Lecture : 2

Linear Algebra :-

~ is the mathematical discipline that deals with vectors and matrices, more generally vector spaces and linear transformations.

Vector space is a set of vectors toghter with a set of one-dimentional quantities known as ~~scale~~ scalars while preserving the ordinary arithmetic properties (such as associativity, commutativity, ~~distributi~~ ~~dis~~ ~~dis~~ distributivity, and so forth).

In vector space any two vectors can be added togther to give another vector and vectors ~~can~~ can be multiplied by numbers to give 'shorter' or 'longer' vectors.

Linear Transformation are function that map one vector to another. Two major operations of LT are stretching and rotating.

L.T. preserve vector addition and ~~multiplication~~ scalar multiplication. Means, if T is a linear transformation sending vector v to $T(v)$, then for any vectors v and w, and any scaler c, the transformation must satisfy the properties,

$$T(v+w) = T(v) + T(w) \text{ and}$$

$$T(cv) = cT(v).$$

when doing computation, L.T. are treated as matrix.

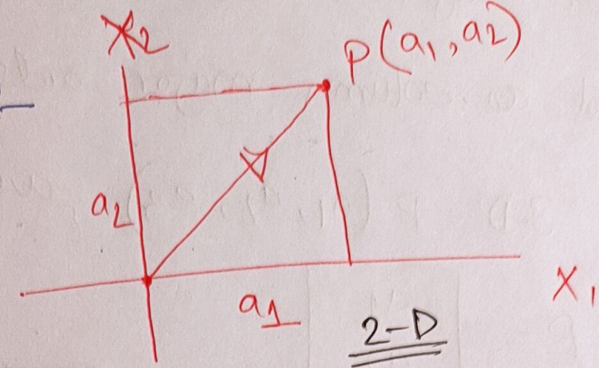
Matrix rules (i) $n\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} na & nb \\ nc & nd \end{bmatrix}$

(ii) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$

(iii) $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} g & h \\ i & j \\ k & l \end{bmatrix} = \begin{bmatrix} ag+bi+ck & ah+bj+cl \\ dg+ei+fk & dh+ej+fl \end{bmatrix}$

Basic concept of Linear Algebra :-

* **Vector :-**

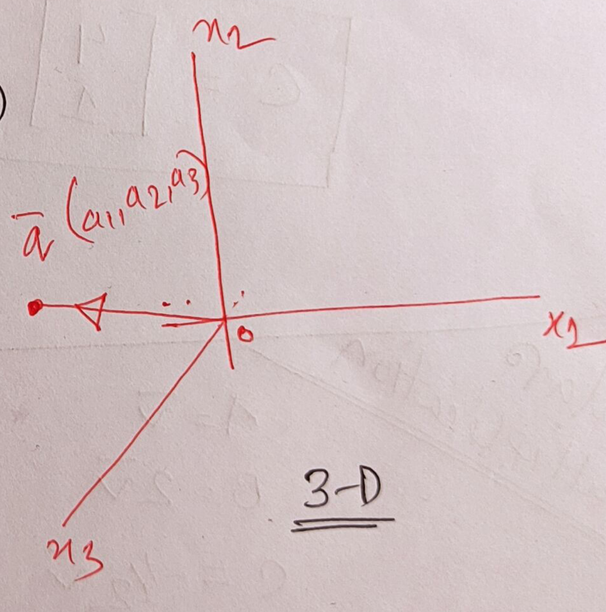


here, vector $\bar{P} = (a_1, a_2)$ so we can express it using;

$$\bar{P} = [a_1, a_2]$$

here, $\bar{q}$ vector (a_1, a_2, a_3) are like.

$$\bar{q} = [a_1, a_2, a_3]$$



So if we consider n-D ~~so we can write~~ has a vector $\bar{z}$ where the n-components are $a_1, a_2, a_3, \ldots, a_n$ so we can con:

$$\bar{z} = [a_1, a_2, a_3, \ldots, a_n]$$

in linear algebra we can write the vector also in ~~col~~ column major order.

