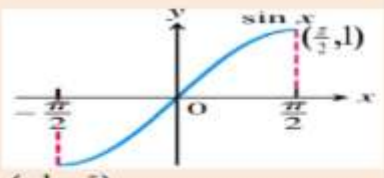
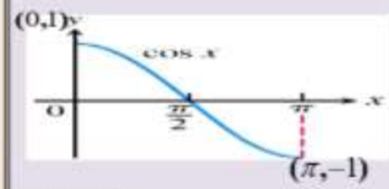
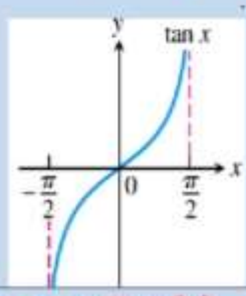
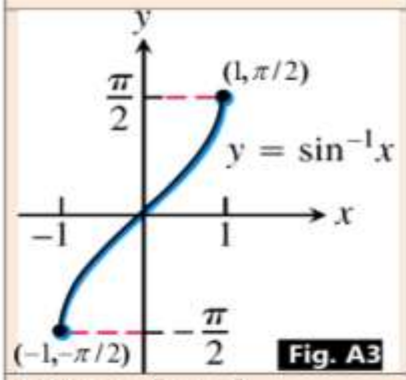
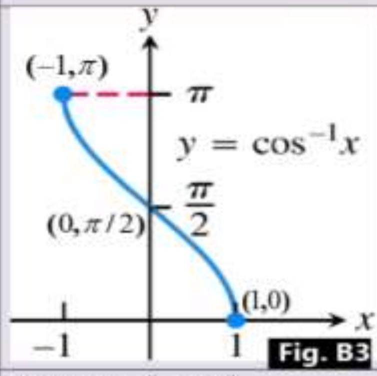
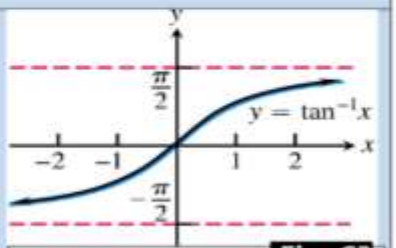
CHAPTER - 2

Derivatives of Trancendental Functions

2. **Inverse trigonometric functions** arise when we want to calculate angles from side measurements in triangles. They also provide useful antiderivatives and appear frequently in the solutions of differential equations.
When we try to find the inverse trigonometric functions, we have a slight difficulty: Because the trigonometric functions are not one-to-one, they don't have inverse functions. The difficulty is overcome by restricting the domains of these functions so that they become one-to-one. The following table shows how these functions are defined and graphed.

$\sin^{-1} x \neq (1/\sin x)$	$\cos^{-1} x \neq (1/\cos x)$	$\tan^{-1} x \neq (1/\tan x)$
Fig A1 shows that the sine function is not one-to-one (use H.L.T.). But the restricted function $f=\sin x, -\pi/2 \le x \le \pi/2$ is one-to-one (see Fig A2).	Fig B1 shows that the sine function is not one-to-one (use H.L.T.). But the restricted function $f=\cos x, 0 \le x \le \pi$ is one-to-one (see Fig B2).	Fig C1 shows that the sine function is not one-to-one (use H.L.T.). But the restricted function $f=\tan x, -\pi/2 < x < \pi/2$ is one-to-one (see Fig C2)
 Fig. A2	 Fig. B2	 Fig. C2
$\sin^{-1} x = \arcsin(x)$	$\cos^{-1} x = \arccos(x)$	$\tan^{-1} x = \arctan(x)$
 Fig. A3	 Fig. B3	 Fig. C3 Take this note: $\lim_{x\to\pm\infty} (\tan^{-1} x) = \pm\pi/2$
Domain: $[-1,1]$ Range: $[-\pi/2, \pi/2]$	Domain: $[-1,1]$ Range: $[0,\pi]$	Domain: R Range: $]-\pi/2, \pi/2[$
$\sin^{-1}(\sin x) = x$ if $x \in [-\pi/2, \pi/2]$ $\sin(\sin^{-1} x) = x$ if $x \in [-1,1]$	$\cos^{-1}(\cos x) = x,$ if $x \in [0,\pi]$ $\cos(\cos^{-1} x) = x,$ if $x \in [-1,1]$	$\tan^{-1}(\tan x) = x,$ if $x \in]-\pi/2, \pi/2[$ $\tan(\tan^{-1} x) = x,$ if $x \in R$

3 .Hyperbolic Functions

are exponential functions that share similar properties to trigonometric functions.

