Algorithms Homework 1

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1 Homework 1

Part A: Combinatorics.

1. How many different ways are there to order a list containing 100 distinct elements?

Answer: The first element can be chosen from 100, the second one can be chosen from 99, third 98, etc. We then obtain: 100!

2. A palindrome is a word that can be read in both directions, such as 'madam' or 'noon'. How many 7-letter palindromes can be formed using the letters of the alphabet $\mathcal{A}$? Start by choosing the alphabet and write down the number of letters, e.g. the English alphabet has 26 letters. (Please only choose alphabets that have > 10 and < 50 letters). Then calculate the number of palindromes.

Answer: Let us assume we are using the english language that has 26 letters. We can write the word as follows: $x_1x_2x_3x_4x_5x_6x_7$ where we then require for it to be a palindrome that $x_1 = x_7$, $x_2 = x_6$, $x_3 = x_5$. Then we can choose x_i from 26 possible and we obtain: $26 \cdot 26 \cdot 26 \cdot 26 \cdot 1 \cdot 1 \cdot 1 = 26^4$ total number of palindromes. Obviously this is a theorethical value as many of these constructed words are all palindromes but they are not actual words.

3. A graph is complete if any pair of its vertices is connected by an edge. How many edges are there in a complete graph with 5 vertices? What about 25 vertices and n vertices?

Answer: We require that each node in the graph has connections to all other nodes for it to be a complete graph. Assuming we have n vertices, and we look at a specific node, then it will have edges connected to every other vertex and hence $n-1$ edges. Doing this for all edges we obtain $(n \cdot (n-1))/2$, where the division by 2 comes from the fact that each edge has a connection between two vertices and we do not wish to double count any edge. If we have 5 vertices we then have $20/2 = 10$ edges and 300 for 25 vertices following the same formula.

4. How many different ways are there to order the letters contained in the word "engineering"?

Answer: $\frac{11!}{3!3!2!2!1!} = 277200$, the word is of length 11 so if each were distinct then we would have 11!. However for example the letter 'e' has frequency 3 in this word and can be rearranged in 3! ways without changing the resulting word. Hence we divide by 3! and so we divide by factorial of the frequency of the letter to obtain the total.

Part B: Asymptotic order

1. Take the following list of functions and arrange them in ascending order of growth rate. That is, if function $f(n)$ comes before function $g(n)$ in your list, then it should be the case that $f(n)$ is $\mathcal{O}(g(n))$

$$
\begin{aligned}
f_1(n) &= 5^n \\
f_2(n) &= n^{1.5} \\
f_3(n) &= 2^{2^n} \\
f_4(n) &= n^{300} \\
f_5(n) &= 2^{n^2} \\
f_6(n) &= n(\log n)^2 \\
f_7(n) &= n \log \log n \\
f_8(n) &= n^{\log n}
\end{aligned}
$$

Answer: The order I believe to be correct is $f_7, f_6, f_2, f_4, f_8, f_1, f_5, f_3$

2. For any two functions $f(n)$ and $g(n)$ that immediately follow each other in your list, prove that $f(n)$ is $\mathcal{O}(g(n))$.

Answer: We can prove that $f_i(n)$ is $\mathcal{O}(f_j(n))$ from the list in the answer in question 1, and then by transitivity we have proven what is asked in the question.

Definition 1. *$f(n)$ is $\mathcal{O}(g(n))$ if $\exists c, n_0$, $\forall n \geq n_0 : f(n) \leq c \cdot g(n)$*

Let us start with proving $f_7(n)$ is $\mathcal{O}(f_6(n))$. $n(log(log(n)) \leq n(log(n) \leq n(log(n))^2$ and we can choose $c = 1, n_0 = 1$ to satisfy this.

$f_6(n)$ is $\mathcal{O}(f_2(n))$ because $n(log(n))^2 \leq n \cdot n^{0.4} \leq n^{1.5}$ for sufficiently large n_0 since $n^{0.4}$ grows faster than logarithm function and $c = 1$.

$f_2(n)$ is $\mathcal{O}(f_4(n))$ is trivial because the exponent is a higher degree.

$f_4(n)$ is $\mathcal{O}(f_8(n))$. We can see here that if $log(n) \geq 300$ then the relationship will hold, so we let $n_0 = \lceil log^{-1}(300) \rceil$ and then it becomes clear that $n^{300} \leq n^{log(n)}$ where $c = 1$.

$f_8(n)$ is $\mathcal{O}(f_1(n))$, we have that $n^{log(n)} = 5^{log_5(n)^{log(n)}} = 5^{log_5(n) log(n)}$ and since the logarithm to any exponent grows slower than a polynomial such as n the statement holds true.

$f_1(n)$ is $\mathcal{O}(f_5(n))$ because $5^n = 2^{log_2(5^n)} = 2^{nlog_2(5)} = 2^{kn}$, $k = log_2(5)$ and then the statement is true since $kn \leq n^2$ and hence $2^{kn} \leq 2^{n2}$.

$f_5(n)$ is $\mathcal{O}(f_3(n))$ is clear since the exponent $n^2 \leq 2^n$ for sufficiently large n_0.