Assignment 2 - Random Process Simulation

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1 Task 1: One Day Rental

John's Truck One Day Rental does business in North Carolina, South Carolina and Virginia. As with most rental agencies, customers may return the vehicle that they have rented at any of the company's franchises throughout the three state area. In order to keep track of the movement of its vehicles, the company has accumulated the following data: 50% of the trucks rented in North Carolina are returned to North Carolina locations, 30% are dropped off in Virginia, and 20% in South Carolina. Of those rented in South Carolina, 40% are returned to South Carolina, 40% are returned in North Carolina, and 20% in Virginia. Of trucks rented in Virginia, 50% are returned in Virginia, 40% in North Carolina, and 10% in South Carolina. Consider a randomly chosen vehicle from this rental agency. We want to find the proportion of time that it stays in each state in long run (assuming that each rental is one day rental)?

(a) Consider time period of 1 year (365 days). Use simulations to estimate the proportion of time spent by the vehicle in each state. Base your estimates on 1000 runs.

(b) Use Matlab to compute theoretical solution of the problem. Present necessary reasoning guaranteeing existence of the solution.

1.1 Theory

Let $\{X_n, n = 1, 2, ...\}$ *denote a Markov Chain witch is a sequence of dependent random variables. Where n is a discrete time. Let values of* X_n *to be in a set* $\{1, 2, \ldots, N\}$ *This values are called states and the set of them the states space. Assume the Markov property holds*

$$(X_{n+1} = j | X_n = i, X_{n-1} = i_1, \ldots) = P\left(X_{n+1} = j | X_n = i\right) = p_{ij} \tag{1}$$

Probabilities p_{ij} *are called the transition probabilities.*

Definition 1 (Transition Matrix)
The transition probabilities can be described in the following transition matrix:

$$\mathbb{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22} & \cdots & p_{2N} \\ \vdots & \vdots & & \vdots \\ p_{N1} & p_{N2} & \cdots & p_{NN} \end{bmatrix} \tag{2}$$

Let's introduce two important Markov chains.

Definition 2 (Irreducible Markov Chain)
The Markov Chain $X_n, n = 0, 1, 2, \dots$ with space state $\{1, 2, \dots, N\}$ is called irreducible if each state $j, j \in \{1, 2, \dots, N\}$ can be reached in a from any other state.

Definition 3 (Aperiodic Markov Chain)
The Markov Chain $X_n, n = 0, 1, 2, \dots$ with space state $\{1, 2, \dots, N\}$ is aperiodic if for all states $j, j \in \{1, 2, \dots, N\}$ there exists n such that $P(X_n = j) > 0$ and $P(X_{n+1} = j | X_0 = j) > 0$.

Theorem 1 *Let $X_n, n = 1, 2, \dots$ with space state $\{1, 2, \dots, N\}$ be an irreducible and aperiodic Markov Chain. Then there exist an asymptotic distribution $\pi = (\pi_1, \pi_2, \dots, \pi_N)$ such that $\lim_{n\to\infty} P(X_n = j | X_0 = i) = \pi_j$ independently of i. The unique distribution π is a solution of equation $\pi\mathbb{P} = \pi$. With additional condition $\sum_{i=1}^{N} \pi_i = 1$.*

To generate uniform discrete random numbers over $\{1, 2...., n\}$ Let

$$X = \lceil n * U \rceil \tag{3}$$

Where $U \sim \mathscr{U}(0, 1)$.

1.2 Method and Solution

The task description describes a Markov Chain denoted as $\{X_n, n = 1, 2, 3\}$ The text can be interpret and divided into three business locations in three different states.

State 1 = North Carolina
State 2 = Virginia
State 3 = South Carolina

With the given probabilities a transition matrix can be created as in definition 1. Denoted as

$$\mathbb{P} = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.5 & 0.1 \\ 0.4 & 0.2 & 0.4 \end{bmatrix} \tag{4}$$

The start location in the Markov chain will be randomly chosen and the result will work as a seed for the simulation. By using inverse transform method the different steps in the Markov chain can easily be simulated. This is described below under **Algorithm** 1. Every step will be saved to a vector. Finally the mean will be returned as seen in **Algorithm** 2. This will be simulated a thousand times to approximate the time the vehicle have spent in each state as seen in **Algorithm** 3.