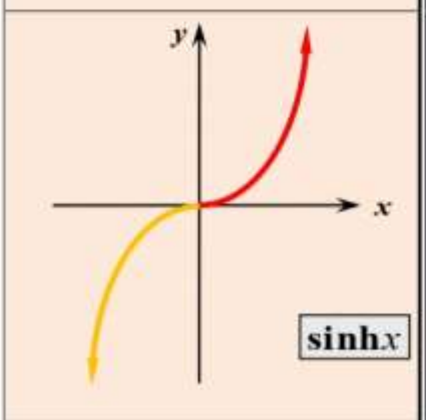
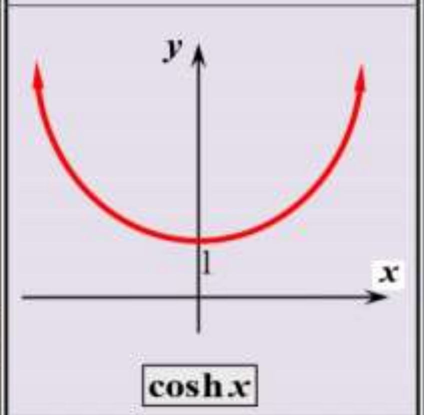
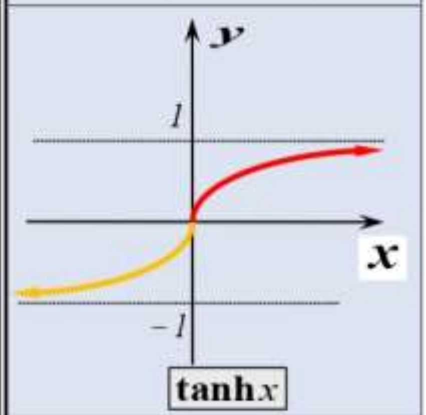
The hyperbolic functions simplify many mathematical expressions and they are important in applications. For instance, they are used in problems such as computing the tension in a cable suspended by its two ends, as in an electric transmission line. They describe the motions of waves in elastic solids and the temperature distributions in metal cooling fins. They also play an important role in finding solutions to differential equations (math. 3).

The reason why it's called Hyperbolic because it obeys the equation $x^2 - y^2 = 1$ (which is the Rectangular hyperbola)

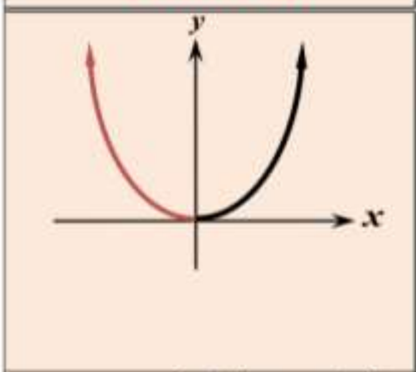
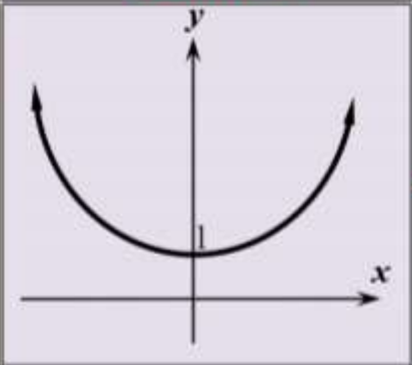
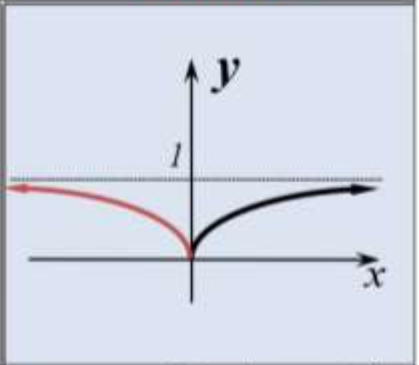
In the following

- We define the three main hyperbolic functions ($\sinh x$, $\cosh x$, $\tanh x$), and sketch their graphs.
- We also discuss some identities relating these functions
- and mention their reciprocal functions ($\operatorname{csch} x$, $\operatorname{sech} x$, $\coth x$)
- and inverse functions ($\sinh^{-1} x$, $\cosh^{-1} x$, $\tanh^{-1} x$).

