Alexandria University
Faculty of Engineering
Mechanical Department

Numerical Heat Transfer
Technical Report

No (1): Consider a large plate of thickness 5 cm and thermal conductivity of 0.2 W/m. K, in which heat is generated uniformly at a constant rate of 10000 W/m^3. The right side of the plate is well insulated, while the other side is subjected to convection to an environment at 25 °C with a heat transfer coefficient of 20 W/ m^2 .K and incident radiation heat flux of 150 W/m^2.

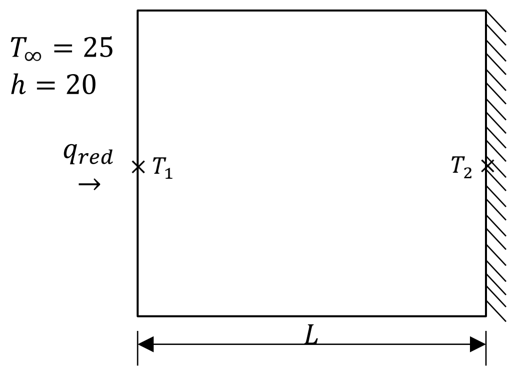
Write the boundary conditions for the heat transfer process, and then estimate the temperature distribution and the rate of heat transfer per unit area (W/ m^2) from both sides of the plate. Also define the value and position of maximum temperature.

Solve the problem:

a) Analytical.
b) Numerical (manual 10 iteration).
c) Programming by MATLAB (error 0.0001).

And compare between the results by plot them on the same (T-x) diagram.

Firstly: Analytical Solution



$$\frac{\partial^2 T}{\partial x^2} + \frac{g}{k} = 0$$

$$\frac{\partial^2 T}{\partial x^2} = -\frac{g}{k} \rightarrow \frac{\partial T}{\partial x} = -\frac{g}{k}x + C_1$$

$$T = -\frac{g}{2k}x^2 + C_1 x + C_2$$

- The boundary conditions:

1. $Insulated\ at\ x\ =\ L$

$$\therefore q_{cond}|_{x=l}\ = 0$$

$$\therefore -k\left(\frac{\partial T}{\partial x}\right)|_{x=l}\ =\ 0\ \rightarrow \left(\frac{\partial T}{\partial x}\right)|_{x=l} =\ -\frac{g}{k}L + C_1 = 0$$

$$\therefore\ C_1 = \frac{g}{k}L = \frac{1000}{.2} \times 0.05 = 2500$$

2. $at\ x\ =\ 0\ \ (Convection\ and\ Radiation)$

$$q_{cond}\ =\ q_{conv}\ +\ q_{rad}$$

$$\therefore\ -k\left(\frac{dT}{dx}\right)|_x = h(T_\infty -\ T_0) +\ q_{rad}\ \rightarrow \left(\frac{dT}{dx}\right)|_{x=0} = C_1\ ,T(0) =\ C_2$$

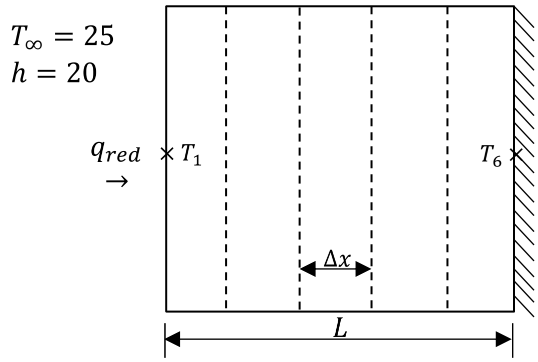
$$\therefore\ -0.2\ \times\ 25000\ =\ 20(25\ -\ C_2) +\ 150 \qquad \therefore C_2 = 57.5$$

$\therefore The\ temperature\ distribution\ is$

$$T\ =\ -\frac{g}{2k}x^2 +\ C_1 x\ +\ C_2 = \frac{-10000}{0.2}x^2\ +\ 2500\ x\ +\ 57.5$$

$$\boxed{\therefore\ T\ =\ -25000\ x^2\ +\ 2500\ x\ +\ 57.5}$$

Secondly: Numerical Solution



$$Number\ of\ nodes\ =\ 6$$

$$\Delta x = \frac{L}{6-1} = \frac{0.05}{5} = 0.01\ m$$

- $Interior\ nodes\ (Equations)\ (T_2\ \rightarrow\ T_5)$

$$Q_{in} -\ Q_{out}\ =\ 0\ \ \therefore Q_{m-1\ ,m}\ +\ -Q_{m\ ,m+1}\ =\ 0$$

$$\frac{(-\ T_{\mathrm{m}})}{\left(\frac{\Delta x}{kA}\right)} + \ g(\Delta x\ \times A)\ - \frac{(T_m\ -\ T_{m+1})}{\left(\frac{\Delta x}{kA}\right)}\ =\ 0$$

