$, and since $n$ is odd there exists an integer $q$ such that $n = 2q + 1$.

Thus $mn = (2p)(2q + 1) = 2r$, where $r$ is an integer. By definition of even, then, $m \cdot n$ is even, as was to be shown."

*Proof.* The issue is just like in Exercise 17. The proof uses the $r$ value without establishing the existence of $r$ first.

"If $m \cdot n$ is even..." has an unjustified assumption because we haven't proved that $m \cdot n$ is even yet (that's what we are *trying to prove*), so its conclusion "...$m \cdot n = 2r$" has not been proven.

Therefore the part "$mn = (2p)(2q + 1) = 2r$, where $r$ is an integer" is unjustified as well.

**Discussion:**

This is a fairly common form of circular reasoning: assuming what we have to prove. It happens because, at the beginning of the proof, we want to mention to the reader what we want to prove.

What we want to prove has a short, condensed definition (in this case "being even"), so we write out the full definition of what it is that we are *trying to prove* (in this case "the existence of an integer $r$ such that $\ldots = 2r$"). Again, the purpose of this is to articulate to the reader what we are *trying to prove*.

But then we forget that and continue as if that was already an established fact. The act of writing out the full definition of what we are trying to prove is not the same as actually having proved it.

Using "if $m \cdot n$ is even..." in this case is the problem; it has the *feeling* of using modus ponens on an already established implication with an established premise. But we are just writing out the full definition, instead of using modus ponens.

So it would be better to write: "*We want to prove that $m \cdot n$ is even. In other words, we want to prove that there is an integer $r$ such that $m \cdot n = 2r$.*" Using the words "We want to prove that..." instead of "If..." goes a long way to avoid this common mistake. This way we can "unpack" the definition of what we are trying to prove without assuming it.

Another related problem is to first unpack the definition of what we are trying to prove, then try to "prove backwards". Say we want to prove $A$, and we unpack the definition to B. So we have to prove $B$. But instead, we start by assuming $B$ is true, and apply some algebra or logic to it, to arrive at something else, say $E$, that is true:

$$A \to \text{unpack definition} \to B \to \text{middle steps} \to C \to D \to E = \text{something true!}$$

But this would only prove that $B$ implies $E$. In order to establish the truth of $B$ (and hence of $A$), we would actually have to prove that $E$ implies $B$! So all the "steps" from $B$ to $E$ would have to be "reversible", in other words, logical equivalences (biconditionals):

$$A \leftrightarrow \text{unpack definition} \leftrightarrow B \leftrightarrow \text{middle steps} \leftrightarrow C \leftrightarrow D \leftrightarrow E = \text{something true!}$$

But that is rarely the case! □

## 2.19 Exercise 19

**Theorem:** The sum of any two even integers equals $4k$ for some integer $k$.

"**Proof:** Suppose $m$ and $n$ are any two even integers. By definition of even, $m = 2k$ for some integer $k$ and $n = 2k$ for some integer $k$. By substitution,

$$m + n = 2k + 2k = 4k.$$

This is what was to be shown."

*Proof.* The problem here is the same as in Exercise 16. The mistake in the "proof" is that the same symbol, $k$, is used to represent two different quantities. By setting $m = 2k$

and $n = 2k$, the proof implies that $n = m$, and thus it deduces the conclusion only for this one situation.

When $m = 4$ and $n = 20$, for instance, the proof indicates that $n = m = 4$, but actually $n = 20$. In other words, the proof does not deduce the conclusion for an arbitrarily chosen even integer $m$ and an arbitrarily chosen even integer $n$, and hence it is invalid. □

**In $20 - 38$ determine whether the statement is true or false. Justify your answer with a proof or a counterexample, as appropriate. In each case use only the definitions of the terms and the assumptions listed on page 161, not any previously established properties.**

## 2.20 Exercise 20

The product of any two odd integers is odd.

*Proof.* True. Suppose $m$ and $n$ are any odd integers. *[We must show that $mn$ is odd.]* By definition of odd, $n = 2r + 1$ and $m = 2s + 1$ for some integers $r$ and $s$.

Then

$$
\begin{aligned}
mn &= (2r+1)(2s+1) && \text{by substitution} \\
&= 4rs + 2r + 2s + 1 \\
&= 2(2rs + r + s) + 1 && \text{by algebra}
\end{aligned}
$$

Now $2rs + r + s$ is an integer because products and sums of integers are integers and 2, $r$, and $s$ are all integers. Hence $mn = 2 \cdot (\text{some integer}) + 1$, and so, by definition of odd, $mn$ is odd. □

## 2.21 Exercise 21

The negative of any odd integer is odd.

*Proof.* True. Assume $n$ is any odd integer. *We want to prove $-n$ is odd.*

By definition of odd, $n = 2r + 1$ for some integer $r$.

Then $-n = -(2r + 1) = -2r - 1 = -2r - 2 + 2 - 1 = 2(-r - 1) + 1$.

Let $k = -r - 1$. Then $k$ is an integer because it is the difference of two integers.

Therefore $-n = 2k + 1$ where $k$ is an integer, hence by definition of odd, $-n$ is odd. □

## 2.22 Exercise 22

For all integers $a$ and $b$, $4a + 5b + 3$ is even.

*Proof.* False. Counterexample: Let $a = 1$ and $b = 0$.

Then $4a + 5b + 3 = 4 \cdot 1 + 5 \cdot 0 + 3 = 7$, which is odd.

*[This is one counterexample among many. Can you find a way to characterize all counterexamples?]* □

## 2.23 Exercise 23

The product of any even integer and any integer is even.

*Proof.* True. Suppose $m$ is any even integer and $n$ is any integer. *[We want to prove $m \cdot n$ is even.]*

By definition of even, $m = 2k$ for some integer $k$.

Then, $m \cdot n = (2k) \cdot n = 2kn = 2(kn)$.

Let $r = kn$. Then $r$ is an integer because it is the product of two integers.

Therefore $m \cdot n = 2r$ where $r$ is an integer. So by definition of even, $m \cdot n$ is even. □

## 2.24 Exercise 24

If a sum of two integers is even, then one of the summands is even. (In the expression $a + b$, $a$ and $b$ are called **summands**.)

*Proof.* False. Counterexample: Let $m = 1$ and $n = 3$.

Then $m + n = 4$ is even, but neither summand $m$ nor summand $n$ is even. □

## 2.25 Exercise 25

The difference of any two even integers is even.

*Proof.* True. Assume $m$ and $n$ are any two even integers. *[We want to prove $m - n$ is even.]*

By definition of even, $m = 2r$ and $n = 2s$ for some integers $r, s$.

Then $m - n = 2r - 2s = 2(r - s)$. Let $k = r - s$. Then $k$ is an integer because it is the difference of two integers.

Therefore $m - n = 2k$ where $k$ is an integer. So $m - n$ is even by definition of even. □

## 2.26 Exercise 26

For all integers $a, b$, and $c$, if $a, b$, and $c$ are consecutive, then $a + b + c$ is even.

*Proof.* False. Counterexample: Let $a = 2, b = 3, c = 4$. They are consecutive integers but $a + b + c = 9$ which is not even. □

## 2.27 Exercise 27

The difference of any two odd integers is even.

*Proof.* True. Assume $m, n$ are any two odd integers. *[We want to prove $m - n$ is even.]*

By definition of odd, $m = 2r + 1, n = 2s + 1$ for some integers $r, s$.

Then $m - n = 2r + 1 - (2s + 1) = 2r + 1 - 2s - 1 = 2r - 2s = 2(r - s)$.

Let $k = r - s$. Then $k$ is an integer because it is the difference of two integers.

So $m - n = 2k$ where $k$ is an integer. Hence $m - n$ is even by definition of even. □

## 2.28 Exercise 28

For all integers $n$ and $m$, if $n - m$ is even then $n^3 - m^3$ is even.

*Proof.* True. Assume $n, m$ are any integers such that $n - m$ is even. *[Want to prove that $n^3 - m^3$ is even.]*

By definition of even, $n - m = 2r$ for some integer $r$. By algebra,

$n^3 - m^3 = (n - m)(n^2 + nm + m^2) = 2r(n^2 + nm + m^2) = 2(r(n^2 + nm + m^2))$.