To find the proportion of time the vehicle been in each state theoretically we need to find a stationary or asymptotic distribution. To do that the Markov Chain needs to meet two conditions, irreducibility and aperiodicity.

With use of the definition for irreducibility. Theoretically we found that every state can be reached from the other states as shown below.

$1 \to 2 \to 3 \to 2 \to 1$

With use of the definition for aperodic Markov Chain. Theoretically we show that every state can reach it self in n and $n+1$ steps as shown below.

State 1: $1\to 1(n=1)$ and $1\to 1 \to 1(n+1=2)$
State 2: $2\to 2(n=1)$ and $2\to 2 \to 2(n+1=2)$
State 3: $3\to 3(n=1)$ and $3\to 3 \to 3(n+1=2)$

Hence a asymptotic distribution exists. To find it we use Theorem 2 and solve $\pi\mathbb{P} = \pi$ with additional equation $\pi_1+\pi_2+\pi_3 = 1$, which will guarantee uniqueness of the solution. In our case, we have

$$[\pi_1, \pi_2, \pi_3] \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.5 & 0.1 \\ 0.4 & 0.2 & 0.4 \end{bmatrix} = [\pi_1, \pi_2, \pi_3] \tag{5}$$

which corresponds to

$$\begin{cases} 0.5\pi_1 + 0.3\pi_2 + 0.2\pi_3 = \pi_1 \\ 0.4\pi_1 + 0.5\pi_2 + 0.1\pi_3 = \pi_2 \\ 0.4\pi_1 + 0.2\pi_2 + 0.4\pi_3 = \pi_3 \end{cases} \tag{6}$$

We substitute the third row with $\pi_1 + \pi_2 + \pi_3 = 1$. To obtain

$$\begin{cases} 0.5\pi_1 + 0.3\pi_2 + 0.2\pi_3 = \pi_2 \\ 0.4\pi_1 + 0.5\pi_2 + 0.1\pi_3 = \pi_3 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases} \tag{7}$$

Which is equal to

$$\begin{cases} -0.5\pi_1 + 0.3\pi_2 + 0.2\pi_3 = 0 \\ 0.4\pi_1 - 0.5\pi_2 + 0.1\pi_3 = 0 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases} \tag{8}$$

Now we can create a matrix A and π

$$A = \begin{bmatrix} -0.5 & 0.3 & 0.2 \\ 0.4 & -0.5 & 0.1 \\ 1 & 1 & 1 \end{bmatrix}, \pi = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} \tag{9}$$

The right sides of the the equation we have a vector denoted as,

$$V = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{10}$$

We know from basic linear algebra that

$$A\pi = V. \tag{11}$$

Hence to find π we take inverse of A on both sides

$$A^{-1}A\pi = A^{-1}V. \tag{12}$$

Hence

$$\pi = A^{-1}V. \tag{13}$$

Then using Matlab for calculating the result.

1.2.1 Algorithms

Algorithm 1 Inverse transform method

1: F = cumulative sum of $\mathbb{P}(X_{i-1},(:))$
2: Generate $U \sim \mathscr{U}(0,1)$
3: Find first index where U<F and store in Y
4: Return Y found

Algorithm 2 Markov chain simulation

1: Create Transition Matrix $\mathbb{P}$ and set $T = 365$.
2: Simulate a random number from $U \sim \mathscr{U}(0,1)$
3: Set $X_1 = \lceil 3 * U \rceil$
4: For $i = 2, 3, ..., T-1, T$
5: Simulate step X_i by using Algorithm 1 with parameters $\mathbb{P}(X_{i-1},(:))$
6: Take mean of $X = j$, $j = 1, 2, 3$
7: Return result

Algorithm 3 Finding approximation

1: Initiate i = 1
2: Use algorithm 3
3: Save result to a vector
4: If i $\leq$ 1000, set i = i + 1 and return to 2 else go to 5
5: Return mean of the vector.

