formula for sum of divisors

If one knows the factorization of a number, one can compute the sum of the positive divisors of that number without having to write down all the divisors of that number. To do this, one can use a formula which is obtained by summing a geometric series.

Theorem 1. *Suppose that n is a positive integer whose factorization into prime factors is $\prod_{i=1}^{k} p_i^{m_i}$, where the p_i's are distinct primes and the multiplicities m_i are all at least 1. Then the sum of the divisors of n equals*

$$\prod_{i=1}^{k} \frac{p_i^{m_i+1}-1}{p_i-1}$$

and the sum of the proper divisors of n equals

$$\prod_{i=1}^{k} \frac{p_i^{m_i+1}-1}{p_i-1} - \prod_{i=1}^{k} p_i^{m_i}.$$

Proof. A number will divide n if and only if prime factors are also prime factors of n and their multiplicity is less than to or equal to their multiplicities in n. In other words, a divisors n can be expressed as $\prod_{i=1}^{k} p_i^{\mu_i}$ where $0 \le \mu_i \le m_i$. Then the sum over all divisors becomes the sum over all possible choices for the μ_i's:

$$\sum_{d|n} d = \sum_{0 \le \mu_i \le m_i} \prod_{i=1}^{k} p_i^{\mu_i}$$

This sum may be expressed as a multiple sum like so:

$$\sum_{\mu_1=0}^{m_1} \sum_{\mu_2=0}^{m_2} \cdots \sum_{\mu_k=0}^{m_k} \prod_{i=1}^{k} p_i^{\mu_i}$$

This sum of products may be factored into a product of sums:

$$\prod_{i=1}^{k} \left(\sum_{\mu_i=0}^{m_i} p_i^{\mu_i} \right)$$

Each of these sums is a geometric series; hence we may use the formula for sum of a geometric series to conclude

$$\sum_{d|n} d = \prod_{i=1}^{k} \frac{p_i^{m_i+1}-1}{p_i-1}.$$

If we want only proper divisors, we should not include n in the sum, so we obtain the formula for proper divisors by subtracting n from our formula.
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As an illustration, let us compute the sum of the divisors of 1800. The factorization of our number is $2^3 \cdot 3^2 \cdot 5^2$. Therefore, the sum of its divisors equals

$$\left(\frac{2^4-1}{2-1}\right)\left(\frac{3^3-1}{3-1}\right)\left(\frac{5^3-1}{5-1}\right) = \frac{15 \cdot 26 \cdot 124}{2 \cdot 4} = 6045.$$