CONCEPT BOOSTER

1. Meaning of $x \to a$

The symbol $x \to a$ is called as x tends to a or x approaches to a. It implies that x takes values closer and closer to a but not equal to a.

2. Neighbourhood of a Point

Any open interval containing a point a as its mid-point is called a neighbourhood of a. A positive number δ is called a neighbourhood of a, if $a - \delta < x < \delta + a$.

3. Limit of a Function

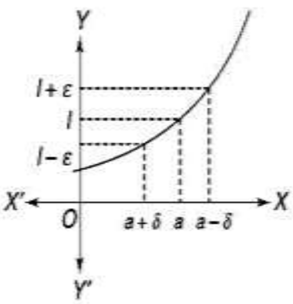
If there is a number l such that x approaches to a, either from the right or from the left, $f(x)$ approaches l, then l is called the limit of $f(x)$ as x approaches to a.

A number l is said to be a limiting value only if it is finite and real, otherwise we say that limit does not exist.

In a piece-wise defined function or more than one function, we shall use the concept of right hand limit as well as left hand limit. In a function(s), if right hand limit as well as left hand limit both are exist and their values are same, then limit exist, otherwise not.

4. Formal Definition of a Limit

A number l is said to be limit of the function $f(x)$ at $x = a$, for any positive number $\in > 0$, there corresponds a positive number δ such that $|f(x) - l| < \in$, $\forall x \in D_f$, $x \neq a$, $|x - a| < \delta$.



5. Concept of Infinity

Let us consider that n assumes sucessively the values 1, 2, 3, ... Then n gets larger and larger and there is no limit to the extent of its increase. It is convenient to that n tends to infinity. When we say that n tends to infinity, we simply mean it that n is supposed to assume a series of values which increas beyond limit.

A function $y = f(x)$ which approaches infinity as $x \to a$ does not have a limit in the ordinary sense.

6. Concept of Limit

The concept of limit is used to discuss the behaviour of a function close to a certain point

Let $$f(x) = \frac{x^2 - 1}{x - 1}, x \neq 1$$

Clearly the given function is not defined at $x = 1$.

It is defined only when $x \neq 1$, that means, either $x > 1$ or $x < 1$.

Case I: When $x > 1$ (just slightly more than 1)

$x > 1$	$f(x) = x + 1$
1.1	2.1
1.01	2.01
1.001	2.001
1.0001	2.0001

Thus, $x \to 1^+ \Rightarrow f(x) \to 2^+$

We define it the right hand limit of a function

Case II: When $x < 1$ (just slightly less than 1)

$x < 1$	$f(x) = x + 1$
.9	1.9
.99	1.99
.999	1.999
.9999	1.9999

8.1 Algebraic Limit

(i) **Direct substitution method**

In this method, we can directly sustitute the number at which the limit is to be find. After substitution, if we get a finite value, that is the limiting value of a function.

Example 4. $\lim_{x\to 1}(x^2+3x+4)=8.$

Example 5. $\lim_{x\to 1}(x^3-5x+4)=0.$

(ii) **Factorisation method**

In this method, we shall find out a common factor from the numerator as well as a denominator. Cancel the common factor and then directly substitute the number at which the limit is to be find.

Example 6. $\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right)$

$$=\lim_{x\to 1}\left(\frac{(x+1)(x-1)}{x-1}\right)=\lim_{x\to 1}(x+1)=2$$

Example 7. $\lim_{x\to 0}\left(\frac{1-x^{-\frac{1}{3}}}{1-x^{-\frac{2}{3}}}\right)$

$$=\lim_{x\to 0}\left(\frac{1-x^{-\frac{1}{3}}}{\left(1+x^{-\frac{1}{3}}\right)\left(1-x^{-\frac{1}{3}}\right)}\right)$$

$$=\frac{1}{2}$$

(iii) **Rationalisation Method**

In this method, our first aim will be, remove the radical sign. This is particularly used when either the numerator or denominator or both involve the fractional powers.

