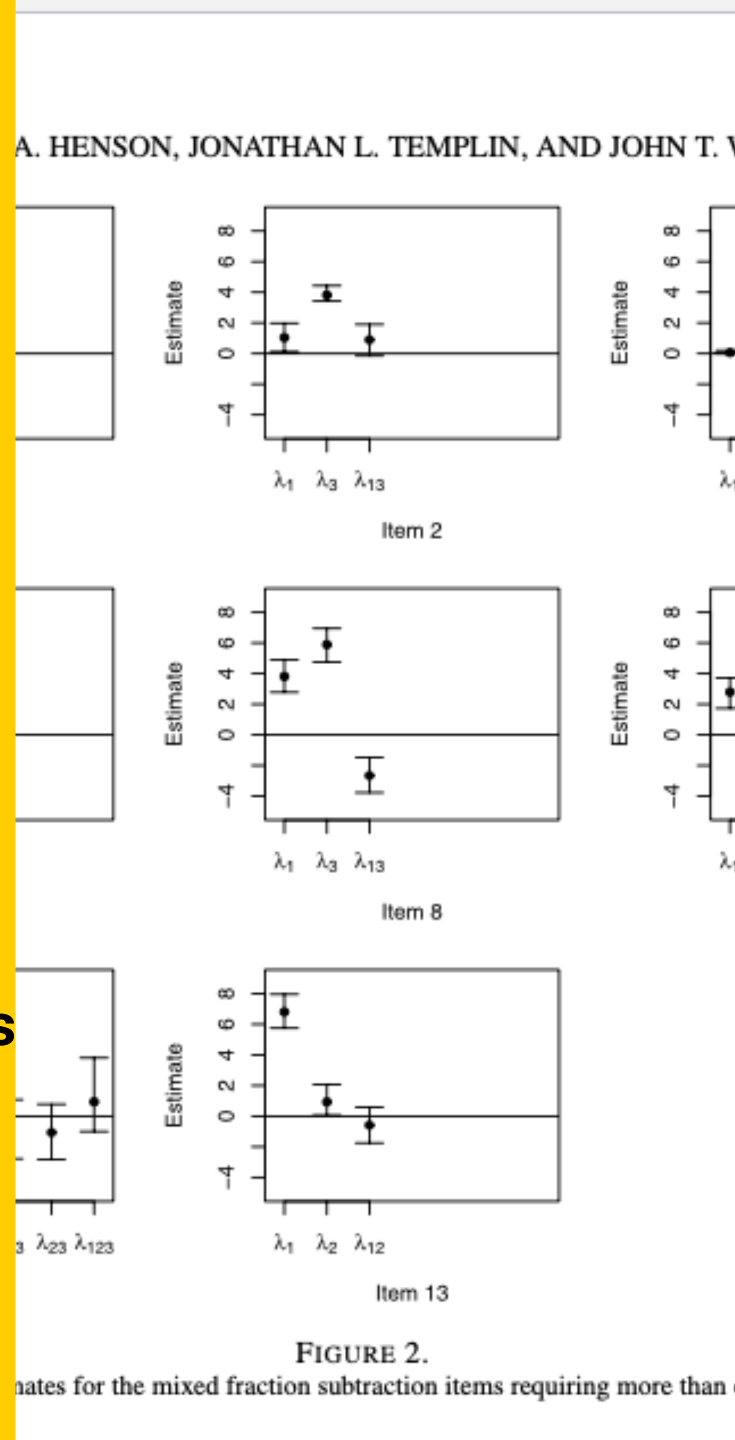
Diagnostic Classification Models

Multidimensional Measurement Models

Lecture 11: November 8, 2023



Lecture Objectives

- Discuss relevant mathematical prerequisites for understanding diagnostic measurement models
- Introduce the loglinear cognitive diagnosis model – a general measurement model for DCMs
- Show some models the LCDM subsumes

Development of Psychometric Models

- Over the past 20+ years, numerous DCMs have been developed
  - We will focus on DCMs that use latent variables for attributes
  - This lecture focus on the Loglinear Cognitive Diagnosis Model
- Each DCM makes assumptions about how mastered attributes combine/interact to produce an item response
  - Compensatory/disjunctive/additive models
  - Non-compensatory/conjunctive/non-additive models
- With so many models, analysts have been unsure which model would best fit their purpose
  - Difficult to imagine all items following same assumptions

General Models for Diagnosis

- Recent developments have produced very general diagnostic models
  - General Diagnostic Model (**GDM**; von Davier, 2005)
  - Loglinear Cognitive Diagnosis Model (**LCDM**; Henson, Templin, & Willse, 2009)
    - Focus of this session
  - Generalized DINA Model (G-DINA; de la Torre, 2011)
    - Is equivalent to the LCDM

- The LCDM provides great modeling flexibility
  - Subsume all other latent variable DCMs
  - Allow both additive and non-additive relationships between attributes/items
  - Sync with other psychometric models allowing for greater understanding of modeling process

Lecture Overview

- Background information
  - ANOVA models and the LCDM
- Logits explained
- The LCDM
  - Parameter structure
  - One-item demonstration
- LCDM general form
- Linking the LCDM to other earlier-developed DCMs

Notation Used Throughout Lecture

- **Attributes**: $a = 1, \dots, A$
- **Respondents**: $r = 1, \dots, R$
- **Attribute Profiles**: $\boldsymbol{\alpha}_r = [\alpha_{r1}, \alpha_{r2}, \dots, \alpha_{rA}]$
  - Each attribute α_{ra} today is defined as being 0 or 1: $\alpha_{ra} \in \{0,1\}$
- **Latent Classes**: $c = 1, \dots, C$
  - We have $C = 2^A$ latent classes – one for each possible attribute profile
  - An attribute profile is a specific permutation of all A attributes
- **Items**: $i = 1, \dots, I$
  - Restricted to dichotomous item responses (either 0 or 1): $Y_{ri} \in \{0,1\}$
- **Q-matrix**: Elements q_{ia} are indicators an item i measures attribute a
  - q_{ia} is either 0 (does not measure a) or 1 (measures a): $q_{ia} \in \{0,1\}$

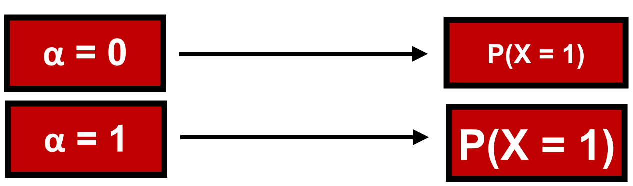
Background Information: ANOVA models

Background Information – ANOVA

- The LCDM models the probability of a correct response to an item as a function of the latent attributes of a respondent

- The latent attributes are categorical, meaning a respondent can have one of a finite set of many possible statuses
  - Each status corresponds to a predicted probability of a correct response

- As such, the LCDM is very similar to an ANOVA model
  - Predicting the a dependent variable as a function of the "experimental group" of a respondent



ANOVA Refresher

- As a refresher on ANOVA, lets imagine that we are interested in the factors that have an effect on *work output* (denoted by Y)

- We design a two-factor study where *work output* may be affected by:
  - Lighting of the workplace
    - High or Low
  - Temperature
    - Cold or Warm

- This experimental design is known as a 2-Way ANOVA

ANOVA Model

- Here is the 2 x 2 Factorial design:

	Low Lighting	High Lighting
Cold Temperature	$\bar{Y}_{Cold,Low}$	$\bar{Y}_{Cold,High}$
Warm Temperature	$\bar{Y}_{Warm,Low}$	$\bar{Y}_{Warm,High}$

- The ANOVA model for a respondent's work output is

$$Y_r = \mu + A_t + B_l + (AB)_{tl} + \varepsilon_r$$

ANOVA Model

- The ANOVA model allows us to test for the presence of
  - A main effect associated with *Temperature* (A_t)
    - Where $A_{Cold} + A_{Warm} = 0$
  - A main effect associated with *Lighting* (B_l)
    - Where $B_{Low} + B_{High} = 0$
  - An interaction effect associated with *Temperature* and *Lighting* $(AB)_{tl}$
    - Where $(AB)_{Cold,Low} + (AB)_{Cold,High} + (AB)_{Warm,Low} + (AB)_{Warm,High} = 0$

$$Y_r = \mu + A_t + B_l + (AB)_{tl} + \varepsilon_r$$

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ANOVA with Dummy Coded Variables

- The ANOVA model can also be re-written using two dummy-coded variables

$Warm_r =$

- 0 for respondents in ***cold temperature*** condition
- 1 for respondents in ***warm temperature*** condition

$High_r =$

- 0 for respondents in ***low lighting*** condition
- 1 for respondents in ***high lighting*** condition