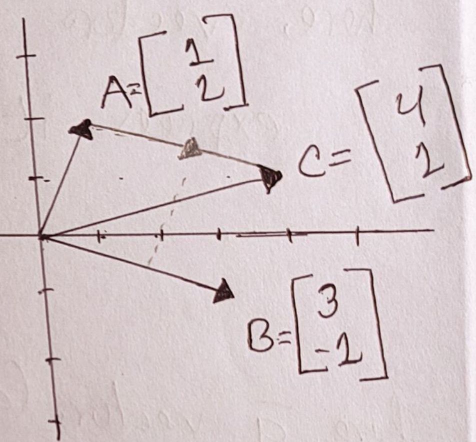
So, for 3-D $P(a_1, a_2, a_3)$ we can express like

$$P = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Addition

$$C = A + B = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\boxed{C = \begin{bmatrix} 4 \\ 1 \end{bmatrix}}$$

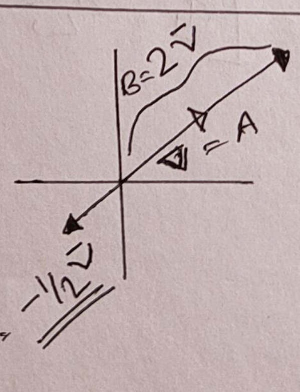


Scalar multiplication

$A = \bar{v}$

$B = 2\bar{v}$

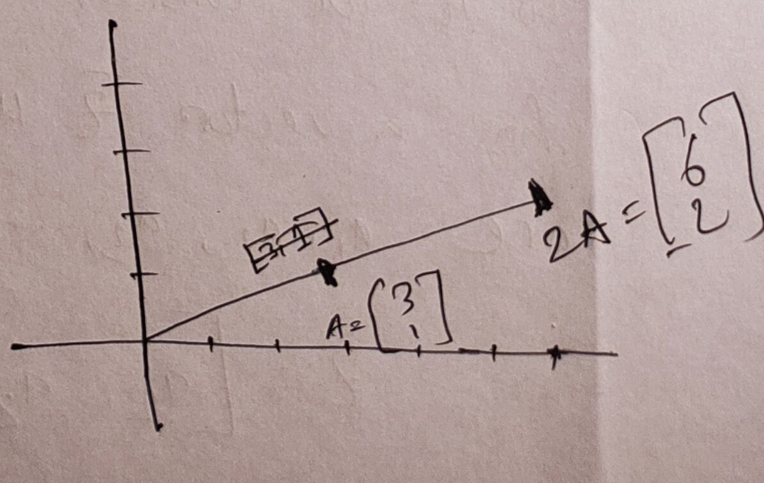
$C = -1/2\,\bar{v}$



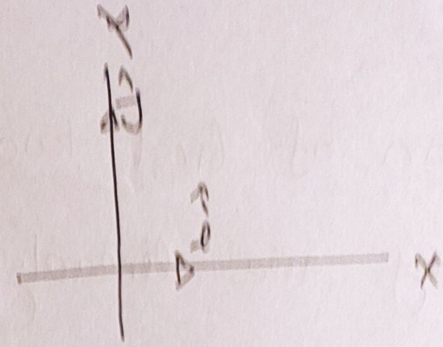
$$A = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\therefore 2A = 2\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$



Basis vectors:-



⇒ in XY-co-ordinate system ~~i and j~~ $\hat{i}$ and $\hat{j}$ are two basic vectors.

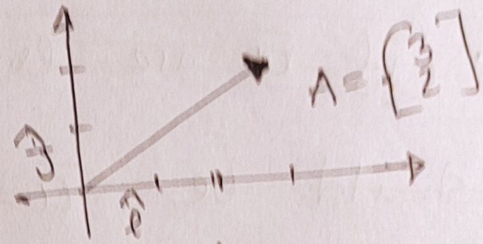


fig-2

in fig-2 we can say that vector A is the addition of two basis vectors with some scaler multiplication.

So, $A = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = (3)\hat{i} + (2)\hat{j}$

*** Span of vectors:-** The set of all possible vectors ~~combination~~ that can be found with a linear combination of a given pair of ~~vector~~. those vector is call the span of a vector.

*** Linearly dependent:-** In 3-D space where the 3rd vector is sitting of the span of first two then we can say that they are linearly dependent.

~~So,~~ For, example 3-vectors are $\vec{u}$, $\bar{v}$, and $\bar{w}$. ~~if $\bar{w}$ are linearly dependent then~~

if ~~should be~~, $\bar{u} = a\bar{v} + b\bar{w}$ or

~~$\bar{u} = \bar{v}$~~

if $\bar{v}$ and $\bar{w}$ are linearly dependent

then $\bar{u} = a\bar{v} + b\bar{w}$ [because $\bar{v}$ and $\bar{w}$ are in a same line]

Linear transformation:-

L.T. has two basic properties.