Example 8. Evaluate: $\lim_{x\to 0}\left(\frac{\sqrt{x^2+x+1}-\sqrt{x^2+1}}{x}\right)$

$$=\lim_{x\to 0}\left(\frac{x^2+x+1-x^2-1}{x\left(\sqrt{x^2+x+1}+\sqrt{x^2+1}\right)}\right)$$

$$=\lim_{x\to 0}\left(\frac{x}{x\left(\sqrt{x^2+x+1}+\sqrt{x^2+1}\right)}\right)$$

$$=\lim_{x\to 0}\left(\frac{1}{\left(\sqrt{x^2+x+1}+\sqrt{x^2+1}\right)}\right)$$

$$=\frac{1}{2}$$

(iv) **Standard Result method**

In this method we shall use the formula

$$\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right)=na^{n-1}.$$

Example 9. $\lim_{x\to 2}\left(\frac{x^5-32}{x-2}\right)$

$$=\lim_{x\to 2}\left(\frac{x^5-2^5}{x-2}\right)=5.2^{5-1}=5\times 16=80$$

Example 10. $\lim_{x\to 3}\left(\frac{x^5-243}{x^3-27}\right)$

$$=\lim_{x\to 3}\left(\frac{\frac{x^5-3^5}{x-3}}{\frac{x^3-3^3}{x-3}}\right)=\frac{5.3^{5-1}}{3.3^{3-1}}=15$$

Example 11. Evaluate: $\lim_{x\to 0}\left(\frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}\right)$

We have $\lim_{x\to 0}\left(\frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}\right)$

$$=\frac{3-\sqrt{5}}{1-\sqrt{5}}.$$

(v) **Infinity Method**

(a) Form: $\left(\frac{\infty}{\infty}\right)$. In this method, we shall write down the given expression in the form of a rational function. i.e. $\frac{f(x)}{g(x)}$.

Then divide the numerator and denominator by the highest power of x and then use

$$\lim_{x\to\infty}\left(\frac{1}{x^n}\right)=0,\ x>1$$

Note: $$\lim_{x\to\infty}x^n=\begin{cases}\infty & : x>1\\ 0 & : 0<x<1\\ 1 & : x=1\\ \text{not defined} & : x<0\end{cases}$$

(b) Form ($\infty \pm \infty$): In this method, first we reduce the function $f(x)$ into a rational function by rationalisation i.e. multiply the numerator and denominator by its conjugate and then apply $\lim_{x\to\infty}\left(\frac{1}{x^n}\right) = 0,\ x > 1.$

2 Non-algebraic Limit

efore solving the value of a limit, please read throughly ıch of the following formulae.

(a) Trigonometric limit

(i) $\lim_{x\to 0} \sin x = 0$

(ii) $\lim_{x\to 0} \cos x = 1$

(iii) $\lim_{x\to 0} \tan x = 0$

(iv) $\lim_{x\to 0} \left(\frac{\sin x}{x}\right) = 1$

(v) $\lim_{x\to 0} \left(\frac{\sin(x-a)}{x-a}\right) = 1$

(vi) $\lim_{x\to 0} \left(\frac{\tan x}{x}\right) = 1$

(vii) $\lim_{x\to a} \left(\frac{\tan(x-a)}{(x-a)}\right) = 1$

(viii) $\lim_{x\to \infty} \left(\frac{\sin x}{x}\right) = 0$

(ix) $\lim_{x\to \infty} \left(\frac{\cos x}{x}\right) = 0$

(x) $\lim_{x\to \infty} \left(\frac{\tan x}{x}\right) = 0$

(b) Inverse Trigonometric limit

(i) $\lim_{x\to 0} \left(\frac{\sin^{-1} x}{x}\right) = 1$

(ii) $\lim_{x\to 0} \left(\frac{\tan^{-1} x}{x}\right) = 1$

(c) Exponential limit

(i) $\lim_{x\to 0} a^x = 1,\ a \neq 1,\ a > 0.$

(ii) $\lim_{x\to 0} e^x = 1$

(iii) $\lim_{x\to 0} \left(\frac{a^x - 1}{x}\right) = \log_e a$

(iv) $\lim_{x\to 0} \left(\frac{e^x - 1}{x}\right) = 1$

(d) Logarithmic limit

(i) $\lim_{x\to 0} \left(\frac{\log(1+x)}{x}\right) = 1$

(e) Miscellaneous limit

(i) ***L'Hospital Rule***

Let $f(x)$ and $g(x)$ be two real functions and $a \in \mathbb{R}$. If $f(a) = 0 = g(a)$ or $f(a) = \infty = g(a)$, then

$$\lim_{x\to 0}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to 0}\left(\frac{f'(x)}{g'(x)}\right) = \lim_{x\to 0}\left(\frac{f''(x)}{g''(x)}\right),$$

untill and unless the form of $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ is removed.

