Algorithms Homework 6

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1 Homework 6

1.1 Question 1

We wish to prove that 3SAT-bis $\leq_p$ CCS (Compound Chemical Safety)

1) Let us show that CCS $\in NP$

Given a solution R^A and R^B we satisfy CCS by the criterias $R^A \cup R^B = R$, $R^A \cap R^B = \emptyset$, with $K_i \nsubseteq R^A$ and $K_i \nsubseteq R^B$. We just wish to show these can be done in polynomial time so let us not think about how to do them most efficiently. Finding union can be done by naive approach in $\mathcal{O}(n^2)$ and similarly for intersection of two sets. To find that $K_i \nsubseteq R^A$ we can go through each element in K_i and check if it exists in R^A, if all are True or all are False we conclude it is not satisfied. Since $|K_i|$ is smaller than $|R| = n$ we can also say this operation is done in $\mathcal{O}(n^2)$. Each of these three criterias are found indepedently and as argued all can be found in $\mathcal{O}(n^2)$, resulting in polynomial time verification. Hence 3SAT-bis is in NP.

2) Let us provide a translation/reduction from 3SAT-bis to CCS. As input to the translation we have all clauses $C = \{C_1, C_2, ..., C_t\}$.

Algorithm 1 Translation3SATbis_CCS(C)

$K = \{\}$, $R = \{\}$
for C_i in C **do**
    $K_i = \{\}$
    for x_i in C_i **do**
        Add x_i to K_i
        Add x_i to R
        if x_i is a negated term **then**
            Add $\{x_i, \overline{x_i}\}$ to K
    Add K_i to K
return K, R

This translation is polynomial and more specifically $\mathcal{O}(3t) = \mathcal{O}(t)$ where t is the number of clauses. We have the 3SAT-bis problem and we go through each of the clauses translating each of the variables x_j into a compound r_j where it is added to R. Each r_j for a specific clause is also added to K_i corresponding to the fact that not all three can either be 1 or all 0, i.e, the compounds cannot all be produced in the same plant. Problem arises since there is no requirement on the original 3SAT-bis problem and it can have negated term but then we solve this by simply adding another forbidden compound combination of $\{x_i, \overline{x_i}\}$ to K.

3) Solution to original 3SAT-bis problem **if and only if** solution to CCS problem

Proof.

- $\Rightarrow$ Given a solution X to the 3SAT-bis problem, we wish to show that the translated CCS problem also is satisfied. We can go through the given solution X and check that if x_i is 1 then we add it to R^A and else we add it to R^B. If a variable x_i appears negated in any clause then add it to the opposite of x_i, i.e if x_i was produced in R^A, $\overline{x_i}$ is produced in R^B. This satisfies $R^A \cup R^B = R$ since X includes all variables $x_1, ..., x_n$ and we go through each variable we conclude that each is produced in plant A or B. By the construction of x_j it is either produced in plant A or B and hence $R^A \cap R^B = \emptyset$. Solution to the 3SAT-bis problem implies each clause cannot have disjuncts either all true or all false, and hence the third criterion is satisfied since each subset cannot be either all in A (corresponding to all set to true), or all in B (corresponding to all set to false). Thus all three criterion by the translation are fullfilled given a solution of 3SAT-bis problem and the CCS problem is also satisfied.

- $\Leftarrow$ Given a solution of the constructed instance of CCS we wish to show that is also solves the original 3SAT-bis problem. We go through each in R^A and set the respective x_j to 1, and similarly for each in R^B set x_j to 0. We know that $K_i \nsubseteq R^A$ and $K_i \nsubseteq R^B$ and hence each clause cannot be all set to true or all false. By the fact that each clause cannot all be the same, we know $\exists$ at least one that equals true which makes the resulting clause true. Since this is for all clauses, the entire boolean formula must evaluate to true. Thus the solution for the instance of CCS also solved the original 3SAT-bis problem.

□

What we showed is that a specific instance of CCS where for all $i : |K_i| = 3$, the 3SAT-bis problem can be translated to CCS and that these problems are equivalent for this specific instance. Thus the CCS problem is at least as hard as 3SAT-bis and this implies CCS is NPC problem.

1.2 Question 2

If we know that $X \in P, Y \in NP, Z \in NPC$ and also that $Y \leq_p X, Y \leq_p Z$. Then the first statement says that X is at least as hard as Y so this constraints $Y \in P$. The second statement says that Z is at least as hard as Y, so Z is as hard or harder than P. We cannot draw any conclusions from this, so the conclusions are $X \in P, Y \in P, Z \in NPC$.