Let $k = r(n^2 + nm + m^2)$. Then $r$ is an integer because it is a sum and product of integers.

So $n^3 - m^3 = 2k$ where $k$ is an integer. So by definition of even, $n^3 - m^3$ is even. □

## 2.29 Exercise 29

For every integer $n$, if $n$ is prime then $(-1)^n = -1$.

*Proof.* False. Counterexample: Let $n = 2$.

Then $n$ is prime, but $(-1)^n = (-1)^2 = 1 \neq -1$. □

## 2.30 Exercise 30

For every integer $m$, if $m > 2$ then $m^2 - 4$ is composite.

*Proof.* False. Counterexample: Let $m = 3$. Then $m^2 - 4 = 3^2 - 4 = 9 - 4 = 5$ is prime. not composite. □

## 2.31 Exercise 31

For every integer $n$, $n^2 - n + 11$ is a prime number.

*Proof.* False. Let $n = 11$. Then $n^2 - n + 11 = 11^2 - 11 + 11 = 11^2$ is not prime. □

## 2.32 Exercise 32

For every integer $n$, $4(n^2+n+1)-3n^2$ is a perfect square.

*Proof.* True. Suppose $n$ is any integer. Then by algebra

$$4(n^2+n+1)-3n^2 = 4n^2+4n+4-3n^2 = n^2+4n+4 = (n+2)^2$$

Now $(n+2)^2$ is a perfect square because $n+2$ is an integer (being a sum of $n$ and 2). Hence $4(n^2+n+1)-3n^2$ is a perfect square, as was to be shown. □

## 2.33 Exercise 33

Every positive integer can be expressed as a sum of three or fewer perfect squares.

*Proof.* False. Counterexample: 7 cannot be written as a sum of three of fewer perfect squares: $7 = 2^2+1^2+1^2+1^2$. □

## 2.34 Exercise 34

(Two integers are **consecutive** if, and only if, one is one more than the other.) Any product of four consecutive integers is one less than a perfect square.

*Proof.* True. Suppose $a, b, c, d$ are any four consecutive integers. *[Want to prove: there is an integer $k$ such that $abcd = k^2-1$.]*

By definition of consecutive, there is an integer $n$ such that $a = n, b = n+1, c = n+2, d = n+3$. Then

$$abcd = n(n+1)(n+2)(n+3) = n(n+3)(n+1)(n+2) = (n^2+3n)(n^2+3n+2)$$

*[Here we notice a pattern. The two factors differ by 2. So it is reminiscent of $(x-1)(x+1) = x^2-1^2$ isn't it?]*

By some more algebra,

$$(n^2+3n)(n^2+3n+2) = (n^2+3n+1-1)(n^2+3n+1+1) = (n^2+3n+1)^2-1^2$$

Let $k = n^2+3n+1$. Then $k$ is an integer because it is a sum and product of integers. Therefore $abcd = k^2-1$ where $k$ is an integer, *[as was to be shown].* □

## 2.35 Exercise 35

If $m$ and $n$ are any positive integers and $mn$ is a perfect square, then $m$ and $n$ are perfect squares.

*Proof.* False. Counterexample: let $m = n = 2$. Then $mn = 2^2$ is a perfect square. But neither $m$ nor $n$ is a perfect square. □

## 2.36 Exercise 36

The difference of the squares of any two consecutive integers is odd.

*Proof.* True. Assume $a, b$ are any two consecutive integers.

By definition of consecutive, $a = n$ and $b = n + 1$ for some integer $n$.

Then $b^2 - a^2 = (n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1$.

So $b^2 - a^2 = 2k + 1$ where $n$ is an integer. Therefore by definition of odd, $b^2 - a^2$ is odd. □

## 2.37 Exercise 37

For all nonnegative real numbers $a$ and $b$, $\sqrt{ab} = \sqrt{a}\sqrt{b}$. (Note that if $x$ is a nonnegative real number, then there is a unique nonnegative real number $y$, denoted $\sqrt{x}$, such that $y^2 = x$.)

*Proof.* True. Assume $a$ and $b$ are any two nonnegative real numbers. By the information given to us in the parentheses:

1. There is a unique nonnegative real number denoted $\sqrt{ab}$ such that $(\sqrt{ab})^2 = ab$.
2. There is a unique nonnegative real number denoted $\sqrt{a}$ such that $(\sqrt{a})^2 = a$.
3. There is a unique nonnegative real number denoted $\sqrt{b}$ such that $(\sqrt{b})^2 = b$.

Since $ab = a \cdot b$, we have by substitution: $(\sqrt{ab})^2 = (\sqrt{a})^2 \cdot (\sqrt{b})^2$.

By algebra, $(\sqrt{ab})^2 = [(\sqrt{a}) \cdot (\sqrt{b})]^2 = (\sqrt{a}\sqrt{b})^2$. Therefore $(\sqrt{ab})^2 - (\sqrt{a}\sqrt{b})^2 = 0$.

By factoring we get $(\sqrt{ab} - \sqrt{a}\sqrt{b})(\sqrt{ab} + \sqrt{a}\sqrt{b}) = 0$.

So: either $\sqrt{ab} - \sqrt{a}\sqrt{b} = 0$, or $\sqrt{ab} + \sqrt{a}\sqrt{b} = 0$ (by the Zero Product Property T11).

If $\sqrt{ab} - \sqrt{a}\sqrt{b} = 0$, then $\sqrt{ab} = \sqrt{a}\sqrt{b}$ *[as was to be shown.]*

If $\sqrt{ab} + \sqrt{a}\sqrt{b} = 0$, then since both $\sqrt{ab}$ and $\sqrt{a}\sqrt{b}$ are nonnegative, they must be both 0, hence $\sqrt{ab} = \sqrt{a}\sqrt{b}$ again *[as was to be shown.]* □

## 2.38 Exercise 38

For all nonnegative real numbers $a$ and $b$, $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$.

*Proof.* False. Counterexample: Let $a = b = 1$. Then

$$\sqrt{a+b} = \sqrt{1+1} = \sqrt{2} \neq 2 = 1 + 1 = \sqrt{1} + \sqrt{1} = \sqrt{a} + \sqrt{b}$$

□

## 2.39 Exercise 39

Suppose that integers $m$ and $n$ are perfect squares. Then $m+n+2\sqrt{mn}$ is also a perfect square. Why?

*Proof.* Assume $m$ and $n$ are perfect squares (so they are nonnegative real numbers). By definition of perfect square, $m = r^2$ and $n = s^2$ for some integers $r, s$. Using Exercise 37 $\sqrt{mn} = \sqrt{m}\sqrt{n}$, we get:

$m + n + 2\sqrt{mn} = r^2 + s^2 + 2\sqrt{m}\sqrt{n} = r^2 + s^2 + 2rs = (r + s)^2$.

Let $k = r + s$. $k$ is an integer because it is a sum of integers. So $m + n + 2\sqrt{mn} = k^2$ where $k$ is an integer, therefore $m + n + 2\sqrt{mn}$ is a perfect square by definition.

□

## 2.40 Exercise 40

If $p$ is a prime number, must $2^p - 1$ also be prime? Prove or give a counterexample.

*Proof.* No. Counterexample: $p = 11$ is prime, but $2^p - 1 = 2^{11} - 1 = 2047 = 13 \cdot 89$ is not prime. □

## 2.41 Exercise 41

If $n$ is a nonnegative integer, must $2^{2n} + 1$ be prime? Prove or give a counterexample.

*Proof.* No. Counterexample: Let $n = 3$. Then $2^{2n} + 1 = 2^6 + 1 = 65 = 13 \cdot 5$ is not prime. □

# 3 Exercise Set 4.3

**The numbers in $1 - 7$ are all rational. Write each number as a ratio of two integers.**

## 3.1 Exercise 1

$$-\frac{35}{6}$$

*Proof.* $\frac{-35}{6} = \frac{-35}{6}$ □

## 3.2 Exercise 2

4.6037

*Proof.* $4.6037 = \frac{46037}{10000}$ □

## 3.3 Exercise 3

$\frac{4}{5} + \frac{2}{9}$

*Proof.* $\frac{4}{5} + \frac{2}{9} = \frac{4 \cdot 9 + 5 \cdot 2}{5 \cdot 9} = \frac{46}{45}$ $\square$

## 3.4 Exercise 4

$0.37373737\ldots$

*Proof.* Let $x = 0.373737\ldots$

Then $100x = 37.373737\ldots$, so $100x - x = 37.373737\ldots - 0.373737\ldots = 37$.