1.3 Results

From our simulation we obtained that the vehicle spent 44.29 % of it's time in North Carolina, 35.23 % of it's time in Virginia and 20.48 % of it's time in South Carolina.

Solving the task theoretically we obtained

$$\pi = \begin{bmatrix} 0.4444 & 0.3492 & 0.2063 \end{bmatrix} \tag{14}$$

Hence the result gives us that the vehicle spent spent 44.44 % of it's time in North Carolina, 34.92 % of it's time in Virginia and 20.63 % of it's time in South Carolina.

1.4 Conclusions

We see when we solve the task by simulation the result gets very close to the exact theoretically result. However this is an approximation and the result will vary every time we simulate it. What we can say for sure is that the car will spend exactly the the time we found out in the theoretically result in each business location if the days rented would go towards infinity. To improve the simulation we could have increased the number of simulations to get a more exact result.

2 Task 2: Overture to asset pricing

Consider an asset which price evolution can be modelled with a Geometric Brownian motion with the risk free rent r = 0.04 and volatility $\sigma = 0.1$. Assume that at time t = 0 the price of the asset is $S_0 = 100$ SEK. Consider the following strategy:

- You buy one of this assets and monitor it's price development during time $[0, 10]$.
- If at any time of the interval (0,10) the price goes above 150, you immediately sell the asset and earn 50.
- If it doesn't happen you sell the asset at time t = 10, regardless of it's price at this time (even if it means that you loose money). Use the simulation to determine the probability that you will sell the stock before time t = 10. Further, use simulation to determine the expected profit of that strategy. In both cases provide point estimates and the 95% confidence intervals. Use $\Delta t = 0.001$ in simulation of underlying Brownian Motion.

2.1 Theory

Definition 1 Brownian motion
A random process $\{W(t), t \geq 0\}$ is called a Brownian motion with parameter σ^2 if the following is satisfied:

1. $W(0) = 0$
2. $W(t)$ *has independent increments*
3. $W(t)$ *has stationary increments, such that, for* $0 \leq s \leq t$

$$W(t) - W(s) \sim \mathcal{N}\left(0, \sigma^2(t-s)\right) \tag{15}$$

A Brownian motion is a process where both the time and domain is continuous. For a simulation of the price of a asset it is better to use the Geometric Brownian motion since it will only simulate positive values.

Definition 2 (Geometric Brownian Motion)
The dynamic of an asset price with initial price S_0, on a market with a risk free rent r can be described by

$$S(t) = S_0 e^{\left(r-\sigma^2/2\right)t+\sigma W(t)}, \quad t \geq 0 \tag{16}$$

where W(t) is the standard Brownian motion and σ is the asset specific parameter called volatility. Process $\{S(t), t \geq 0\}$ is called a Geometric Brownian motion.

2.2 Method and Solution

We know how to simulate Brownian motion with **Algorithm** 4 which satisfies all conditions required by **Definition 1**. With the simulation of the standard Brownian motion which is Brownian motion with $\sigma = 1$, we can use it in the simulation Geometric Brownian Motion (16) and simulate a possible development of the asset price. The transformation of Standard Brownian motion into Geometric Brownian motion is detailed in **Algorithm** 5.

By simulating these developments of asset prices several times, they can then be used to simulate probabilities of asset price exceeding 150, and expected profit. This is concluded in **Algorithm** 6.