$$\therefore T_{m-1} - \ 2T_{\mathrm{m}}\ +\ T_{m+1}\ \ = -\frac{-g(\Delta x)^2}{k} = -5\ \rightarrow (m = 2, m = 3, m = 4)$$

- $Exterior\ nodes\ (Equations)\ (T_1\ ,T_6)$

  The Boundary Conditions:

1. $At\ x\ =\ L\ =\ 0.05\ m\ \rightarrow (Insulated) \therefore g\left(\frac{\Delta x}{2}\right)\ =\ -k\frac{(T_5-T_6)}{\Delta x}$

$$\therefore 50 = \frac{0.2}{0.01}(-T_5 + T_6) \rightarrow \frac{50}{20} = -T_5 + T_6 \quad \therefore T_6 = T_5 + \frac{5}{2}$$

2. $At\ x\ =\ 0\ (Convection\ and\ Radiation)$:

$$q_{in} -\ q_{out}\ =\ 0$$

$$\therefore Ah\ (T_\infty -\ T_1) +\ q_{rad}A\ + g\ \left(\frac{\Delta x}{2}\ A\right) - \frac{T_1 -\ T_2}{\left(\frac{\Delta x}{kA}\right)} =\ 0$$

$$\therefore h\ (T_\infty -\ T_1) +\ q_{rad}\ +\ g\,\frac{\Delta x}{2}\ = \frac{T_1 -\ T_2}{\left(\frac{\Delta x}{kA}\right)}$$

$$\therefore h\,T_\infty - hT_1 + q_{red} + g\frac{\Delta x}{2} = \frac{kT_1}{\Delta x} - \frac{kT_2}{\Delta x}$$

$$\therefore 20 \times 25 - 20T_1 + 150 + \frac{10000 * 0.01}{2} = \frac{0.2}{0.01}T_1 - \frac{0.2}{0.01}T_2$$

$$500 - 20T_1 + 150 + 50 = 20T_1 - 20T_2 \rightarrow 40T_1 - 20T_2 = 700$$

$$The\ Equations \rightarrow \boxed{40T_1 - 20T_2 = 700}$$

$$T_\mathrm{m} = \frac{T_{m-1} + T_{m+1} + 5}{2}$$

$$T_6 = T_5 + \frac{5}{2}$$

Iterations	*1*	*2*	*3*	*4*	*5*	*6*
Initial Guess	*60*	*70*	*80*	*90*	*100*	*110*
1	*52.5*	*72.5*	*82.5*	*92.5*	*102.5*	*102.5*
2	*53.75*	*70*	*85*	*95*	*100*	*105*
3	*52.5*	*71.875*	*85*	*95*	*102.5*	*102.5*
4	*53.437*	*71.25*	*85.937*	*96.25*	*101.25*	*105*
5	*53.25*	*72.1875*	*86.25*	*96.093*	*103.125*	*103.75*
3	*53.593*	*72.1875*	*86.61*	*97.187*	*102.42*	*105.62*
7	*53.593*	*72.6172*	*87.187*	*97.0321*	*103.706*	*104.92*
8	*53.808*	*72.8906*	*87.324*	*98.016*	*103.478*	*106.406*
9	*53.945*	*73.0664*	*87.968*	*97.9001*	*104.726*	*105.97*
10	*54.033*	*73.157*	*87.983*	*98.84*	*104.438*	*107.226*

$$For\ the\ maximum\ Temperature\ \therefore \frac{dT}{dx} = 0$$

$$\therefore -\frac{g}{k}x + C_1 = 0 \quad \therefore x = (-C_1)\left(\frac{-k}{g}\right) = \left(\frac{g}{k}L\right)\left(\frac{k}{g}\right) = L = 0.05$$