Thus $99x = 37$, and hence $x = \frac{37}{99}$. $\square$

## 3.5 Exercise 5

$0.56565656\ldots$

*Proof.* Let $x = 0.565656\ldots$

Then $100x = 56.565656\ldots$ and so $100x - x = 56.565656\ldots - 0.565656\ldots = 56$.

Thus $99x = 56$, and hence $x = \frac{56}{99}$. $\square$

## 3.6 Exercise 6

$320.5492492492\ldots$

*Proof.* Let $x = 320.5492492492\ldots$.

Then $10000x = 3205492.492492\ldots$ and $10x = 3205.492492\ldots$, so

$10000x - 10x = 3205492.492492\ldots - 3205.492492\ldots = 3205492 - 3205 = 3202287$.

Thus $9990x = 3202287$, and hence $x = \frac{3202287}{9990}$. $\square$

## 3.7 Exercise 7

$52.4672167216721\ldots$

*Proof.* Let $x = 52.467216721\ldots$

Then $100000x = 5246721.67216721\ldots$ and $10x = 524.67216721\ldots$, so

$100000x - 10x = 5246721.67216721\ldots - 524.67216721\ldots = 5246721 - 524 = 5246197$.

Thus $99990x = 5246197$, and hence $x = \frac{5246197}{99990}$. $\square$

## 3.8 Exercise 8

The zero product property, says that if a product of two real numbers is 0, then one of the numbers must be 0.

### 3.8.1 (a)

Write this property formally using quantifiers and variables.

*Proof.* $\forall$ real numbers $x, y$, if $xy = 0$ then $x = 0$ or $y = 0$. □

### 3.8.2 (b)

Write the contrapositive of your answer to part (a).

*Proof.* $\forall$ real numbers $x, y$, if $x \neq 0$ and $y \neq 0$ then $xy \neq 0$. □

### 3.8.3 (c)

Write an informal version (without quantifier symbols or variables) for your answer to part (b).

*Proof.* The product of two nonzero real numbers is nonzero. □

## 3.9 Exercise 9

Assume that $a$ and $b$ are both integers and that $a \neq 0$ and $b \neq 0$. Explain why $(b - a)/(ab^2)$ must be a rational number.

*Proof.* Given that $a$ and $b$ are integers, both $b - a$ and $ab^2$ are integers (since differences and products of integers are integers). Also, by the zero product property, $ab^2 \neq 0$ because neither $a$ nor $b$ is zero. Hence $(b - a)/(ab^2)$ is a quotient of two integers with a nonzero denominator, and so it is rational. □

## 3.10 Exercise 10

Assume that $m$ and $n$ are both integers and that $n \neq 0$. Explain why $(5m - 12n)/(4n)$ must be a rational number.

*Proof.* Given that $m$ and $n$ are integers, both $5m - 12n$ and $4n$ are integers (since differences and products of integers are integers). Also, by the zero product property, $4n \neq 0$ because neither 4 nor $n$ is zero. Hence $(5m - 12n)/(4n)$ is a quotient of two integers with a nonzero denominator, and so it is rational. □

## 3.11 Exercise 11

Prove that every integer is a rational number.

*Proof.* Suppose $n$ is any *[particular but arbitrarily chosen]* integer. Then $n = n \cdot 1$, and so $n = n/1$ by dividing both sides by 1. Now $n$ and 1 are both integers, and $1 \neq 0$. Hence $n$ can be written as a quotient of integers with a nonzero denominator, and so $n$ is rational. □

## 3.12 Exercise 12

Let $S$ be the statement "The square of any rational number is rational." A formal version of $S$ is "For every rational number $r$, $r^2$ is rational." Fill in the blanks in the proof for $S$.

**Proof:** Suppose that $r$ is (a) ____ . By definition of rational, $r = a/b$ for some (b) ____ with $b \neq 0$. By substitution,

$$r^2 = (c)\_\_\_\_ = a^2/b^2.$$

Since $a$ and $b$ are both integers, so are the products $a^2$ and (d) ____ . Also $b^2 \neq 0$ by the (e) ____ . Hence $r^2$ is a ratio of two integers with a nonzero denominator, and so (f) ____ by definition of rational.

*Proof.* (a) any *[particular but arbitrarily chosen]* rational number

(b) integers $a$ and $b$

(c) $(a/b)^2$

(d) $b^2$

(e) zero product property

(f) $r^2$ is rational □

## 3.13 Exercise 13

Consider the following statement: The negative of any rational number is rational.

### 3.13.1 (a)

Write the statement formally using a quantifier and a variable.

*Proof.* $\forall$ real number $r$, if $r$ is rational then $-r$ is rational.

Or: $\forall r$, if $r$ is a rational number then $-r$ is rational.

Or: $\forall$ rational number $r$, $-r$ is rational. □

### 3.13.2 (b)

Determine whether the statement is true or false and justify your answer.

*Proof.* The statement is true. Suppose $r$ is a *[particular but arbitrarily chosen]* rational number. *[We must show that $-r$ is rational.]* By definition of rational, $r = a/b$ for some integers $a$ and $b$ with $b \neq 0$. Then by substitution and algebra,

$$-r = -\frac{a}{b} = \frac{-a}{b}$$

Now since $a$ is an integer, so is $-a$ (being the product of $-1$ and $a$). Hence $-r$ is a quotient of integers with a nonzero denominator, and so $-r$ is rational *[as was to be shown]*. □

## 3.14 Exercise 14

Consider the statement: The cube of any rational number is a rational number.

### 3.14.1 (a)

Write the statement formally using a quantifier and a variable.

*Proof.* $\forall$ rational $r$, $r^3$ is rational. □

### 3.14.2 (b)

Determine whether the statement is true or false and justify your answer.

*Proof.* The statement is true. Suppose $r$ is a *[particular but arbitrarily chosen]* rational number. *[We must show that $r^3$ is rational.]* By definition of rational, $r = a/b$ for some integers $a$ and $b$ with $b \neq 0$. Then by substitution and algebra,

$$r^3 = \left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$$

Now since $a, b$ are integers, so are $a^3$ and $b^3$ (being the products of $a$ and $b$). Moreover, since $b \neq 0$, by the Zero Product Property, $b^3 \neq 0$.

Hence $r^3$ is a quotient of integers with a nonzero denominator, and so $r^3$ is rational *[as was to be shown]*. □

**Determine which of the statements in $15 - 19$ are true and which are false. prove each true statement directly from the definitions, and give a counterexample for each false statement. For a statement that is false, determine whether a small change would make it true. If so, make the change and prove the new statement. Follow the directions for writing proofs on page 173.**

## 3.15 Exercise 15

The product of any two rational numbers is a rational number.

*Proof.* Suppose $r$ and $s$ are rational numbers. By definition of rational, $r = a/b$ and $s = c/d$ for some integers $a, b, c$, and $d$ with $b \neq 0$ and $d \neq 0$. Then by substitution and algebra,

$$rs = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Now $ac$ and $bd$ are both integers (being products of integers) and $bd \neq 0$ (by the zero product property). Hence $rs$ is a quotient of integers with a nonzero denominator, and so, by definition of rational, $rs$ is rational. □

## 3.16 Exercise 16

The quotient of any two rational numbers is a rational number.

*Proof.* Counterexample: Let $r$ be any rational number and $s = 0$. Then $r$ and $s$ are both rational, but the quotient of $r$ divided by $s$ is not a real number and therefore is not a rational number.

*Revised statement to be proved:* For all rational numbers $r$ and $s$, if $s \neq 0$ then $r/s$ is rational.

Suppose $r, s$ are rational numbers such that $s \neq 0$. *[Want to prove $r/s$ is rational.]*

By definition of rational, $r = a/b$ and $s = c/d$ for some integers $a, b, c, d$ where $b \neq 0, d \neq 0$. Since $s \neq 0$ we also have $c \neq 0$. Then by algebra

$$\frac{r}{s} = \frac{a/b}{c/d} = \frac{ad}{bc}$$

Now $ad$ and $bc$ are integers because they are products of integers. Since $b \neq 0$ and $c \neq 0$, by Zero Product Property $bc \neq 0$.