L'Hospital rule is applicable only when the limits are in the form of $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$.

Note: Indeterminate form:

(i) $\left(\frac{0}{0}\right)$	(ii) $\left(\frac{\infty}{\infty}\right)$
(iii) $(\infty, -\infty)$	(iv) $(0, \infty)$
(v) (∞^0)	(vi) (0^0)
(vii) (1^∞)	(viii) (∞^0)

1 L'Hospital's Rule

Another useful application of mean value theorems is *L'Hospital's Rule*. It helps us to evaluate limits of "*indeterminate forms*" such as $\frac{0}{0}$. Let's look at the following example. Recall that we have proved in week 3 (using the sandwich theorem and a geometric argument)

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

We say that the limit above has indeterminate form $\frac{0}{0}$ since both the numerator and denominator goes to 0 as $x\to 0$. Roughly speaking, L'Hospital's rule says that under such situation, we can differentiate the numerator and denominator first and then take the limit. The result, if exists, should be equal to the original limit. For example,

$$\lim_{x\to 0}\frac{(\sin x)'}{(x)'}=\lim_{x\to 0}\frac{\cos x}{1}=1,$$

which is equal to the limit before we differentiate!

Theorem 1.1 (L'Hospital's Rule) *Let $f,g:(a,b)\to\mathbb{R}$ be differentiable functions in (a,b) and fix an $x_0\in(a,b)$. Assume that*

(i) $f(x_0)=0=g(x_0)$.

(ii) $\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=L$ *(i.e. the limit exists and is finite).*

Then, we have

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=L.$$

Example 1.2 *Consider the limit*

$$\lim_{x\to 0}\frac{\sin^2 x}{1-\cos x},$$

this is a limit of indeterminate form $\frac{0}{0}$. Therefore, we can apply L'Hospital's Rule to obtain

$$\lim_{x\to 0}\frac{\sin^2 x}{1-\cos x}=\lim_{x\to 0}\frac{(\sin^2 x)'}{(1-\cos x)'},$$

if the limit on the right hand side exists. Since the right hand side is the same as

$$\lim_{x\to 0}\frac{2\sin x\cos x}{\sin x}=\lim_{x\to 0}(2\cos x)=2.$$

Therefore, we conclude that $\lim_{x\to 0}\frac{\sin^2 x}{1-\cos x}=2$.

Exercise: Calculate the limit in Example 1.2 without using L'Hospital's Rule (hint: $\sin^2 x = 1-\cos^2 x$).

Sometimes we have to apply L'Hospital's Rule a few times before we can evaluate the limit directly. This is illustrated by the following two examples.

Example 1.3 *Consider the limit*

$$\lim_{x\to 0}\frac{x-\sin x}{x^3},$$

this is of the form "$\frac{0}{0}$*". Therefore, by L'Hospital's rule*

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}=\lim_{x\to 0}\frac{1-\cos x}{3x^2},$$

if the right hand side exists. The right hand side is still in the form "$\frac{0}{0}$*", therefore we can apply L'Hospital's Rule again*

$$\lim_{x\to 0}\frac{1-\cos x}{3x^2}=\lim_{x\to 0}\frac{\sin x}{6x},$$

if the right hand side exists. But now the right hand side can be evaluated:

$$\lim_{x\to 0}\frac{\sin x}{6x}=\frac{1}{6}\lim_{x\to 0}\frac{\sin x}{x}=\frac{1}{6}.$$

As a result, if we trace backwards, we conclude that the original limit exists and

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}=\frac{1}{6}.$$

Example 1.4 *Consider the limit*

$$\lim_{x\to 0}\frac{e^x-x-1}{1-\cosh x}.$$

Applying L'Hospital's Rule twice, we can argue as in Example 1.3 that

$$\lim_{x\to 0}\frac{e^x-x-1}{1-\cosh x}=\lim_{x\to 0}\frac{e^x-1}{-\sinh x}=\lim_{x\to 0}\frac{e^x}{-\cosh x}=\frac{1}{-1}=-1.$$

After seeing these examples, let us now go back to give a proof of L'Hospital's Rule.