ANOVA with Dummy Coded Variables

	$High_r$ = 0 Low Lighting	*$High_r$ = 1* High Lighting
$Warm_r$ = 0 Cold Temperature	$\bar{Y}_{Cold,Low}$	$\bar{Y}_{Cold,High}$
$Warm_r$ = 1 Warm Temperature	$\bar{Y}_{Warm,Low}$	$\bar{Y}_{Warm,High}$

$$Y_r = \beta_0 + \beta_t Warm_r + \beta_l High_r + \beta_{t*l} Warm_r * High_r + \varepsilon_r$$

ANOVA Effects Explained

$$Y_r = \beta_0 + \beta_t Warm_r + \beta_l High_r + \beta_{t*l} Warm_r * High_r + \varepsilon_r$$

- β_0 is the mean for the cold and low light condition (reference group)
  - The intercept
- β_t is the difference in the average response for warm temperature for a business with low lights $High_r = 0$ (Conditional Main Effect)
- β_l is the difference in the average response for high lights for a business with cold temperature $Warm_r = 0$ (Conditional Main Effect)
- β_{t*l} is additional change in average that is not explained by the shift in temperature and shift and lights, when both occur (2-Way Interaction)
- Respondents from in the same condition have the same predicted value

ANOVA and the LCDM

- The ANOVA model and the LCDM take the same modeling approach
  - Predict a response using dummy coded variables
    - In LCDM dummy coded variables are latent attributes
  - Using a set of main effects and interactions
    - Links attributes to item response
  - We may look for ways to reduce the model
    - Removing non-significant interactions and/or main effects

Differences Between LCDM and ANOVA

- The LCDM and the ANOVA model differ in two ways:
  - Instead of a continuous outcome such as work output the LCDM models a function of the probability of a correct response
    - The logit of a correct response (defined next)
  - Instead of observed "factors" as predictors the LCDM uses discrete *latent* variables (the attributes being measured)
- Attributes are given dummy codes (act as latent factors)
  - $\alpha_{ra} = 1$ if respondent *r* has **mastered** attribute *a*
  - $\alpha_{ra} = 0$ if respondent *r* has **not mastered** attribute *a*
- The LCDM treats the attributes as *crossed* experimental factors: all combinations are assumed to exist
  - This assumption can be (and will be) modified

Logits Explained

Model Background

- The LCDM models the log-odds of a correct response conditional on a respondent's attribute pattern $\boldsymbol{\alpha}_r$
  - The log-odds is called a logit

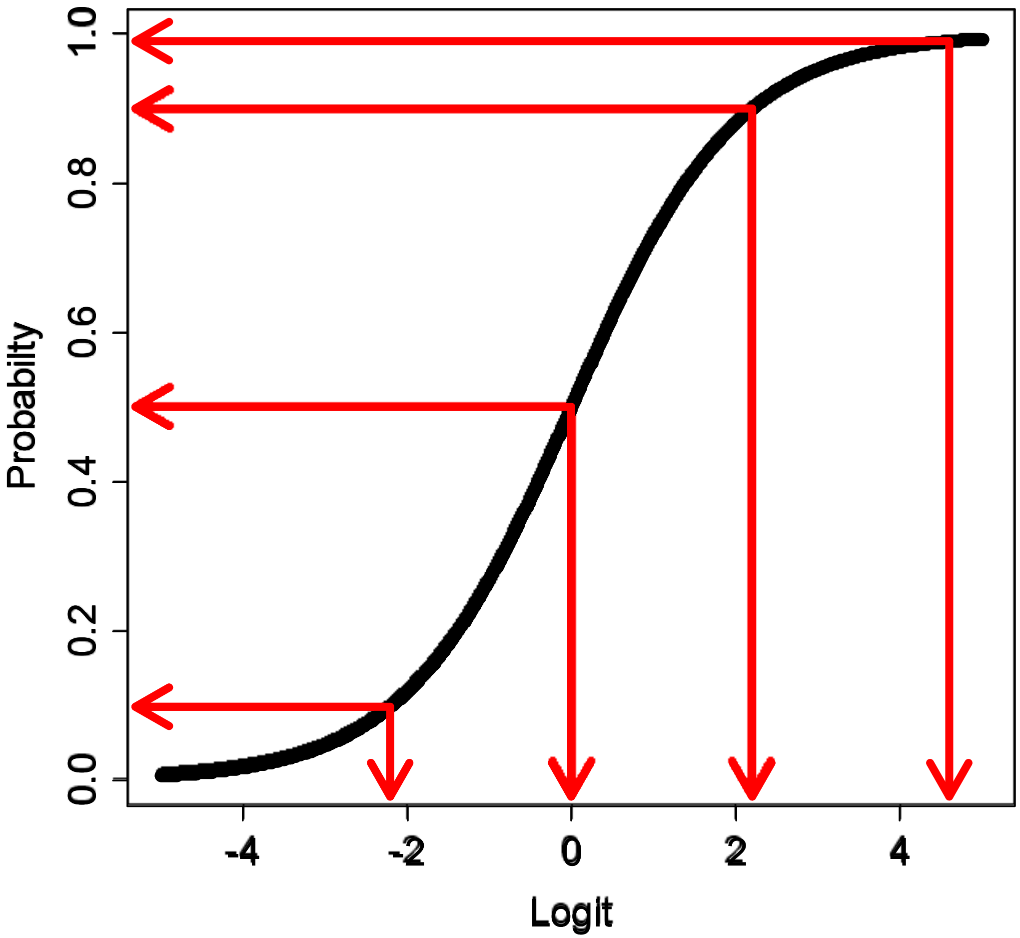
$$Logit(Y_{ri} = 1|\boldsymbol{\alpha}_r) = \log\left(\frac{P(Y_{ri} = 1|\boldsymbol{\alpha}_r)}{1 - P(Y_{ri} = 1|\boldsymbol{\alpha}_r)}\right)$$

  - Here $\log(\cdot)$ is the natural log

- The logit is used because the responses are binary
  - Items are either answered correctly (1) or incorrectly (0)

- The linear model with an identity link and Gaussian error is inappropriate for categorical data
  - Can lead to impossible predictions (i.e., probabilities greater than 1 or less than 0)

More on Logits



Probability	Logit
0.5	0.0
0.9	2.2
0.1	-2.2
0.99	4.6

From Logits to Probabilities

- Logits are useful as they are unbounded continuous variables
  - But they are hard to interpret
  - Better as a probability

- The inverse logit function coverts the unbounded logit to a probability
  - This is also the form of an IRT model (and logistic regression)

$$P(Y_{ri}=1|\boldsymbol{\alpha}_r)=\frac{\exp\left(Logit(Y_{ri}=1|\boldsymbol{\alpha}_r)\right)}{1+\exp\left(Logit(Y_{ri}=1|\boldsymbol{\alpha}_r)\right)}$$

  - Here, $\exp(\cdot)=2.718282$: the inverse function of the natural log (Euler's number)
  - Sometimes this is written:

$$P(Y_{ri}=1|\boldsymbol{\alpha}_r)=\frac{\exp\left(Logit(Y_{ri}=1|\boldsymbol{\alpha}_r)\right)}{1+\exp\left(Logit(Y_{ri}=1|\boldsymbol{\alpha}_r)\right)}=\left[1+\exp\left(-Logit(Y_{ri}=1|\boldsymbol{\alpha}_r)\right)\right]^{-1}$$

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The LCDM

The Loglinear Cognitive Diagnosis Model

Building the LCDM

- To demonstrate the LCDM, consider the item 2+3-1=?
  from a basic math example
  - Measures addition (attribute 1: α_{r1}) and subtraction (attribute 2: α_{r2})

- Only attributes defined by the Q-matrix are modeled for an item

- The LCDM provides the logit of a correct response as a function of the latent attributes mastered by a respondent:

$$Logit(Y_{ri} = 1|\boldsymbol{\alpha}_r) = \lambda_{i,0} + \lambda_{i,1,(1)}\alpha_{r1} + \lambda_{i,1,(2)}\alpha_{r2} + \lambda_{i,2,(1,2)}\alpha_{r1}\alpha_{r2}$$

LCDM Explained

$$\boxed{Logit(Y_{ri} = 1|\boldsymbol{\alpha}_r)} = \boxed{\lambda_{i,0}} + \lambda_{i,1,(1)}\alpha_{r1} + \lambda_{i,1,(2)}\alpha_{r2} + \lambda_{i,2,(1,2)}\alpha_{r1}\alpha_{r2}$$

- $Logit(Y_{ri} = 1|\boldsymbol{\alpha}_r)$ is the logit of a correct response to item *i* by respondent *r*

- $\lambda_{i,0}$ is the intercept
  - The logit for non-masters of addition and subtraction
  - The reference group is respondents who have not mastered ***either*** attribute ($\alpha_{r1} = 0$ and $\alpha_{r2} = 0$)

LCDM Explained

$$Logit(Y_{ri} = 1|\boldsymbol{\alpha}_r) = \lambda_{i,0} + \boxed{\lambda_{i,1,(1)}}\alpha_{r1} + \boxed{\lambda_{i,1,(2)}}\alpha_{r2} + \boxed{\lambda_{i,2,(1,2)}}\alpha_{r1}\alpha_{r2}$$