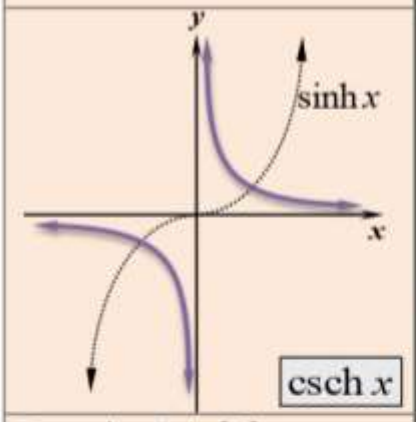
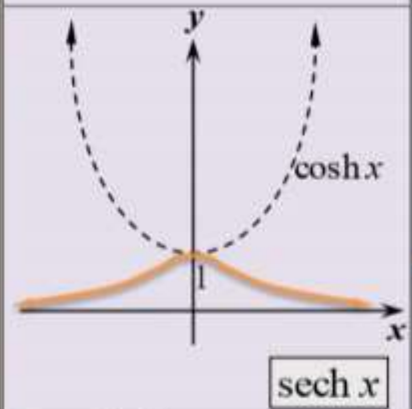
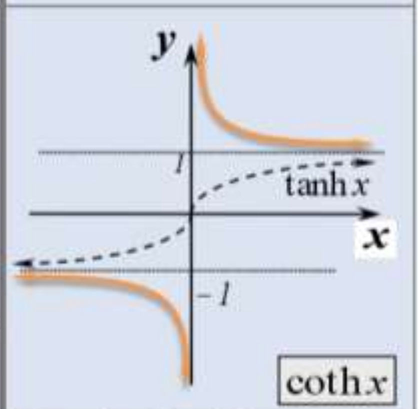
The Three Main Hyperbolic Functions:

Hyperbolic sine : $\sinh x$	Hyperbolic cosine: $\cosh x$	Hyperbolic tan: $\tanh x$
Read as 'shine', or 'sinch'	Read as 'kosh'	Read as 'tansh'
Definition: $y = \sinh x = (e^x - e^{-x})/2$	**Definition:** $y = \cosh x = (e^x + e^{-x})/2$	**Definition:** $y = \tanh x = (\sinh x/\cosh x)$ $= (e^x - e^{-x})/(e^x + e^{-x})$
It is an odd function because: $\sinh(-x) = (e^{-x} - e^x)/2$ $= -(e^x - e^{-x})/2 = -\sinh x$	It is an even function because: $\cosh(-x) = (e^{-x} + e^x)/2$ $= (e^x + e^{-x})/2 = \cosh x$	It is an odd function because: $\tanh(-x) = \left(\frac{\sinh(-x)}{\cosh(-x)}\right)$ $= \left(\frac{-\sinh x}{\cosh x}\right) = -\tanh x$
▪ $\lim_{x\to\infty}(\sinh x)$ $= \lim_{x\to\infty}((e^x - e^{-x})/2) = \infty$ • $\sinh 0 = \frac{e^0 - e^{-0}}{2} = 0$	▪ $\lim_{x\to\infty}(\cosh x)$ $= \lim_{x\to\infty}((e^x + e^{-x})/2) = \infty$ • $\cosh 0 = \frac{e^0 + e^{-0}}{2} = 1$	▪ $\lim_{x\to\infty}(\tanh x)$ $= \lim_{x\to\infty}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$ $= \lim_{x\to\infty}\left(\frac{1-(e^{-x})^2}{1+(e^{-x})^2}\right) = \mathbf{1}$
		
Domain: R, Range : R Zeroes: $x = \{0\}$ $\lim_{x\to\pm\infty}(\sinh x) = \pm\infty$	Domain: R, Range : $[1, \infty[$ Zeroes: $x = \{\}$ $\lim_{x\to\pm\infty}(\cosh x) = +\infty$	Domain: R, Range : $]-1,1[$ Zeroes: $x = \{0\}$ $\lim_{x\to\pm\infty}(\tanh x) = \pm 1$

