Regression

Master Data Science
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Formulae, R Reference and Tables

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Contents

1 Mathematics

1.1 Derivatives of Functions

- constant, multiplying with a constant: $(c)' = 0\,, \quad (c \cdot f(x))' = c \cdot f'(x)$
- power function: $(x^n)' = n\, x^{n-1}$
- logarithms and exponential functions:

$$(\ln(x))' = \frac{1}{x}\,, \quad (\log_a(x))' = \frac{1}{\ln(a)} \cdot \frac{1}{x}\,, \quad (e^x)' = e^x\,, \quad (a^x)' = \ln(a) \cdot a^x$$

- sine and cosine: $(\sin(x))' = \cos(x)\,, \quad (\cos(x))' = -\sin(x)$
- sum rule: $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
- product rule: $(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x)$ short: $(uv)' = u'v + uv'$
- quotient rule: $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$ short: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$
- chain rule: $(g(h(x))' = g'(h(x)) \cdot h'(x)$

1.2 Multidimensional Functions

- gradient vector and Hessian matrix:

$$\mathcal{D}_f(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial}{\partial x_1} f(\boldsymbol{x}) \\ \vdots \\ \frac{\partial}{\partial x_p} f(\boldsymbol{x}) \end{pmatrix} \quad \text{and} \quad \mathbf{H}_f(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial^2}{\partial x_1 \partial x_1} f(\boldsymbol{x}) & \dots & \frac{\partial^2}{\partial x_1 \partial x_p} f(\boldsymbol{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial x_1 \partial x_p} f(\boldsymbol{x}) & \dots & \frac{\partial^2}{\partial x_p \partial x_p} f(\boldsymbol{x}) \end{pmatrix}$$

1.3 Linear Algebra

Vectors

- inner product: $\boldsymbol{u}^\top \boldsymbol{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$
- norm (length): $\|\boldsymbol{u}\| = \sqrt{\boldsymbol{u}^\top \boldsymbol{u}}$

Matrices

- transpose: $\mathbf{A}^\top = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}^\top = \begin{pmatrix} a_{11} & \dots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{mn} \end{pmatrix}$

  properties:

$$(\mathbf{A} + \mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top\,, \quad (c\,\mathbf{A})^\top = c\,\mathbf{A}^\top\,, \quad (\mathbf{A}^\top)^\top = \mathbf{A}, \quad (\mathbf{A} \cdot \mathbf{B})^\top = \mathbf{B}^\top \cdot \mathbf{A}^\top$$

- system of equations: $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$
- matrix product: $\mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} \boldsymbol{a}_1^\top \\ \vdots \\ \boldsymbol{a}_m^\top \end{pmatrix} \cdot (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k) = \begin{pmatrix} \boldsymbol{a}_1^\top \boldsymbol{b}_1 & \ldots & \boldsymbol{a}_1^\top \boldsymbol{b}_k \\ \vdots & \ddots & \vdots \\ \boldsymbol{a}_m^\top \boldsymbol{b}_1 & \ldots & \boldsymbol{a}_m^\top \boldsymbol{b}_k \end{pmatrix}$

  properties:

$$(\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C}\,, \quad s(\mathbf{A} \cdot \mathbf{B}) = (s\mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (s\mathbf{B})$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}\,, \quad (\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C}$$

Square Matrices

- symmetric matrix: $\mathbf{A}^\top = \mathbf{A}\,,$ identity matrix: $\mathbf{I}_n = \mathbf{I} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$
- trace: $\mathrm{trace}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$

  properties:

$$\mathrm{trace}(\mathbf{A} + \mathbf{B}) = \mathrm{trace}(\mathbf{A}) + \mathrm{trace}(\mathbf{B})\,, \quad \mathrm{trace}(c\,\mathbf{A}) = c\,\mathrm{trace}(\mathbf{A})$$

$$\mathrm{trace}(\mathbf{A}) = \mathrm{trace}(\mathbf{A}^\top)\,, \quad \mathrm{trace}(\mathbf{ABC}) = \mathrm{trace}(\mathbf{CAB}) = \mathrm{trace}(\mathbf{BCA})$$

- 2×2 determinant: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$
- 3×3 determinant: (Sarrus' rule)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

- determinant properties:

  multiply a row or column by a scalar $\Rightarrow\ s \cdot \det(\mathbf{A})$

  exchanging two rows or columns $\Rightarrow\ -\det(\mathbf{A})$.

  add a multiple of a row/column to another row/column $\Rightarrow$ no change

  $\det(\mathbf{A}^\top) = \det(\mathbf{A})$

  $\det(s\,\mathbf{A}) = s^n\ \det(\mathbf{A})$

  row/column completely zero $\Rightarrow\ \det(\mathbf{A}) = 0$

  two rows/columns identical $\Rightarrow\ \det(\mathbf{A}) = 0$

  $\det(\mathbf{A}) \cdot \det(\mathbf{B}) = \det(\mathbf{B}) \cdot \det(\mathbf{A}) = \det(\mathbf{A} \cdot \mathbf{B}) = \det(\mathbf{B} \cdot \mathbf{A})$

- inverse matrix: $\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}$
- 2×2 inverse: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{ad - cb} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

- inverse matrix properties:

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}\,, \quad \det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}\,, \quad (\mathbf{A}^\top)^{-1} = (\mathbf{A}^{-1})^\top$$

$$(s\,\mathbf{A})^{-1} = \frac{1}{s}\mathbf{A}^{-1}\,, \quad (\mathbf{A}\cdot\mathbf{B})^{-1} = \mathbf{B}^{-1}\cdot\mathbf{A}^{-1}$$

- equivalent $n \times n$ matrix properties:

$$\det(\mathbf{A}) \neq 0 \iff \mathbf{A} \text{ invertible} \iff \operatorname{rank}(\mathbf{A}) = n$$

$$\iff \text{rows/columns of } \mathbf{A} \text{ are linearly independent}$$

$$\iff \mathbf{A}\boldsymbol{x} = \boldsymbol{b} \quad \text{has exactly one solution} \quad \boldsymbol{x} = \mathbf{A}^{-1}\boldsymbol{b}$$

- eigenvalues, eigenvectors: $\mathbf{A}\boldsymbol{v} = \lambda\boldsymbol{v}$,
  characteristic polynomial to determine λ: $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

- spectral decomposition of a symmetric matrix $\mathbf{A}$: $\mathbf{A} = \boldsymbol{\Gamma}\boldsymbol{\Lambda}\boldsymbol{\Gamma}^\top = \sum_{j=1}^{d} \lambda_j \boldsymbol{\gamma}_j \boldsymbol{\gamma}_j^\top$

  where Λ $(d\times d)$ is the diagonal matrix of all eigenvalues λ_j and $\boldsymbol{\Gamma}$ $(d\times d)$ is an orthogonal matrix containing the eigenvectors $\boldsymbol{\gamma}_j$ as columns

  properties:

  orthogonality: $\boldsymbol{\gamma}_j^\top\boldsymbol{\gamma}_j = 1\,,\quad \boldsymbol{\gamma}_j^\top\boldsymbol{\gamma}_k = 0$ for $j \neq k$

  if $\mathbf{A}$ has full rank d, then all $\lambda_j \neq 0$

  $\operatorname{trace}(\mathbf{A}) = \sum_j \lambda_j\,,\quad \det(\mathbf{A}) = \prod_j \lambda_j\,,\quad \mathbf{A}^{-1} = \Gamma\Lambda^{-1}\Gamma^\top$

  $\boldsymbol{\Lambda} > \mathbf{0} \iff \mathbf{A} > \mathbf{0}\,,\quad \boldsymbol{\Lambda} \geq \mathbf{0} \iff \mathbf{A} \geq \mathbf{0}$

2 Probability

2.1 Probability of Events

- Laplace probability: $P(A) = \dfrac{n(A)}{n(\Omega)} = \dfrac{\text{number of elementary events in } A}{\text{number of all elementary events}}$

- Kolmogorov Axioms:

  (A1) For any event A holds $P(A) \geq 0$.

  (A2) The sure event Ω has probability $P(\Omega) = 1$.

  (A3) For any two disjoint events A, B holds $P(A \cup B) = P(A) + P(B)$.