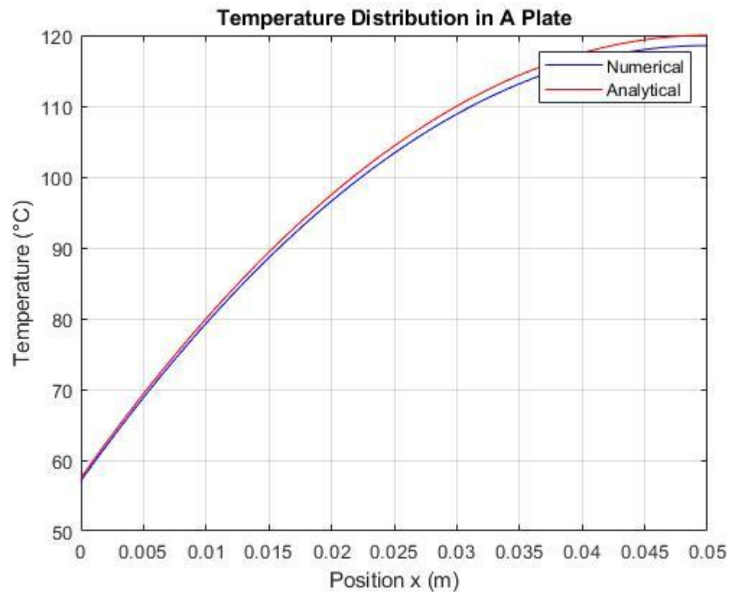
$$\therefore\ the\ maximum\ Temperature\ at\ x = 0.05$$

$$T_{max} = -2500\,(0.05)^2 + 2500\,x + 57.5 = 120\,°C$$

$$For\ the\ heat\ transfer\ per\ unit\ area\ \left(\frac{W}{m^2}\right)$$

$$q|_{x=0} = -k\frac{dT}{dx}|_{x=0} = -k\ C_1 = -k \times 2500 = -500\,W/m^2$$

Thirdly: MATLAB Solution




```
%%prep work space%%
clc, clear

%% Givens %%
L = 0.05; %in m%
K = 0.2; %in W/m.k%
g = 10000; %in W/m^3%
T_inf = 25; %in C%
h = 20; %W/m2.k%
q_rad = 150; %W/m2%

% Nodes and Element Length %
M = 100;
dx = L/(M-1);

% Initial Guess %
T = ones(M,1) * T_inf;

% Iteration Settings %
error = 1;
iter = 0;

% Calculations %
while error > 0.0001
    T_old = T;

    % Interior Nodes %
    for i = 2:M-1
        T(i) = 0.5 * (T(i+1) + T(i-1) + (g * dx^2)/K);
    end

    % Left Boundry: %
    T(1) = (2*T(2) + 2*dx*(h*T_inf + q_rad)/K) / (2 + 2*dx*h/K);

    % Right Boundry: insulated (dT/dx = 0) %
    T(M) = T(M-1);

    % Error Check %
    error = max(abs(T - T_old));
    iter = iter + 1;
end

% Analytical Solution %
x = linspace(0, L, M);
T_anal = -g/(2*K) * x.^2 + g*L/K * x + 57.5;

% Maximum Temperature and Position %
[T_max, idx] = max(T);
x_max = x(idx);
T_max_anal = max(T_anal);

% Heat Transfer Rate%
q_left = g*L/K * -K ;
q_right = 0;

% Displaying Results %
fprintf('Maximum Temperature from Numerical Solution: %2f °C at x = %.4f m\n', T_max,
x_max);
fprintf('Maximum Temperature from Analytical Solution: %2f °C\n', T_max_anal);
fprintf('Heat Transfer Rate at x = 0: %.2f W/m^2\n', q_left);
fprintf('Heat Transfer Rate at x = L: %.2f W/m^2\n', q_right);

%% Plotting Results %%
figure;
plot(x, T, '-b', 'DisplayName', 'Numerical');
hold on;
plot(x, T_anal, '-r', 'DisplayName', 'Analytical')
xlabel('Position x (m)');
ylabel('Temperature (°C)');
title('Temperature Distribution in A Plate');
legend('show');
grid on;
```

```
Maximum Temperature from Numerical Solution: 118.548454 °C at x = 0.0495 m
Maximum Temperature from Analytical Solution: 120.000000 °C
Heat Transfer Rate at x = 0: -500.00 W/m^2
Heat Transfer Rate at x = L: 0.00 W/m^2
```

No (2): A large flat plate of thickness 12 cm has a thermal conductivity of 50 W/m·K and a thermal diffusivity of 3.94 × 10^{-5} m^2/s. The plate is initially at a uniform temperature of 20°C. At time t = 0, the right surface of the plate is subjected to a constant heat flux of 100 kW/m^2, while the left surface is exposed to a fresh water at 30°C, with a convective heat transfer coefficient of 700 $W/m^2 \cdot K$.

a) Use the explicit finite difference method (for a non-lumped system) to determine the temperature distribution across the plate thickness at 5, 10, 15, 20, 25, and 30 minutes after the start of heating.

b) Plot the minimum temperature in the plate as a function of time.

c) Estimate the total amount of heat stored in the plate after 30 minutes, in kJ.