- $\lambda_{i,1,(1)}$= **<u>conditional</u> main effect** for addition (attribute 1)
  - The increase in the logit for mastering addition (for someone who has *<u>not mastered</u>* subtraction)
- $\lambda_{i,1,(2)}$= **<u>conditional</u> main effect** for subtraction (attribute 2)
  - The increase in the logit for mastering subtraction (for someone who has *<u>not mastered</u>* addition)
- $\lambda_{i,2,(1,2)}$= is the **2-way interaction** between addition and subtraction (attributes 1 and 2)
  - Change in the logit for mastering ***both*** addition & subtraction

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Understanding LCDM Notation

- The LCDM item parameters have several subscripts:

$$\lambda_{\boxed{i},\boxed{e},\boxed{(a_1,\ldots)}}$$

- Subscript #1 – i: the item to which parameters belong

- Subscript #2 – e: the level of the effect
  - 0 is the intercept
  - 1 is the main effect
  - 2 is the two-way interaction
  - 3 is the three-way interaction

- Subscript #3 –$(a_1, \ldots)$: the attributes to which the effect applies
  - Same number of attributes listed as number in Subscript #2

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LCDM: A Numerical example

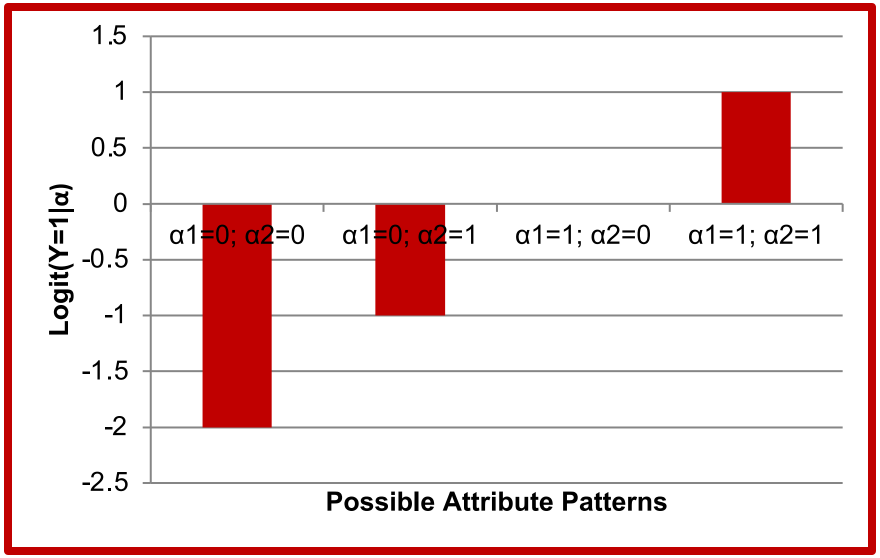
LCDM with Example Numbers

- Imagine we obtained the following estimates for the item $2 + 3 - 1 = ?$:

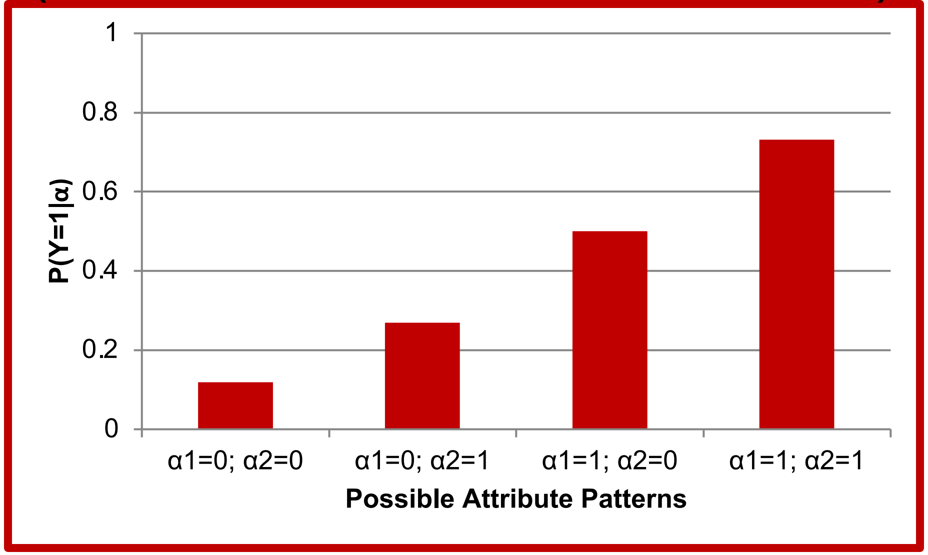
Parameter	Estimate	Effect Name
$\lambda_{i,0}$	-2	Intercept
$\lambda_{i,1,(1)}$	2	Addition Conditional Main Effect
$\lambda_{i,1,(2)}$	1	Subtraction Conditional Main Effect
$\lambda_{i,2,(1,2)}$	0	Addition/Subtraction Interaction

α_{r1}	α_{r2}	LCDM Logit Function	Logit	Probability
0	0	$\lambda_{i,0}+\lambda_{i,1,(1)}\times(0)+\lambda_{i,1,(2)}\times(0)+\lambda_{i,2,(1,2)}\times(0)\times(0)$	-2	0.12
0	1	$\lambda_{i,0}+\lambda_{i,1,(1)}\times(0)+\lambda_{i,1,(2)}\times(1)+\lambda_{i,2,(1,2)}\times(0)\times(1)$	-1	0.27
1	0	$\lambda_{i,0}+\lambda_{i,1,(1)}\times(1)+\lambda_{i,1,(2)}\times(0)+\lambda_{i,2,(1,2)}\times(1)\times(0)$	0	0.50
1	1	$\lambda_{i,0}+\lambda_{i,1,(1)}\times(1)+\lambda_{i,1,(2)}\times(1)+\lambda_{i,2,(1,2)}\times(1)\times(1)$	1	0.73

Logit Response Function



Probability Response Function (Item Characteristic Bar Chart)



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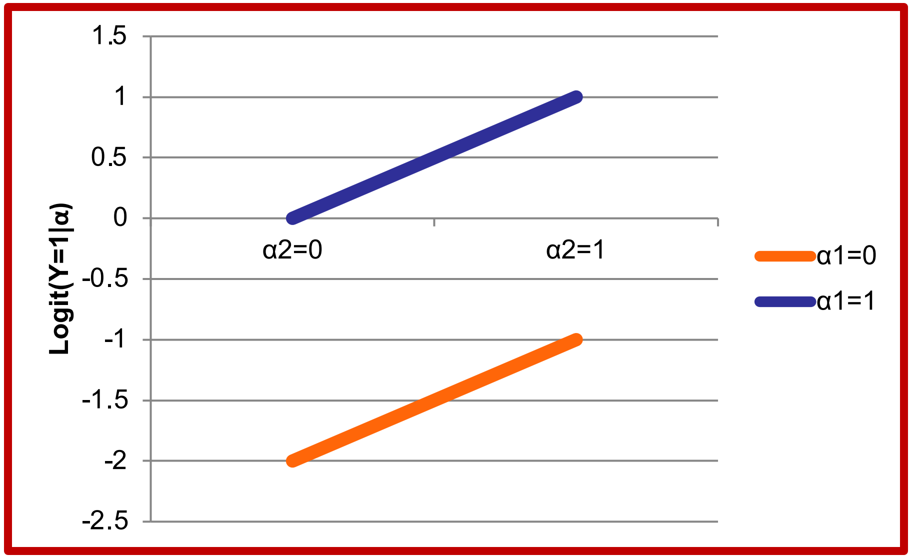
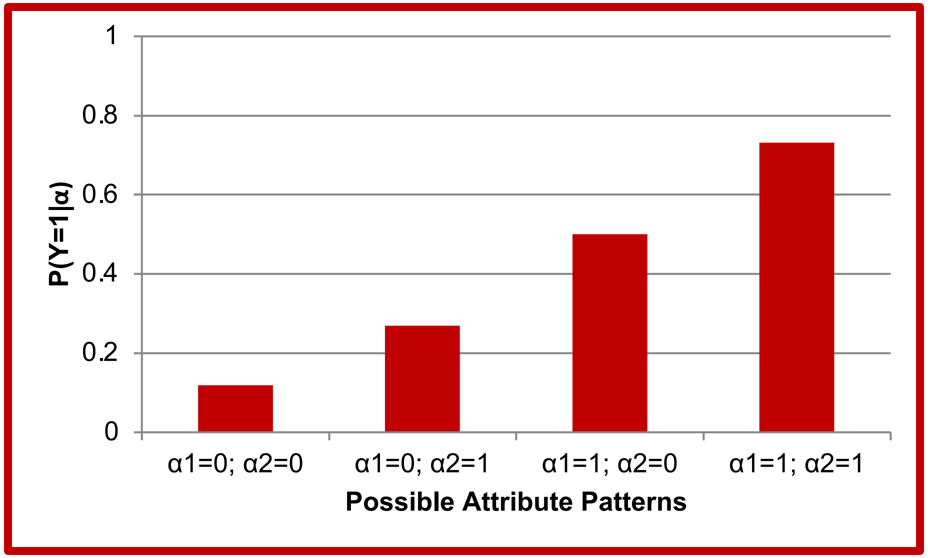
LCDM Interaction Plots