- properties of P:

  $P(A^C) = 1 - P(A)$

  $P(\varnothing) = 0$

  $A \subseteq B \;\Rightarrow\; P(A) \leq P(B)$

$P(B \setminus A) = P(B) - P(A \cap B)$

inclusion–exclusion principle: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

- sums of probabilities:

  for n mutually disjoint events: $P(A_1 \cup \ldots \cup A_n) = P(A_1) + \ldots + P(A_n)$

  for any n events:

$$
\begin{aligned}
P\left(\bigcup_{i=1}^{n} A_i\right) &= \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) \mp \ldots \\
&= \sum_{k=1}^{n} (-1)^{k+1} \sum_{I \subseteq \{1,\ldots,n\}, n(I)=k} P\left(\bigcap_{i \in I} A_i\right)
\end{aligned}
$$

- conditional probability: $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$ if $P(A) > 0$

- stochastically independent events:

$$P(B \mid A) = P(B \mid A^C) = P(B) \iff P(A \cap B) = P(A) \cdot P(B)$$

- 2×2 table:

	B	B^C	$\sum$
A	$P(A \cap B)$	$P(A \cap B^C)$	$P(A)$
A^C	$P(A^C \cap B)$	$P(A^C \cap B^C)$	$P(A^C)$
$\sum$	$P(B)$	$P(B^C)$	1

- products of probabilities:

  for n (stochastically) independent events: $P(A_1 \cap \ldots \cap A_n) = P(A_1) \ldots P(A_n)$

  for any n events:

$$
\begin{aligned}
P(A_1 \cap \ldots \cap A_n) &= P(A_n \mid A_1 \cap \ldots \cap A_{n-1}) \cdot P(A_{n-1} \mid A_1 \cap \ldots \cap A_{n-2}) \\
&\quad \cdot \ldots \cdot P(A_2 \mid A_1) \cdot P(A_1)
\end{aligned}
$$

- Bayes' formula: $P(B|A) = \dfrac{P(A|B) \cdot P(B)}{P(A)}$

- total probability: consider B and mutually disjoint A_i with $P(A_1 \cup \ldots \cup A_n) = \Omega$, then:

$$P(B) = \sum_{i=1}^{n} P(B \mid A_i) \cdot P(A_i)$$

- Bayes' theorem: $P(A_k \mid B) = \dfrac{P(B \mid A_k) \cdot P(A_k)}{\sum_{i=1}^{n} P(B \mid A_i) \cdot P(A_i)}$

2.2 Random Variables

- pdf vs. cdf: $F(x) = P(X \leq x)$, i.e.

$$F(x) = \sum_{j:x_j \leq x} f(x) \quad \text{or} \quad F(x) = \int_{-\infty}^{\infty} f(x)\,dx$$

- expectation of a function $g(X)$:

$$E\big(g(X)\big) = \sum_j g(x_j)\,f(x_j) \quad \text{or} \quad E\left(g(X)\right) = \int_{-\infty}^{\infty} g(x)f(x)\,dx$$

- variance: $\mathrm{Var}(X) = E\left(X - E(X)\right)^2 = E(X^2) - \big(E(X)\big)^2$
- expectation of a linear function: $E(a_0 + a_1X_1 + \ldots a_nX_n) = a_0 + a_1E(X_1) + \ldots + a_nE(X_n)$
- covariance: $\mathrm{Cov}(X,Y) = E\Big(\big(X - E(X)\big)\big(Y - E(Y)\big)\Big) = E(XY) - E(X)E(Y)$
- correlation: $\mathrm{Corr}(X,Y) = \dfrac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)} \cdot \sqrt{\mathrm{Var}\,Y}}$
- (stochastically) independent random variables $X_1, \ldots, X_n$:

$$f(x_1, \ldots, x_n) = f_{X_1}(x_1) \cdot \ldots \cdot f_{X_n}(x_n)$$

- variance of a linear function:

  for any two random variables

$$\mathrm{Var}(a + bX + cY) = b^2\,\mathrm{Var}(X) + c^2\,\mathrm{Var}(Y) + 2bc\,\mathrm{Cov}(X,Y)$$

  for n (stochastically) independent random variables

$$\mathrm{Var}(a_0 + a_1X_1 + \ldots a_nX_n) = \sum_j a_j^2\,\mathrm{Var}(X_j)$$

  for any n random variables

$$\mathrm{Var}(a_0 + a_1X_1 + \ldots a_nX_n) = \sum_j a_j^2\,\mathrm{Var}(X_j) + 2\sum_{j<k} a_ja_k\,\mathrm{Cov}(X_j, X_k)$$

- marginal pdf of X (calculated from the joint distribution of X and Y):

$$f_X(x) = \sum_k f(x, y_k) \quad \text{or} \quad f_X(x) = \int f(x,y)\,dy$$

- conditional pdf of X given $Y = y$: $f(x|y) = f_{X|Y=y}(x) = \dfrac{f(x,y)}{f_Y(y)}$
- median, quantiles of a continuous random variable: $x_{0.5} = F^{-1}(0.5)\,, x_p = F^{-1}(p)$

2.3 Random Vectors

- expectation, variance–covariance matrix:

$$
E\boldsymbol{X} = \begin{pmatrix} EX_1 \\ \vdots \\ EX_n \end{pmatrix}, \quad \operatorname{Var}(\boldsymbol{X}) = \begin{pmatrix} \operatorname{Var}(X_1) & \operatorname{Cov}(X_1, X_2) & \dots & \operatorname{Cov}(X_1, X_n) \\ \operatorname{Cov}(X_1, X_2) & \operatorname{Var}(X_2) & \dots & \operatorname{Cov}(X_2, X_n) \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{Cov}(X_1, X_n) & \operatorname{Cov}(X_2, X_n) & \dots & \operatorname{Var}(X_n) \end{pmatrix}
$$

properties: if $\boldsymbol{\mu} = E\boldsymbol{X}$ and $\boldsymbol{\Sigma} = \operatorname{Var}(\boldsymbol{X})$

$$
\begin{aligned}
&E(\boldsymbol{c} + \mathbf{A}\boldsymbol{X}) = \boldsymbol{c} + \mathbf{A}\boldsymbol{\mu} \\
&\operatorname{Var}(\boldsymbol{c} + \mathbf{A}\boldsymbol{X}) = \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top \\
&E(\boldsymbol{X}^\top \mathbf{B}\boldsymbol{X}) = \boldsymbol{\mu}^\top \mathbf{B}\boldsymbol{\mu} + \operatorname{trace}(\mathbf{B}\boldsymbol{\Sigma}) \\
&\operatorname{Cov}(\mathbf{A}\boldsymbol{X}, \mathbf{D}\boldsymbol{X}) = \mathbf{A}\boldsymbol{\Sigma}\mathbf{D}^\top
\end{aligned}
$$

2.4 Frequently used Distributions

Discrete	parameters	values	$f(x)$	$E(X)$	$\mathrm{Var}(X)$
Bernoulli $B(1,p)$	$0<p<1$	$X \in \{0,1\}$	$p^x(1-p)^{1-x}$	p	$p(1-p)$
binomial $B(m,p)$	$0<p<1$ $m \in \mathbb{N}$	$X \in \{0,\dots,m\}$	$\binom{m}{x} p^x(1-p)^{m-x}$	$m\,p$	$m\,p(1-p)$
hypergeometric $H(N,M,n)$	$N,M,n \in \mathbb{N}$ $M,n \le N$	$X \in \{\max(0,n+M-N),$ $\dots,\min(n,M)\}$	$\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$	$n \cdot \frac{M}{N}$	$n \cdot \frac{N-n}{N-1} \cdot \frac{M}{N} \cdot \left(1-\frac{M}{N}\right)$
geometric $Geo(p)=NB(1,p)$	$0<p<1$	$X \in \mathbb{N}$	$p(1-p)^{x-1}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
negative binomial $NB(r,p)$	$0<p<1$ $r \in \mathbb{N}$	$X \in \{r,r+1,\dots\}$	$\binom{x-1}{r-1} p^r(1-p)^{x-r}$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$
Poisson $Poi(\lambda)$	$\lambda>0$	$X \in \{0,1,2,\dots\}$	$\frac{\lambda^x}{x!}e^{-\lambda}$	λ	λ