The following table explains the absolute of the three hyperbolic functions

$\mid\sinh x\mid$	$\mid\cosh x\mid$	$\mid\tanh x\mid$
		
$\lvert\sinh x\rvert = \begin{cases} \sinh x, & x \ge 0 \\ -\sinh x, & x < 0 \end{cases}$	$\lvert\cosh x\rvert = \cosh x, \quad x \in R$	$\lvert\tanh x\rvert = \begin{cases} \tanh x, & x \ge 0 \\ -\tanh x, & x < 0 \end{cases}$

The following table explains the reciprocal of the three functions

$1/\sinh x =$ **csch x**	$1/\cosh x =$ **sech x**	$1/\tanh x =$ **coth x**
$= 2/(e^x - e^{-x})$	$= 2/(e^x + e^{-x})$	$= (e^x + e^{-x})/(e^x - e^{-x})$
		
Domain: $R - \{0\}$ Range : $R - \{0\}$ Zeroes: $x = \{\,\}$ $\lim_{x\to\pm\infty} (\operatorname{csch} x) = 0$ $\lim_{x\to 0^\pm} (\operatorname{csch} x) = \pm\infty$	Domain: R Range : $]0,1]$ Zeroes: $x = \emptyset = \{\,\}$ $\lim_{x\to\pm\infty} (\operatorname{sech} x) = 0$	Domain: $R - \{0\}$ Range : $R - [-1,1]$ Zeroes: $x = \emptyset = \{\,\}$ $\lim_{x\to\pm\infty} (\coth x) = \pm 1$ $\lim_{x\to 0^\pm} (\coth x) = \pm\infty$

Hyperbolic Identities: For every formula for the trigonometric functions, there is a similar (not necessary identical) formula for the hyperbolic functions.

$\cosh x + \sinh x = e^{x}$ & $\cosh x - \sinh x = e^{-x}$
$\cosh^2 x - \sinh^2 x = 1$, $1 - \tanh^2 x = \operatorname{sech}^2 x$, $\coth^2 x - 1 = \operatorname{csch}^2 x$
$\operatorname{csch} x = 1/\sinh x$, $\operatorname{sech} x = 1/\cosh x$, $\coth x = 1/\tanh x$,
$\tanh x = \frac{\sinh x}{\cosh x}$, $\coth x = \frac{\cosh x}{\sinh x}$
$\cosh(a \pm b) = \cosh a \cosh b \pm \sinh a \sinh b$
$\sinh(a \pm b) = \sinh a \cosh b \pm \cosh a \sinh b$
$\cosh 2a = \cosh^2 a + \sinh^2 a = 2\cosh^2 a - 1 = 2\sinh^2 a + 1$
$\sinh 2a = 2 \sinh a \cosh a$
$\tanh 2a = \frac{2\tanh a}{1+\tanh^2 a}$
$\sinh a \cosh b = \frac{1}{2}[\sinh(a-b) + \sinh(a+b)]$
$\cosh a \cosh b = \frac{1}{2}[\cosh(a-b) + \cosh(a+b)]$
$\sinh a \sinh b = \frac{1}{2}[\cosh(a+b) - \cosh(a-b)]$

Prove that: i) $\cosh x \pm \sinh x = e^{(\pm x)}$

ii) $\cosh^2 x - \sinh^2 x = 1$

iii) $\cosh^2 x + \sinh^2 x = \cosh 2x$

Solution

i) $\cosh x + \sinh x = \left(\frac{e^x+e^{-x}}{2}\right) + \left(\frac{e^x-e^{-x}}{2}\right) = \frac{2e^x}{2} = e^x$