- The LCDM interaction term can be investigated via plots
- **No interaction**: parallel lines for the logit
  - Compensatory RUM (Hartz, 2002)

Logit Response Function



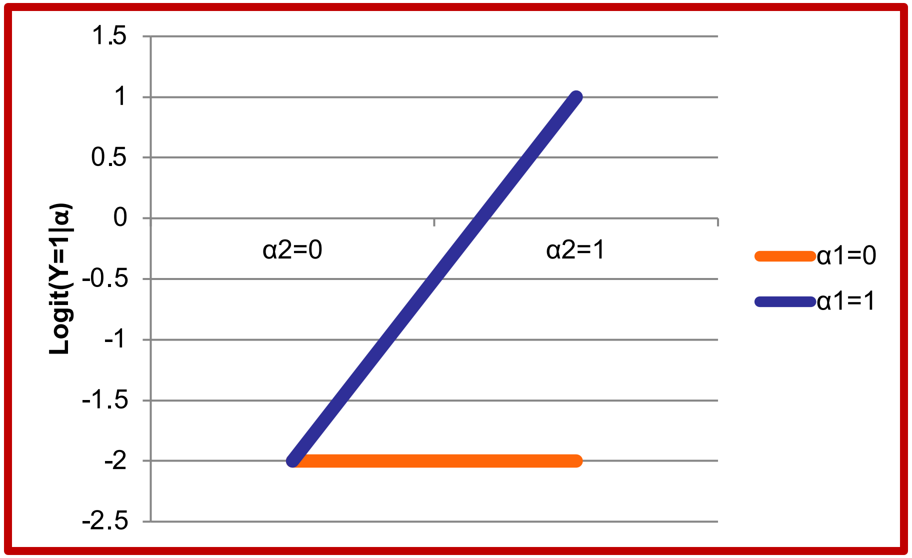
Probability Response Function (Item Characteristic Bar Chart)



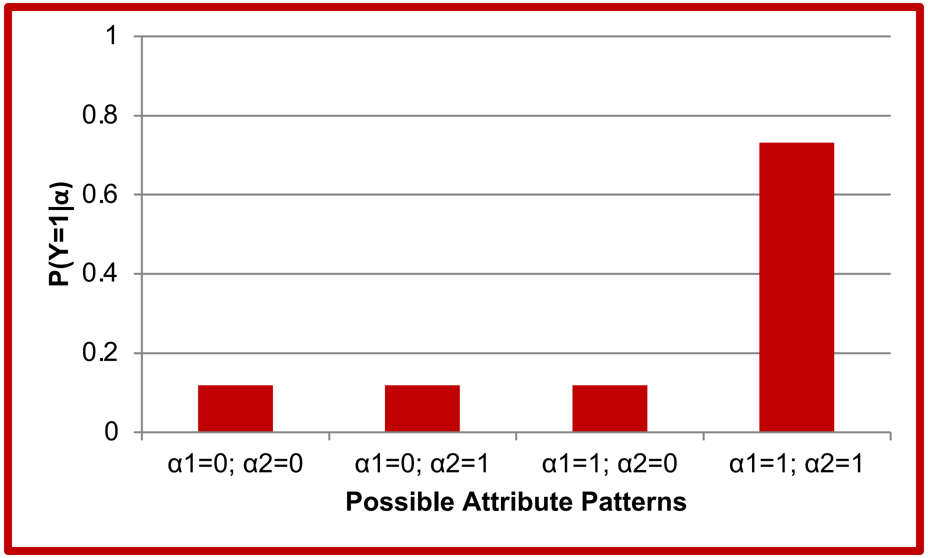
Strong Positive Interactions

- **Positive interaction**: over-additive logit model
  - Conjunctive model (i.e., all-or-none)
  - DINA model (Haertel, 1989; Junker & Sijtsma, 1999)

Logit Response Function



Probability Response Function (Item Characteristic Bar Chart)



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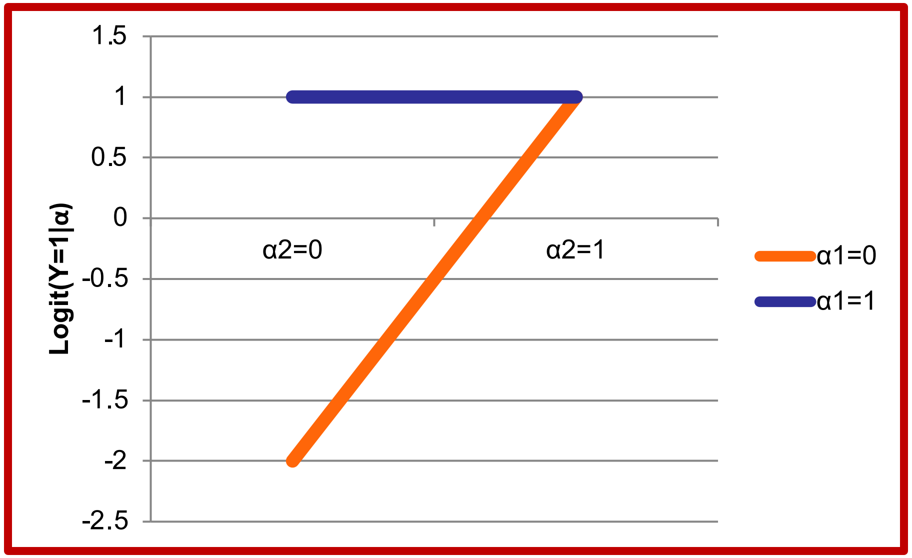
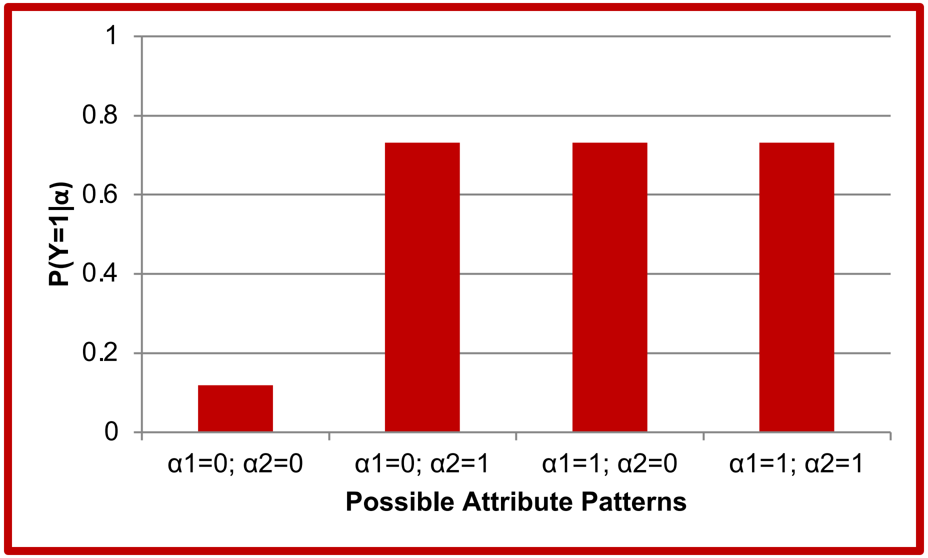
Strong Negative Interactions

- **Negative interaction**: under-additive logit model
  - Disjunctive model (i.e., one-or-more)
  - DINO model (Templin & Henson, 2006)

Logit Response Function



Probability Response Function (Item Characteristic Bar Chart)