Continuous	parameters	values	$f(x)$	$F(x)$	$E(X)$	$\mathrm{Var}(X)$
uniform $U[a,b]$	$a,b \in \mathbb{R}$ $a<b$	$X \in [a,b]$	$\frac{1}{b-a}$	$\frac{x-a}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
normal $N(\mu,\sigma^2)$	$\mu \in \mathbb{R}$ $\sigma>0$	$X \in \mathbb{R}$	$\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$\Phi\left(\frac{x-\mu}{\sigma}\right)$	μ	σ^2
exponential $Exp(\lambda)$	$\lambda>0$	$X \in \mathbb{R}$ $X \ge 0$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
lognormal $LN(\mu,\sigma^2)$	$\mu \in \mathbb{R}$ $\sigma>0$	$X \in \mathbb{R}$ $X>0$	$\frac{1}{\sqrt{2\pi}\,\sigma x}\exp\left(-\frac{(\ln(x)-\mu)^2}{2\sigma^2}\right)$	$\Phi\left(\frac{\ln(x)-\mu}{\sigma}\right)$	$\exp\left(\mu+\frac{\sigma^2}{2}\right)$	$\exp(2\mu+\sigma^2)$ $\cdot\left(\exp(\sigma^2)-1\right)$

3 Linear Models

Simple Linear Regression

- model: $Y = \beta_0 + \beta_1 x + \varepsilon\,,$ data: pairs $(x_i, y_i)\,,\; i = 1, \ldots, n$
- parameter estimation by least squares (LSE):

$$Q(\beta_0, \beta_1) = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2 \to \min_{\beta_0, \beta_1 \in \mathbb{R}} \qquad \Rightarrow \quad \widehat{\beta}_0 = \overline{y} - \widehat{\beta}_1 \overline{x}\,, \quad \widehat{\beta}_1 = \frac{s_{xy}}{s_x^2}$$

- coefficient of determination:

$$R^2 = \frac{\sum\limits_{i=1}^{n} (\widehat{y}_i - \overline{y})^2}{\sum\limits_{i=1}^{n} (y_i - \overline{y})^2} = \frac{s_{\widehat{Y}}^2}{s_Y^2} \quad \in [0, 1]$$

in simple linear regression holds: $R^2 = r_{XY}^2$

Multiple Linear Regression

- model: $Y = \beta_0 + \beta_1 x_1 + \ldots + \beta_p x_p + \varepsilon\,,$ data: vectors $(x_{i1}, \ldots, x_{ip}, y_i)\,,\; i = 1, \ldots, n$
- in vector–matrix form: $\boldsymbol{Y} = \mathcal{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$

$$\boldsymbol{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}, \quad \mathcal{X} = \begin{pmatrix} 1 & x_{11} & \ldots & x_{1p} \\ 1 & x_{21} & \ldots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & \ldots & x_{np} \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_1^\top \\ \boldsymbol{x}_2^\top \\ \vdots \\ \boldsymbol{x}_n^\top \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}, \quad \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

- parameter estimation by least squares (LSE):

$$Q(\boldsymbol{\beta}) = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_{i1} - \ldots - \beta_p x_{ip})^2 = \|\boldsymbol{y} - \mathcal{X}\boldsymbol{\beta}\|^2 \to \min_{\boldsymbol{\beta} \in \mathbb{R}^p}$$

$$\Rightarrow \quad \widehat{\boldsymbol{\beta}} = (\mathcal{X}^\top \mathcal{X})^{-1} \mathcal{X}^\top \boldsymbol{y}\,, \quad \widehat{\boldsymbol{y}} = \mathcal{X}(\mathcal{X}^\top \mathcal{X})^{-1} \mathcal{X}^\top \boldsymbol{y} = \mathbf{P}\boldsymbol{y}\,, \quad \widehat{\boldsymbol{\varepsilon}} = \boldsymbol{y} - \widehat{\boldsymbol{y}} = (\mathbf{I} - \mathbf{P})\boldsymbol{y}$$

$$\mathrm{RSS} = \sum_{i=1}^{n} (y_i - \widehat{y}_i)^2 = \sum_{i=1}^{n} \widehat{\varepsilon}_i^2 = \boldsymbol{y}^\top (\mathbf{I} - \mathbf{P})\boldsymbol{y}\,, \quad \widehat{\sigma}^2 = \frac{\mathrm{RSS}}{n - p - 1}$$

- coefficient of determination:

$$R^2 = \frac{\sum\limits_{i=1}^{n} (\widehat{y}_i - \overline{y})^2}{\sum\limits_{i=1}^{n} (y_i - \overline{y})^2} = \frac{s_{\widehat{Y}}^2}{s_Y^2} = 1 - \frac{\mathrm{RSS}}{\mathrm{TSS}} \quad \in [0, 1]$$

it holds: $R^2 = r_{Y\widehat{Y}}^2$

- prediction: $\widehat{y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 x_{i1} + \ldots \widehat{\beta}_p x_{ip} = \widehat{\boldsymbol{\beta}}^\top \boldsymbol{x}_i = \boldsymbol{x}_i^\top \widehat{\boldsymbol{\beta}} \quad \Longleftrightarrow \quad \widehat{\boldsymbol{y}} = \mathcal{X}\widehat{\boldsymbol{\beta}}$

- properties under fixed design and $E\boldsymbol{\varepsilon} = \mathbf{0}$, $\mathrm{Var}(\boldsymbol{\varepsilon}) = \sigma^2\mathbf{I}$:

$$\begin{aligned} E\widehat{\boldsymbol{\beta}} &= \boldsymbol{\beta}\,, \quad E\widehat{\sigma}^2 \;=\; \sigma^2 \\ \mathrm{Var}(\widehat{\boldsymbol{\beta}}) &= \sigma^2(\mathcal{X}^\top\mathcal{X})^{-1} \\ \widehat{\boldsymbol{\beta}} &\text{ is } \text{ BLUE} \end{aligned}$$

- properties under fixed design and normality, i.e. $\boldsymbol{\varepsilon} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$:
  - (i) $\widehat{\boldsymbol{\beta}} \sim N\left(\boldsymbol{\beta}, \sigma^2(\mathcal{X}^\top\mathcal{X})^{-1}\right)$
  - (ii) $\widehat{\boldsymbol{\beta}}$ and thus $\widehat{\boldsymbol{Y}} = \mathcal{X}\widehat{\boldsymbol{\beta}}$ are (stoch.) independent of $\mathrm{RSS} = \boldsymbol{Y}^\top(\mathbf{I}-\mathbf{P})\boldsymbol{Y}$
  - (iii) $\dfrac{1}{\sigma^2}\,\mathrm{RSS} \sim \chi^2_{n-p-1}$
  - (vi) $\dfrac{\widehat{\beta}_j - \beta_j}{\widehat{\sigma}\sqrt{c_{jj}}} \sim t_{n-p-1}$ ($j = 0, \ldots, p$ and c_{jj} is the jth diagonal element of $(\mathcal{X}^\top\mathcal{X})^{-1}$)
- maximum likelihood estimates under normality:

$$\widehat{\boldsymbol{\beta}} = (\mathcal{X}^\top\mathcal{X})^{-1}\mathcal{X}^\top\boldsymbol{Y}\,, \quad \widetilde{\sigma}^2 = \frac{\mathrm{RSS}}{n}$$

- standardized residuals: $\dfrac{\widehat{\varepsilon}_i}{\widehat{\sigma}\sqrt{1-p_{ii}}}$ (where p_{ii} are diagonal elements of $\mathbf{P}$)

t Test for Coefficients

- $H_0 : \beta_j = 0$ vs. $H_1 : \beta_j \neq 0$

  reject H_0 if $\quad |t_{value,j}| = \left|\dfrac{\widehat{\beta}_j}{\widehat{\sigma}\sqrt{c_{jj}}}\right| > t_{n-p-1,1-\alpha/2}$

  $p_{value,j} = 2 - 2F_{t_{n-p-1}}(|t_{value,j}|)$

Confidence Regions

- $(1-\alpha)$ confidence interval for β_j: $\quad \left[\widehat{\beta}_j - t_{n-p-1,1-\alpha/2}\,\widehat{\sigma}\sqrt{c_{jj}}\,;\, \widehat{\beta}_j + t_{n-p-1,1-\alpha/2}\,\widehat{\sigma}\sqrt{c_{jj}}\right]$
- $(1-\alpha)$ confidence ellipsoid for $\boldsymbol{\beta}$:

$$\frac{(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta})^\top(\mathcal{X}^\top\mathcal{X})(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta})}{(p+1)\widehat{\sigma}^2} \sim F_{p+1,n-p-1}$$

$$\Rightarrow \quad \left\{\,\boldsymbol{\beta} \mid (\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta})^\top(\mathcal{X}^\top\mathcal{X})(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}) \leq (p+1)\widehat{\sigma}^2 F_{p+1,n-p-1,1-\alpha}\,\right\}$$