Proof of L'Hospital's Rule: Recall Cauchy's Mean Value Theorem which says that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)}$$

for some $\xi \in (a,b)$. Therefore, since $f(x_0)=g(x_0)=0$, we have

$$\frac{f(x)}{g(x)}=\frac{f(x)-f(x_0)}{g(x)-g(x_0)}=\frac{f'(\xi)}{g'(\xi)}$$

for some ξ between x and x_0. Notice that as $x \to x_0$, we must also have $\xi \to x_0$. Therefore, we have

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{\xi\to x_0}\frac{f'(\xi)}{g'(\xi)}.$$

This proves the L'Hospital's Rule.

2 Other indeterminate forms

When we evaluate limits, there are other possible "indeterminate forms", for example

$$\frac{0}{0}, \qquad 0\cdot\infty, \qquad \frac{\infty}{\infty}, \qquad 0^0. \tag{2.1}$$

Note that these forms above are just *formal* expressions which does not have very precise mathematical meanings as ∞ is not a real number.

Convention: We distinguish two "infinities" by writing

$$\infty := +\infty \qquad \text{and} \qquad -\infty := -\infty.$$

Remark 2.1 *Not all expressions involving* 0 *and* ∞ *would result in an indeterminate form. For example,*

$$0^\infty=0, \qquad \infty^\infty=\infty, \qquad \infty+\infty=\infty, \qquad \infty\cdot\infty=\infty.$$

In this section, we will see that all the indeterminate forms in (2.1) can actually be rewritten into the standard form $\frac{0}{0}$. Symbolically we have $1/0=\infty$. Therefore,

$$0\cdot\infty=0\cdot\frac{1}{0}=\frac{0}{0},$$

$$\frac{\infty}{\infty} = \frac{1/0}{1/0} = \frac{0}{0}.,$$

$$0^0 = \exp(0 \ln 0) = \exp(0 \cdot (-\infty)) = \exp(-\frac{0}{0}).$$

We should emphasize that the "calculations" above are just formal. They indicate the general idea of transforming the limits rather than actual arithmetic of numbers. Using these ideas, we can actually handle all the determinate forms in (2.1) by the L'Hospital's Rule. We have

Theorem 2.2 (L'Hospital's Rule) *The same conclusion holds if we replace (i) by*

$$\lim_{x \to x_0} f(x) = \pm\infty = \lim_{x \to x_0} g(x).$$

Remark 2.3 *The theorem also holds in the case $x_0 = \pm\infty$ and for one-sided limits as well.*

We postpone the proof of Theorem 2.2 until the end of this section but we will first look at a few applications.

Example 2.4 *Consider the one-side limit*

$$\lim_{x \to 0+} x \ln x.$$

This is of the form $0 \cdot (-\infty)$. However, we can rewrite it as

$$x \ln x = \frac{\ln x}{1/x},$$

which is of the form $\frac{-\infty}{\infty}$ as $x \to 0^+$. Therefore, we can apply Theorem 2.2 to conclude that

$$\lim_{x \to 0+} \frac{\ln x}{1/x} = \lim_{x \to 0+} \frac{1/x}{-1/x^2} = \lim_{x \to 0+} (-x) = 0.$$

Therefore, we have $\lim_{x \to 0+} x \ln x = 0$. In words, this means that as $x \to 0^+$, the linear function x is going to 0 faster than the logarithm function $\ln x$ going to $-\infty$.

Example 2.5 *Sometimes we have to apply L'Hospital's Rule a few times. For example,*

$$\lim_{x \to +\infty} \frac{x^2}{e^x} = \lim_{x \to +\infty} \frac{2x}{e^x} = \lim_{x \to +\infty} \frac{2}{e^x} = 0.$$

Similarly, we can prove that

$$\lim_{x\to+\infty}\frac{x^k}{e^x}=0, \qquad \textit{for any } k.$$

In other words, as $x\to+\infty$, the exponential function e^x is going to ∞ faster than any polynomial of x.