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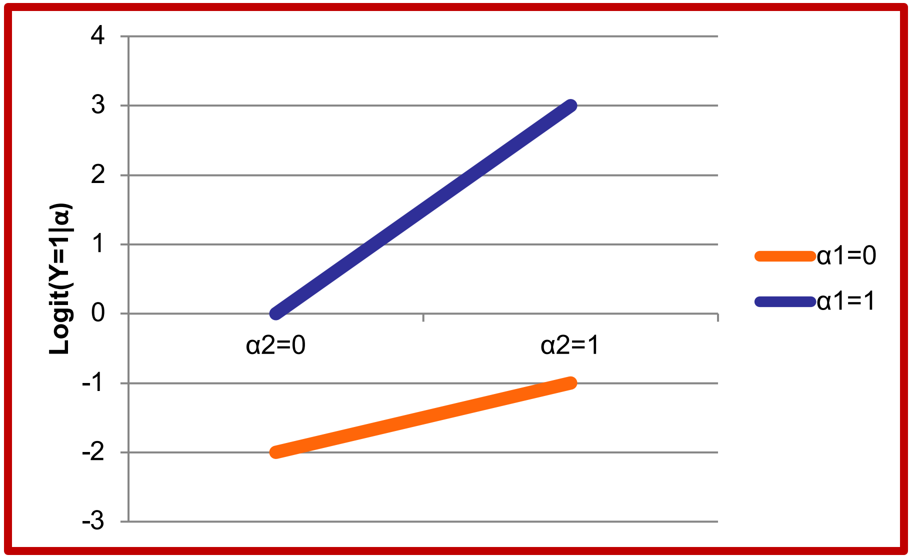
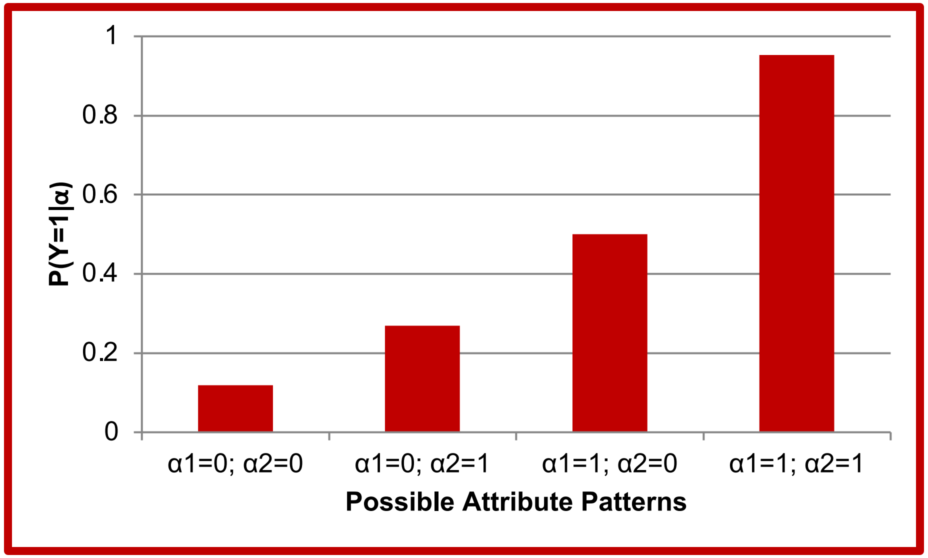
Less Extreme Interactions

- Extreme interactions are unlikely in practice
- Below: positive interaction with positive main effects

Logit Response Function



Probability Response Function (Item Characteristic Bar Chart)



General FORM of the LCDM

More General Versions of the LCDM

- The LCDM is based on the General Diagnostic Model by von Davier (GDM; 2005)
  - The GDM allows for both categorical and continuous latent variables

- For items measuring more than two attributes, higher level interactions are possible
  - Difficult to estimate in practice

- The LCDM appears in the psychometric literature in a more general form
  - See Henson, Templin, & Willse (2009)

General Form of the LCDM

- The LCDM specifies the probability of a correct response as a function of a set of attributes and a Q-matrix:

$$P(Y_{ri} = 1|\boldsymbol{\alpha}_r) = \frac{\exp\left(\boldsymbol{\lambda}_i^T \boldsymbol{h}(\boldsymbol{q}_i, \boldsymbol{\alpha}_r)\right)}{1 + \exp\left(\boldsymbol{\lambda}_i^T \boldsymbol{h}(\boldsymbol{q}_i, \boldsymbol{\alpha}_r)\right)}$$

Unpacking the General Form of the LCDM: Components α_r and q_i

- The key to understanding the general form of the LCDM is to understand that it is a general equation that makes any possible number of attributes be measured by an item

- To put this into context, we will continue with our basic mathematics example
  - Overall, four attributes measured: Addition (α_{r1}), Subtraction (α_{r2}), Multiplication (α_{r3}), Division (α_{r4})
    - Attribute profile vector for respondent r: $\boldsymbol{\alpha}_r = [\alpha_{r1} \quad \alpha_{r2} \quad \alpha_{r3} \quad \alpha_{r4}]$ (size 1 x 4)

  - Item i: $2 + 3 - 1 = ?$
    - Measures Addition (α_{r1}) and Subtraction (α_{r2})
    - Q-matrix row vector for item i: $\boldsymbol{q}_i = [1 \quad 1 \quad 0 \quad 0]$

Unpacking the General Form of the LCDM: Parameter Vector $\boldsymbol{\lambda}_i$

- From the general LCDM notation, $\boldsymbol{\lambda}_i$ is a vector of all possible item parameters for item i
  - All possible: if all A Q-matrix entries in $\boldsymbol{q}_i$ were equal to 1 (so size is $2^A \times 1$)
  - Not all parameters will be estimated if some $q_{ia} = 0$

- Vector to the right is for a four-attribute example

$$\boldsymbol{\lambda}_i = \begin{bmatrix} \lambda_{i,0} \\ \lambda_{i,1,(1)} \\ \lambda_{i,1,(2)} \\ \lambda_{i,1,(3)} \\ \lambda_{i,1,(4)} \\ \lambda_{i,2,(1,2)} \\ \lambda_{i,2,(1,3)} \\ \lambda_{i,2,(1,4)} \\ \lambda_{i,2,(2,3)} \\ \lambda_{i,2,(2,4)} \\ \lambda_{i,2,(3,4)} \\ \lambda_{i,3,(1,2,3)} \\ \lambda_{i,3,(1,2,4)} \\ \lambda_{i,3,(1,3,4)} \\ \lambda_{i,3,(2,3,4)} \\ \lambda_{i,4,(1,2,3,4)} \end{bmatrix}_{(16\times 1)}$$

Intercept

Possible Main Effects

Possible Two-Way Interactions

Possible Higher-Order Interactions

Unpacking the General Form of the LCDM: Helper Function $\boldsymbol{h}(\boldsymbol{q}_i, \boldsymbol{\alpha}_r)$

- From the general LCDM notation $\boldsymbol{h}(\boldsymbol{q}_i, \boldsymbol{\alpha}_r)$ is a vector-valued function
  - Vector valued = result is a vector
  - Provides whether or not specific parameter from $\boldsymbol{\lambda}_i$ should be part of item response function

$\boldsymbol{h}(\boldsymbol{q}_i, \boldsymbol{\alpha}_r)$	$\boldsymbol{\lambda}_i$
$$\boldsymbol{h}(\boldsymbol{q}_i, \boldsymbol{\alpha}_r) = \begin{bmatrix} 1 \\ (q_{i1}\alpha_{r1}) \\ (q_{i2}\alpha_{r2}) \\ (q_{i3}\alpha_{r3}) \\ (q_{i4}\alpha_{r4}) \\ (q_{i1}\alpha_{r1})(q_{i2}\alpha_{r2}) \\ (q_{i1}\alpha_{r1})(q_{i3}\alpha_{r3}) \\ (q_{i1}\alpha_{r1})(q_{i4}\alpha_{r4}) \\ (q_{i2}\alpha_{r2})(q_{i3}\alpha_{r3}) \\ (q_{i2}\alpha_{r2})(q_{i4}\alpha_{r4}) \\ (q_{i3}\alpha_{r3})(q_{i4}\alpha_{r4}) \\ (q_{i1}\alpha_{r1})(q_{i2}\alpha_{r2})(q_{i3}\alpha_{r3}) \\ (q_{i1}\alpha_{r1})(q_{i2}\alpha_{r2})(q_{i4}\alpha_{r4}) \\ (q_{i1}\alpha_{r1})(q_{i3}\alpha_{r3})(q_{i4}\alpha_{r4}) \\ (q_{i2}\alpha_{r2})(q_{i3}\alpha_{r3})(q_{i4}\alpha_{r4}) \\ (q_{i1}\alpha_{r1})(q_{i2}\alpha_{r2})(q_{i3}\alpha_{r3})(q_{i4}\alpha_{r4}) \end{bmatrix}_{(16\times 1)}$$	$$\boldsymbol{\lambda}_i = \begin{bmatrix} \lambda_{i,0} \\ \lambda_{i,1,(1)} \\ \lambda_{i,1,(2)} \\ \lambda_{i,1,(3)} \\ \lambda_{i,1,(4)} \\ \lambda_{i,2,(1,2)} \\ \lambda_{i,2,(1,3)} \\ \lambda_{i,2,(1,4)} \\ \lambda_{i,2,(2,3)} \\ \lambda_{i,2,(2,4)} \\ \lambda_{i,2,(3,4)} \\ \lambda_{i,3,(1,2,3)} \\ \lambda_{i,3,(1,2,4)} \\ \lambda_{i,3,(1,3,4)} \\ \lambda_{i,3,(2,3,4)} \\ \lambda_{i,4,(1,2,3,4)} \end{bmatrix}_{(16\times 1)}$$