- $(1-\alpha)$ confidence interval for $\boldsymbol{\beta}^\top\boldsymbol{x}_0$:

$$\widehat{y}_0 = \widehat{\boldsymbol{\beta}}^\top\boldsymbol{x}_0 \sim N\left(\boldsymbol{\beta}^\top\boldsymbol{x}_0, \sigma^2\boldsymbol{x}_0^\top(\mathcal{X}^\top\mathcal{X})^{-1}\boldsymbol{x}_0\right)$$

$$\Rightarrow \quad \left[\widehat{y}_0 - t_{n-p-1,1-\alpha/2}\,\widehat{\sigma}\sqrt{\boldsymbol{x}_0^\top(\mathcal{X}^\top\mathcal{X})^{-1}\boldsymbol{x}_0}\,;\, \widehat{y}_0 + t_{n-p-1,1-\alpha/2}\,\widehat{\sigma}\sqrt{\boldsymbol{x}_0^\top(\mathcal{X}^\top\mathcal{X})^{-1}\boldsymbol{x}_0}\right]$$

F Test for Model Comparison

- F test for comparing with the constant model:

$$\begin{aligned} H_0 &: \beta_1 = \ldots = \beta_p = 0 \\ H_1 &: \exists \beta_j \neq 0 \; (j = 1, \ldots, p) \end{aligned}$$

reject H_0 if $\quad F_{value} = \dfrac{n-p-1}{p} \dfrac{(\mathrm{RSS}_{H_0} - \mathrm{RSS}_{H_1})}{\mathrm{RSS}_{H_1}} > F_{p,n-p-1,1-\alpha}$

$p_{value} = 1 - F_{F_{p,n-p-1}}(F_{value})$

- F test for comparing two nested models:

$$\begin{aligned} H_0 &: \text{smaller model} \\ H_1 &: \text{larger model} \end{aligned}$$

reject H_0 if $\quad F_{value} = \dfrac{(\mathrm{RSS}_{H_0} - \mathrm{RSS}_{H_1}) / (\mathrm{df}_0 - \mathrm{df}_1)}{\mathrm{RSS}_{H_1} / \mathrm{df}_1} > F_{\mathrm{df}_0 - \mathrm{df}_1, \mathrm{df}_1, 1-\alpha} = F_{p_1 - p_0, n-p_1-1, 1-\alpha}$

$p_{value} = 1 - F_{F_{p_1-p_0,n-p_1-1}}(F_{value})$

Modell Choice

- unbiased estimate of the mean sqaures error (MSE):

$$\text{Mallows' } C_p(s) = \|\boldsymbol{Y} - \mathbf{P}_s \boldsymbol{Y}\|^2 + 2\,\mathrm{trace}(\mathbf{P}_s)\widehat{\sigma}^2 - n\widehat{\sigma}^2$$

- Akaike's information criterion:

$$\mathrm{AIC}(s) = n \log\left(\frac{1}{n}\|\boldsymbol{Y} - \mathbf{P}_s \boldsymbol{Y}\|^2\right) + 2\,\mathrm{Spur}(\mathbf{P}_s)$$

more general AIC:

$$\mathrm{AIC} = -2 \cdot \text{log-likelihood} + 2 \cdot \text{no. of coefficients}$$

- Schwarz' Bayesian information criterion:

$$\mathrm{BIC}(s) = n \log\left(\frac{1}{n}\|\boldsymbol{Y} - \mathbf{P}_s \boldsymbol{Y}\|^2\right) + \log(n)\,\mathrm{Spur}(\mathbf{P}_s)$$

Coding Factors as Dummy Variables

- choose a reference category, say r
- add a column to $\mathcal{X}$ for each further category c, defined by

$$x_{i,c} = \begin{cases} 1 & \text{if } x_i = c \\ 0 & \text{otherwise} \end{cases}$$

4 Generalized Linear Models

- logistic regression (logit model): $E(Y|\boldsymbol{x}) = F(\boldsymbol{\beta}^\top \boldsymbol{x})$ with $F(u) = \dfrac{1}{1+e^{-u}}$
- probit model: $E(Y|\boldsymbol{x}) = \Phi(\boldsymbol{\beta}^\top \boldsymbol{x})$ with $\Phi(u) = F_{N(0,1)}$
- generalized linear model (GLM): $E(Y|\boldsymbol{x}) = G(\boldsymbol{\beta}^\top \boldsymbol{x})$ where G is the inverse link function

	notation	$y \in$	$b(\theta)$	$\mu(\theta)$	canonical link $\theta(\mu)$	variance $V(\mu)$	$a(\psi)$
Bernoulli	$B(1,\mu)$	$\{0,1\}$	$\log(1+e^\theta)$	$\dfrac{e^\theta}{(1+e^\theta)}$	logit	$\mu(1-\mu)$	1
Binomial	$B(k,\mu)$	$\{0,\dots,k\}$	$k\log(1+e^\theta)$	$\dfrac{ke^\theta}{(1+e^\theta)}$	logit	$\mu\left(1-\frac{\mu}{k}\right)$	1
Poisson	$Poi(\mu)$	$\mathbb{N}_0$	$\exp(\theta)$	$\exp(\theta)$	$\log$	μ	1
Negative binomial	$NB(\mu,k)$	$\mathbb{N}_0$	$-k\log\left(1-e^\theta\right)$	$\dfrac{ke^\theta}{(1-e^\theta)}$	$\log\left(\dfrac{\mu}{k+\mu}\right)$	$\mu+\dfrac{\mu^2}{k}$	1
Normal	$N(\mu,\sigma^2)$	$\mathbb{R}$	$\theta^2/2$	θ	identical	1	σ^2
Exponential	$Exp(\mu)$	$\mathbb{R}^+$	$-\log(-\theta)$	$-1/\theta$	reciprocal	μ^2	1
Gamma	$G(\mu,\nu)$	$\mathbb{R}^+$	$-\log(-\theta)$	$-1/\theta$	reciprocal	μ^2	$1/\nu$
Inverse normal	$IN(\mu,\sigma^2)$	$\mathbb{R}^+$	$-\sqrt{-2\theta}$	$\dfrac{-1}{\sqrt{-2\theta}}$	squared reciprocal	μ^3	σ^2

A R Reference

A.1 General

`q()` quit R

`ls(), rm(list = ls())`
list / remove all objects in / from workspace

`help(<func>)` or alternatively
`?<func>` to show help on `<func>`

`apropos('<keyword>')`
show functions that contain `keyword`

`require, library` load a package

`system.time`
measure CPU times of a command sequence, e.g. `system.time(n <- 100000; x <- rnorm(n))`

Relations, Operations

`==, !=`
equal to, unequal to

`<, >, <=, >=`
less than, greater than (or equal to)

`-, +, *, /`
elementwise operations

`%*%` matrix product

`%%, %/%` modulo, integer division

`!` logical “NOT”

`&, &&` logical “AND” (`&&` operates only on the first component)

`|, ||` logical “OR” (`||` operates only on the first component)

Constants, Special Values

`pi` $\pi = 3.141593...$

`TRUE, FALSE`
logical values (numerically 1, 0)

`letters, LETTERS`
vector of the 26 small / capital letters

Mathematical Functions

`a^x, sqrt(x)`
exponent (basis a), square root

`abs(x)` absolute value

`sin(x), cos(x), tan(x)`
trigonometric functions

`exp(x), log(x)` exponent (basis e), natural logarithm (math. $\ln$)

`factorial(n), choose(n,k)`
factorial, binomial coefficient

`gamma(x), lgamma(x)`
gamma- and log-gamma functions

Vectors and Matrices

`c` create vector

`rep` repeat elements

`rev` revert elements

`seq`
sequence of values, e.g.
`seq(-3,3,by=0.5` or
`seq(0,1,length=10)`

`matrix` create matrix

`cbind`
combine column vectors into matrix

`rbind`
combine row vectors into matrix

`length`
: length of a vector (or number of elements in a matrix)

`dim` dimension of a matrix or array

`[]` select elements from vector

`[,]` select elements from matrix e.g. `m[1,2]`, `m[1:2,]`, `m[,2:3]`, `m[1:2,2:3]`