Let $m = ad$ and $n = bc$. So $m$ and $n$ are integers with $n \neq 0$, and $r/s = m/n$. Therefore by definition of rational, $r/s$ is rational. □

## 3.17 Exercise 17

The difference of any two rational numbers is a rational number.

*Proof.* True. Suppose $r, s$ are rational numbers. *[Want to prove $r - s$ is rational.]*

By definition of rational, $r = a/b$ and $s = c/d$ for some integers $a, b, c, d$ where $b \neq 0, d \neq 0$. Then by algebra

$$r - s = \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

Now $ad - bc$ and $bd$ are integers because they are products and differences of integers. Since $b \neq 0$ and $d \neq 0$, by Zero Product Property $bd \neq 0$.

Let $m = ad - bc$ and $n = bd$. So $m$ and $n$ are integers with $n \neq 0$, and $r - s = m/n$. Therefore by definition of rational, $r - s$ is rational. $\square$

## 3.18 Exercise 18

If $r$ and $s$ are any two rational numbers, then $\frac{r+s}{2}$ is rational.

*Proof.* True. The proof is very similar to Exercises 16 and 17. The crucial steps are

$$\frac{r+s}{2} = \frac{\frac{a}{b} + \frac{c}{d}}{2} = \frac{(ad+bc)/bd}{2} = \frac{ad+bc}{2bd}$$

and noticing that $ad + bc$ and $2bd$ are integers, and $2bd \neq 0$. $\square$

## 3.19 Exercise 19

For all real numbers $a$ and $b$, if $a < b$ then $a < \frac{a+b}{2} < b$. (You may use the properties of inequalities in T17-T27 of Appendix A.)

*Proof.* True. Suppose $a, b$ are any two real numbers such that $a < b$. *[We need to prove two inequalities: $a < \frac{a+b}{2}$ and $\frac{a+b}{2} < b$.]*

Since $a < b$ we have $a + a < a + b$ by T19. So $2a < a + b$. Then $a < \frac{a+b}{2}$ by T20. This proves the first inequality.

Since $a < b$ we have $a + b < b + b$ by T19. So $a + b < 2b$. Then $\frac{a+b}{2} < b$ by T20. This proves the second inequality. $\square$

## 3.20 Exercise 20

Use the results of exercises 18 and 19 to prove that given any two rational numbers $r$ and $s$ with $r < s$, there is another rational number between $r$ and $s$. An important consequence is that there are infinitely many rational numbers in between any two distinct rational numbers. See Section 7.4.

*Proof.* Assume $r$ and $s$ are any two rational numbers with $r < s$. *[Want to prove: $r < t < s$ for some rational number $t$.]*

Let $t = \frac{a+b}{2}$. By Exercise 18, $t$ is a rational number. By Exercise 19, $r < t < s$, *[as was to be shown.]* $\square$

**Use the properties of even and odd integers that are listed in Example 4.3.3 to do exercises 21 – 23. Indicate which properties you use to justify your reasoning.**

## 3.21 Exercise 21

True or false? If $m$ is any even integer and $n$ is any odd integer, then $m^2 + 3n$ is odd. Explain.

*Proof.* True.

$m$ is even. An even integer times an even integer is even, therefore $m^2$ is even.

3 and $n$ are both odd. An odd integer times an odd integer is odd, therefore $3n$ is odd.

$m^2$ is even. $3n$ is odd. An even integer plus an odd integer is odd, therefore $m^2 + 3n$ is odd. □

## 3.22 Exercise 22

True or false? If $a$ is any odd integer, then $a^2 + a$ is even. Explain.

*Proof.* True. $a^2 + a = a(a + 1)$. Since $a$ is odd, $a + 1$ is even. Odd times even is even, therefore $a(a + 1)$ is even. So $a^2 + a$ is even. □

## 3.23 Exercise 23

True or false? If $k$ is any even integer and $m$ is any odd integer, then $(k+2)^2-(m-1)^2$ is even. Explain.

*Proof.* True.

$k$ is even, so $k + 2$ is even. Even squared is even, so $(k + 2)^2$ is even.

$m$ is odd, so $m - 1$ is even. Even squared is even, so $(m - 1)^2$ is even.

Even minus even is even, so $(k + 2)^2 - (m - 1)^2$ is even.

*Another solution.* By algebra:

$(k + 2)^2 - (m - 1)^2 = (k + 2 - (m - 1))(k + 2 + m - 1) = (k - m + 3)(k + m + 1)$

Now $k - m + 3$ is even $-$ odd $+$ odd $=$ even. Even times anything is even, therefore $(k - m + 3)(k + m + 1)$ is even. So $(k + 2)^2 - (m - 1)^2$ is even. □

**Derive the statements in $24 - 26$ as corollaries of theorems 4.3.1, 4.3.2, and the results of exercises 12, 13, 14, 15, and 17.**

## 3.24 Exercise 24

For any rational numbers $r$ and $s$, $2r + 3s$ is rational.

*Proof.* Suppose $r$ and $s$ are any rational numbers. By Theorem 4.3.1, both 2 and 3 are rational, and so, by Exercise 15, both $2r$ and $3s$ are rational. Hence, by Theorem 4.3.2, $2r + 3s$ is rational. □

## 3.25 Exercise 25

If $r$ is any rational number, then $3r^2 - 2r + 4$ is rational.

*Proof.* Suppose $r$ is any rational number. By Exercise 12, $r^2$ is rational. By Theorem 4.3.1, $2, 3, 4$ are all rational. By Exercise 15, $3r^2$ and $2r$ are rational. By Exercise 17, $3r^2 - 2r$ is rational. So by Theorem 4.3.2, $3r^2 - 2r + 4$ is rational. □

## 3.26 Exercise 26

For any rational number $s$, $5s^3 + 8s^2 - 7$ is rational.

*Proof.* Assume $s$ is any rational number. By Theorem 4.3.1, $5, 8, 7$ are rational, and by Exercise 13, $-7$ is rational. By Exercise 14, $s^3$ is rational. By Exercise 12, $s^2$ is rational. By Exercise 15, $5s^3$ and $8s^2$ are rational. Therefore by Theorem 4.3.2, $5s^3 + 8s^2 - 7$ is rational. □

## 3.27 Exercise 27

It is a fact that if $n$ is any nonnegative integer, then

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = \frac{1 - (1/2^{n+1})}{1 - (1/2)}$$

(A more general form of this statement is proved in Section 5.2.) Is the right-hand side of this equation rational? If so, express it as a ratio of two integers.

*Proof.*

$$x = \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = \frac{\frac{2^{n+1} - 1}{2^{n+1}}}{\frac{1}{2}} = \frac{2^{n+1} - 1}{2^{n+1}} \cdot \frac{2}{1} = \frac{2^{n+1} - 1}{2^n}$$

Now $2^{n+1} - 1$ and $2^n$ are both integers (since $n$ is a nonnegative integer) and $2^n \neq 0$ by the zero product property. Therefore, $x$ is rational. □

## 3.28 Exercise 28

Suppose $a, b, c$, and $d$ are integers and $a \neq c$. Suppose also that $x$ is a real number that satisfies the equation

$$\frac{ax + b}{cx + d} = 1.$$

Must $x$ be rational? If so, express $x$ as a ratio of two integers.

*Proof.*

$$\frac{ax + b}{cx + d} = 1 \to ax + b = cx + d \to ax - cx + b - d = 0 \to x(a - c) = d - b \to x = \frac{d - b}{a - c}$$

$x$ is rational because both $d - b$ and $a - c$ are rational, and because $a - c \neq 0$ (since $a \neq c$). □

## 3.29 Exercise 29

Suppose $a, b$, and $c$ are integers and $x, y$, and $z$ are nonzero real numbers that satisfy the following equations:

$$\frac{xy}{x+y} = a, \frac{xz}{x+z} = b, \frac{yz}{y+z} = c$$

Is $x$ rational? If so, express it as ratio of two integers.

