Psychometric Prerequisites

Multidimensional Measurement Models (Fall 2023): Lecture 3

# Today’s Lecture

* Context matters: Your data sets
  + Educational standards
* Psychometric Prerequisites
  + Q-matrices
  + Latent Variable Measurement Models
    - Binary IRT models
    - Polytomous IRT models
    - Confirmatory Factor models
  + Latent Variable Structural Models
* Model Identification

## Context Matters: Your Data Sets

The data sets you will use in this class are from educational assessments:

* Large, urban district in a rectangular midwestern state
* District-wide benchmark assessments (given three times per year)
* Results used for helping students progress through the curriculum
* Subjects assessed: Mathematics and English Language Arts (ELA; where ELA == Reading only)

Note: These are not the *actual* assessments but are similar in structure and content and created by a bootstrapping process

## Data Set Format

Each data set is saved as in the RData format

* Can be imported into R with the load() function
* Data contain a list object named temp
  + The temp object has a Q-matrix and the response matrix

load("9-10th\_Grade\_Q2.RData")  
  
ls()

[1] "has\_annotations" "temp"

names(temp)

[1] "Qmatrix" "responseMatrix"

Qmatrix = temp$Qmatrix  
responseMatrix = temp$responseMatrix

## Q-matrices

Q-matrices are a way to specify which latent variables are measured by which items

* The Q-matrix is a binary matrix
  + Each row is an item
  + Each column is a latent variable
  + Each entry is a one or a zero
    - One means the latent variable is measured by the item
    - Zero means the latent variable is not measured by the item

head(Qmatrix)

RL.9-10.4 RL.9-10.6 RL.9-10.1 RL.9-10.2 RI.9-10.6 RI.9-10.1 RI.9-10.4  
item1 1 0 0 0 0 0 0  
item2 0 1 0 0 0 0 0  
item3 0 0 1 0 0 0 0  
item4 0 1 0 0 0 0 0  
item5 0 0 0 1 0 0 0  
item6 0 0 0 0 1 0 0  
 A.REI.1 A.REI.2 A.REI.8 A.CED.2 G.CO.7 G.SRT.4 G.CO.2  
item1 0 0 0 0 0 0 0  
item2 0 0 0 0 0 0 0  
item3 0 0 0 0 0 0 0  
item4 0 0 0 0 0 0 0  
item5 0 0 0 0 0 0 0  
item6 0 0 0 0 0 0 0

## The Latent Variables

Latent variables are built by specification:

* What is their distribution? (nearly all are normally distributed)
* How do you specify their scale: mean and standard deviation? (step two in our model analysis steps)

You create latent variables by specifying which items measure which latent variables in an analysis model

* Called different names by different fields:
  + Alignment (educational measurement)
  + Factor pattern matrix (factor analysis)
  + Q-matrix (Question matrix; diagnostic models and multidimensional IRT)

## Q-Matrices in Class Data

The column names in the class data Q-matrix reflect different educational standards being assessed

* The assessments were created to assess these standards directly
* Each standard has a unique code
  + A guide to what the code references is found here <https://commoncore.tcoe.org/Content/Public/doc/tcoe_understanding_standards_codes.pdf>

Question for discussion: What are the latent variables we must estimate in the class data?

## From Q-matrix to Model

The alignment provides a specification of which latent variables are measured by which items

* Sometimes we say items “load onto” factors

The mathematical definition of either of these terms is simply whether or not a latent variable appears as a predictor for an item

* For instance, item one appears to measure nongovernment conspiracies, meaning its alignment (row vector of the Q-matrix) would be: ::: {.cell}

Qmatrix[1,]

RL.9-10.4 RL.9-10.6 RL.9-10.1 RL.9-10.2 RI.9-10.6 RI.9-10.1 RI.9-10.4 A.REI.1   
 1 0 0 0 0 0 0 0   
 A.REI.2 A.REI.8 A.CED.2 G.CO.7 G.SRT.4 G.CO.2   
 0 0 0 0 0 0

:::

The model for the first item is then built with only the factors measured by the item as being present:

# Psychometric Measurement Models

Psychometric Measurement Models to model the relationship between observed variables and latent variables

* Observed variables are the items in the assessment
* Latent variables are the factors measured by the items

The models discussed today all model the expected value (mean) of the observed variables as a function of the latent variables

## Binary IRT Models

Binary IRT models are used when the observed variables are binary (0/1) variables

* The distribution of the observed variables is a Bernoulli distribution

Where:

* is the response for person on item
* is the probability of a correct response on item and is the expected value of the Bernoulli distribution

## Binary IRT Models

There are many different binary IRT models, the most common fall under the family of models identified by the number of parameters in the model

* Often, the parameterization used for binary models is discrimination/difficulty when only one latent variable is present

Where:

* is the item difficulty (parameter one)
* is the item discrimination (parameter two)
* is the lower asymptote (guessing parameter; parameter three)
* is the upper asymptote (discrimination parameter; parameter four)

## More Binary IRT Models

The model on the previous page is the four-parameter logistic model (4PL)

* The logistic function is a link function mapping probability of a correct response (the expected value of the item) to the real number line
* The inverse logistic function is the inverse link function mapping the real number line to the probability of a correct response

Many link functions exist, including:

* Normal ogive (probit) link function
* Log link function
* Log-log link function
* Cumulative log-log
* Complementary log-log

## Slope Intercept Form

Depending on the context, the slope/intercept parameterization can also be used:

* Here, (when is standardized)
  + The item intercept is the expected log odds of a correct response when the latent variable is zero
  + The item loading/slope/discrimination is the expected change in log odds of a correct response for a one unit change in the latent variable
* The slope/intercept parameterization is often used in the context of linear models

## Multidimensional IRT Models

We then can add the Q-matrix for expanding the model to multiple latent variables:

Where:

* is the number of latent variables
* is the Q-matrix entry for item and latent variable

Note: Not all items need to measure all latent variables (some )

## Polytomous IRT Models

Polytomous IRT models are used for discrete categorical observed variables

* The assumed distribution for observed variables is a categorical distribution
* A categorical distribution is a multinomial distribution with a single trial

## Categorical distribution

The PMF of the categorical distribution is:

Where:

* is the number of categories for item
* is the response for person on item in category
* is the probability of a response on item in category and is the expected value of the categorical distribution
* is the constraint that the probabilities sum to one

## Graded Response Models

The graded response model is an ordered logistic regression model where:

Where:

With:

* Ordered intercepts:

## Nominal Response Models

The nominal response model is an unordered logistic regression model where:

With:

* Unordered intercepts:
  + One constraint must be implemented to identify the model, typically either
    - One
* Unordered slopes:
  + One constraint *per dimension* must be implemented to identify the model, typically either
    - One

## Confirmatory Factor Models

Confirmatory factor models are used when the observed variables are continuous (interval/ratio) variables

* The distribution of the observed variables is a normal distribution
* The PDF of the normal distribution is:

## Confirmatory Factor Models

The CFA model is then:

Where:

* is the response for person on item
* is the error term for person on item
* is the variance of the error term
* is the item intercept
* is the item loading/slope for item and latent variable
* is the latent variable score for person and latent variable
* is the number of latent variables
* is the Q-matrix entry for item and latent variable

# More Q-matrix Stuff

We could show the model with the Q-matrix entries:

* contains all *possible* factor loadings for item 1 (size )
* contains the factor scores for person (size )
* is a diagonal matrix of ones times the vector of Q-matrix entries for item 1

The matrix product then gives:

The Q-matrix functions like a partial version of the model (predictor) matrix that we saw in linear models

## Latent Variable Interactions

Although infrequently shown in psychometric measurement models of the IRT/CFA family, latent variable interactions can be modeled

For example, a binary item measuring two latent variables could be modeled as:

Notes:

* The interaction between latent variables is modeled as a product of the latent variables
* Diagnostic classification models is one latent variable model family that prominantly uses latent variable interactions
* We will discuss latent variable interactions in more detail later in the semester

# Latent Variable Structural Models

Latent variable structural models are models for the distributions of the latent variables

* When saturated, these are just distribution parameters that are estimated
* But, models for the latent variables can be built
  + Simultaneous equations (structural equations)

## Our Example: Multivariate Normal Distribution

For our example, we will assume the set of traits follows a multivariate normal distribution

Where:

* is the number of latent variables (dimensions)
* is the covariance matrix of the latent variables
  + is the inverse of the covariance matrix
* is the mean vector of the latent variables
* is the matrix determinant function
* is the matrix transpose operator

Alternatively, we would specify ; but, we cannot always estimate and

# Identification of Latent Traits, Part 1

Psychometric models require two types of identification to be valid:

1. Empirical Identification

* The minimum number of items that must measure each latent variable
* From CFA: three observed variables for each latent variable (or two if the latent variable is correlated with another latent variable)

Bayesian priors can help to make models with fewer items than these criteria suggest estimable

* The parameter estimates (item parameters and latent variable estimates) often have MCMC convergence issues and should not be trusted
* Use the CFA standard in your work

## Identification of Latent Traits, Part 2

Psychometric models require two types of identification to be valid:

1. Scale Identification (i.e., what the mean/variance is for each latent variable)

* The additional set of constraints needed to set the mean and standard deviation (variance) of the latent variables
* Two main methods to set the scale:
  + Marker item parameters
    - For variances: Set the loading/slope to one for one observed variable per latent variable
      * Can estimate the latent variable’s variance (the diagonal of )
    - For means: Set the item intercept to one for one observed variable perlatent variable
      * Can estimate the latent variable’s mean (in )
  + Standardized factors
    - Set the variance for all latent variables to one
    - Set the mean for all latent variables to zero
    - Estimate all unique off-diagonal correlations (covariances) in

## More on Scale Identification

Bayesian priors can let you believe you can estimate more parameters than the non-Bayesian standards suggest

* For instance, all item parameters and the latent variable means/variances

Like empirical identification, these estimates are often unstable and are not recommended

Most common:

* Standardized latent variables
  + Used for scale development and/or when scores are of interest directly
* Marker item for latent variables and zero means
  + Used for cases where latent variables may become outcomes (and that variance needs explained)

## Wrapping Up

* Psychometric models are models for the relationship between observed variables and latent variables
* Psychometric models are built from the Q-matrix
* Psychometric models require two types of identification to be valid:
  + Empirical Identification
  + Scale Identification

## Next Time

* Alternative models for model comparison
* Model fit