Art of Problem Solving

Legendre's Formula

Legendre's Formula states that

$$e_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor = \frac{n - S_p(n)}{p-1}$$

where p is a prime and $e_p(n!)$ is the exponent of p in the prime factorization of $n!$ and $S_p(n)$ is the sum of the digits of n when written in base p.

Contents

Examples

Find the largest integer k for which 2^k divides $27!$

Solution 1

Using the first form of Legendre's Formula, substituting $n = 27$ and $p = 2$ gives

$$\begin{aligned} e_2(27!) = &\left\lfloor \frac{27}{2} \right\rfloor + \left\lfloor \frac{27}{2^2} \right\rfloor + \left\lfloor \frac{27}{2^3} \right\rfloor + \left\lfloor \frac{27}{2^4} \right\rfloor \\ =&13 + 6 + 3 + 1 \\ =&23 \end{aligned}$$

which means that the largest integer k for which 2^k divides $27!$ is $\boxed{23}$.

Solution 2

Using the second form of Legendre's Formula, substituting $n = 27$ and $p = 2$ gives

$$e_2(27!) = \frac{27 - S_2(27)}{2-1} = 27 - S_2(27)$$

The number 27 when expressed in Base-2 is 11011. This gives us $S_2(27) = 1 + 1 + 0 + 1 + 1 = 4$. Therefore,

$$e_2(27!) = 27 - S_2(27) = 27 - 4 = 23$$

which means that the largest integer k for which 2^k divides $27!$ is $\boxed{23}$.

Proofs

Part 1

We use a counting argument.

We could say that $e_p(n!)$ is equal to the number of multiples of p less than n, or $\left\lfloor \frac{n}{p} \right\rfloor$. But the multiples of p^2 are only counted once, when they should be counted twice. So we need to add $\left\lfloor \frac{n}{p^2} \right\rfloor$ on. But this only counts the multiples of p^3 twice, when we need to count them thrice. Therefore we must add a $\left\lfloor \frac{n}{p^3} \right\rfloor$ on. We continue like this to get $e_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor$. This makes sense, because the terms of this series tend to 0.

Part 2

Let the base p representation of n be

$$e_x e_{x-1} e_{x-2} \dots e_0$$

where the e_i are digits in base p. Then, the base p representation of $\left\lfloor \frac{n}{p^i} \right\rfloor$ is

$$e_x e_{x-1} \dots e_{x-i}.$$

Note that the infinite sum of these numbers (which is $e_p(n!)$) is

$$\begin{aligned}\sum_{j=1}^{x} e_j \cdot (p^{j-1} + p^{j-2} + \cdots + 1) &= \sum_{j=1}^{x} e_j \left(\frac{p^j - 1}{p - 1} \right) \\ &= \frac{\sum_{j=1}^{x} e_j p^j - \sum_{j=1}^{x} e_j}{p-1} \\ &= \frac{(n - e_0) - (S_p(n) - e_0)}{p-1} \\ &= \frac{n - S_p(n)}{p-1}.\end{aligned}$$

Problems

Introductory

- How many zeros are at the end of the base-15 representation of $50!$?
- $\binom{4042}{2021} + \binom{4043}{2022}$ can be written as $n \cdot 10^x$ where n and x are positive integers. What is the largest possible value of x? (BorealBear)
- Find the sum of digits of the largest positive integer n such that $n!$ ends with exactly 100 zeros. (demigod)

Olympiad

- Let b_m be numbers of factors 2 of the number $m!$ (that is, $2^{b_m} | m!$ and $2^{b_m+1} \nmid m!$). Find the least m such that $m - b_m = 1990$. (Turkey TST 1990)