More on the Helper Function $\boldsymbol{h}(\boldsymbol{q_i}, \boldsymbol{\alpha_r})$

- Consider the item i: $2 + 3 - 1 = ?$ with $q_i = [1 \quad 1 \quad 0 \quad 0]$

$\boldsymbol{h}(\boldsymbol{q}_i, \boldsymbol{\alpha}_r)$	$\boldsymbol{\lambda}_i$
$$\boldsymbol{h}(\boldsymbol{q}_i, \boldsymbol{\alpha}_r) = \begin{bmatrix} 1 \\ (q_{i1}\alpha_{r1}) = \alpha_{r1} \\ (q_{i2}\alpha_{r2}) = \alpha_{r2} \\ (q_{i3}\alpha_{r3}) = 0 \\ (q_{i4}\alpha_{r4}) = 0 \\ (q_{i1}\alpha_{r1})(q_{i2}\alpha_{r2}) = \alpha_{r1}\alpha_{r2} \\ (q_{i1}\alpha_{r1})(q_{i3}\alpha_{r3}) = 0 \\ (q_{i1}\alpha_{r1})(q_{i4}\alpha_{r4}) = 0 \\ (q_{i2}\alpha_{r2})(q_{i3}\alpha_{r3}) = 0 \\ (q_{i2}\alpha_{r2})(q_{i4}\alpha_{r4}) = 0 \\ (q_{i3}\alpha_{r3})(q_{i4}\alpha_{r4}) = 0 \\ (q_{i1}\alpha_{r1})(q_{i2}\alpha_{r2})(q_{i3}\alpha_{r3}) = 0 \\ (q_{i1}\alpha_{r1})(q_{i2}\alpha_{r2})(q_{i4}\alpha_{r4}) = 0 \\ (q_{i1}\alpha_{r1})(q_{i3}\alpha_{r3})(q_{i4}\alpha_{r4}) = 0 \\ (q_{i2}\alpha_{r2})(q_{i3}\alpha_{r3})(q_{i4}\alpha_{r4}) = 0 \\ (q_{i1}\alpha_{r1})(q_{i2}\alpha_{r2})(q_{i3}\alpha_{r3})(q_{i4}\alpha_{r4}) = 0 \end{bmatrix}_{(16\times 1)}$$	$$\boldsymbol{\lambda}_i = \begin{bmatrix} \lambda_{i,0} \\ \lambda_{i,1,(1)} \\ \lambda_{i,1,(2)} \\ \lambda_{i,1,(3)} \\ \lambda_{i,1,(4)} \\ \lambda_{i,2,(1,2)} \\ \lambda_{i,2,(1,3)} \\ \lambda_{i,2,(1,4)} \\ \lambda_{i,2,(2,3)} \\ \lambda_{i,2,(2,4)} \\ \lambda_{i,2,(3,4)} \\ \lambda_{i,3,(1,2,3)} \\ \lambda_{i,3,(1,2,4)} \\ \lambda_{i,3,(1,3,4)} \\ \lambda_{i,3,(2,3,4)} \\ \lambda_{i,4,(1,2,3,4)} \end{bmatrix}_{(16\times 1)}$$

Putting It All Together: The Matrix Product $\boldsymbol{\lambda}_i^T \boldsymbol{h}(\boldsymbol{q}_i, \boldsymbol{\alpha}_r)$

- The term in the exponent is the logit we have been using all along:

- In general:

$Logit(Y_{ri} = 1|\boldsymbol{\alpha}_r)$

$$\boldsymbol{\lambda}_i^T \boldsymbol{h}(\boldsymbol{q}_i, \boldsymbol{\alpha}_r) = \lambda_{i,0} + \sum_{a=1}^{A} \lambda_{i,1,(a)}(q_{ia}\alpha_{ra}) + \sum_{a=1}^{A-1} \sum_{b=a+1}^{A} \lambda_{i,2,(a,b)}(q_{ia}\alpha_{ra})(q_{ib}\alpha_{rb}) + \cdots$$

Intercept

Main Effects

Two-Way Interactions

Higher Interactions

- For our example item:

$$\left(\boldsymbol{\lambda}_i^T\right)_{(1\times16)} \boldsymbol{h}(\boldsymbol{q}_i, \boldsymbol{\alpha}_r)_{(16\times1)} = \lambda_{i,0} + \lambda_{i,1,(1)}\alpha_{r1} + \lambda_{i,1,(2)}\alpha_{r2} + \lambda_{i,2,(1,2)}\alpha_{r1}\alpha_{r2}$$

- Result is a scalar (1×1)

Subsumed models

Previously Popular DCMs

- Because the advent of the GDM and LCDM has been fairly recent, other earlier DCMs are still in use

- Such DCMs are much more restrictive than the LCDM
  - Not discussed at length here
  - It is anticipated that field will adapt to more general forms

- Each of these models can be fit using the LCDM
  - Fixing certain model parameters

- Shown for reference purposes
  - See Henson, Templin, & Willse (2009) for more detail

Other DCMs with the LCDM

- The Big 6 - DCMs with latent variables:
  - **DINA** (Deterministic Inputs, Noisy ‘AND’ Gate)
    - Haertel (1989); Junker and Sijtsma (1999)
  - **NIDA** (Noisy Inputs, Deterministic ‘AND’ Gate)
    - Maris (1995)
  - **RUM** (Reparameterized Unified Model)
    - Hartz (2002)
  - **DINO** (Deterministic Inputs, Noisy ‘OR’ Gate)
    - Templin & Henson (2006)
  - **NIDO** (Noisy Inputs, Deterministic ‘OR’ Gate)
    - Templin (2006)
  - **C-RUM** (Compensatory Reparameterized Unified Model)
    - Hartz (2002)

Other DCMs with the LCDM

LCDM Parameters	Non-compensatory Models			Compensatory Models		
	DINA	NIDA	NC-RUM	DINO	NIDO	C-RUM
Main Effects	Zero	Positive	Positive	Positive	Positive	Positive
Interactions	Positive	Positive	Positive	Negative	Zero	Zero
Parameter Restrictions	Across Attributes	Across Items	---	Across Attributes	Across Items	---

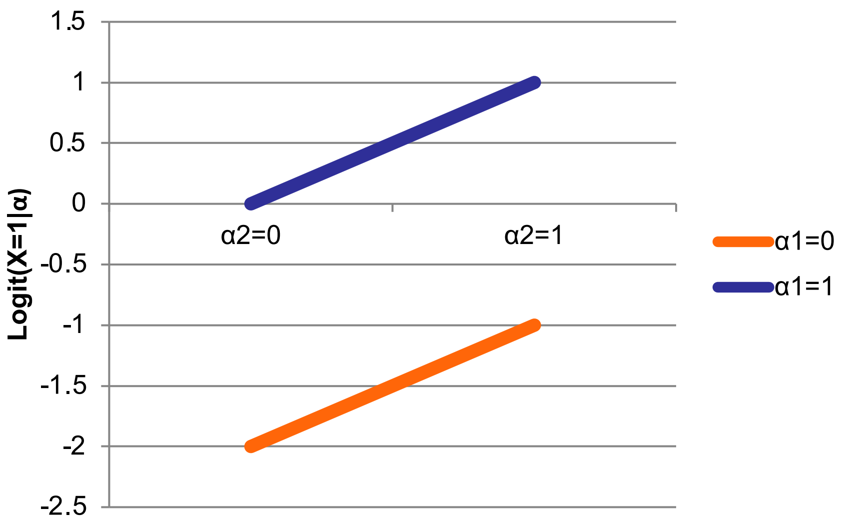
Adapted from: Rupp, Templin, and Henson (2010)

Compensatory RUM (Hartz, 2002)

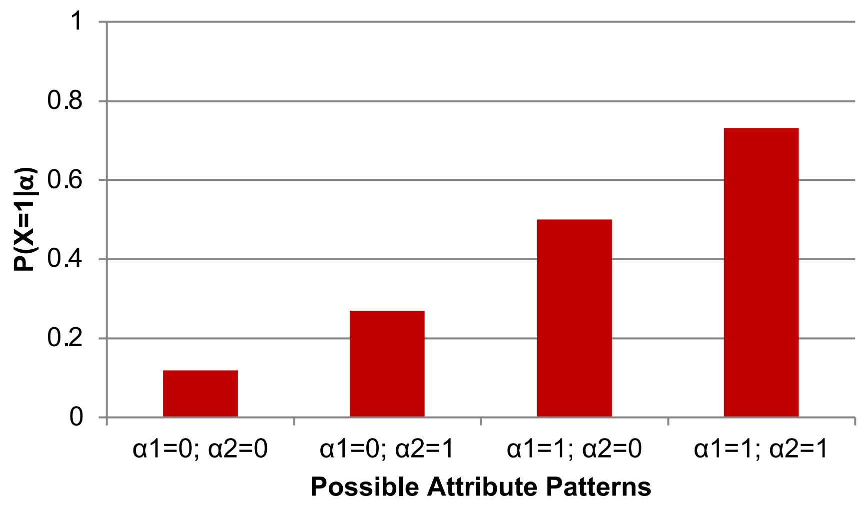
- No interactions in model
- **No interaction**: parallel lines for the logit

$$Logit(Y_{ri} = 1|\boldsymbol{\alpha}_r) = \lambda_{i,0} + \lambda_{i,1,(1)}\alpha_{r1} + \lambda_{i,1,(2)}\alpha_{r2}$$