We simulate N independent increments on the interval [0, 10] of the standard Brownian Motion

$$X_i \sim \mathcal{N}(0, \Delta t) \tag{17}$$

W(0) is set equal to 0 and our Brownian motion W(t) is

$$W(k\Delta t) = \sum_{i=1}^{k} X_i. \tag{18}$$

The simulated Brownian motion W is then used in Definition 2 to transform it into a Geometric Brownian Motion S(t) using **Algorithm** 5 with the given risk free rent, r$=$ 0.04, volatility $\sigma = 0.1$ and $\Delta t = 0.001$ since it is impossible to simulate the exact simulation with continuous time.

$$S(t) = 100e^{\left(0.04-0.1^2/2\right)t+0.1W(t)}, \quad t \geq 0. \tag{19}$$

A Geometric Brownian Motion is said to have a lognormal distribution.

$$S(t) \sim log\mathcal{N}(ln(S_o) + (r - \sigma^2/2)t,\ \sigma^2 t) \tag{20}$$

Which in our case becomes

$$S(t) \sim log\mathcal{N}(ln(100) + (0.04 - 0.1^2/2)t,\ 0.1^2 t) \tag{21}$$

To find out how often the asset will exceed 150 **Algorithm** 6 is used where a indicator function of the simulated Geometric Brownian Motions is created. The mean of the function is calculated which gives us the probability that the asset exceeds 150.

To find out the expected profit using this strategy the initial price is subtracted from the indicator function and then the mean of it is calculated. Corresponding 95 % confidence interval for the two mean values is also computed.

2.2.1 Algorithms

Algorithm 4 Simulation of Brownian Motion

Input T, σ_2, Δt.
Initialize $N = \frac{T}{\Delta t}$, vector $t = [0, \Delta t, 2\Delta t, ..., T]$.
Simulate N random numbers $\sim N(0,1)$ and store in Z.
Set $X = Z \cdot \sigma_2 \cdot \Delta t$.
Initialize $W(0) = 0$, and store cumulative sum of X into W.
Return vector $\mathbf{W}$.

Algorithm 5 Simulation of Geometric Brownian Motion

Input initial capital S_0, risk free rent r, volatility σ_1, Δt and T.
Initialize $t = [0, \Delta t, 2\Delta t, ..., T]$.
Simulate Standard Brownian Motion $\mathbf{W}$ from Algorithm 4 with $\sigma_2 = 1$.
Transform Standard Brownian Motion $\mathbf{W}$ to $\mathbf{S} = S_0 e^{(r-\sigma_1^2/2)t+\sigma_1 \mathbf{W}}$.
Return vector $\mathbf{S}$.

Algorithm 6 Simulation of expected profit and probability of asset over $150

Initialize $S_0 = 100$, $T = 10$, $\sigma_1 = 0.1$, $\Delta t = 0.001$, $r = 0.04$ and sell = [ ].
For $i = 1, 2, ..., N-1, N$:
  Simulate Geometric Brownian Motion in vector $\mathbf{S}$ from **Algorithm** 5.
  If $S \geq (S_0 + 50)$ sell stock and store stock price in vector sell.
  Else sell at $T = 10$ and store stock price in vector sell.
Calculate indicator function $I = \text{sell} \geq (S_0 + 50)$.
Calculate $Y = (\text{sell} - S_0)$.
Return mean of I and it's corresponding 95% confidence interval.
Return mean of Y and it's corresponding 95% confidence interval.

2.3 Results

The probability in this simulation that the asset exceeds 150 and is sold before the end of time is 0.5724 with the 95 % confidence interval is [0.5627, 0.5821].
The expected profit by using this strategy is $32.9562 with the 95 % confidence interval [32.4884, 33.4240]. The simulations can be seen in the graph below.