*Proof.* Taking the reciprocals of both sides of the first equation:

$$\frac{x+y}{xy} = \frac{1}{a}$$

Now we split the first fraction into two, and simplify:

$$\frac{x+y}{xy} = \frac{x}{xy} + \frac{y}{xy} = \frac{1}{y} + \frac{1}{x}$$

Therefore $\frac{1}{y} + \frac{1}{x} = \frac{1}{a}$. By performing the same steps on the other two equations, we see that $\frac{1}{z} + \frac{1}{x} = \frac{1}{b}$ and $\frac{1}{z} + \frac{1}{y} = \frac{1}{c}$.

For the sake of simplicity let's do some renaming: let

$$X = 1/x,\ Y = 1/y,\ Z = 1/z,\ A = 1/a,\ B = 1/b,\ C = 1/c$$

So the three new equations we derived above become:

$$\begin{array}{ccccc} Y & + & X & = & A & (1) \\ Z & + & X & = & B & (2) \\ Z & + & Y & = & C & (3) \end{array}$$

From (1) we get (4): $Y = A - X$ and from (2) we get (5): $Z = B - X$.

Substituting (4) and (5) back into (3) we get: $(B - X) + (A - X) = C$.

So $B + A - 2X = C$, then $B + A - C = 2X$ and $\frac{1}{2}(B + A - C) = X$. Using our old variable names, we get:

$$\frac{1}{2}\left(\frac{1}{b} + \frac{1}{a} - \frac{1}{c}\right) = \frac{1}{x}$$

Rewriting:

$$\frac{1}{2b} + \frac{1}{2a} - \frac{1}{2c} = \frac{1}{x}$$

Getting a common denominator, then adding:

$$\frac{ac}{2abc} + \frac{bc}{2abc} - \frac{ab}{2abc} = \frac{ac+bc-ac}{2abc} = \frac{1}{x}$$

Finally, taking reciprocals of both sides, we get $x$ as a ratio of two integers:

$$\frac{2abc}{ac+bc-ac} = x$$

□

## 3.30 Exercise 30

Prove that if one solution for a quadratic equation of the form $x^2+bx+c=0$ is rational (where $b$ and $c$ are rational), then the other solution is also rational. (Use the fact that if the solutions of the equation are $r$ and $s$, then $x^2+bx+c=(x-r)(x-s)$.)

*Proof.* Assume $x^2+bx+c=0$ has two solutions $r,s$ where one of them, $r$, is rational. *[Want to prove: s is also rational.]*

We are given the fact that $x^2+bx+c=(x-r)(x-s)$. This holds true for all real numbers $x$. Solving for $s$, we get

$$\frac{x^2+bx+c}{x-r}=x-s \implies \frac{x^2+bx+c}{x-r}-x=-s \implies -\frac{x^2+bx+c}{x-r}+x=s$$

This equality is true for all real $x$ except $x=r$ (because then division by $x-r$ would be illegal). So, let's substitute an $x$ value that is rational and different than $r$, say $r+1$. Then we get:

$$-\frac{(r+1)^2+b(r+1)+c}{r+1-r}+r+1=s \implies -(r+1)^2+b(r+1)+c+r+1=s$$

Now we use the facts established in the Exercises that sums, negatives, products and squares of rational numbers are rational. Since $r,b,c,1$ are all rational, this implies that $s$ is rational. □

## 3.31 Exercise 31

Prove that if a real number $c$ satisfies a polynomial equation of the form

$$r_3x^3+r_2x^2+r_1x+r_0=0$$

where $r_0,r_1,r_2,r_3$ are rational numbers, then $c$ satisfies an equation of the form

$$n_3x^3+n_2x^2+n_1x+n_0=0$$

where $n_0,n_1,n_2,n_3$ are integers.

*Proof.* Suppose $c$ is a real number such that $r_3c^3+r_2c^2+r_1c+r_0=0$, where $r_0,r_1,r_2,r_3$ are rational numbers.

By definition of rational, $r_0=a_0/b_0, r_1=a_1/b_1, r_2=a_2/b_2, r_3=a_3/b_3$ for some integers $a_0,a_1,a_2,a_3$ and some nonzero integers $b_0,b_1,b_2,b_3$. By substitution,

$$\begin{aligned} r_3c^3+r_2c^2+r_1c+r_0 &= \frac{a_3}{b_3}c^3+\frac{a_2}{b_2}c^2+\frac{a_1}{b_1}c+\frac{a_0}{b_0} \\ &= \frac{b_0b_1b_2a_3}{b_0b_1b_2b_3}c^3+\frac{b_0b_1b_3a_2}{b_0b_1b_2b_3}c^2+\frac{b_0b_2b_3a_1}{b_0b_1b_2b_3}c+\frac{b_1b_2b_3a_0}{b_0b_1b_2b_3} \\ &= 0. \end{aligned}$$

Multiplying both sides by $b_0b_1b_2b_3$ gives

$$b_0b_1b_2a_3 \cdot c^3 + b_0b_1b_3a_2 \cdot c^2 + b_0b_2b_3a_1 \cdot c + b_1b_2b_3a_0 = 0$$

Let $n_3 = b_0b_1b_2a_3, n_2 = b_0b_1b_3a_2, n_1 = b_0b_2b_3a_1, n_0 = b_1b_2b_3a_0$. Then $n_3, n_2, n_1, n_0$ are all integers (being products of integers). Hence $c$ satisfies the equation

$$n_3 \cdot c^3 + n_2 \cdot c^2 + n_1 \cdot c + n_0 = 0$$

where $n_3, n_2, n_1, n_0$ are all integers, *[as was to be shown.]* □

**Definition:** A number $c$ is called a **root** of a polynomial $p(x)$ if, and only if, $p(c) = 0$.

## 3.32 Exercise 32

Prove that for every real number $c$, if $c$ is a root of a polynomial with rational coefficients, then $c$ is a root of a polynomial with integer coefficients.

*Proof.* The proof is extremely similar to Exercise 31. Assume $p(c) = 0$ where:

$c$ is any real number, and $p(x) = \frac{a_n}{b_n}x^n + \cdots + \frac{a_1}{b_1}x + \frac{a_0}{b_0}$ is a polynomial with rational coefficients (so $a_0, \ldots, a_n$ are all integers and $b_0, \ldots, b_n$ are all nonzero integers).

So $c$ satisfies the equation

$$\frac{a_n}{b_n}c^n + \cdots + \frac{a_1}{b_1}c + \frac{a_0}{b_0} = 0$$

Multiply both sides by $L = b_0b_1 \cdots b_{n-1}b_n$:

$$\frac{a_nL}{b_n}c^n + \cdots + \frac{a_1L}{b_1}c + \frac{a_0L}{b_0} = 0$$

Notice that $\frac{a_nL}{b_n}, \ldots, \frac{a_0L}{b_0}$ are all integers (because all the denominators can be cancelled out with one of the factors of $L$). So $c$ is the root of a polynomial

$$q(x) = \frac{a_nL}{b_n}x^n + \cdots + \frac{a_1L}{b_1}x + \frac{a_0L}{b_0}$$

where $q(x)$ has all integer coefficients, *[as was to be shown.]* □

**Use the properties of even and odd integers that are listed in example 4.3.3 to do exercises 33 and 34.**

## 3.33 Exercise 33

When expressions of the form $(x - r)(x - s)$ are multiplied out, a quadratic polynomial is obtained. For instance, $(x - 2)(x - (-7)) = (x - 2)(x + 7) = x^2 + 5x - 14$.

### 3.33.1 (a)

What can be said about the coefficients of the polynomial obtained by multiplying out $(x-r)(x-s)$ when both $r$ and $s$ are odd integers? When both $r$ and $s$ are even integers? When one of $r$ and $s$ is even and the other is odd?

*Proof.* Note that $(x-r)(x-s) = x^2 - (r+s)x + rs$.

If both $r$ and $s$ are odd, then $r+s$ is even and $rs$ is odd. So the coefficient of $x^2$ is 1 (odd), the coefficient of $x$ is even, and the constant coefficient, $rs$, is odd.

If both $r$ and $s$ are even, then $r+s$ is even and $rs$ is even. So the coefficient of $x^2$ is 1 (odd), the coefficient of $x$ is even, and the constant coefficient, $rs$, is even.