Solve the problem using MATLAB programming.

Firstly: Question a)

Givens:

$L = 0.12\,, \alpha = 3.9x10^5, h = 700\,, T_\infty = 30\,,\; q_{in} = 100 \times 10^3, T_{initial} = 20$

$$\Delta x = \frac{0.12}{5} = 0.024$$

$$\tau = \frac{\alpha \Delta t}{\Delta x^2} \le 0.5\,, \qquad Take\ \Delta t = 5s\,, \tau = 0.342$$

- *Interior nodes*

$$T_2^{i+1} = \tau T_1^i + (1 - 2\tau) T_2^i + \tau\, T_3^i$$

$$T_3^{i+1} = \tau T_2^i + (1 - 2\tau) T_3^i + \tau\, T_4^i$$

$$T_4^{i+1} = \tau T_3^i + (1 - 2\tau) T_4^i + \tau\, T_5^i$$

$$T_5^{i+1} = \tau T_4^i + (1 - 2\tau) T_5^i + \tau\, T_6^i$$

- *Exterior nodes*

    1) $At\ x = 0:$

$$E_{in} - E_{out} = \frac{dE_{sys}}{dt}$$

$$h\left(T_\infty - T_1^i\right) - k\frac{T_1^i - T_2^i}{\Delta x^2} = \frac{k}{\alpha} \times \frac{\Delta t}{2} \times \frac{T_1^{i+1} - T_1^i}{\Delta t}$$

$$\frac{2h\alpha\Delta t}{k\Delta x}\left(T_\infty - T_1^i\right) + \frac{2\alpha\Delta t}{\Delta x^2}\left(T_2^i - T_1^i\right) + T_1^i = T_1^{i+1}$$

$$T_1^{i+1} = \left(1 - 2\tau - \frac{2\tau h \Delta x}{k}\right) T_1^i + 2\tau\, T_2^i + \frac{2\tau h \Delta x}{k} T_\infty$$

2) $At\ x = L$:

$$E_{in} - E_{out} = \frac{dE_{sys}}{dt}$$

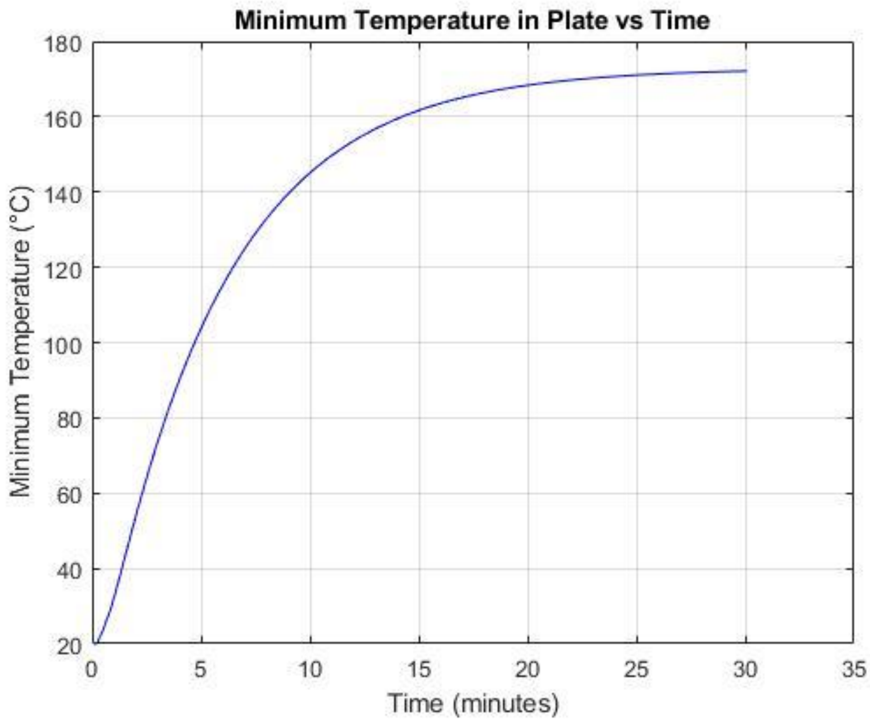
$$k\frac{T_5^i - T_6^i}{\Delta x} + q_{in} = \frac{k}{\alpha}\frac{\Delta x}{2}\frac{T_6^{i+1} - T_6^i}{\Delta t}$$