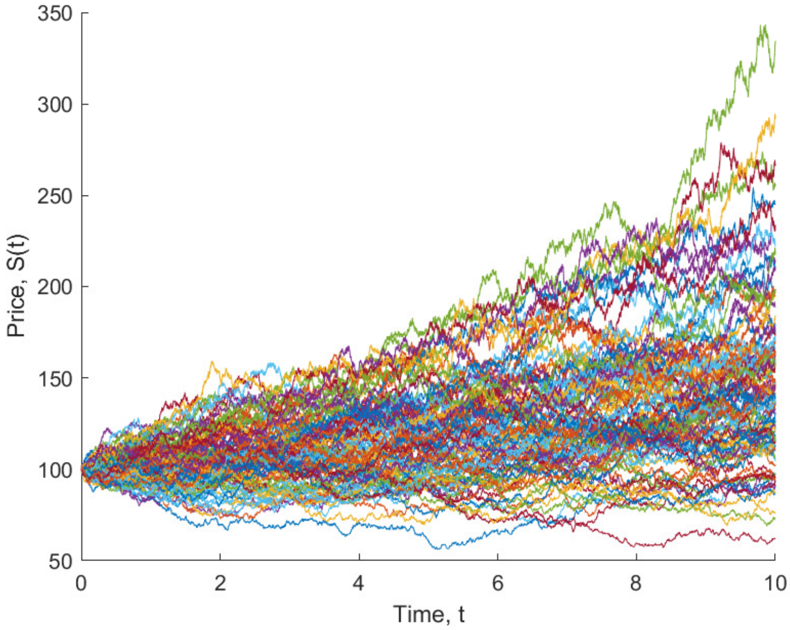

Figure 1 – Plot illustrating simulations of Geometric Brownian Motion.

2.4 Conclusions

From these simulations we can see that the given strategy is profitable. We also see that by increasing the sell limit for when the asset is sold, we can see that the profit increases. If we don't use a limit at all, and always sell by time 10, one can observe that the profit will in general be greater.

From equation (16) we can see that depending on the values of r and σ, the process over time will on average either increase or decrease. In this case the volatility is greater than the risk free rent, and therefore the simulations will over time in most cases increase. From that, we can conclude that higher profit will always be obtained the longer we wait until we sell. Optionally, to earn the most profit, we would wait infinitely long time, which in practice of course is impossible.

3 Task 3: Lecture starts in 5

Consider a non-homogeneous Poisson process $\Big\{N(t), t \geq 0\Big\}$ with intensity function

$$\lambda(t) = \begin{cases} 3 + \frac{t(10-t)}{10} & t \in [0, 10] \\ 0 & t > 10 \end{cases} \tag{22}$$

(a) Use superposition property to decompose the process into a homogeneous and non-homogeneous Poisson process.

(b) Simulate the homogeneous component of the process.

(c) Simulate the non-homogeneous component of the process using thinning algorithm.

(d) Use the results from b) and c) to simulate $\{N(t), t \in [0, 10]\}$.

(e) Use simulations to estimate $E(N(10))$. Does your estimate coincide with theoretical value?

3.1 Theory

Definition 1 (Homogeneous Poisson Process)
Suppose that events are occurring at random time points and let N(t) denote the number of events in interval [0, t). These events create a ***homogeneous Poisson process*** *with rate* $\lambda > 0$ if

1. $N(0) = 0$
2. *The number of events occurring in disjoint intervals are independent (independent increments)*
3. *The distribution of number of events that occurs in a given interval depends only on length of the interval and not on its location (stationary increments)*
4. *On short intervals, the probability of events is proportional to the interval length.*

$$\lim_{h \to 0} \frac{P(N(h) = 1)}{h} = \lambda \quad [\text{For } h \approx 0, P(N(h) = 1) \approx \lambda h]$$

5. *No two events at the same time*

$$\lim_{h \to 0} \frac{P(N(h) \geq 1)}{h} = 0 \quad [\text{For } h \approx 0, P(N(h) \geq 1) \approx 0]$$

Theorem 1 (Property of homogeneous of Poisson process)
Consider a Poisson process $\{N(t),\, t \geq 0\}$ *with intensity* λ. *Then:*

1. Number of occurrences in time interval of length $t > 0$ is Poisson distributed with parameter λt, i.e

$$N(s+t) - N(s) \sim Po(\lambda t)\ \forall_{s>0} \tag{23}$$

2. The inter arrival times are independent and identically exponentially distributed with parameter λ.

Definition 2 (Non-homogeneous Poisson process)
Suppose that events are occurring at random time points and let $N(t)$ denote the number of events in interval $[0, t)$. These events create a ***non-homogeneous Poisson process with*** *intensity $\lambda(t) \geq 0$ if*