`t` matrix transpose

`diag`
: diagonal of a matrix / generate diagonal matrix from vector

`dimnames`
: row / column names of matrix or dataframe

`which.min, which.max`
: index of smallest / largest elements

`which`
: extracts indices following a condition, e.g. `which(a >5)`

`rowSums, colSums`

`det` determinant

`eigen` eigen values and vectors

`solve`
: solve a system of linear equations (`solve(A,b)`) or determine the invers (`solve(A)`)

`outer`
: outer product for a function, z.B. `outer(1:5,7:9,"+")`

`apply`
: row- or columnwise evaluation of a function on a matrix

Lists

`list` create list

`[[]]` select list component, e.g. `mylist[[2]]`

`$` select list component by name, e.g. `mylist$date`

`names` names of all list components

`lapply, sapply` variants of `apply` for lists

A.2 Data

Data Input and Output

`setwd`
: set the working directory, e.g. `setwd("d:/MyDir")`

`read.csv, read.csv2`
: read a .csv file (English/German)

`write.csv, write.csv2`
: write a .csv file (English/German)

`data.frame`
: data matrix (mixed data types possible)

`$` select component by its name, e.g. `mydata$date`

`attach, detach`
: load / remove components of a data matrix into / from workspace

Data Manipulation

`sort` sort a vector

`order`
: find the ranks in a vector, e.g. `v[order[v]]` equals `sort(v)`

`cut` cut a vector into intervals

`factor, ordered`
: convert numerical data into factors

`table` frequency table

`tapply` variant of `apply` for data matrices

Strings

`paste`
: concatenate strings (converts nemerical values automatically)

`substr`
: extract substring, e.g. `"abc"` by `substr("abcde",1,3)`

A.3 Statistics

Chacteristic Numbers

`mean, median, quantile`

`min, max`
: minimum, maximum

`var, sd, cov, cor`
: variance, standard deviation, covariance, correlation

`summary`
: several characteristic numbers of a data matrix

`fivenum`
: five number summary of a data vector

`rowMeans, colMeans`
: row and column means of a matrix

`table` frequency table

Distributions

`dnorm, pnorm, qnorm, rnorm`
: pdf, cdf, quantiles and pseudo random numbers for the normal distribution

other distributions:

continuous uniform	`{r|d|p|q}unif`
exponential	`{r|d|p|q}exp`
lognormal	`{r|d|p|q}lnorm`
binomial	`{r|d|p|q}binom`
Poisson	`{r|d|p|q}pois`
hypergeometric	`{r|d|p|q}hyper`
geometric	`{r|d|p|q}geom`
negative binomial	`{r|d|p|q}nbinom`
logistic	`{r|d|p|q}logis`
χ^2	`{r|d|p|q}chisq`
t	`{r|d|p|q}t`
F	`{r|d|p|q}f`

`set.seed`
: fix the seed of the random number generator

Hypotheses Tests

`chisq.test, fisher.test`
: test for independence in a cross table (contingency table)

`binom.test, prop.test`
: test for p using Bernoulli data

`t.test`
: one or two sample t tests for the mean

`wilcox.test`
: Wilcoxon test a a nonparamtric alternative to the t test

`var.test` χ^2 tests for the variance (variance comparison)

`bartlett.test`
: variance comparison for more than two samples

`cor.test`
: t test for the correlation

`ks.test`
: Kolmogoroff Smirnoff test for a specific continuous distribution (or for comparing two continuous distributions)

`shapiro.test`
: Shapiro Wilk test for normality

A.4 Graphics

R Builtin Function plot

`plot`

`lines, points` add curve / points

`abline` add a straight line

`text, legend, title`
: add text / legend / plot title

`par(mfrow=c(r,c))`
: split display into r rows and c columns

`rug` add rug plot

`polygon` fill polygone with color

Parameters for plot

`type`
: change type, e.g. `='l'` (line), `='p'` (points), `='b'` (both), …

`col`
: change color, e.g. `col='red'` or `col=rainbow(25)`

`lwd, lty` change line width and type

`pch` change symbol

`cex`
: change factor for symbol or text size

`main` set plot title

`xlab, ylab`
: change labels for the axes

`xlim, ylim`
: change ranges of the axes

`log` use axes in log scale (e.g. `log='x'`)

more options under `?plot` and `?par`

Specific Plots

`barplot, pie`

`mosaicplot, spineplot`

`boxplot`
: optionally groupwise by e.g. `boxplot(y ~ g)`

`hist` histogram

`pairs` scatterplot matrix

`ecdf` empirical cdf

`qqnorm, qqplot` quantile–quantile plot

`curve`

`matplot` plot several curves simultaneusly

`contour` plot contours (bivariate function)

`persp` perspective plot in 3D

`persp3D`
: perspective plot in 3D (needs package `rgl` to use OpenGL)

A.5 Programming

`if`
: conditional execution, i.e. `if (` <cond> `) {` <command sequence> `}` or `if (` <cond> `) {` <command sequence> `} else {` <other command sequence> `}`

`for, while, repeat`
: different loops e.g. `for (i in 1:n){` <command sequence> `}` or `while (` <cond> `) {` <command sequence> `}` or `repeat {` <command sequence> `; if (`<cond>`) break }`

`function`
: function defintion, `f <- function(`<parameters>`) {` <command sequence>`; return(`<result>`) }`

A.6 Creating Documents

`xtable` from package `xtable`:
: create/save tables or other objects in LaTeXor HTML

`postscript, pdf, png, ...`
: save plots in different formats

A.7 Regression Models

`lm, glm`
: fit a linear or generalized linear model

`abline`
: add the regression line for a simple linear model

- List of functions to extract components from a model.

  Application example: `model <- lm(y ~ x); summary(model)`

Function	Description
`summary`	summary output (see also `?summary.lm`)
`coef`	estimated coefficients
`residuals`	residuals $\widehat{\varepsilon}_i$
`fitted`	predicted values $\widehat{y}_i$
`predict`	predicted values, useful for new data (see also `?predict.lm`)
`anova`	test for comparing two nested models
`plot`	some diagnostic plots
`confint`	confidence intervals for the coefficients
`deviance`	residual sum of squares RSS
`vcov`	estimated covariance matrix (of the coefficients)
`logLik`	log-likelihood (under normality assumption)
`AIC`	Akaike's information criterion (for model choice)

- Further terms can be obtained from `summary`.

  Application example: `summary(model)$call`

Function	Description
`call`	call of `lm`
`terms`	information on the explanatory variables
`residuals`	residuals $\widehat{\varepsilon}_i$
`coefficients`	table of coefficients, standard errors, t values and p values
`sigma`	estimated standard deviation $\widehat{\sigma}$
`df`	degrees of freedom
`r.squared`	coefficient of determination R^2
`adj.r.squared`	adjusted coefficient of determination
`fstatistic`	F statistic with acc. degress of freedom
`cov.unscaled`	unscaled covariance matrix (results in `vcov` when multiplied with $\widehat{\sigma}^2$)

- Examples for building models:

Formula	Description
`y ~ 1`	constant model (intercept only) $y_i = \beta_0 + \varepsilon_i$
`y ~ x`	simple linear model $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$
`y ~ x - 1`	regression without intercept ("through the origin") $y_i = \beta_1 x_i + \varepsilon_i$
`y ~ x + I(x^2)`	linear and quadratic term $y_i = \beta_1 x_i + \varepsilon_i$
`y ~ x + I(x^2)+ I(x^3)`	linear and quadratic and cubic term $y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \beta_2 x_i^3 + \varepsilon_i$
`y ~ x + I(sin(x))+ I(cos(x))`	sine and cosine transformations $y_i = \beta_0 + \beta_1 \sin(x_i) + \beta_2 \cos(x_i) + \varepsilon_i$
`y ~ x1 + x2`	two regressors (explanatory variables) $y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \varepsilon_i$
`y ~ x1 * x2`	two regressors plus interaction $y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i1} x_{i2} + \varepsilon_i$
`y ~ x1 + x2 + x1:x2`	again, two regressors plus interaction $y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i1} x_{i2} + \varepsilon_i$
`y ~ (x1 + x2)^2`	another way, two regressors plus interaction $y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i1} x_{i2} + \varepsilon_i$
`y ~ (x1 + x2 + x3)^2`	three regressors and all bivariate interactions $y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 + \beta_3 x_{i3}$ $+ \beta_4 x_{i1} x_{i2} + \beta_5 x_{i1} x_{i3} + \beta_6 x_{i2} x_{i3} + \varepsilon_i$
`y ~ x1 * x2 * x3`	three regressors and all possible interactions $y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 + \beta_3 x_{i3}$ $+ \beta_4 x_{i1} x_{i2} + \beta_5 x_{i1} x_{i3} + \beta_6 x_{i2} x_{i3} + \beta_7 x_{i1} x_{i2} x_{i3} + \varepsilon_i$