(i) All lines remain lines.

(ii) Origin remain fixed.

4

considers, two basis vectors $\hat{i}$ and $\hat{j}$ each land.

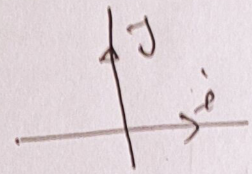


fig-1

Let assum vectors.

$V = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and placed

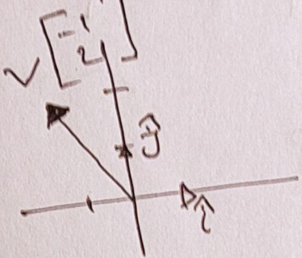
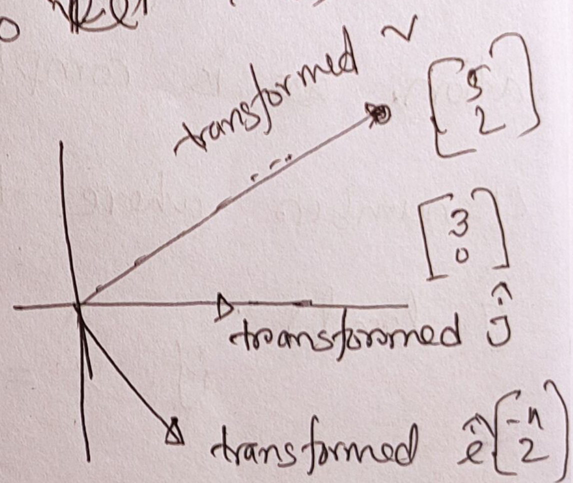


fig-2.

So, we can write:

$$V = (-1)\,\hat{i} + (2)\,\hat{j}$$

if we transform v vector to ~~T~~ T(v) ~~and~~



its placed in fig-3.

So,

~~Transformed (v)~~

$$T(\bar{v}) = -1\left(T(\hat{i})\right) + 2\left(T(\hat{j})\right).$$

$$\text{So, } T(\bar{v}) = -1\begin{bmatrix} -1 \\ -2 \end{bmatrix} + 2.\begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -2 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

So, more generally $\hat{i} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\hat{j} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{first vector}} \rightarrow \underbrace{x \begin{bmatrix} 1 \\ -2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{\text{where the vector lands}} = \begin{bmatrix} x + 3y \\ -2x + 0y \end{bmatrix}$$

So, we can say that in 2-D linear transformation, ~~we~~ is completely describes by using 4-numbers where the $\hat{i}$ lands and where the $\hat{j}$ lands. if. $\hat{i} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\hat{j} = \begin{bmatrix} c \\ d \end{bmatrix}$

then we can represent it into 2x2 matrices.

$$M = \begin{bmatrix} \underbrace{a \quad}_{\text{for } \hat{i}} & \underbrace{c \quad}_{\text{for } \hat{j}} \\ b & d \end{bmatrix}$$

Now. what if we want to transform vector $A = \begin{bmatrix} x \\ y \end{bmatrix}$.

Using the privious observation we can say that:

$$T(A) = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

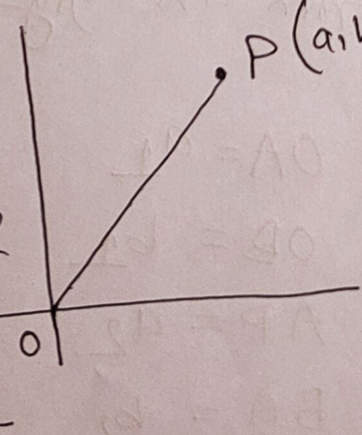
$$= x \begin{bmatrix} a \\ b \end{bmatrix} + y \begin{bmatrix} c \\ d \end{bmatrix}$$

$$= \begin{bmatrix} ax + cy \\ bx + dy \end{bmatrix}$$

Actually the result is matrix multiplications.