The following example shows that L'Hospital's Rule may not always work:

$$\lim_{x\to\infty}\frac{\sinh x}{\cosh x}=\lim_{x\to\infty}\frac{\cosh x}{\sinh x}=\lim_{x\to\infty}\frac{\sinh x}{\cosh x},$$

which gets back to the original limit we want to evaluate! So L'Hospital's Rule leads us nowhere in such situation. For this example, we have to do some cancellations first,

$$\lim_{x\to\infty}\frac{\sinh x}{\cosh x}=\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}=\lim_{x\to\infty}\frac{1-e^{-2x}}{1+e^{-2x}}=1.$$

3 Some tricky examples of L'Hospital's Rule

Sometimes it is not very obvious how we should transform a limit into a "standard" indeterminate form.

Example 3.1 *Evaluate that limit*

$$\lim_{x\to\infty} x\sin\frac{1}{x}.$$

We can choose to transform it to either

$$x\sin\frac{1}{x}=\frac{\sin(1/x)}{1/x} \qquad \text{or} \qquad x\sin\frac{1}{x}=\frac{x}{1/\sin(1/x)}.$$

The first one has the form "$\frac{0}{0}$" and the second one has the form "$\frac{\infty}{\infty}$" as $x\to\infty$. Therefore, we can apply L'Hospital's Rule to both cases. For the first case, we have

$$\lim_{x\to\infty}\frac{\sin(1/x)}{1/x}=\lim_{x\to\infty}\frac{-\frac{1}{x^2}\cos\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to\infty}\cos\frac{1}{x}=1.$$

However, for the second case, we have

$$\lim_{x\to\infty}\frac{x}{1/\sin(1/x)}=\lim_{x\to\infty}\frac{1}{\frac{1}{x^2}\frac{\cos(1/x)}{\sin^2(1/x)}},$$

which doesn't seem to simplify after L'Hospital's Rule. Therefore, sometimes we have to choose a good way to transform the limit before we apply the L'Hospital's Rule. A general rule of thumb here is that the expression should get simpler after taking the derivatives.

Example 3.2 *Evaluate the limit*

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right).$$

This limit has the indeterminate form "$\infty-\infty$", which we haven't mentioned. There is in fact no general way to evaluate limits of such forms. But for this particular example, we can transform it as

$$\frac{1}{\sin x}-\frac{1}{x}=\frac{x-\sin x}{x\sin x},$$

which has the standard indeterminate form "$\frac{0}{0}$". Therefore, we can apply L'Hospital's Rule a few times to get

$$\lim_{x\to 0}\frac{x-\sin x}{x\sin x}=\lim_{x\to 0}\frac{1-\cos x}{\sin x+x\cos x}=\lim_{x\to 0}\frac{\sin x}{2\cos x-x\sin x}=0.$$

Example 3.3 *Evaluate the limit*

$$\lim_{x\to\infty} x^{\frac{1}{x}}.$$

Recall that if $a>0, b$ are *real* numbers, we define $a^b:=\exp(b\ln a)$. Therefore,

$$\lim_{x\to\infty} x^{\frac{1}{x}}=\lim_{x\to\infty}\exp\left(\frac{1}{x}\ln x\right)=\exp\left(\lim_{x\to\infty}\frac{\ln x}{x}\right)=\exp\left(\lim_{x\to\infty}\frac{1/x}{1}\right)=e^0=1.$$

Note that we can move the limit into the function "exp" since the exponential function "exp" is continuous.

We end this section with a proof of Theorem 2.2.

Proof of Theorem 2.2: The idea is that if $f(x_0)=\pm\infty=g(x_0)$, then we have $\frac{1}{f(x_0)}=0=\frac{1}{g(x_0)}$. Therefore, we can apply L'Hospital's Rule to conclude that

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{1/g(x)}{1/f(x)}=\lim_{x\to x_0}\frac{-g'(x)/g(x)^2}{-f'(x)/f(x)^2}.$$

Example 1

Evaluate the limit $\lim_{x\to 3} \frac{x^2+x-12}{x^2-9}$ using

(a) algebraic manipulation (factor and cancel)

Solution

$$\lim_{x\to 3} \frac{x^2+x-12}{x^2-9} = \lim_{x\to 3} \frac{(x-3)(x+4)}{(x-3)(x+3)} = \lim_{x\to 3} \frac{x+4}{x+3} = \frac{7}{6}$$