$\cosh x - \sinh x = \left(\frac{e^x+e^{-x}}{2}\right) - \left(\frac{e^x-e^{-x}}{2}\right) = \frac{2e^{-x}}{2} = e^{-x}$

ii) $\cosh^2 x - \sinh^2 x = (\cosh x + \sinh x)(\cosh x - \sinh x) = e^x \; e^{-x} = e^0 = 1$

iii) $\cosh^2 x + \sinh^2 x = \frac{1}{2}\,[(\cosh^2 x + \sinh^2 x) + (\cosh^2 x + \sinh^2 x)]$

$$= \frac{1}{2}\,[(\cosh^2 x + 2\sinh x\cosh x + \sinh^2 x) + (\cosh^2 x - 2\sinh x\cosh x + \sinh^2 x)]$$

$$= \frac{1}{2}[(\cosh x + \sinh x)^2 + (\cosh x - \sinh x)^2]$$

$$= \frac{1}{2}[(e^x)^2 + (e^{-x})^2] = \frac{1}{2}[e^{2x} + e^{-2x}] = \cosh 2x$$

4 . Derivatives of Hyperbolic Functions

Rules of differentiating the hyperbolic functions are listed in the following theorem.

THEOREM

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\sinh x$	$\cosh x$	$\sinh(u(x))$	$\cosh(u(x)) \times u'(x)$
$\cosh x$	$\sinh x$	$\cosh(u(x))$	$\sinh(u(x)) \times u'(x)$
$\tanh x$	$\operatorname{sech}^2 x$	$\tanh(u(x))$	$\operatorname{sech}^2(u(x)) \times u'(x)$
$\coth x$	$-\operatorname{csch}^2 x$	$\coth(u(x))$	$-\operatorname{csch}^2(u(x)) \times u'(x)$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$	$\operatorname{sech}(u(x))$	$-\operatorname{sech}(u(x)) \tanh(u(x)) \times u'(x)$
$\operatorname{csch} x$	$-\operatorname{csch} x \coth x$	$\operatorname{csch}(u(x))$	$-\operatorname{csch}(u(x)) \coth(u(x)) \times u'(x)$

Note: *the analogy of the differentiation formulas of the hyperbolic functions with the differentiation formulas for trigonometric functions, but the signs are different in some cases (the derivative of the last three functions is negative).*

Proof

The first two of these formulas are easily verified using the definition of $\sinh x$ and $\cosh x$:

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{1}{2}\frac{d}{dx}(e^x - e^{-x}) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{1}{2}\frac{d}{dx}(e^x + e^{-x}) = \frac{1}{2}(e^x - e^{-x}) = \sinh x$$

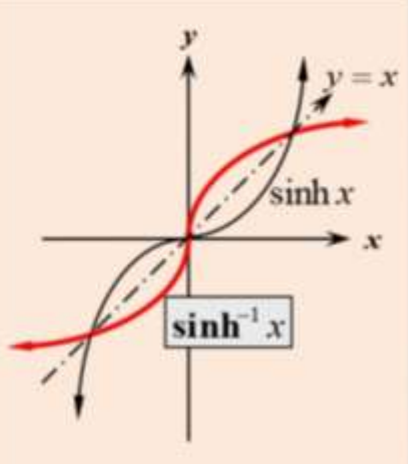
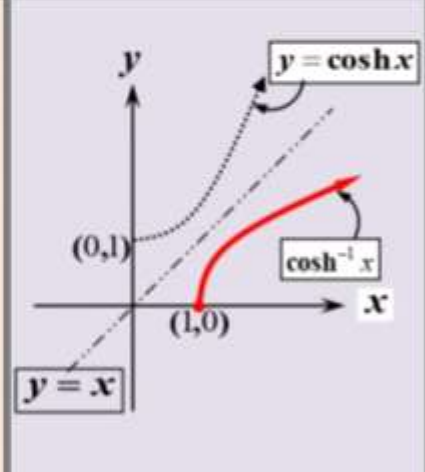
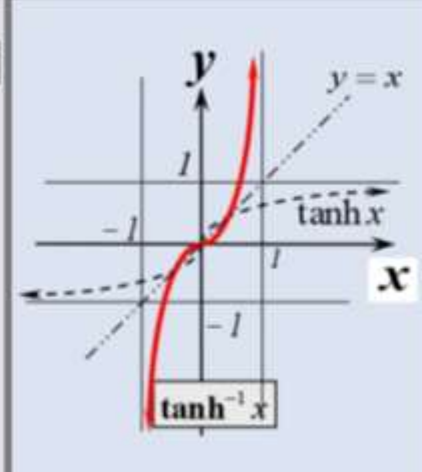
These two formulas can be used to drive the others:

$$\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\coth x) = \frac{d}{dx}\left(\frac{\cosh x}{\sinh x}\right) = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = \frac{d}{dx}\left(\frac{1}{\cosh x}\right) = \frac{0 - \sinh x}{\cosh^2 x} = -\frac{1}{\cosh x} \times \frac{\sinh x}{\cosh x} = -\operatorname{sech} x \tanh x$$