* Distance of a point from origin:

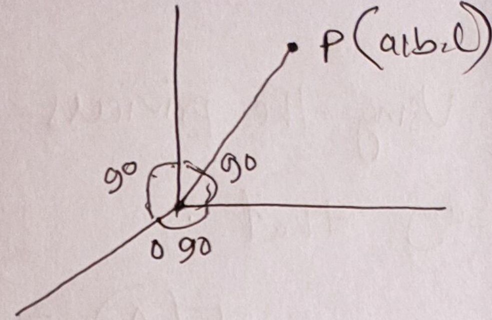
⇒ 2-D plane:



The straight line distance from O to P is $|OP|$. define as. $|OP| = \sqrt{a^2 + b^2}$.

⇒ 3-D plane:- for 3-D

distance $|OP| = \sqrt{a^2+b^2+c^2}$.



⇒ n-D plane: if $P(a_1, a_2, a_3, \ldots, a_n)$ is a point in ~~vector~~ n-dimentional plane then $|OP| = \sqrt{a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2}$.

* Distance between 2-vectors/points:

⇒ for 2-D:-

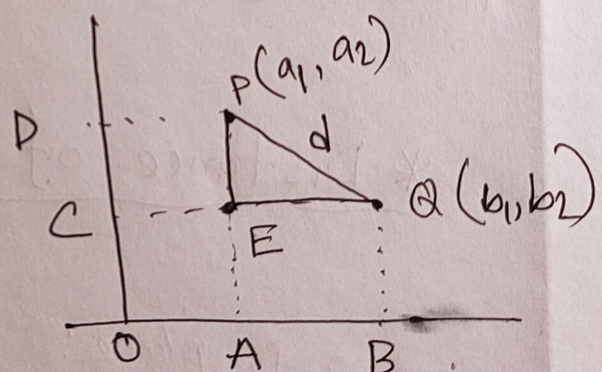


fig-1

From the fig-1

$OA = a_1$

$OB = b_1$

$AP = a_2$

$BQ = b_2$

$\therefore EQ = CQ - CE$
$= OB - OA$
$= b_1 - a_1$

Same, $PE = PA - AE$
$= a_2 - BQ$
$= a_2 - b_2$

So, $d = \sqrt{PE^2 + EQ^2}$
$= \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2}$

6

for 3-D space:- distance between $P(a_1, a_2, a_3)$ and $Q(b_1, b_2, b_3)$ is d and define as:

$$d = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$$

for n-D space: if $P(a_1, a_2, a_3, \dots, a_n)$ and $Q(b_1, b_2, b_3, \dots b_3)$ are two points in n-dim. space then distance between them is d and define by $d = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 + \dots + (a_n - b_n)^2}$.

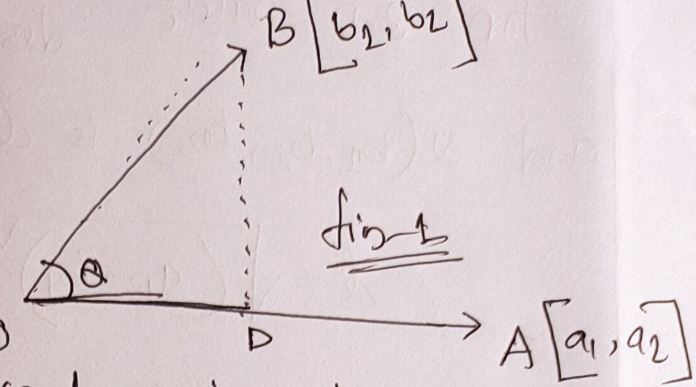
$$= \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}$$

Row & Column vector:-

row vector, $A = \begin{bmatrix} a_1, a_2, a_3, \dots, a_n \end{bmatrix}_{(1 \times n)}$

Column vector. $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}_{(n \times 1)}$

Projection :-



projection of a vector to another vector is the saddow ~~from~~ of first vector to its the second vector.

Projection of $\vec{OB}$ vector on $\vec{OA}$ vector is OD.

In fig-1, $OA = a_1$

So, $\cos Q = \dfrac{OD}{OB}$ [if we consider $\triangle OBD$]

or, $OD = OB \cdot \cos Q$

$\therefore OD = |OB| \cdot \cos Q$

$$\boxed{OD = \sqrt{b_1^2 + b_2^2} \cdot \cos Q}$$

So, the projection of OB over OA is $\sqrt{b_1^2 + b_2^2} \cdot \cos \theta$.