Logit Response Function



Item Characteristic Bar Chart



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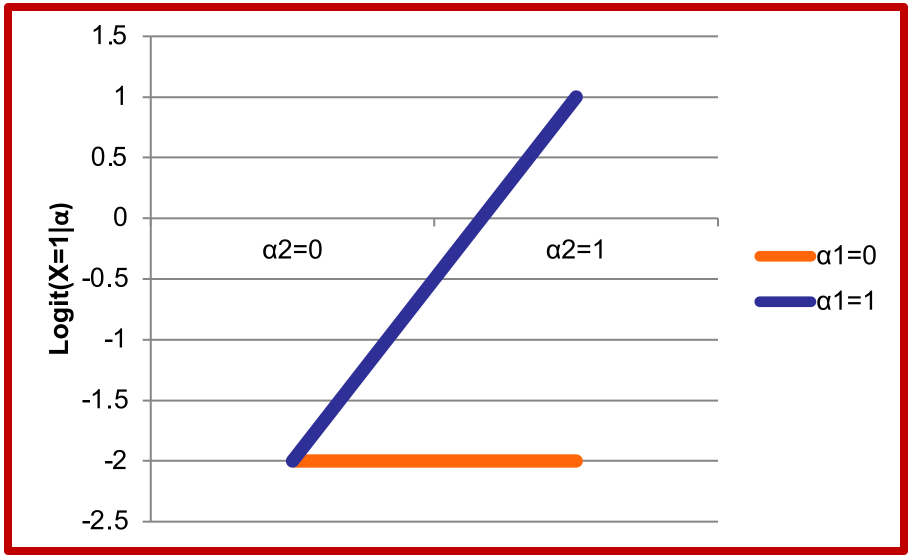
DINA Model (Haertel, 1989; Junker & Sijstma, 1999)

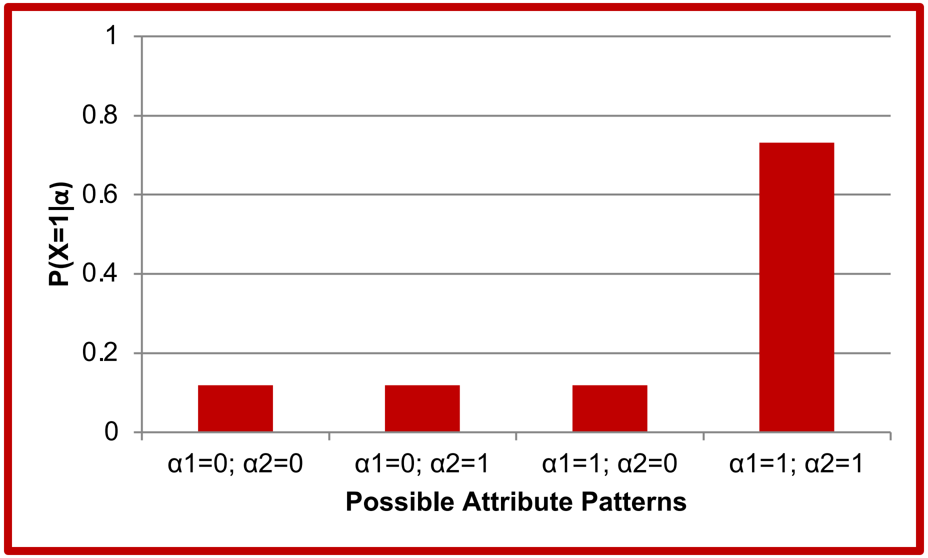
- **Positive interaction**: over-additive logit model
  - Highest interaction parameter is non-zero
  - All main effects (and lower interactions) zero

$$Logit(Y_{ri} = 1|\boldsymbol{\alpha}_r) = \lambda_{i,0} + \lambda_{i,2,(1,2)}\alpha_{r1}\alpha_{r2}$$

Logit Response Function



Item Characteristic Bar Chart



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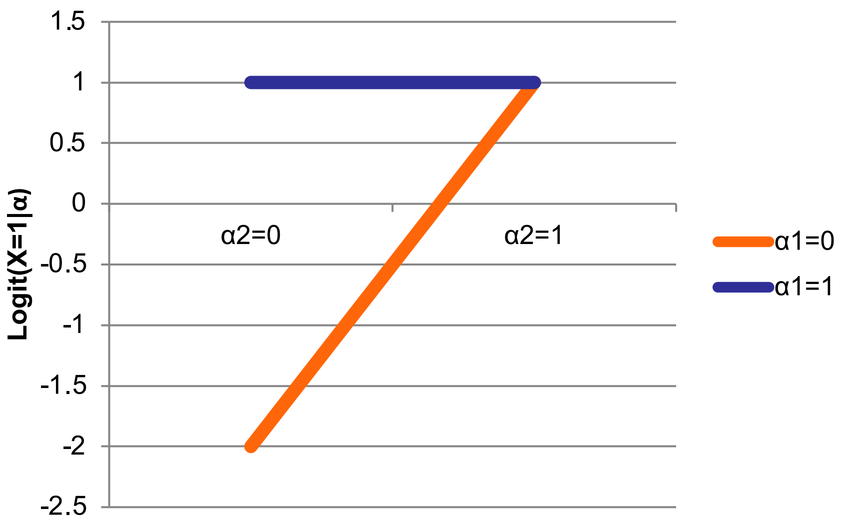
DINO Model (Templin & Henson, 2006)

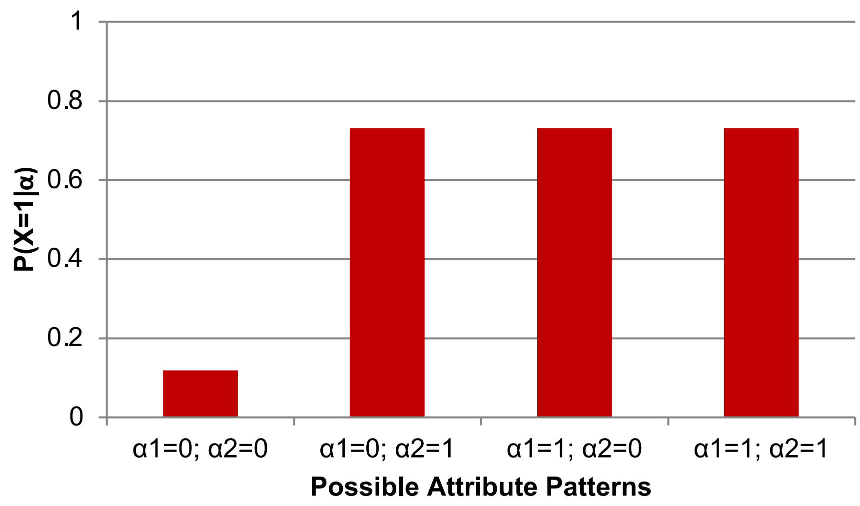
- **Negative interaction**: under-additive logit model
  - All main effects equal
  - Interaction terms are -1 sum of corresponding lower effects

$$Logit(Y_{ri} = 1|\boldsymbol{\alpha}_r) = \lambda_{i,0} + \lambda_{i,1}\alpha_{r1} + \lambda_{i,1}\alpha_{r2} - \lambda_{i,1}\alpha_{r1}\alpha_{r2}$$

Logit Response Function



Item Characteristic Bar Chart



The LCDM as a Constrained Latent Class Analysis Model

Four-Attribute Example as a LCA Model

- Imagine an LCA model where an exponential number of classes are defined:
  - Total Attributes: 4
  - Total Classes: 16
  - Model: LCA
  - Q-matrix: None
  - Equivalence Classes Per Item: None

- There is nothing to keep some skill pattern response profiles from exceeding others
  - Label for attributes is meaningless