- Additional functions for regression:

`options(contrasts=c('contr.treatment','contr.poly'))`
: code factor into dummy variables (see `?contrasts` for other ways of coding)

`stepAIC` from package `MASS`:
: perform stepwise model choice on the basis of AIC (or BIC)

`Anova` from package `car`:
: other ways for performing ANOVA (type II and III for example)

`influencePlot` from package `car`:
: analyze the influence of single observations

`allEffects` from package `effects`:
: calculate and plot effects and interactions

B Tables

B.1 Gaussian CDF Φ

application example: $Z \sim N(0,1), P(Z \leq -1.23) = \Phi(-1.23) = 0.10935$

	0.00	-0.01	-0.02	-0.03	-0.04	-0.05	-0.06	-0.07	-0.08	-0.09
0.0	0.50000	0.49601	0.49202	0.48803	0.48405	0.48006	0.47608	0.47210	0.46812	0.46414
-0.1	0.46017	0.45620	0.45224	0.44828	0.44433	0.44038	0.43644	0.43251	0.42858	0.42465
-0.2	0.42074	0.41683	0.41294	0.40905	0.40517	0.40129	0.39743	0.39358	0.38974	0.38591
-0.3	0.38209	0.37828	0.37448	0.37070	0.36693	0.36317	0.35942	0.35569	0.35197	0.34827
-0.4	0.34458	0.34090	0.33724	0.33360	0.32997	0.32636	0.32276	0.31918	0.31561	0.31207
-0.5	0.30854	0.30503	0.30153	0.29806	0.29460	0.29116	0.28774	0.28434	0.28096	0.27760
-0.6	0.27425	0.27093	0.26763	0.26435	0.26109	0.25785	0.25463	0.25143	0.24825	0.24510
-0.7	0.24196	0.23885	0.23576	0.23270	0.22965	0.22663	0.22363	0.22065	0.21770	0.21476
-0.8	0.21186	0.20897	0.20611	0.20327	0.20045	0.19766	0.19489	0.19215	0.18943	0.18673
-0.9	0.18406	0.18141	0.17879	0.17619	0.17361	0.17106	0.16853	0.16602	0.16354	0.16109
-1.0	0.15866	0.15625	0.15386	0.15151	0.14917	0.14686	0.14457	0.14231	0.14007	0.13786
-1.1	0.13567	0.13350	0.13136	0.12924	0.12714	0.12507	0.12302	0.12100	0.11900	0.11702
-1.2	0.11507	0.11314	0.11123	0.10935	0.10749	0.10565	0.10383	0.10204	0.10027	0.09853
-1.3	0.09680	0.09510	0.09342	0.09176	0.09012	0.08851	0.08691	0.08534	0.08379	0.08226
-1.4	0.08076	0.07927	0.07780	0.07636	0.07493	0.07353	0.07215	0.07078	0.06944	0.06811
-1.5	0.06681	0.06552	0.06426	0.06301	0.06178	0.06057	0.05938	0.05821	0.05705	0.05592
-1.6	0.05480	0.05370	0.05262	0.05155	0.05050	0.04947	0.04846	0.04746	0.04648	0.04551
-1.7	0.04457	0.04363	0.04272	0.04182	0.04093	0.04006	0.03920	0.03836	0.03754	0.03673
-1.8	0.03593	0.03515	0.03438	0.03362	0.03288	0.03216	0.03144	0.03074	0.03005	0.02938
-1.9	0.02872	0.02807	0.02743	0.02680	0.02619	0.02559	0.02500	0.02442	0.02385	0.02330
-2.0	0.02275	0.02222	0.02169	0.02118	0.02068	0.02018	0.01970	0.01923	0.01876	0.01831
-2.1	0.01786	0.01743	0.01700	0.01659	0.01618	0.01578	0.01539	0.01500	0.01463	0.01426
-2.2	0.01390	0.01355	0.01321	0.01287	0.01255	0.01222	0.01191	0.01160	0.01130	0.01101
-2.3	0.01072	0.01044	0.01017	0.00990	0.00964	0.00939	0.00914	0.00889	0.00866	0.00842
-2.4	0.00820	0.00798	0.00776	0.00755	0.00734	0.00714	0.00695	0.00676	0.00657	0.00639
-2.5	0.00621	0.00604	0.00587	0.00570	0.00554	0.00539	0.00523	0.00508	0.00494	0.00480
-2.6	0.00466	0.00453	0.00440	0.00427	0.00415	0.00402	0.00391	0.00379	0.00368	0.00357
-2.7	0.00347	0.00336	0.00326	0.00317	0.00307	0.00298	0.00289	0.00280	0.00272	0.00264
-2.8	0.00256	0.00248	0.00240	0.00233	0.00226	0.00219	0.00212	0.00205	0.00199	0.00193
-2.9	0.00187	0.00181	0.00175	0.00169	0.00164	0.00159	0.00154	0.00149	0.00144	0.00139
-3.0	0.00135	0.00131	0.00126	0.00122	0.00118	0.00114	0.00111	0.00107	0.00104	0.00100
-3.1	0.00097	0.00094	0.00090	0.00087	0.00084	0.00082	0.00079	0.00076	0.00074	0.00071
-3.2	0.00069	0.00066	0.00064	0.00062	0.00060	0.00058	0.00056	0.00054	0.00052	0.00050
-3.3	0.00048	0.00047	0.00045	0.00043	0.00042	0.00040	0.00039	0.00038	0.00036	0.00035
-3.4	0.00034	0.00032	0.00031	0.00030	0.00029	0.00028	0.00027	0.00026	0.00025	0.00024
-3.5	0.00023	0.00022	0.00022	0.00021	0.00020	0.00019	0.00019	0.00018	0.00017	0.00017
-3.6	0.00016	0.00015	0.00015	0.00014	0.00014	0.00013	0.00013	0.00012	0.00012	0.00011
-3.7	0.00011	0.00010	0.00010	0.00010	0.00009	0.00009	0.00008	0.00008	0.00008	0.00008
-3.8	0.00007	0.00007	0.00007	0.00006	0.00006	0.00006	0.00006	0.00005	0.00005	0.00005
-3.9	0.00005	0.00005	0.00004	0.00004	0.00004	0.00004	0.00004	0.00004	0.00003	0.00003
-4.0	0.00003	0.00003	0.00003	0.00003	0.00003	0.00003	0.00002	0.00002	0.00002	0.00002
-4.1	0.00002	0.00002	0.00002	0.00002	0.00002	0.00002	0.00002	0.00002	0.00001	0.00001
-4.2	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001
-4.3	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001
-4.4	0.00001	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

application example: $Z \sim N(0,1), P(Z \leq 2.22) = \Phi(2.22) = 0.98679$

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999
4.2	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999
4.3	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999	0.99999
4.4	0.99999	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

B.2 Selected Quantiles of the Gaussian

application example: 99.5% quantile of the Gaussian $N(0,1)$ *distribution* $z_{0.995} = 2.57583$

α	z_α
0.600	0.25335
0.750	0.67449
0.800	0.84162
0.900	1.28155
0.950	1.64485
0.975	1.95996
0.990	2.32635
0.995	2.57583