(b) L'Hopital's Rule

Solution

Since direct substitution gives $\frac{0}{0}$ we can use L'Hopital's Rule to give

$$\lim_{x\to 3} \frac{x^2+x-12}{x^2-9} \overset{H}{=} \lim_{x\to 3} \frac{2x+1}{2x} = \frac{7}{6}$$

Example 2

Evaluate the limit $\lim_{x\to 0} \frac{\sin 3x}{\tan 4x}$ using

(a) the basic trigonometric limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$ together with appropriate changes of variables

Solution

Write the limit as

$$\lim_{x\to 0} \frac{\sin 3x}{\tan 4x} = \left(\lim_{x\to 0} \frac{\sin 3x}{x}\right)\left(\lim_{x\to 0} \frac{x\cos 4x}{\sin 4x}\right)$$

In the first limit let $u = 3x$ and in the second let $v = 4x$. Then the limit is

$$\lim_{x\to 0} \frac{\sin 3x}{\tan 4x} = \left(\lim_{u\to 0} \frac{3\sin u}{u}\right)\left(\lim_{v\to 0} \frac{v\cos v}{4\sin v}\right)$$
$$= \frac{3}{4}\left(\lim_{u\to 0} \frac{\sin u}{u}\right)\left(\lim_{v\to 0} \frac{v}{\sin v}\right)\left(\lim_{v\to 0} \cos v\right) = \frac{3}{4}$$

(b) L'Hopital's Rule

Solution

Since direct substitution gives $\frac{0}{0}$ we can use L'Hopital's Rule to give

$$\lim_{x\to 0} \frac{\sin 3x}{\tan 4x} \overset{H}{=} \lim_{x\to 0} \frac{3\cos 3x}{4\sec^2 4x} = \frac{3}{4}$$

Example 3

Evaluate the limit $\lim_{x\to\frac{\pi}{2}} \left(x-\frac{\pi}{2}\right)\tan x$ using L'Hopital's Rule.

Solution

Write the limit as

$$\lim_{x\to\frac{\pi}{2}} \left(x-\frac{\pi}{2}\right)\tan x = \lim_{x\to\frac{\pi}{2}} \frac{x-\frac{\pi}{2}}{\cot x}$$

Then direct substitution gives $\frac{0}{0}$ so we can use L'Hopital's Rule to give

$$\lim_{x\to\frac{\pi}{2}} \left(x-\frac{\pi}{2}\right)\tan x \overset{\text{H}}{=} \lim_{x\to\frac{\pi}{2}} \frac{1}{(-\csc^2 x)} = -1$$

Example 4

Evaluate the limit $\lim_{x\to 1} \frac{\sqrt{2-x}-x}{x-1}$ using L'Hopital's Rule.

Solution

Since direct substitution gives $\frac{0}{0}$ use L'Hopital's Rule to give

$$\lim_{x\to 1} \frac{\sqrt{2-x}-x}{x-1} \overset{\text{H}}{=} \lim_{x\to 1} \frac{-\frac{1}{\sqrt{2-x}}-1}{1} = -\frac{3}{2}$$

Note that this result can also be obtained by rationalizing the numerator by multiplying top and bottom by the root conjugate $\sqrt{2-x}+x$.

Example 5

Evaluate the limit $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ using L'Hopital's Rule.

Solution

Since direct substitution gives $\frac{0}{0}$ use L'Hopital's Rule to give

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} \overset{\text{H}}{=} \lim_{x\to 0} \frac{\sin x}{2x}$$

Again direct substitution gives $\frac{0}{0}$ so use L'Hopital's Rule a second time to give

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} \overset{\text{H}}{=} \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$$

Example 6

Evaluate the limits at infinity

(a) $\lim_{x\to\infty} \frac{e^x}{x}$

Solution

Direct "substitution" gives $\frac{\infty}{\infty}$ so we can use L'Hopital's Rule to give

$$\lim_{x\to\infty} \frac{e^x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

(b) $\lim_{x\to\infty} x^2 e^{-x}$

Solution

Write the limit as

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x}$$

Then direct "substitution" gives $\frac{\infty}{\infty}$ so we can use L'Hopital's Rule to give

$$\lim_{x\to\infty} x^2 e^{-x} \overset{H}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$$