The following table explains **the inverse** of the hyperbolic functions

$\sinh^{-1} x \neq (1/\sinh x)$	$\cosh^{-1} x \neq (1/\cosh x)$	$\tanh^{-1} x \neq (1/\tanh x)$
Is the inverse of $\sinh x$, $x \in R$	Is the inverse of $\cosh x, x \geq 0$	Is the inverse of $\tanh x, x \in R$
		
Domain: R Range : R Zeroes: $\{0\}$ $\lim\limits_{x\to\pm\infty} (\sinh^{-1} x) = \pm\infty$	Domain: $[1, \infty[$ Range : $[0, \infty[$ Zeroes: $\{1\}$ $\lim\limits_{x\to\infty} (\cosh^{-1} x) = \infty$	Domain: $]-1,1[$ Range : R Zeroes: $\{0\}$ $\lim\limits_{x\to 1^-} (\tanh^{-1} x) = \infty$ $\lim\limits_{x\to(-1)^+} (\tanh^{-1} x) = -\infty$
Odd function	Neither odd nor even	Odd function
■ $\sinh^{-1}(\sinh x) = x$ if $x \in R$ ■ $\sinh(\sinh^{-1} x) = x$ if $x \in R$	■ $\cosh^{-1}(\cosh x) = x$ if $x \in [0, \infty[$ ■ $\cosh(\cosh^{-1} x) = x$ if $x \in [1, \infty[$	■ $\tanh^{-1}(\tanh x) = x$ if $x \in R$ ■ $\tanh(\tanh^{-1} x) = x$ if $x \in]-1,1[$

5 . Derivatives of Inverse Transcendental Functions

$(\tan^{-1} x)'$, $(\cot^{-1} x)'$ $= \frac{\pm 1}{1+x^2}$	$(\tanh^{-1} x)'$ $= \frac{1}{1-x^2}$	$(\coth^{-1} x)'$ $= \frac{-1}{x^2-1}$
$(\sin^{-1} x)'$, $(\cos^{-1} x)'$ $= \frac{\pm 1}{\sqrt{1-x^2}}$	$(\sinh^{-1} x)'$ $= \frac{1}{\sqrt{1+x^2}}$	$\cosh^{-1} x$ $= \frac{1}{\sqrt{x^2-1}}$
$(\sec^{-1} x)'$, $(\csc^{-1} x)'$ $= \frac{\pm 1}{x\sqrt{x^2-1}}$	$(\text{sech}^{-1} x)'$ $= \frac{-1}{x\sqrt{1-x^2}}$	$(\text{csch}^{-1} x)'$ $= \frac{-1}{x\sqrt{1+x^2}}$

Proof

1) Prove that $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

Let $y = \tan^{-1} x \xrightarrow{\tan} \tan y = \tan(\tan^{-1} x) \Longrightarrow \tan y = x$ ……… (Eq. 1)

$\xrightarrow{\frac{d}{dx}\text{(Eq.1)}} \sec^2 y \;\; y' = 1 \Longrightarrow y' = 1/\sec^2 y$ ……… (Eq. 2)

$\because 1 + \tan^2 y = \sec^2 y \xrightarrow{\text{(Eq.1)}} 1 + x^2 = \sec^2 y$ ……… (Eq. 3)

$\text{(Eq. 3) } in \text{ (Eq. 2)} \Longrightarrow y' = \frac{1}{(1+x^2)}$ #

2) Prove that $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

Let $y = \sin^{-1} x \xrightarrow{\sin} \sin y = \sin(\sin^{-1} x) \Longrightarrow \sin y = x$ ……… (Eq. 1)