1. $N(0) = 0$

2. *The number of events occurring in disjoint intervals are independent (independent increments)*

3. $$\lim_{h \to 0} \frac{P(N(t+h) - N(t) = 1}{h} = \lambda(t) \tag{24}$$

4. $$\lim_{h \to 0} \frac{P(N(t+h) - N(t) > 1}{h} = 0 \tag{25}$$

For a non-homogeneous Poisson process $\{N(t), t \geq 0\}$ with intensity $\lambda(t)$ we define $\Lambda(t)$ by

$$\Lambda(t) = \int_0^t \lambda(s) ds \tag{26}$$

Theorem 2 (Property of Non-homogeneous Poisson process)

1. *Let $\{N(t), t \geq 0\}$ be a non-homogeneous Poisson process with intensity $\lambda(t)$. Number of occurrences in time interval $(s, t), t > s$ if Poisson distributed with parameter $\Lambda(t) - \Lambda(s)$ i.e*

$$N(t) - N(s) \sim Po(\Lambda(t) - \Lambda(s)) \tag{27}$$

2. *Suppose that events are occurring according to a homogeneous Poisson process having rate λ, and probability that event at time t is counted is given by p(t) (independently from the process). Then the process of counted events is a non-homogeneous Poisson process with intensity function $\lambda(t) = p(t)\lambda$*

3. *If we have two indepedent non-homogeneous Poission processes with intensity $\lambda_1(t)$ and $\lambda_2(t)$, then the superposition of that process is a non-homogeneous Poission process with intensity $\lambda_1(t) + \lambda_2(t)$*

Theorem 3 (Superposition property)
Let $\{N_1(t),\ t \geq 0\}$ and $\{N_2(t),\ t \geq 0\}$ be two independent Poisson processes with intensities $\lambda_1(t)$ and $\lambda_2(t)$ respectively. Then $N(t) = N_1(t) + N_2(t)$ is a Poisson process with intensity $\lambda = \lambda_1 + \lambda_2$.

3.2 Method and Solution

3,a)

Given from problem description in (Eq. 22) we know that in interval $t \in [0, 10]$ that

$$\lambda(t) = 3 + \frac{t(10-t)}{10}. \tag{28}$$

we know from Theorem 2, statement 3 that

$$\lambda_1 = 3 \tag{29}$$
$$\lambda_2 = \frac{t(10-t)}{10} \tag{30}$$
$$\lambda(t) = \lambda_1(t) + \lambda_2(t) = 3 + \frac{t(10-t)}{10}. \tag{31}$$

We let

$$\{N_1(t), t \geq 0\} \text{ Denote the number of events in interval } [0, t) \text{ with rate } \lambda_1. \tag{32}$$
$$\{N_2(t), t \geq 0\} \text{ Denote the number of events in interval } [0, t) \text{ with rate } \lambda_2. \tag{33}$$

Now using Theorem 3, the superposition property tells us that we can write

$$N(t) = N_1(t) + N_2(t), \tag{34}$$

with intensity λ_1 and λ_2. We also note here that λ_1 is a constant. $N_1(t)$ will therefore describe the number of events in $[0, t)$ for a homogeneous Poisson process and $N_2(t)$ a non-homogeneous Poisson process.

3,b)

We note that if we wish to simulate $N_1(t)$ then we know that $\lambda_1 = 3$ and that we wish to simulate from $[0, T]$ where $T = 10$. To simulate from this homogeneous Poisson process then we can use **Algorithm 7**.

3,c)

The non-homogeneous component we have concluded is $N_2(t)$ which we can simulate using the thinning algorithm which is written in **Algorithm** 8. To be able to simulate using this algorithm we need to find λ_{max}. In this case we wish to simulate from $[0, T]$ where $T = 10$.