(c) $\lim_{x\to\infty} \left(\sqrt{x^2+x+1} - x\right)$

Solution

Direct "substitution" gives the indeterminate form $\infty - \infty$. Write the limit as

$$\lim_{x\to\infty} \left(\sqrt{x^2+x+1} - x\right) = \lim_{x\to\infty} x\left(\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} - 1\right) = \lim_{x\to\infty} \frac{\left(\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} - 1\right)}{\frac{1}{x}}$$

Now direct "substitution" gives $\frac{0}{0}$ so we can use L'Hopital's Rule to give

$$\lim_{x\to\infty} \left(\sqrt{x^2+x+1} - x\right) \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{2}\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^{-1/2}\left(-\frac{1}{x^2}-\frac{2}{x^3}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1+\frac{2}{x}}{2\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^{1/2}} = \frac{1}{2}$$

Example 7

Evaluate the limit $\lim_{x\to 2^+} \frac{\ln(x-2)}{\ln(x^2-4)}$ using L'Hopital's Rule.

Solution

Direct substitution gives $\frac{\infty}{\infty}$ so we can use L'Hopital's Rule

$$\lim_{x\to 2^+} \frac{\ln(x-2)}{\ln(x^2-4)} \overset{H}{=} \lim_{x\to 2^+} \frac{\frac{1}{x-2}}{\frac{2x}{x^2-4}} = \lim_{x\to 2^+} \frac{x^2-4}{2x(x-2)} = \lim_{x\to 2^+} \frac{x^2-4}{2x^2-4x}$$

$$\overset{H}{=} \lim_{x\to 2^+} \frac{2x}{4x-4} = 1$$

Example 8

Evaluate the limit $\lim_{x\to\infty}(\ln x - x)$ using L'Hopital's Rule.

Solution

Direct "substitution" gives the indeterminate form $\infty - \infty$. Write the limit as

$$\lim_{x\to\infty}(\ln x - x) = \lim_{x\to\infty}\ln\left(xe^{-x}\right)$$

Then, as in Example 6 (a) and (b),

$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty}\frac{x}{e^x} \overset{H}{=} \lim_{x\to\infty}\frac{1}{e^x} = 0$$

Now let $u = xe^{-x}$, then $x \to \infty \Rightarrow u \to 0^+$. Hence

$$\lim_{x\to\infty}(\ln x - x) = \lim_{u\to 0^+}\ln u = -\infty$$

Example 9

Evaluate the limit $\lim_{x\to 0^+}(\sin x)^{\sqrt{x}}$ using L'Hopital's Rule.

Solution

Direct substitution gives the indeterminate form 0^0. First find the natural logarithm of the limit, as

$$\ln\left(\lim_{x\to 0^+}(\sin x)^{\sqrt{x}}\right) = \lim_{x\to 0^+}\sqrt{x}\ln(\sin x) = \lim_{x\to 0^+}\frac{\ln(\sin x)}{\frac{1}{\sqrt{x}}}$$

Now direct substitution gives $\frac{\infty}{\infty}$, so we can use L'Hopital's Rule

$$\lim_{x\to 0^+}\frac{\ln(\sin x)}{\frac{1}{\sqrt{x}}} \overset{H}{=} \lim_{x\to 0^+}\frac{\frac{\cos x}{\sin x}}{\left(-\frac{1}{2}\right)x^{-3/2}}$$

$$= -2\lim_{x\to 0^+}\frac{x^{3/2}\cos x}{\sin x} = -2\left(\lim_{x\to 0^+}\sqrt{x}\cos x\right)\left(\lim_{x\to 0^+}\frac{x}{\sin x}\right) = 0$$

since

$$\lim_{x\to 0^+}\sqrt{x}\cos x = 0$$

and we can use L'Hopital's Rule on the second limit to give

$$\lim_{x\to 0^+}\frac{x}{\sin x} \overset{H}{=} \lim_{x\to 0^+}\frac{1}{\cos x} = 1$$

Hence

$$\ln\left(\lim_{x\to 0^+}(\sin x)^{\sqrt{x}}\right) = 0$$

so that

$$\lim_{x\to 0^+}(\sin x)^{\sqrt{x}} = e^0 = 1$$

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