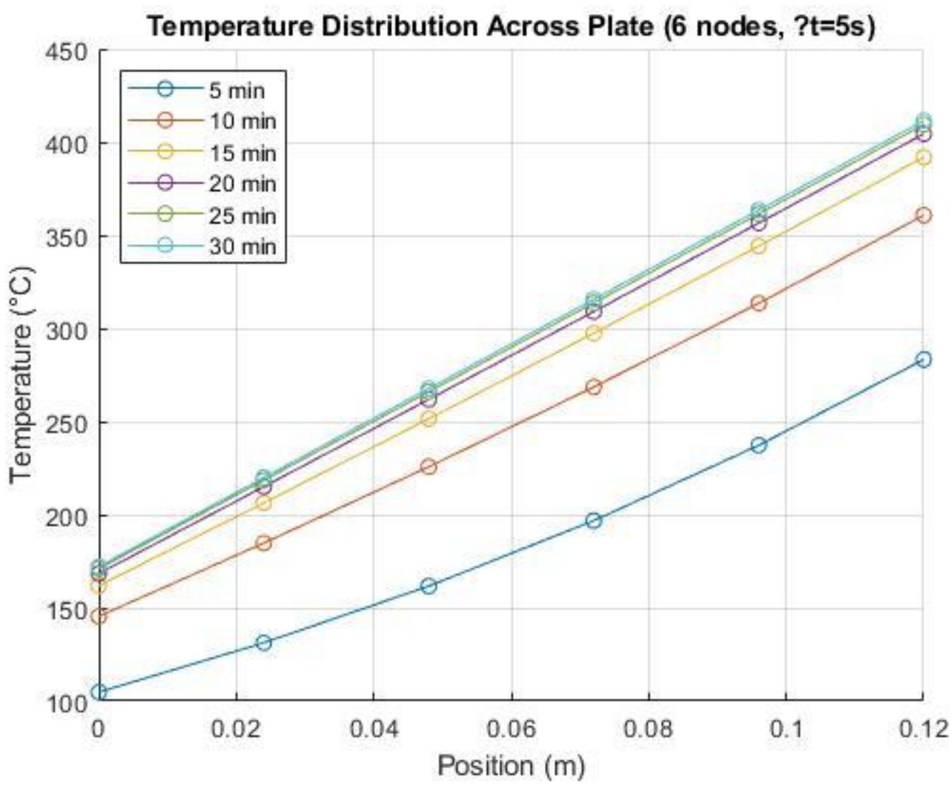
$$\therefore \frac{2\alpha\Delta t}{\Delta x^2}\left(T_5^i - T_6^i\right) + \frac{2\alpha\Delta t}{k\Delta x}q_{in} + T_6^i = T_6^{i+1}$$

$$\therefore T_6^{i+1} = 2\tau\, T_5^i + (1\ -\ 2\tau)T_6^i + \frac{2\tau\Delta x}{k}q_{in}$$

	node 1	node 2	node 3	node 4	node 5	node 6
time 5 min	100.3757	125.5418	154.7052	188.6561	227.9837	273.0518
time 10 min	142.3957	180.7994	220.8831	262.9789	307.3344	354.1023
time 15 min	160.0553	204.0223	248.6954	294.2141	340.6824	388.1646
time 20 min	167.4770	213.7821	260.3839	307.3411	354.6974	402.4798
time 25 min	170.5961	217.9264	265.2962	312.8578	360.5874	408.4959
time 30 min	171.9069	219.6075	267.3606	315.1764	363.0627	411.0242

Secondly: Question b)





Thirdly: Question c)

$$Total\ Heat\ Stored = (T - T_{initial})\Delta x\frac{k}{\alpha} = \boxed{49655.71\ Kj/m^2}$$