If one of $r$ and $s$ is even and the other is odd, then $r+s$ is odd and $rs$ is even. So the coefficient of $x^2$ is 1 (odd), the coefficient of $x$ is odd, and the constant coefficient, $rs$, is even. □

### 3.33.2 (b)

It follows from part (a) that $x^2 - 1253x + 255$ cannot be written as a product of two polynomials with integer coefficients. Explain why this is so.

*Proof.* Assume $x^2 - 1253x + 255 = (x-r)(x-s)$ where $r, s$ are real numbers. So $r+s = 1253$ and $rs = 255$. If $r, s$ are both integers, then since $rs = 255$, $r$ and $s$ must be both odd. But this is impossible, because then $r+s$ would be even, but 1253 is not even! Therefore $r, s$ cannot be both integers. □

## 3.34 Exercise 34

Observe that

$$(x-r)(x-s)(x-t) = x^3 - (r+s+t)x^2 + (rs+rt+st)x - rst$$

### 3.34.1 (a)

Derive a result for cubic polynomials similar to the result in part (a) of exercise 33 for quadratic polynomials.

*Proof.* If $r, s, t$ are all odd, then the constant coefficient is odd, the coefficient of $x$ is odd, the coefficient of $x^2$ is odd, and the coefficient of $x^3$ is odd (it's 1).

If exactly one of $r, s, t$ is even, then the constant coefficient is even, the coefficient of $x$ is odd, the coefficient of $x^2$ is even, and the coefficient of $x^3$ is odd (it's 1).

If exactly two of $r, s, t$ are even, then the constant coefficient is even, the coefficient of $x$ is even, the coefficient of $x^2$ is odd, and the coefficient of $x^3$ is odd (it's 1).

If $r, s, t$ are all even, then the constant coefficient is even, the coefficient of $x$ is even, the coefficient of $x^2$ is even, and the coefficient of $x^3$ is odd (it's 1). □

### 3.34.2 (b)

Can $x^3 + 7x^2 - 8x - 27$ be written as a product of three polynomials with integer coefficients? Explain.

*Proof.* Assume $x^3 + 7x^2 - 8x - 27 = (x - r)(x - s)(x - t)$ where $r, s, t$ are integers.

The coefficient of $x^2$ is 7, which is odd. So by part (a), either $r, s, t$ are all odd, or exactly two of them are even.

They can't be all odd, because then the coefficient of $x$ would be odd, but it's $-8$ which is even.

If exactly two of them are even, then the constant coefficient would have to be even, but it's $-27$ which is odd.

So it's impossible for $r, s, t$ to be all integers. □

**In 35 – 39 find the mistakes in the "proofs" that the sum of any two rational numbers is a rational number.**

## 3.35 Exercise 35

"**Proof:** Any two rational numbers produce a rational number when added together. So if $r$ and $s$ are particular but arbitrarily chosen rational numbers, then $r + s$ is rational."

*Proof.* This "proof" assumes what is to be proved. □

## 3.36 Exercise 36

"**Proof:** Let rational numbers $r = \frac{1}{4}$ and $s = \frac{1}{2}$ be given. Then $r + s = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$, which is a rational number. This is what was to be shown."

*Proof.* This "proof" argues from a single example. It does not establish the result for *[arbitrarily chosen]* rational numbers. □

## 3.37 Exercise 37

"**Proof:** Suppose $r$ and $s$ are rational numbers. By definition of rational, $r = a/b$ for some integers $a$ and $b$ with $b \neq 0$, and $s = a/b$ for some integers $a$ and $b$ with $b \neq 0$. Then $r + s = \dfrac{a}{b} + \dfrac{a}{b} = \dfrac{2a}{b}$.

Let $p = 2a$. Then p is an integer since it is a product of integers. Hence $r + s = p/b$, where $p$ and $b$ are integers and $b \neq 0$. Thus $r + s$ is a rational number by definition of rational. This is what was to be shown."

*Proof.* By setting both $r$ and $s$ equal to $a/b$, this incorrect proof violates the requirement that $r$ and $s$ be arbitrarily chosen rational numbers. If both $r$ and $s$ equal $a/b$, then $r = s$. □

### 3.38 Exercise 38

"**Proof:** Suppose $r$ and $s$ are rational numbers. Then $r = a/b$ and $s = c/d$ for some integers $a, b, c$, and $d$ with $b \neq 0$ and $d \neq 0$ (by definition of rational). Then

$$r = \frac{a}{b} + \frac{c}{d}$$

But this is a sum of two fractions, which is a fraction. So $r + s$ is a rational number since a rational number is a fraction."

*Proof.* This "proof" does not establish that "the sum of two fractions is a fraction". Also, "a rational number is a fraction" is ambiguous. A rational number is a quotient of two integers (with nonzero denominator). But "a fraction" could be a fraction of non-integer numbers too. □

### 3.39 Exercise 39

"**Proof:** Suppose $r$ and $s$ are rational numbers. If $r + s$ is rational, then by definition of rational $r + s = a/b$ for some integers $a$ and $b$ with $b \neq 0$.

Also since $r$ and $s$ are rational, $r = i/j$ and $s = m/n$ for some integers $i, j, m$, and $n$ with $j \neq 0$ and $n \neq 0$. It follows that

$$r + s = \frac{i}{j} + \frac{m}{n} = \frac{a}{b}$$

which is a quotient of two integers with a nonzero denominator. Hence it is a rational number. This is what was to be shown."