7//

Dot producto of two vectors:-

The dot product of two vectors is the multiplication of the magnitude of one vectors with the projection of others.

if $\bar{A}$ and $\bar{B}$ are two vectors then the dot producto of $\bar{A}$ and $\bar{B}$.

$$\bar{A}\cdot\bar{B} = |A|\cdot(\text{the projection of } B \text{ over } A).$$

$$\bar{A}\cdot\bar{B} = |A|\cdot|B|\cos\theta \longrightarrow (i)$$

Now, let consider two vector $\bar{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ So, the dot product is.

$$\bar{A}\cdot\bar{B} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$= (a_1\hat{i}\cdot b_1\hat{i}) + (a_1\hat{i}\cdot b_2\hat{j}) + (a_1\hat{i}\cdot b_3\hat{k}) + (a_2\hat{j}\cdot b_1\hat{i}) + (a_2\hat{j}\cdot b_2\hat{j}) + (a_2\hat{j}\cdot b_3\hat{k}) + (a_3\hat{k}\, b_1\hat{i}) + (a_3\hat{k}\cdot b_2\hat{j}) + (a_3\hat{k}\cdot b_3\hat{k})$$

$$\bar{A}\cdot\bar{B} = (a_1 b_1)\cdot(\hat{i}\cdot\hat{i}) + (a_1\cdot b_2)(\hat{i}\cdot\hat{j}) + (a_1 b_3)(\hat{i}\cdot\hat{k}) + (a_2 b_1)(\hat{j}\cdot\hat{i}) + (a_2 b_2)(\hat{j}\cdot\hat{j}) + (a_2 b_3)(\hat{j}\cdot\hat{k}) + (a_3 b_1)\cdot(\hat{k}\cdot\hat{i}) + (a_3 b_2)(\hat{k}\cdot\hat{j}) + (a_3 b_3)(\hat{k}\cdot\hat{k}) \longrightarrow (ii)$$

here, unit vectors $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

$$\hat{i}\cdot\hat{i} = |\hat{i}||\hat{i}|\cdot\cos 0^\circ$$
$$= 1$$

Again, $\hat{i}\cdot\hat{j} = |\hat{i}|\cdot|\hat{j}|\cdot\cos 90^\circ = 0$

here, ~~the~~ the dot product of (i, j), (j, k), (k, i) are 0. because the perpendicular unit vectors has angle 90°.

So, modifying (ii) become.

$$\bar{A}\cdot\bar{B} = a_1b_1 + a_2b_2 + a_3b_3$$

So, from (i).

$$\bar{A}\cdot\bar{B} = |A|\cdot|B|\cdot\cos\theta$$

$$\Rightarrow a_1b_1 + a_2b_2 + a_3b_3 = |A|\cdot|B|\cdot\cos\theta$$

$$\Rightarrow \cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|A|\cdot|B|}$$

$$\boxed{\therefore \theta = \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\cdot\sqrt{b_1^2 + b_2^2 + b_3^2}}\right)}$$

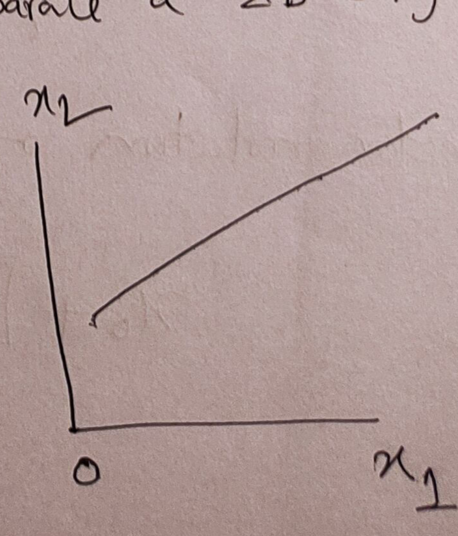
Now, for two vector $\overline{A} = [a_1, a_2, a_3, \ldots, a_n]$ and $\overline{B} = [b_1, b_2, b_3, \ldots, b_n]$ the θ is define as.

$$\theta = \cos^{-1}\left(\frac{\cancel{a_1b_1}\ a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_nb_n}{\sqrt{a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2 + \cdots + b_n^2}}\right)$$

$$= \cos^{-1}\left(\frac{\sum_{i=1}^{n} a_i b_i}{\sqrt{\sum_{i=2}^{n} a_i^2 \cdot b_i^2}}\right)$$

$$\boxed{\therefore \theta = \cos^{-1}\left(\frac{\sum_{i=1}^{n} a_i b_i}{\sqrt{\sum_{i=1}^{n} (a_i \cdot b_i)^2}}\right)}$$

Line :- Line can separate a 2-D surface into two region.