MATLAB Code

```
% Parameters
L = 0.12; % thickness (m)
k = 50; % thermal conductivity (W/m·K)
alpha = 3.94e-5; % thermal diffusivity (m²/s)
h = 700; % convection coefficient (W/m²·K)
T_inf = 30; % water temperature (°C)
q_flux = 100e3; % heat flux (W/m²)
T_initial = 20; % initial temperature (°C)

% Discretization
dx = 0.024; % spatial step (m)
dt = 5; % time step (s)
x = 0:dx:L; % spatial nodes (6 nodes total)
N = length(x); % number of nodes = 6

% Stability check
Fo = alpha*dt/dx^2;
if Fo > 0.5
    error('Stability criterion not satisfied! Reduce time step.');
end
fprintf('Fourier number Fo = %.3f (stable)\n', Fo);

% Simulation time setup
sim_time = 30*60; % 30 minutes in seconds
output_times = [5, 10, 15, 20, 25, 30]*60; % in seconds

% Initialize temperature array
T = ones(1,N)*T_initial;
T_record = zeros(length(output_times), N);
time_indices = zeros(1, length(output_times));

% Time stepping
current_time = 0;
output_counter = 1;
min_temp_history = [];

while current_time <= sim_time
    % Store current temperature for time stepping
    T_old = T;

    % Interior nodes (i=2 to N-1)
    for i = 2:N-1
        T(i) = T_old(i) + Fo*(T_old(i+1) - 2*T_old(i) + T_old(i-1));
    end
```

```
    end

    % Left boundary (node 1, convection)
    T(1) = (k*T_old(2)/dx + h*T_inf)/(k/dx + h);

    % Right boundary (node N, constant heat flux)
    T(N) = T_old(N-1) + q_flux*dx/k;

    % Record minimum temperature
    min_temp_history = [min_temp_history, min(T)];

    % Check if current time matches output times
    if output_counter <= length(output_times) && current_time >= output_times(output_counter)
        T_record(output_counter,:) = T;
        time_indices(output_counter) = current_time;
        fprintf('Time = %d min:\n', current_time/60);
        fprintf('Node positions (m): %s\n', sprintf('%.3f ', x));
        fprintf('Temperatures (°C): %s\n\n', sprintf('%.1f ', T));
        output_counter = output_counter + 1;
    end

    current_time = current_time + dt;
end

% Plotting temperature distributions
figure;
hold on;
colors = lines(length(output_times));
for i = 1:size(T_record,1)
    plot(x, T_record(i,:), 'o-', 'Color', colors(i,:), 'DisplayName', sprintf('%d min', output_times(i)/60));
end
xlabel('Position (m)');
ylabel('Temperature (°C)');
title('Temperature Distribution Across Plate (6 nodes, Δt=5s)');
legend('Location', 'northwest');
grid on;

% Plotting minimum temperature vs time
figure;
plot((1:length(min_temp_history))*dt/60, min_temp_history, 'b-');
xlabel('Time (minutes)');
ylabel('Minimum Temperature (°C)');
title('Minimum Temperature in Plate vs Time');
grid on;
```

```
% Calculate total heat stored after 30 minutes
rho_cp = k/alpha; % ρ*c_p (J/m³·K)
total_heat = sum(T - T_initial)*dx*rho_cp; % J/m²
total_heat_kJ = total_heat/1000; % kJ/m²

fprintf('\nTotal heat stored after 30 minutes: %.2f kJ/m²\n', total_heat_kJ);
```

```
Fourier number Fo = 0.342 (stable)
Time = 5 min:
Node positions (m): 0.000 0.024 0.048 0.072 0.096 0.120
Temperatures (°C): 104.8 131.2 161.7 196.8 237.3 283.3

Time = 10 min:
Node positions (m): 0.000 0.024 0.048 0.072 0.096 0.120
Temperatures (°C): 145.4 184.8 225.7 268.6 313.5 360.7

Time = 15 min:
Node positions (m): 0.000 0.024 0.048 0.072 0.096 0.120
Temperatures (°C): 161.8 206.3 251.5 297.4 344.2 391.9

Time = 20 min:
Node positions (m): 0.000 0.024 0.048 0.072 0.096 0.120
Temperatures (°C): 168.4 215.0 261.9 309.0 356.5 404.4

Time = 25 min:
Node positions (m): 0.000 0.024 0.048 0.072 0.096 0.120
Temperatures (°C): 171.1 218.5 266.0 313.7 361.5 409.5

Time = 30 min:
Node positions (m): 0.000 0.024 0.048 0.072 0.096 0.120
Temperatures (°C): 172.1 219.9 267.7 315.6 363.5 411.5


Total heat stored after 30 minutes: 49655.71 kJ/m²
```