Equal $P(X_{ij} = 1|c)$ Denoted By Symbol

		Item		
c	α	1	2	3...
1	[0000]	$\pi_{1,1}$	$\pi_{2,1}$	$\pi_{3,1}$
2	[0001]	$\pi_{1,2}$	$\pi_{2,2}$	$\pi_{3,2}$
3	[0010]	$\pi_{1,3}$	$\pi_{2,3}$	$\pi_{3,3}$
4	[0011]	$\pi_{1,4}$	$\pi_{2,4}$	$\pi_{3,4}$
5	[0100]	$\pi_{1,5}$	$\pi_{2,5}$	$\pi_{3,5}$
6	[0101]	$\pi_{1,6}$	$\pi_{2,6}$	$\pi_{3,6}$
7	[0110]	$\pi_{1,7}$	$\pi_{2,7}$	$\pi_{3,7}$
8	[0111]	$\pi_{1,8}$	$\pi_{2,8}$	$\pi_{3,8}$
9	[1000]	$\pi_{1,9}$	$\pi_{2,9}$	$\pi_{3,9}$
10	[1001]	$\pi_{1,10}$	$\pi_{2,10}$	$\pi_{3,10}$
11	[1010]	$\pi_{1,11}$	$\pi_{2,11}$	$\pi_{3,11}$
12	[1011]	$\pi_{1,12}$	$\pi_{1,12}$	$\pi_{1,12}$
13	[1100]	$\pi_{1,13}$	$\pi_{1,13}$	$\pi_{1,13}$
14	[1101]	$\pi_{1,14}$	$\pi_{1,14}$	$\pi_{1,14}$
15	[1110]	$\pi_{1,15}$	$\pi_{2,15}$	$\pi_{3,15}$
16	[1111]	$\pi_{1,16}$	$\pi_{2,16}$	$\pi_{3,16}$

Four-Attribute Example with LCDM

Class #	α	2+3-1	4/2	3 x (4 - 2)
1	[0000]			
2	[0001]			
3	[0010]			
4	[0011]			
5	[0100]			
6	[0101]			
7	[0110]			
8	[0111]			
9	[1000]			
10	[1001]			
11	[1010]			
12	[1011]			
13	[1100]			
14	[1101]			
15	[1110]			
16	[1111]			

Total Attributes: 4

Total Classes: 16

Model: LCDM

Q-matrix:

Item	Add α_1	Sub α_2	Mult α_3	Div α_4
$2+3-1$	1	1	0	0
$\frac{4}{2}$	0	0	0	1
$3\times(4-2)$	0	1	1	0

Equivalence Classes Per Item: $(2^{q_{i\cdot}})$

Item	$2+3-1$	$\frac{4}{2}$	$3\times(4-2)$
Classes	4	2	4

LCDM Item Response Model: Replacing LCA $\boldsymbol{\pi}$

- The LCDM item response model equation serves to take the place of the π_{ic} parameter (class-specific item difficulty) in LCA models

$$\pi_{ic} = P\left(X_{pi} = 1 \mid \boldsymbol{\alpha}_p = \boldsymbol{\alpha}_c \right) = \frac{\exp\left(h(\boldsymbol{\alpha}_p, \boldsymbol{q}_i)\right)}{1 + \exp\left(h(\boldsymbol{\alpha}_p, \boldsymbol{q}_i)\right)}$$

- For the 2+3-1 item, the LCDM linear predictor (in the exp) is

$$\lambda_{i,0} + \lambda_{i,1(1)}\alpha_{p1} + \lambda_{i,1,(2)}\alpha_{p2} + \lambda_{i,2,(1,2)}\alpha_{p1}\alpha_{p2}$$

- Thus, the four cells in the column of the table, representing equal probabilities give linear predictors of:
  - Rows 1, 2, 3, 4 (where $\alpha_{p1} = 0$ and $\alpha_{p2} = 0$): $\lambda_{i,0}$
  - Rows 5, 6, 7, 8 (where $\alpha_{p1} = 0$ and $\alpha_{p2} = 1$): $\lambda_{i,0} + \lambda_{i,1(2)}$
  - Rows 9, 10, 11, 12 (where $\alpha_{p1} = 1$ and $\alpha_{p2} = 0$): $\lambda_{i,0} + \lambda_{i,1(1)}$
  - Rows 13, 14, 15, 16 (where $\alpha_{p1} = 1$ and $\alpha_{p2} = 1$): $\lambda_{i,0} + \lambda_{i,1(1)} + \lambda_{i,1,(2)} + \lambda_{i,2,(1,2)}$

LCDM Linear Predictors

- What are the linear predictors for the $\frac{4}{2}$ item?
  - Rows 1, 3, 5, 7, 9, 11, 13, 15 (where $\alpha_{p4} = 0$):
  - Rows 2, 4, 6, 8, 10, 12, 14, 16 (where $\alpha_{p4} = 1$):
- What are the linear predictors for the $3 \times (4 - 2)$ item?
  - Rows 1, 2, 9, 10 (where $\alpha_{p2} = 0$ and $\alpha_{p3} = 0$):
  - Rows 3, 4, 11, 12 (where $\alpha_{p2} = 0$ and $\alpha_{p3} = 1$):
  - Rows 5, 6, 13, 14 (where $\alpha_{p2} = 0$ and $\alpha_{p3} = 1$):
  - Rows 7, 8, 15, 16 (where $\alpha_{p2} = 1$ and $\alpha_{p3} = 1$):

Additional Constraints

- The LCDM places an additional monotonicity constraint on the item parameters

- Profiles with higher numbers of mastered attributes have a probability of a correct response greater than or equal to those with lower numbers of mastered attributes

- Helps to ensure the label for the class is correct

Monotonicity Constraints

- To show how monotonicity constraints work, consider the item $2 + 3 - 1$
- There are four sets of probabilities:
  1. Rows 1, 2, 3, 4 (where $\alpha_{p1} = 0$ and $\alpha_{p2} = 0$): $\lambda_{i,0}$
  2. Rows 5, 6, 7, 8 (where $\alpha_{p1} = 0$ and $\alpha_{p2} = 1$): $\lambda_{i,0} + \lambda_{i,1(2)}$
  3. Rows 9, 10, 11, 12 (where $\alpha_{p1} = 1$ and $\alpha_{p2} = 0$): $\lambda_{i,0} + \lambda_{i,1(1)}$
  4. Rows 13, 14, 15, 16 (where $\alpha_{p1} = 1$ and $\alpha_{p2} = 1$): $\lambda_{i,0} + \lambda_{i,1(1)} + \lambda_{i,1,(2)} + \lambda_{i,2,(1,2)}$
- Set #1 has zero mastered attributes, Sets #2 and 3 have one mastered attribute, and Set #4 has two mastered attributes
- So, the orderings of probabilities must be such that

$$Set\ \#1 \leq [Set\ \#2\ and\ Set\ \#3] \leq Set\ \#4$$

Showing Constraints with LCDM Item Parameters

- To show these constraints as a function of the LCDM item parameters, we first have to realize that the ordering of probabilities is not changed by the logit link function
  - So, we can work with the linear predictors of the items directly

- We then can show that, for $Set\ \#1 \leq Set\ \#2$:

$$\lambda_{i,0} \leq \lambda_{i,0} + \lambda_{i,1,(2)} \rightarrow \lambda_{i,1,(2)} \geq 0$$

- For $Set\ \#2 \leq Set\ \#3$:

$$\lambda_{i,0} + \lambda_{i,1,(2)} \leq \lambda_{i,0} + \lambda_{i,1(1)} + \lambda_{i,1,(2)} + \lambda_{i,2,(1,2)} \rightarrow$$
$$\lambda_{i,2,(1,2)} \geq -\lambda_{i,1,(1)}$$

- Continued on next slide…

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Showing Constraints with LCDM Item Parameters

- For $Set\ \#1 \leq Set\ \#3$:
$$\lambda_{i,0} \leq \lambda_{i,0} + \lambda_{i,1,(1)} \rightarrow \lambda_{i,1,(1)} \geq 0$$
- For $Set\ \#3 \leq Set\ \#4$:
$$\lambda_{i,0} + \lambda_{i,1,(1)} \leq \lambda_{i,0} + \lambda_{i,1(1)} + \lambda_{i,1,(2)} + \lambda_{i,2,(1,2)} \rightarrow$$
$$\lambda_{i,2,(1,2)} \geq -\lambda_{i,1,(2)}$$
- Finally, as we have $\lambda_{i,2,(1,2)} \geq -\lambda_{i,1,(2)}$ and $\lambda_{i,2,(1,2)} \geq -\lambda_{i,1,(1)}$ and we know $\lambda_{i,1,(1)} \geq 0$ and $\lambda_{i,1,(2)} \geq 0$ then:
$$\lambda_{i,2,(1,2)} \geq -1 * \min\left(\lambda_{i,1,(1)}, \lambda_{i,1,(2)}\right)$$
- So, our monotonicity constraints are:
  - $\lambda_{i,1,(1)} \geq 0$
  - $\lambda_{i,1,(2)} \geq 0$
  - $\lambda_{i,2,(1,2)} \geq -1 * \min\left(\lambda_{i,1,(1)}, \lambda_{i,1,(2)}\right)$

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Concluding Remarks

Wrapping Up

- The LCDM uses an ANOVA-like approach to map latent attributes onto item responses
  - Uses main effects and interactions for each attribute
  - Uses a logit link function

- Multiple diagnostic models are subsumed by the LCDM