*Proof.* This "proof" assumes what is to be proved. □

# 4 Exercise Set 4.4

### 4.1 Exercise 1

*Proof.* □

### 4.2 Exercise 2

*Proof.* □

### 4.3 Exercise 3

*Proof.* □

### 4.4 Exercise 4

*Proof.* □

## 4.5 Exercise 5

*Proof.* ∎

## 4.6 Exercise 6

*Proof.* ∎

## 4.7 Exercise 7

*Proof.* ∎

## 4.8 Exercise 8

*Proof.* ∎

## 4.9 Exercise 9

*Proof.* ∎

## 4.10 Exercise 10

*Proof.* ∎

## 4.11 Exercise 11

*Proof.* ∎

## 4.12 Exercise 12

*Proof.* ∎

## 4.13 Exercise 13

*Proof.* ∎

## 4.14 Exercise 14

*Proof.* ∎

## 4.15 Exercise 15

*Proof.* ∎

## 4.16 Exercise 16

*Proof.* ∎

## 4.17 Exercise 17

*Proof.* □

## 4.18 Exercise 18

### 4.18.1 (a)

*Proof.* □

### 4.18.2 (b)

*Proof.* □

## 4.19 Exercise 19

*Proof.* □

## 4.20 Exercise 20

*Proof.* □

## 4.21 Exercise 21

*Proof.* □

## 4.22 Exercise 22

*Proof.* □

## 4.23 Exercise 23

*Proof.* □

## 4.24 Exercise 24

*Proof.* □

## 4.25 Exercise 25

*Proof.* □

## 4.26 Exercise 26

*Proof.* □

## 4.27 Exercise 27

*Proof.* □

## 4.28 Exercise 28

*Proof.* □

## 4.29 Exercise 29

*Proof.* □

## 4.30 Exercise 30

*Proof.* □

## 4.31 Exercise 31

*Proof.* □

## 4.32 Exercise 32

*Proof.* □

## 4.33 Exercise 33

*Proof.* □

## 4.34 Exercise 34

*Proof.* □

## 4.35 Exercise 35

*Proof.* □

## 4.36 Exercise 36

### 4.36.1 (a)

*Proof.* □

### 4.36.2 (b)

*Proof.* □

### 4.36.3 (c)

*Proof.* □

### 4.36.4 (d)

*Proof.* □

## 4.37 Exercise 37

### 4.37.1 (a)

*Proof.* □

### 4.37.2 (b)

*Proof.* □

### 4.37.3 (c)

*Proof.* □

## 4.38 Exercise 38

### 4.38.1 (a)

*Proof.* □

### 4.38.2 (b)

*Proof.* □

### 4.38.3 (c)

*Proof.* □

### 4.38.4 (d)

*Proof.* □

## 4.39 Exercise 39

### 4.39.1 (a)

*Proof.* □

### 4.39.2 (b)

*Proof.* □

## 4.40 Exercise 40

### 4.40.1 (a)

*Proof.* □

### 4.40.2 (b)

*Proof.* □

## 4.41 Exercise 41

*Proof.* □

## 4.42 Exercise 42

### 4.42.1 (a)

*Proof.* □

### 4.42.2 (b)

*Proof.* □

### 4.42.3 (c)

*Proof.* □

## 4.43 Exercise 43

*Proof.* □

## 4.44 Exercise 44

*Proof.* □

## 4.45 Exercise 45

*Proof.* □

## 4.46 Exercise 46

*Proof.* □

## 4.47 Exercise 47

*Proof.* □

### 4.48 Exercise 48

*Proof.* □

### 4.49 Exercise 49

*Proof.* □

### 4.50 Exercise 50

*Proof.* □

## 5 Exercise Set 4.5

### 5.1 Exercise 1

*Proof.* □

### 5.2 Exercise 2

*Proof.* □

### 5.3 Exercise 3

*Proof.* □

### 5.4 Exercise 4

*Proof.* □

### 5.5 Exercise 5

*Proof.* □

### 5.6 Exercise 6

*Proof.* □

### 5.7 Exercise 7

#### 5.7.1 (a)

*Proof.* □

#### 5.7.2 (b)

*Proof.* □

## 5.8 Exercise 8

### 5.8.1 (a)

*Proof.* □

### 5.8.2 (b)

*Proof.* □

## 5.9 Exercise 9

### 5.9.1 (a)

*Proof.* □

### 5.9.2 (b)

*Proof.* □

## 5.10 Exercise 10

### 5.10.1 (a)

*Proof.* □

### 5.10.2 (b)

*Proof.* □

## 5.11 Exercise 11

### 5.11.1 (a)

*Proof.* □

### 5.11.2 (b)

*Proof.* □

### 5.11.3 (c)

*Proof.* □

## 5.12 Exercise 12

*Proof.* □

## 5.13 Exercise 13

*Proof.* □

## 5.14 Exercise 14

*Proof.* □

## 5.15 Exercise 15

*Proof.* □

## 5.16 Exercise 16

*Proof.* □

## 5.17 Exercise 17

*Proof.* □

## 5.18 Exercise 18

### 5.18.1 (a)

*Proof.* □

### 5.18.2 (b)

*Proof.* □

## 5.19 Exercise 19

*Proof.* □

## 5.20 Exercise 20

*Proof.* □

## 5.21 Exercise 21

*Proof.* □

## 5.22 Exercise 22

*Proof.* □

## 5.23 Exercise 23

*Proof.* □

## 5.24 Exercise 24

*Proof.* □

## 5.25 Exercise 25

*Proof.* □

## 5.26 Exercise 26

*Proof.* □

## 5.27 Exercise 27

*Proof.* □

## 5.28 Exercise 28

### 5.28.1 (a)

*Proof.* □

### 5.28.2 (b)

*Proof.* □

## 5.29 Exercise 29

### 5.29.1 (a)

*Proof.* □

### 5.29.2 (b)

*Proof.* □

## 5.30 Exercise 30

### 5.30.1 (a)

*Proof.* □

### 5.30.2 (b)

*Proof.* □

## 5.31 Exercise 31

### 5.31.1 (a)

*Proof.* □

### 5.31.2 (b)

*Proof.* □

### 5.31.3 (c)

*Proof.* □

## 5.32 Exercise 32

*Proof.* □

## 5.33 Exercise 33

*Proof.* □

## 5.34 Exercise 34

*Proof.* □

## 5.35 Exercise 35

*Proof.* □

## 5.36 Exercise 36

*Proof.* □

## 5.37 Exercise 37

*Proof.* □

## 5.38 Exercise 38

*Proof.* □

## 5.39 Exercise 39

*Proof.* □

## 5.40 Exercise 40

*Proof.* □

## 5.41 Exercise 41

*Proof.* □

## 5.42 Exercise 42

*Proof.* □

## 5.43 Exercise 43

*Proof.* □

## 5.44 Exercise 44

### 5.44.1 (a)

*Proof.* □

### 5.44.2 (b)

*Proof.* □

### 5.44.3 (c)

*Proof.* □

## 5.45 Exercise 45

*Proof.* □

## 5.46 Exercise 46

*Proof.* □

## 5.47 Exercise 47

*Proof.* □

## 5.48 Exercise 48

*Proof.* □

### 5.49 Exercise 49

*Proof.* ☐

### 5.50 Exercise 50

*Proof.* ☐

## 6 Exercise Set 4.6

### 6.1 Exercise 1

*Proof.* ☐

### 6.2 Exercise 2

*Proof.* ☐

### 6.3 Exercise 3

*Proof.* ☐

### 6.4 Exercise 4

*Proof.* ☐

### 6.5 Exercise 5

*Proof.* ☐

### 6.6 Exercise 6

*Proof.* ☐

### 6.7 Exercise 7

*Proof.* ☐

### 6.8 Exercise 8

*Proof.* ☐

### 6.9 Exercise 9

*Proof.* ☐

## 6.10 Exercise 10

### 6.10.1 (a)

*Proof.* □

### 6.10.2 (b)

*Proof.* □

## 6.11 Exercise 11

*Proof.* □

## 6.12 Exercise 12

*Proof.* □

## 6.13 Exercise 13

*Proof.* □

## 6.14 Exercise 14

*Proof.* □

## 6.15 Exercise 15

*Proof.* □

## 6.16 Exercise 16

*Proof.* □

## 6.17 Exercise 17

*Proof.* □

## 6.18 Exercise 18

*Proof.* □

## 6.19 Exercise 19

*Proof.* □

## 6.20 Exercise 20

*Proof.* □

## 6.21 Exercise 21

*Proof.* □

## 6.22 Exercise 22

*Proof.* □

## 6.23 Exercise 23

*Proof.* □

## 6.24 Exercise 24

*Proof.* □

## 6.25 Exercise 25

*Proof.* □

## 6.26 Exercise 26

*Proof.* □

## 6.27 Exercise 27

*Proof.* □

## 6.28 Exercise 28

*Proof.* □

## 6.29 Exercise 29

*Proof.* □

## 6.30 Exercise 30

*Proof.* □

## 6.31 Exercise 31

*Proof.* □

### 6.32 Exercise 32

*Proof.* □

### 6.33 Exercise 33

*Proof.* □

# 7 Exercise Set 4.7

## 7.1 Exercise 1

*Proof.* □

## 7.2 Exercise 2

*Proof.* □

## 7.3 Exercise 3

*Proof.* □

## 7.4 Exercise 4

*Proof.* □

## 7.5 Exercise 5

*Proof.* □

## 7.6 Exercise 6

*Proof.* □

## 7.7 Exercise 7

*Proof.* □

## 7.8 Exercise 8

*Proof.* □