B.3 Selected Quantiles of the t Distribution with m DF

application example: 99.5% quantile of the t_{10} *distribution* $t_{10;0.995} = 3.169$

	m										
α	1	2	3	4	5	6	7	8	9	10	α
0.600	0.325	0.289	0.277	0.271	0.267	0.265	0.263	0.262	0.261	0.260	0.600
0.750	1.000	0.817	0.765	0.741	0.727	0.718	0.711	0.706	0.703	0.700	0.750
0.800	1.376	1.061	0.979	0.941	0.920	0.906	0.896	0.889	0.883	0.879	0.800
0.900	3.078	1.886	1.638	1.533	1.476	1.440	1.415	1.397	1.383	1.372	0.900
0.950	6.314	2.920	2.353	2.132	2.015	1.943	1.895	1.860	1.833	1.813	0.950
0.975	12.706	4.303	3.182	2.776	2.571	2.447	2.365	2.306	2.262	2.228	0.975
0.990	31.821	6.965	4.541	3.747	3.365	3.142	2.998	2.897	2.821	2.764	0.990
0.995	63.657	9.925	5.841	4.604	4.032	3.707	3.500	3.355	3.250	3.169	0.995

	m										
α	11	12	13	14	15	16	17	18	19	20	α
0.600	0.260	0.259	0.259	0.258	0.258	0.258	0.257	0.257	0.257	0.257	0.600
0.750	0.697	0.696	0.694	0.692	0.691	0.690	0.689	0.688	0.688	0.687	0.750
0.800	0.876	0.873	0.870	0.861	0.866	0.865	0.863	0.862	0.861	0.860	0.800
0.900	1.363	1.356	1.350	1.345	1.341	1.337	1.333	1.330	1.328	1.325	0.900
0.950	1.796	1.782	1.771	1.761	1.753	1.746	1.740	1.734	1.729	1.725	0.950
0.975	2.201	2.179	2.160	2.145	2.131	2.120	2.110	2.101	2.093	2.086	0.975
0.990	2.718	2.681	2.650	2.625	2.603	2.584	2.567	2.552	2.540	2.528	0.990
0.995	3.106	3.055	3.012	2.977	2.947	2.921	2.898	2.878	2.861	2.845	0.995

	m										
α	21	22	23	24	25	26	27	28	29	30	α
0.600	0.257	0.256	0.256	0.256	0.256	0.256	0.256	0.256	0.256	0.256	0.600
0.750	0.686	0.686	0.685	0.685	0.684	0.684	0.684	0.683	0.683	0.683	0.750
0.800	0.859	0.858	0.858	0.857	0.856	0.856	0.855	0.855	0.854	0.854	0.800
0.900	1.323	1.321	1.320	1.318	1.316	1.315	1.314	1.313	1.311	1.310	0.900
0.950	1.721	1.717	1.714	1.711	1.708	1.706	1.703	1.701	1.699	1.697	0.950
0.975	2.080	2.074	2.069	2.064	2.060	2.056	2.052	2.048	2.045	2.042	0.975
0.990	2.518	2.508	2.500	2.492	2.485	2.479	2.473	2.467	2.462	2.457	0.990
0.995	2.831	2.819	2.807	2.797	2.787	2.779	2.771	2.763	2.756	2.750	0.995

B.4 Selected Quantiles of the Chi Square Distribution with m DF

application example: 99.5% quantile of the χ^2_{10} distribution $\chi^2_{10;0.995} = 25.19$

α	m = 1	2	3	4	5	6	7	8	9	10	α
0.005	0.00	0.01	0.07	0.21	0.41	0.68	0.99	1.34	1.73	2.16	0.005
0.010	0.00	0.02	0.11	0.30	0.55	0.87	1.24	1.65	2.09	2.56	0.010
0.025	0.00	0.05	0.22	0.48	0.83	1.24	1.69	2.18	2.70	3.25	0.025
0.050	0.00	0.10	0.35	0.71	1.15	1.64	2.17	2.73	3.33	3.94	0.050
0.100	0.02	0.21	0.58	1.06	1.61	2.20	2.83	3.49	4.17	4.87	0.100
0.200	0.06	0.45	1.01	1.65	2.34	3.07	3.82	4.59	5.38	6.18	0.200
0.250	0.10	0.58	1.21	1.92	2.67	3.45	4.25	5.07	5.90	6.74	0.250
0.400	0.27	1.02	1.87	2.75	3.66	4.57	5.49	6.42	7.36	8.30	0.400
0.500	0.45	1.39	2.37	3.36	4.35	5.35	6.35	7.34	8.34	9.34	0.500
0.600	0.71	1.83	2.95	4.04	5.13	6.21	7.28	8.35	9.41	10.47	0.600
0.750	1.32	2.77	4.11	5.39	6.63	7.84	9.04	10.22	11.39	12.55	0.750
0.800	1.64	3.22	4.64	5.99	7.29	8.56	9.80	11.03	12.24	13.44	0.800
0.900	2.71	4.61	6.25	7.78	9.24	10.64	12.02	13.36	14.68	15.99	0.900
0.950	3.84	5.99	7.81	9.49	11.07	12.59	14.07	15.51	16.92	18.31	0.950
0.975	5.02	7.38	9.35	11.14	12.83	14.45	16.01	17.53	19.02	20.48	0.975
0.990	6.63	9.21	11.34	13.28	15.09	16.81	18.48	20.09	21.67	23.21	0.990
0.995	7.88	10.60	12.84	14.86	16.75	18.55	20.28	21.95	23.59	25.19	0.995

α	m = 11	12	13	14	15	16	17	18	19	20	α
0.005	2.60	3.07	3.57	4.07	4.60	5.14	5.70	6.26	6.84	7.43	0.005
0.010	3.05	3.57	4.11	4.66	5.23	5.81	6.41	7.01	7.63	8.26	0.010
0.025	3.82	4.40	5.01	5.63	6.26	6.91	7.56	8.23	8.91	9.59	0.025
0.050	4.57	5.23	5.89	6.57	7.26	7.96	8.67	9.39	10.12	10.85	0.050
0.100	5.58	6.30	7.04	7.79	8.55	9.31	10.09	10.86	11.65	12.44	0.100
0.200	6.99	7.81	8.63	9.47	10.31	11.15	12.00	12.86	13.72	14.58	0.200
0.250	7.58	8.44	9.30	10.17	11.04	11.91	12.79	13.68	14.56	15.45	0.250
0.400	9.24	10.18	11.13	12.08	13.03	13.98	14.94	15.89	16.85	17.81	0.400
0.500	10.34	11.34	12.34	13.34	14.34	15.34	16.34	17.34	18.34	19.34	0.500
0.600	11.53	12.58	13.64	14.69	15.73	16.78	17.82	18.87	19.91	20.95	0.600
0.750	13.70	14.85	15.98	17.12	18.25	19.37	20.49	21.60	22.72	23.83	0.750
0.800	14.63	15.81	16.98	18.15	19.31	20.47	21.61	22.76	23.90	25.04	0.800
0.900	17.28	18.55	19.81	21.06	22.31	23.54	24.77	25.99	27.20	28.41	0.900
0.950	19.68	21.03	22.36	23.68	25.00	26.30	27.59	28.87	30.14	31.41	0.950
0.975	21.92	23.34	24.74	26.12	27.49	28.85	30.19	31.53	32.85	34.17	0.975
0.990	24.72	26.22	27.69	29.14	30.58	32.00	33.41	34.81	36.19	37.57	0.990
0.995	26.76	28.30	29.82	31.32	32.80	34.27	35.72	37.16	38.58	40.00	0.995

	m										
α	21	22	23	24	25	26	27	28	39	40	α
0.005	8.03	8.64	9.26	9.89	10.52	11.16	11.81	12.46	13.12	13.79	0.005
0.010	8.90	9.54	10.20	10.86	11.52	12.20	12.88	13.56	14.26	14.95	0.010
0.025	10.28	10.98	11.69	12.40	13.12	13.84	14.57	15.31	16.05	16.79	0.025
0.050	11.59	12.34	13.09	13.85	14.61	15.38	16.15	16.93	17.71	18.49	0.050
0.100	13.24	14.04	14.85	15.66	16.47	17.29	18.11	18.94	19.77	20.60	0.100
0.200	15.44	16.31	17.19	18.06	18.94	19.82	20.70	21.59	22.48	23.36	0.200
0.250	16.34	17.24	18.14	19.04	19.94	20.84	21.75	22.66	23.57	24.48	0.250
0.400	18.77	19.73	20.69	21.65	22.62	23.58	24.54	25.51	26.48	27.44	0.400
0.500	20.34	21.34	22.34	23.34	24.34	25.34	26.34	27.34	28.34	29.34	0.500
0.600	21.99	23.03	24.07	25.11	26.14	27.18	28.21	29.25	30.28	31.32	0.600
0.750	24.93	26.04	27.14	28.24	29.34	30.43	31.53	32.62	33.71	34.80	0.750
0.800	26.17	27.30	28.43	29.55	30.68	31.79	32.91	34.03	35.14	36.25	0.800
0.900	29.62	30.81	32.01	33.20	34.38	35.56	36.74	37.92	39.09	40.26	0.900
0.950	32.67	33.92	35.17	36.42	37.65	38.89	40.11	41.34	42.56	43.77	0.950
0.975	35.48	36.78	38.08	39.36	40.65	41.92	43.19	44.46	45.72	46.98	0.975
0.990	38.93	40.29	41.64	42.98	44.31	45.64	46.96	48.28	49.59	50.89	0.990
0.995	41.40	42.80	44.18	45.56	46.93	48.29	49.64	50.99	52.34	53.67	0.995