$\xrightarrow{\frac{d}{dx}\text{(Eq.1)}} \cos y \;\; y' = 1 \Longrightarrow y' = 1/\cos y$ ……… (Eq. 2)

$$\because \cos^2 y + \sin^2 y = 1 \xrightarrow{(\text{Eq.1})} \cos^2 y + x^2 = 1 \Rightarrow \cos y = \sqrt{1-x^2} \ldots\ldots (\text{Eq.}\,3)$$

$$(\text{Eq.}\,3)\ in\ (\text{Eq.}\,2) \Longrightarrow y' = \frac{1}{\sqrt{1-x^2}} \quad \#$$

3) Prove that $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{d}{dx}(\cos^{-1}(1/x)) = \frac{-1}{\sqrt{1-(1/x)^2}} \times \frac{-1}{x^2} = \frac{1}{x\sqrt{x^2 - x^2(1/x)^2}}$$

$$= \frac{1}{x\sqrt{x^2-1}}$$

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4) Prove that $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$

$$\text{Let } y = \tanh^{-1} x \xrightarrow{\tanh} \tanh y = \tanh(\tanh^{-1} x) \Rightarrow \tanh y = x \ldots\ldots\ldots (\text{Eq.}\,1)$$

$$\xrightarrow{\frac{d}{dx}(\text{Eq.1})} \operatorname{sech}^2 y \ \ y' = 1 \Longrightarrow y' = 1/\operatorname{sech}^2 y \ \ldots\ldots\ldots\ldots\ldots (\text{Eq.}\,2)$$

$$\because 1 - \tanh^2 y = \operatorname{sech}^2 y \xrightarrow{(\text{Eq.1})} 1 - x^2 = \operatorname{sech}^2 y \ \ldots\ldots\ldots\ldots (\text{Eq.}\,3)$$

$$(\text{Eq.}\,3)\ in\ (\text{Eq.}\,2) \Longrightarrow y' = \frac{1}{(1-x^2)} \quad \#$$

5) Prove that $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$

$$\text{Let } y = \sinh^{-1} x \xrightarrow{\sinh} \sinh y = \sinh(\sinh^{-1} x) \Longrightarrow \sinh y = x \ldots\ldots\ldots (\text{Eq.}\,1)$$

$$\xrightarrow{\frac{d}{dx}(\text{Eq.1})} \cosh y \ \ y' = 1 \Longrightarrow y' = 1/\cosh y \ \ldots\ldots\ldots\ldots (\text{Eq.}\,2)$$

$$\because \cosh^2 y - \sinh^2 y = 1 \xrightarrow{(\text{Eq.1})} \cosh^2 y - x^2 = 1 \Rightarrow \cosh y = \sqrt{1+x^2} \ldots\ldots (\text{Eq.}\,3)$$

$$(\text{Eq. 3})\ in\ (\text{Eq. 2}) \implies y' = \frac{1}{\sqrt{1+x^2}} \quad \#$$

6) Prove that $\frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}$

$$\frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{d}{dx}(\cosh^{-1}(1/x)) = \frac{1}{\sqrt{(1/x)^2 - 1}} \times \frac{-1}{x^2} = \frac{-1}{x\sqrt{1-x^2}}$$

Example 10.

Find the derivative of $\quad y = \tan^{-1} x + (\tan x)^{-1} + e^{\tan^{-1} x} + \tan^{-1}(e^x) + \sqrt{\tan^{-1} x}$

Solution

$$y' = \left(\frac{1}{1+x^2}\right) - (\tan^{-2} x)(\sec^2 x) + \left(e^{\tan^{-1} x}\right)\left(\frac{1}{1+x^2}\right) + \left(\frac{1}{1+e^{2x}}\right)(e^x)$$
$$+ \frac{1}{2\sqrt{\tan^{-1} x}}\left(\frac{1}{1+x^2}\right)$$

Example 11.