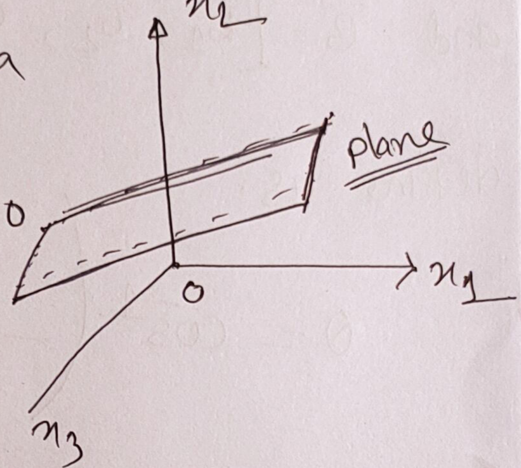


In x_1-x_2 plane the equation is, $a_1x_1 + a_2x_2 + a_0 = 0$

$\longrightarrow$ (1)

Plane :- In 3d-the equivalent of line is plane. The equation of a plane is: $a_1x_1+a_2x_2+a_3x_3+a_0=0$

$$\longrightarrow \text{(II)}$$



Hyper-plane:- Like line and plane hyper plane is the separator of a n-dimentional space. And the equation become,

$$a_1x_1+a_2x_2+a_3x_3+\cdots+a_nx_n+x_0=0$$

$$\longrightarrow \text{(III)}$$

So, ~~the~~ equivalen of eqn-(III) is:

$$x_0+\sum_{i=1}^{n} a_ix_i \longrightarrow \text{(iv)}$$

Vector notation of eqn(iv) become.

$$x_0+\underbrace{\left[a_1, a_2, a_3, \ldots, a_n\right]}_{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \longrightarrow \text{(V)}$$

Q)

if we consider $[a_1, a_2, a_3 \ldots, a_n]$ is a column vector A and $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$ also a column vector X then

the eqn.

become π : ~~$x_0 + X^T A = 0$~~

$$x_0 + A^T X = 0 \longrightarrow (v)$$

Now,

we know the basic eqn of line is.

$$y = mx + c \longrightarrow (vi)$$

where m = slope and c = intersect of line on y-axis.

Lets, go with eqn (i)

$$a_1 x_1 + a_2 x_2 + a_0 = 0$$

$$\Rightarrow a_2 x_2 = -a_1 x_1 - a_0$$

$$\Rightarrow x_2 = -\frac{a_1}{a_2} x_1 - \frac{a_0}{a_2} \longrightarrow (vii)$$

compare (vi) and (vii) we have,

$$c = \frac{-a_0}{a_2}$$

if eqn (vi) go through origin (0,0) ~~then~~

$C = 0$

as $\frac{-a_0}{a_2} = c$ So, $\boxed{a_0 = 0}$

So Now, if any line pass through origin $a_0 = 0$.

From (I) we can write.

$$a_1 x_1 + a_2 x_2 = 0 \longrightarrow (VIII)$$

Similarly any plane go through origin become.

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0 \longrightarrow (IX)$$

Any hyper plane in n-dim pass through origin

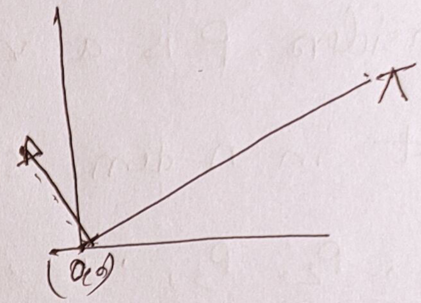
become.

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \dots + a_n x_n = 0 \longrightarrow (X)$$

or, $$\sum_{i=1}^{n} a_i x_i = 0$$

or, $$A^T x = 0 \longrightarrow (XI)$$

Now, consider a plane π which is passing through the origin and the plane π is in a - n-dim.