## 7.9 Exercise 9

### 7.9.1 (a)

*Proof.* □

### 7.9.2 (b)

*Proof.* □

## 7.10 Exercise 10

*Proof.* □

## 7.11 Exercise 11

*Proof.* □

## 7.12 Exercise 12

### 7.12.1 (a)

*Proof.* □

### 7.12.2 (b)

*Proof.* □

## 7.13 Exercise 13

### 7.13.1 (a)

*Proof.* □

### 7.13.2 (b)

*Proof.* □

## 7.14 Exercise 14

### 7.14.1 (a)

*Proof.* □

### 7.14.2 (b)

*Proof.* □

## 7.15 Exercise 15

*Proof.* □

## 7.16 Exercise 16

*Proof.* □

## 7.17 Exercise 17

*Proof.* ∎

## 7.18 Exercise 18

*Proof.* ∎

## 7.19 Exercise 19

*Proof.* ∎

## 7.20 Exercise 20

*Proof.* ∎

## 7.21 Exercise 21

### 7.21.1 (a)

*Proof.* ∎

### 7.21.2 (b)

*Proof.* ∎

## 7.22 Exercise 22

### 7.22.1 (a)

*Proof.* ∎

### 7.22.2 (b)

*Proof.* ∎

## 7.23 Exercise 23

*Proof.* ∎

## 7.24 Exercise 24

*Proof.* ∎

## 7.25 Exercise 25

*Proof.* ∎

## 7.26 Exercise 26

*Proof.* ∎

## 7.27 Exercise 27

*Proof.* ∎

## 7.28 Exercise 28

*Proof.* ∎

## 7.29 Exercise 29

*Proof.* ∎

## 7.30 Exercise 30

### 7.30.1 (a)

*Proof.* ∎

### 7.30.2 (b)

*Proof.* ∎

## 7.31 Exercise 31

### 7.31.1 (a)

*Proof.* ∎

### 7.31.2 (b)

*Proof.* ∎

### 7.31.3 (c)

*Proof.* ∎

## 7.32 Exercise 32

*Proof.* ∎

## 7.33 Exercise 33

*Proof.* ∎

### 7.34 Exercise 34

#### 7.34.1 (a)

*Proof.* ☐

#### 7.34.2 (b)

*Proof.* ☐

#### 7.34.3 (c)

*Proof.* ☐

#### 7.34.4 (d)

*Proof.* ☐

### 7.35 Exercise 35

*Proof.* ☐

### 7.36 Exercise 36

*Proof.* ☐

## 8 Exercise Set 4.8

### 8.1 Exercise 1

*Proof.* ☐

### 8.2 Exercise 2

*Proof.* ☐

### 8.3 Exercise 3

*Proof.* ☐

### 8.4 Exercise 4

*Proof.* ☐

### 8.5 Exercise 5

*Proof.* ☐

## 8.6 Exercise 6

*Proof.* □

## 8.7 Exercise 7

*Proof.* □

## 8.8 Exercise 8

*Proof.* □

## 8.9 Exercise 9

*Proof.* □

## 8.10 Exercise 10

*Proof.* □

## 8.11 Exercise 11

*Proof.* □

## 8.12 Exercise 12

*Proof.* □

## 8.13 Exercise 13

*Proof.* □

## 8.14 Exercise 14

*Proof.* □

## 8.15 Exercise 15

*Proof.* □

## 8.16 Exercise 16

*Proof.* □

## 8.17 Exercise 17

*Proof.* □

## 8.18 Exercise 18

### 8.18.1 (a)

*Proof.* □

### 8.18.2 (b)

*Proof.* □

## 8.19 Exercise 19

### 8.19.1 (a)

*Proof.* □

### 8.19.2 (b)

*Proof.* □

### 8.19.3 (c)

*Proof.* □

## 8.20 Exercise 20

*Proof.* □

## 8.21 Exercise 21

*Proof.* □

## 8.22 Exercise 22

*Proof.* □

## 8.23 Exercise 23

*Proof.* □

## 8.24 Exercise 24

*Proof.* □

## 8.25 Exercise 25

*Proof.* □

## 8.26 Exercise 26

*Proof.* □

## 8.27 Exercise 27

*Proof.* □

## 8.28 Exercise 28

*Proof.* □

## 8.29 Exercise 29

*Proof.* □

## 8.30 Exercise 30

### 8.30.1 (a)

*Proof.* □

### 8.30.2 (b)

*Proof.* □

## 8.31 Exercise 31

*Proof.* □

## 8.32 Exercise 32

*Proof.* □

## 8.33 Exercise 33

*Proof.* □

## 8.34 Exercise 34

### 8.34.1 (a)

*Proof.* □

### 8.34.2 (b)

*Proof.* □

### 8.35 Exercise 35

*Proof.* □

### 8.36 Exercise 36

*Proof.* □

### 8.37 Exercise 37

*Proof.* □

### 8.38 Exercise 38

*Proof.* □

## 9 Exercise Set 4.9

### 9.1 Exercise 1

*Proof.* □

### 9.2 Exercise 2

*Proof.* □

### 9.3 Exercise 3

*Proof.* □

### 9.4 Exercise 4

*Proof.* □

### 9.5 Exercise 5

*Proof.* □

### 9.6 Exercise 6

*Proof.* □

### 9.7 Exercise 7

*Proof.* □

## 9.8 Exercise 8

*Proof.* □

## 9.9 Exercise 9

*Proof.* □

## 9.10 Exercise 10

*Proof.* □

## 9.11 Exercise 11

*Proof.* □

## 9.12 Exercise 12

*Proof.* □

## 9.13 Exercise 13

*Proof.* □

## 9.14 Exercise 14

### 9.14.1 (a)

*Proof.* □

### 9.14.2 (b)

*Proof.* □

## 9.15 Exercise 15

### 9.15.1 (a)

*Proof.* □

### 9.15.2 (b)

*Proof.* □

## 9.16 Exercise 16

### 9.16.1 (a)

*Proof.* □

#### 9.16.2 (b)

*Proof.* ∎

### 9.17 Exercise 17

*Proof.* ∎

### 9.18 Exercise 18

*Proof.* ∎

### 9.19 Exercise 19

*Proof.* ∎

### 9.20 Exercise 20

#### 9.20.1 (a)

*Proof.* ∎

#### 9.20.2 (b)

*Proof.* ∎

### 9.21 Exercise 21

#### 9.21.1 (a)

*Proof.* ∎

#### 9.21.2 (b)

*Proof.* ∎

#### 9.21.3 (c)

*Proof.* ∎

### 9.22 Exercise 22

*Proof.* ∎

### 9.23 Exercise 23

#### 9.23.1 (a)

*Proof.* ∎

#### 9.23.2 (b)

*Proof.* ∎

#### 9.23.3 (c)

*Proof.* ∎

#### 9.23.4 (d)

*Proof.* ∎

#### 9.23.5 (e)

*Proof.* ∎

#### 9.23.6 (f)

*Proof.* ∎

### 9.24 Exercise 24

#### 9.24.1 (a)

*Proof.* ∎

#### 9.24.2 (b)

*Proof.* ∎

#### 9.24.3 (c)

*Proof.* ∎

#### 9.24.4 (d)

*Proof.* ∎

#### 9.24.5 (e)

*Proof.* ∎

#### 9.24.6 (f)

*Proof.* ∎

### 9.25 Exercise 25

*Proof.* ∎

# 10 Exercise Set 4.10

## 10.1 Exercise 1

*Proof.* □

## 10.2 Exercise 2

*Proof.* □

## 10.3 Exercise 3

### 10.3.1 (a)

*Proof.* □

### 10.3.2 (b)

*Proof.* □

## 10.4 Exercise 4

*Proof.* □

## 10.5 Exercise 5

*Proof.* □

## 10.6 Exercise 6

*Proof.* □

## 10.7 Exercise 7

*Proof.* □

## 10.8 Exercise 8

### 10.8.1 (a)

*Proof.* □

### 10.8.2 (b)

*Proof.* □

## 10.9 Exercise 9

*Proof.* □

## 10.10 Exercise 10

*Proof.* □

## 10.11 Exercise 11

*Proof.* □

## 10.12 Exercise 12

*Proof.* □

## 10.13 Exercise 13

*Proof.* □

## 10.14 Exercise 14

*Proof.* □

## 10.15 Exercise 15

*Proof.* □

## 10.16 Exercise 16

*Proof.* □

## 10.17 Exercise 17

*Proof.* □

## 10.18 Exercise 18

*Proof.* □

## 10.19 Exercise 19

*Proof.* □

## 10.20 Exercise 20

*Proof.* □

## 10.21 Exercise 21

*Proof.* □

## 10.22 Exercise 22

*Proof.* □

## 10.23 Exercise 23

### 10.23.1 (a)

*Proof.* □

### 10.23.2 (b)

*Proof.* □

## 10.24 Exercise 24

*Proof.* □

## 10.25 Exercise 25

### 10.25.1 (a)

*Proof.* □

### 10.25.2 (b)

*Proof.* □

## 10.26 Exercise 26

### 10.26.1 (a)

*Proof.* □

### 10.26.2 (b)

*Proof.* □

## 10.27 Exercise 27

### 10.27.1 (a)

*Proof.* □

### 10.27.2 (b)

*Proof.* $\square$

### 10.27.3 (c)

*Proof.* $\square$

## 10.28 Exercise 28

### 10.28.1 (a)

*Proof.* $\square$

### 10.28.2 (b)

*Proof.* $\square$

### 10.28.3 (c)

*Proof.* $\square$

## 10.29 Exercise 29

*Proof.* $\square$

## 10.30 Exercise 30

*Proof.* $\square$

## 10.31 Exercise 31

*Proof.* $\square$

## 10.32 Exercise 32

*Proof.* $\square$