	m										
α	31	32	33	34	35	36	37	38	39	40	α
0.005	14.46	15.13	15.82	16.50	17.19	17.89	18.59	19.29	20.00	20.71	0.005
0.010	15.66	16.36	17.07	17.79	18.51	19.23	19.96	20.69	21.43	22.16	0.010
0.025	17.54	18.29	19.05	19.81	20.57	21.34	22.11	22.88	23.65	24.43	0.025
0.050	19.28	20.07	20.87	21.66	22.47	23.27	24.07	24.88	25.70	26.51	0.050
0.100	21.43	22.27	23.11	23.95	24.80	25.64	26.49	27.34	28.20	29.05	0.100
0.200	24.26	25.15	26.04	26.94	27.84	28.73	29.64	30.54	31.44	32.34	0.200
0.250	25.39	26.30	27.22	28.14	29.05	29.97	30.89	31.81	32.74	33.66	0.250
0.400	28.41	29.38	30.34	31.31	32.28	33.25	34.22	35.19	36.16	37.13	0.400
0.500	30.34	31.34	32.34	33.34	34.34	35.34	36.34	37.34	38.34	39.34	0.500
0.600	32.35	33.38	34.41	35.44	36.47	37.50	38.53	39.56	40.59	41.62	0.600
0.750	35.89	36.97	38.06	39.14	40.22	41.30	42.38	43.46	44.54	45.62	0.750
0.800	37.36	38.47	39.57	40.68	41.78	42.88	43.98	45.08	46.17	47.27	0.800
0.900	41.42	42.58	43.75	44.90	46.06	47.21	48.36	49.51	50.66	51.81	0.900
0.950	44.99	46.19	47.40	48.60	49.80	51.00	52.19	53.38	54.57	55.76	0.950
0.975	48.23	49.48	50.73	51.97	53.20	54.44	55.67	56.90	58.12	59.34	0.975
0.990	52.19	53.49	54.78	56.06	57.34	58.62	59.89	61.16	62.43	63.69	0.990
0.995	55.00	56.33	57.65	58.96	60.27	61.58	62.88	64.18	65.48	66.77	0.995

B.5 Selected Quantiles of the F Distribution with m_1 and m_2 DF

application example: 95% quantile of the $F_{10,15}$ distribution $F_{10,15;0.95} = 2.54$

95% quantiles

	m_2																
m_1	1	2	3	4	5	6	7	8	9	10	15	20	30	40	50	100	m_1
1	161.45	18.51	10.13	7.71	6.61	5.99	5.59	5.32	5.12	4.96	4.54	4.35	4.17	4.08	4.03	3.94	1
2	199.50	19.00	9.55	6.94	5.79	5.14	4.74	4.46	4.26	4.10	3.68	3.49	3.32	3.23	3.18	3.09	2
3	215.71	19.16	9.28	6.59	5.41	4.76	4.35	4.07	3.86	3.71	3.29	3.10	2.92	2.84	2.79	2.70	3
4	224.58	19.25	9.12	6.39	5.19	4.53	4.12	3.84	3.63	3.48	3.06	2.87	2.69	2.61	2.56	2.46	4
5	230.16	19.30	9.01	6.26	5.05	4.39	3.97	3.69	3.48	3.33	2.90	2.71	2.53	2.45	2.40	2.31	5
6	233.99	19.33	8.94	6.16	4.95	4.28	3.87	3.58	3.37	3.22	2.79	2.60	2.42	2.34	2.29	2.19	6
7	236.77	19.35	8.89	6.09	4.88	4.21	3.79	3.50	3.29	3.14	2.71	2.51	2.33	2.25	2.20	2.10	7
8	238.88	19.37	8.85	6.04	4.82	4.15	3.73	3.44	3.23	3.07	2.64	2.45	2.27	2.18	2.13	2.03	8
9	240.54	19.38	8.81	6.00	4.77	4.10	3.68	3.39	3.18	3.02	2.59	2.39	2.21	2.12	2.07	1.97	9
10	241.88	19.40	8.79	5.96	4.74	4.06	3.64	3.35	3.14	2.98	2.54	2.35	2.16	2.08	2.03	1.93	10
15	245.95	19.43	8.70	5.86	4.62	3.94	3.51	3.22	3.01	2.85	2.40	2.20	2.01	1.92	1.87	1.77	15
20	248.01	19.45	8.66	5.80	4.56	3.87	3.44	3.15	2.94	2.77	2.33	2.12	1.93	1.84	1.78	1.68	20
30	250.10	19.46	8.62	5.75	4.50	3.81	3.38	3.08	2.86	2.70	2.25	2.04	1.84	1.74	1.69	1.57	30
40	251.14	19.47	8.59	5.72	4.46	3.77	3.34	3.04	2.83	2.66	2.20	1.99	1.79	1.69	1.63	1.52	40
50	251.77	19.48	8.58	5.70	4.44	3.75	3.32	3.02	2.80	2.64	2.18	1.97	1.76	1.66	1.60	1.48	50
100	253.04	19.49	8.55	5.66	4.41	3.71	3.27	2.97	2.76	2.59	2.12	1.91	1.70	1.59	1.52	1.39	100

99% quantiles

	m_2																
m_1	1	2	3	4	5	6	7	8	9	10	15	20	30	40	50	100	m_1
1	4052.18	98.50	34.12	21.20	16.26	13.75	12.25	11.26	10.56	10.04	8.68	8.10	7.56	7.31	7.17	6.90	1
2	4999.50	99.00	30.82	18.00	13.27	10.92	9.55	8.65	8.02	7.56	6.36	5.85	5.39	5.18	5.06	4.82	2
3	5403.35	99.17	29.46	16.69	12.06	9.78	8.45	7.59	6.99	6.55	5.42	4.94	4.51	4.31	4.20	3.98	3
4	5624.58	99.25	28.71	15.98	11.39	9.15	7.85	7.01	6.42	5.99	4.89	4.43	4.02	3.83	3.72	3.51	4
5	5763.65	99.30	28.24	15.52	10.97	8.75	7.46	6.63	6.06	5.64	4.56	4.10	3.70	3.51	3.41	3.21	5
6	5858.99	99.33	27.91	15.21	10.67	8.47	7.19	6.37	5.80	5.39	4.32	3.87	3.47	3.29	3.19	2.99	6
7	5928.36	99.36	27.67	14.98	10.46	8.26	6.99	6.18	5.61	5.20	4.14	3.70	3.30	3.12	3.02	2.82	7
8	5981.07	99.37	27.49	14.80	10.29	8.10	6.84	6.03	5.47	5.06	4.00	3.56	3.17	2.99	2.89	2.69	8
9	6022.47	99.39	27.35	14.66	10.16	7.98	6.72	5.91	5.35	4.94	3.89	3.46	3.07	2.89	2.78	2.59	9
10	6055.85	99.40	27.23	14.55	10.05	7.87	6.62	5.81	5.26	4.85	3.80	3.37	2.98	2.80	2.70	2.50	10
15	6157.28	99.43	26.87	14.20	9.72	7.56	6.31	5.52	4.96	4.56	3.52	3.09	2.70	2.52	2.42	2.22	15
20	6208.73	99.45	26.69	14.02	9.55	7.40	6.16	5.36	4.81	4.41	3.37	2.94	2.55	2.37	2.27	2.07	20
30	6260.65	99.47	26.50	13.84	9.38	7.23	5.99	5.20	4.65	4.25	3.21	2.78	2.39	2.20	2.10	1.89	30
40	6286.78	99.47	26.41	13.75	9.29	7.14	5.91	5.12	4.57	4.17	3.13	2.69	2.30	2.11	2.01	1.80	40
50	6302.52	99.48	26.35	13.69	9.24	7.09	5.86	5.07	4.52	4.12	3.08	2.64	2.25	2.06	1.95	1.74	50
100	6334.11	99.49	26.24	13.58	9.13	6.99	5.75	4.96	4.41	4.01	2.98	2.54	2.13	1.94	1.82	1.60	100