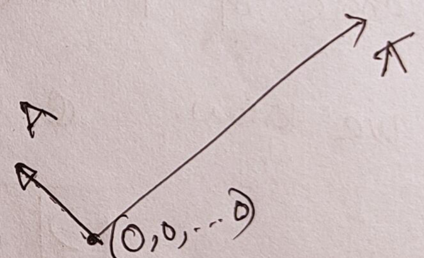
here, $A^T x = 0$

then, $|A| \cdot |x| \cos\theta = 0$

so, $\theta = 90^\circ$

So, we can say that the angle between A and x is 90°.

here,
$$A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$$



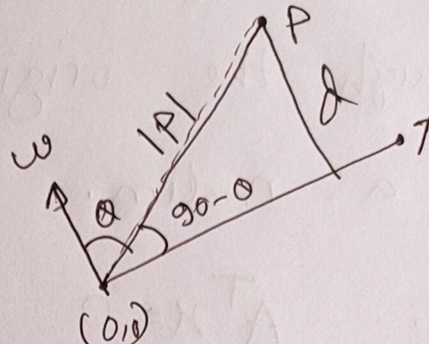
and
$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

if $A \perp x$ then

$$w \cdot x_i = 0 \quad \forall x_i \in \pi \longrightarrow \text{(XI)}$$

from eqn (XI) we can say that, w is the perpendicular object from each point of the plane π.

* Distance ~~from~~ of a point from a plane in n-dim space:



Considers, P is a ~~vectors~~ point in n-dim. space

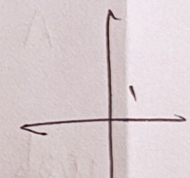
$[P_1, P_2, P_3, \dots, P_n]$.

So, there, angle between w and π is 90°.

Assume. angle between w and P is θ.

So, angle of P and π is $(90^\circ - \theta)$

we know, $\sin(90^\circ - \theta) = \frac{d}{|P|}$



$$\Rightarrow d = \cancel{\frac{|P|}{\sin(90^\circ - \theta)}}$$

$$d = \frac{|P|}{\cos\theta}$$

$$\Rightarrow d = |P| \sin(90^\circ - \theta)$$

$$\therefore d = |P| \cos\theta \longrightarrow (1)$$

Again. $w^T . P = |w| |P| \cos\theta$

$\Rightarrow w^T . P = |w| \, d$ [from eqn (1)]

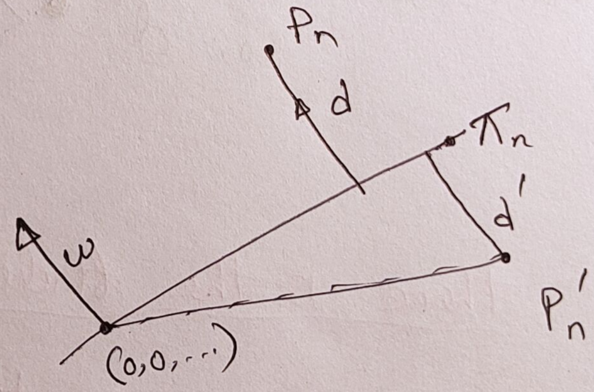
22

$$\Rightarrow d = \frac{\omega^T P}{|\omega|} \xrightarrow{as} (2)$$

ean → (2) is the formula of the distance from any point to ~~the~~ a given plane.

half-space :

Two region when separate by a line (in 2D), or plane (in 3-D), or hyper-plane (in nD) then each of the region is called a half space.



Let cosider,

π_n is an hyper-plane with ~~n-points~~, n-dims.

ω is a vector in the origin. Now for P_n we know,

$$d = \frac{w^T P}{|w|} \longrightarrow (1)$$

Similarly for P_n' we can write that,

$$d' = \frac{w^T P'}{|w|} \longrightarrow (2)$$

here (2), $\omega^T P' = |\omega| \, |P'| . \cos\theta$

~~So, in eqn (2) the value of d' is dep~~

So, in eqn (2) the sign of d' depends on $\cos\theta$. if $\theta > 90$ then $d' < 0$ or if $\theta < 90°$ $d' > 0$.

But, for d' or any other points bellow π_n θ always remain ~~less than~~ greater than $90°$.

So, $d' < 0$ [$\forall$ points bellow π_n.]

There is the finding :- if the dot product of w and any points are greater then '0' than the point is above the plane otherwise bellow the plane.