Find the derivative of $\quad y = \cosh\left(\sinh^{-1}\left(3^{\tan^{-1} x}\right)\right)$

Solution

$$y' = \sinh\left(\sinh^{-1}\left(3^{\tan^{-1} x}\right)\right) \times \frac{1}{\sqrt{1 + \left(3^{\tan^{-1} x}\right)^2}} \times \left[\left(3^{\tan^{-1} x}\right) \times (\ln 3) \times \frac{1}{1+x^2}\right]$$

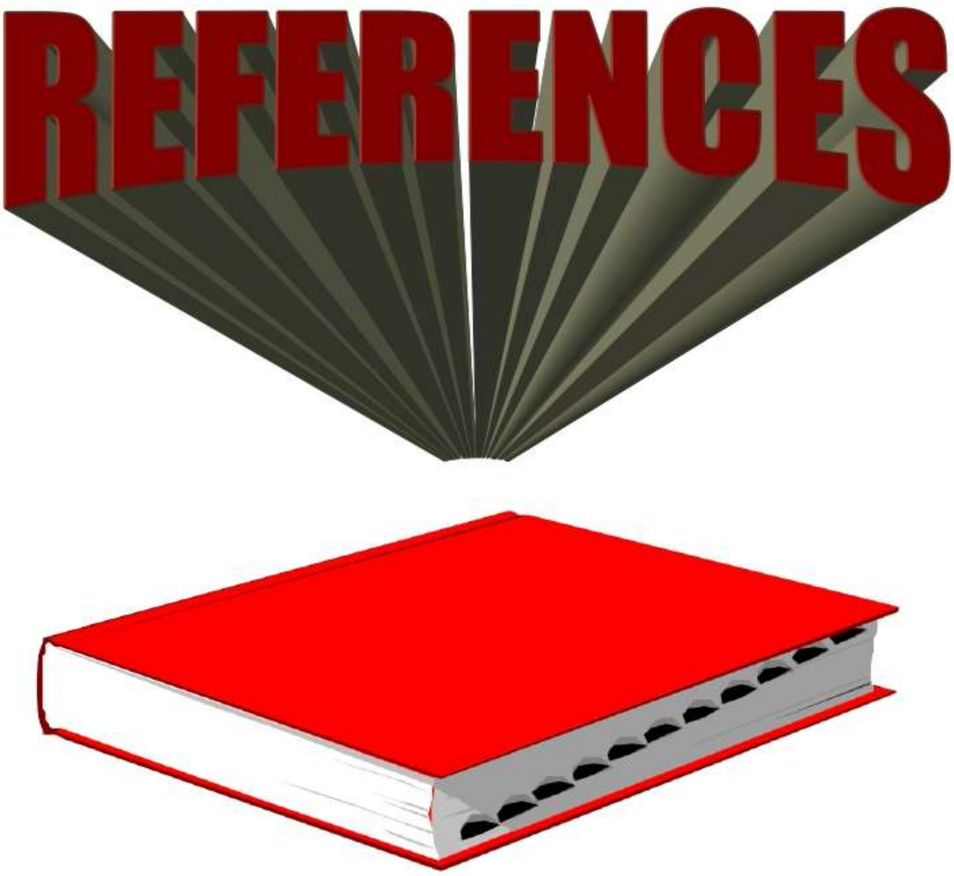
Example 12.

Find the derivative of $\quad y = \sec^{-1}\left((\operatorname{sech} x / (2^x + \cosh x^2)\right)$

Solution

$$y = \cos^{-1}\left((2^x + \cosh x^2)/(\operatorname{sech} x)\right) = \cos^{-1}\left((2^x + \cosh x^2)\cosh x\right)$$

$$y' = \frac{-1}{\sqrt{1 - \left((2^x + \cosh x^2)\cosh x\right)^2}}$$
$$\times \left[\left((2^x + \cosh x^2)\sinh x + (2^x \ \ln 2 + 2x\sinh(x^2))\ \cosh x\right)\right]$$



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2. Thomas-Calculus 12th Edition.
3. Mooculus Calculus, Jim Fowler and Bart Snapp, 2014 .
4. Higher Engineering Mathematics, John Bird, 6th Edition, 2010, John Bird, Published by Elsevier Ltd.
5. Calculus in Context, The Five College Calculus Project James Callahan, 20 Five Colleges, Inc.
6. Dictionary of Mathematics Terms, Douglas Downing, 3rd Edition, 2009 by Barron's Educational Series, Inc.
7. A Dictionary Mathematical English Usage, Jerzy Trzeciak, Available online http://www.impan.pl/Dictionary, Jerzy Trzeciak, Warszawa 2012.
8. English Arabic Technical Computing Dictionary, http://wiki.arabeyes.org, Technical Dictionary.
9. The Concise Oxford Dictionary of Mathematics.