General term of an arithmetic sequence

A general term of an arithmetic sequence $\boldsymbol{T_i}$ is a **function in order**

i.e., it's an expression in one **independent** variable (let's call it 'i') representing the order of each term, i.e., (1^{st} term, 2^{nd} term, 3^{rd} term, ...etc.).

Substituting that independent variable 'i' gives the value of the $\boldsymbol{i_{th}}$ term.

Deriving the general term $\boldsymbol{T_i}$ of an arithmetic sequence

1. **Keep** calculating differences between terms until it becomes a **constant difference**, let this happen after $\boldsymbol{n}$ times of subtracting.
2. **Assume** a polynomial in $\boldsymbol{i}$ of degree $\boldsymbol{n}$ to be the general term $\boldsymbol{T_i}$, i.e., $\boldsymbol{T_i = P_n(i)}$
3. **Substitute** $(n+1)$ terms of the given sequence in $P_n(i)$ and solve the $(n+1)$ equations simultaneously to get the values of coefficients of that polynomial.

Example:

The sequence $1, 10, 23, 40, 61 \ldots$

Let's calculate the differences,

1 10 23 40 61

9 13 17 21

4 4 4 → <u>A constant difference after **2** times of subtracting</u>

Assume $T_i = ai^2 + bi + c$

At $i = 1$ → $a + b + c = 1$

At $i = 2$ → $4a + 2b + c = 10$

At $i = 3$ → $9a + 3b + c = 23$

Solving the equation, we get $a = 2, b = 3, c = -4$

Which is the general term $\boldsymbol{T_i = 2i^2 + 3i - 4}$

Sigma notation $\sum$

$$\sum_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{n}} \boldsymbol{T}_{\boldsymbol{i}}$$

$\boldsymbol{i}$ is the independent variable representing order in the general term $\boldsymbol{T}_{\boldsymbol{i}}$

$\boldsymbol{n}$ represents the **number of terms** to be added after substitution of each $\boldsymbol{i}$

$$\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{C} = \mathrm{n} * \mathrm{C}, \text{where C is a constant}$$

$$\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{i} = \frac{\mathrm{n}(\mathrm{n}+1)}{2}$$

$$\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{i}^2 = \frac{\mathrm{n}(\mathrm{n}+1)(2\mathrm{n}+1)}{6}$$

$$\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{i}^3 = \frac{\mathrm{n}^2(\mathrm{n}+1)^2}{4}$$

From the above equations we can deduce rules for summing Odd and Even numbers

$$For\ ODD\ numbers, \sum_{i=1}^{n} 2i - 1 = n^2$$

$$For\ EVEN\ numbers, \sum_{i=1}^{n} 2i = n^2 + n$$

Ranged sigma $\sum$

The sum of terms of the sequence whose general term is T_i in range $[L, R]$

$$\sum_{i=1}^{\max.\, i\ makes\ T_i \le R} \mathrm{T_i} - \sum_{i=1}^{\max.\, i\ makes\ T_i \le (\mathrm{L}-1)} \mathrm{T_i}$$

We can easily get the maximum i makes $T_i \le X$ using binary search!

```
ll Get_No_Of_Terms (ll X)

{
    ll L = -1, R = 1e9;
    ll N = -1;
    while (L <= R)
    {
        ll mid = L + (R - L) / 2; // To avoid overflow
        // Substitute mid in the General term Ti
        if ((2 * mid * mid - 4 * mid + 5) <= X) // Example
        {
            N = mid;
            L = mid + 1;
        }
        else
            R = mid - 1;
    }
    return N;
}
```

Or simply we can find the floor of the solution to the general term of that sequence

The floor of a number can be easily acquired by casting to an int type

```
ll Sum {};
        // Solving the general term equation. Assume it's Ti = (3i² + 13i - 7)
        ll X = (-13 + sqrtl (169 + 12 * (7 + R))) / 6;
        ll Y = (-13 + sqrtl (169 + 12 * (6 + L))) / 6;

        // ∑ Ti = ∑ (3i² + 13i - 7) = X³ + 8X²
        Sum += (X * X * X + 8 * X * X) - (Y * Y * Y + 8 * Y * Y);
```