Circle:-

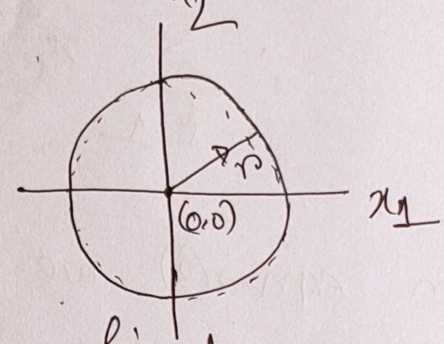


fig-1

The equation of a circle with center in origin is, $x_1^2 + x_2^2 = r^2$

Now, lets consider a point $P(a_1, a_2)$.

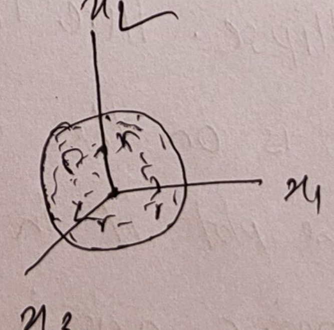
if $a_1^2 + a_2^2 < 0$ then P lies inside the circle.

if $a_1^2 + a_2^2 > 0$ then P lies outside the circle.

if $a_1^2 + a_1^2 = 0$ then P lies into the circle.

Sphere:-

The equivalent object of circle in 3-D is sphere. The equation of ~~stop~~ sphere is, $x_1^2 + x_2^2 + x_3^2 = r^2$



Hyper-sphere:-

An object in n-dim. space with n points is hyper-sphere.

The eqn. is $x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 = r^2$

$$\therefore \sum_{i=1}^{n} x_i^2 = \rho^2 \longrightarrow (3)$$

From eqn → (3) we can say that,

a point $P(P_1, P_2, P_3, \ldots, P_n)$ in an n-dim

hyper-space is:

inside the n-dim hyper-sphere if $\sum_{i=1}^{n} x_i^2 < 0$

outside " n-dim " if $\sum_{i=1}^{n} x_i^2 > 0$

inthe n-dim " if $\sum_{i=1}^{n} x_i^2 = 0$.

Ellipse:-

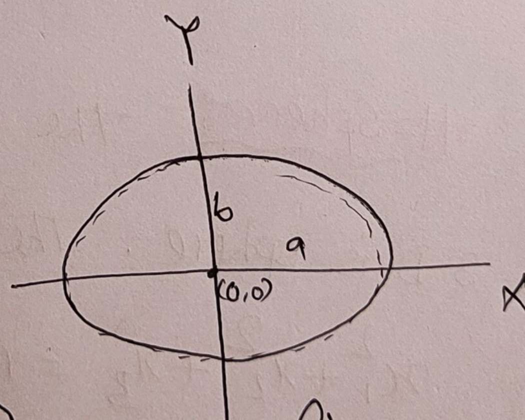


fig-2

The fig-1 represent an ellipse. Here its center is on the origin (0,0) as we plot it in 2-D -plane.

The major axis is → a and minor is → b.

So, the eqn. of an ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \longrightarrow (1)$$

An ellipse divide its region into two regions.

One inside the region other outside the region.

Let us consider a point $P(P_1, P_2)$ in the same plane.

if $\frac{P_1^2}{a^2} + \frac{P_2^2}{b^2} > 1$ then outside the ellipse

if $\frac{P_1^2}{a^2} + \frac{P_2^2}{b^2} < 1$ " inside " "

if $\frac{P_1^2}{a^2} + \frac{P_2^2}{b^2} = 1$ " on the "

Ellipsoid :- In 3-D ellipse is called ellipsoid.

The equation become:

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1$$

Similarly in n-D the ellipse is called hyper-ellipsoid. And the eqn. is

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} + \cdots + \frac{x_n^2}{a_n^2} = 1$$

So, $$\sum_{i=1}^{n} \left(\frac{x_i}{a_i}\right)^2 = 1 \longrightarrow (2)$$

if the left side of eqn. (2) become.

< 1 then a point $P_n(P_1, P_2, P_3 .. P_n)$ inside the hyper-ellipsoid.

> 1 then a point $P_n(P_1\ P_2\ P_3 ... P_n)$ outside the hyper-ellipsoid.

$= 1$ then a point $P_n(P_1, P_2, P_3, .. P_n)$ on the ellipsoid.

——— o ———