To find λ_{max} we need to find the maximum of (Eq. 30). We recognize that this is a relatively simple function and by finding it's derivative equal to 0 and with the second derivative test we are able to confirm it is a maximum point.

$$\lambda_2'(t) = \frac{10 - 2t}{10} \tag{35}$$

$$\lambda_2'(t) = 0 \implies t = 5 \tag{36}$$

$$\lambda_2''(t) = \frac{-1}{5} \implies \lambda_2'(5) \text{ is a maximum point by second derivative test.} \tag{37}$$

Since we now have found the time point for where $\lambda_2(t)$ has a maximum point to find the value we need to evaluate it at $t = 5$

$$\lambda_{max} = \lambda_2(5) = \frac{50 - 25}{10} = 2.5. \tag{38}$$

3,d)

We note from our previous answer in *3,a)* that the total amount of arrivals will be

$$N(t) = N_1(t) + N_2(t) \tag{39}$$

since we know how to simulate both $N_1(t)$ as well as $N_2(t)$ the simulation of $N(t)$ is trivial and is only a addition between the simulation of $N_1(t)$ and $N_2(t)$.

3,e)

Making several simulations of the one in *3,a)* we know that a good estimate of the expected value of $N(t)$ is the average of all simulations that estimate $N(t)$. It is possible to obtain the theoretical expected value which we can use to compare with our estimate that was generated by simulations. Starting with the theoretical we find

$$E(N(10)) = [\text{By (Eq. 34)}] = E(N_1(10) + N_2(10)) = E(N_1(10)) + E(N_2(10)). \tag{40}$$

By Theorem 1, we know that $N_1(10) \sim Po(30)$ and also by Theorem 2, we know $N_2(t) \sim Po(\Lambda(10))$. From definition of homogeneous Poisson process

$$\Lambda(t) = \int_0^t \lambda(s)ds = \int_0^t \frac{10s - s^2}{10} ds = \frac{t^2}{2} - \frac{t^3}{30}. \tag{41}$$

Hence,

$$\Lambda(10) = 16 + \frac{1}{3}. \tag{42}$$

By properties of Poisson processes we find that (Eq. 40) gives that

$$E(N(10)) = 30 + 16 + \frac{1}{3} = 46 + \frac{1}{3}. \tag{43}$$

3.2.1 Algorithms

Algorithm 7 Simulating from Homogeneous Poisson process

1: Set t = 0, I = 0
2: Generate $U \sim \mathscr{U}(0,1)$
3: Set $X = \frac{-1}{\lambda} log(U)$
4: Set $t = t + X$. If $t > T$ STOP
5: $I = I + 1$, $S(I) = t$.
6: Go to step 2.

Algorithm 8 Thinning algorithm

1: Set t = 0, I = 0
2: Generate $U \sim \mathscr{U}(0,1)$
3: Set $X = \frac{-1}{\lambda_{max}} log(U)$
4: Set $t = t + X$. If $t > T$ STOP
5: Generate $U \sim \mathscr{U}(0,1)$
6: If $U \leq \frac{\lambda(t)}{\lambda_{max}}$, set $I = I + 1, S(I) = t$.
7: Go to step 2.

3.3 Results

Simulating $N(t)$ we obtain an estimate of the expected value of $N(t)$ to be

$$\hat{E}(N(10)) \approx 46.5825, \tag{44}$$

this is based on running the simulation of $N(t)$ as described in *3,d)* a total of $n = 10000$ times. Every approximation of $\hat{E}(N(10))$ will recreate different values from the previous but always results in values similar to (Eq. 44). By letting the amount of simulations n for $N(t)$ increase to be much larger than 10000, the estimated expected value will become more and more similar to the theoretical. If one wants to have a closer estimate then it is a simple solution to increase the amount of simulations but requires longer time to simulate.

3.4 Conclusions

We conclude that estimating $N(t)$ by separating into a homogeneous component and a non-homogeneous part where both are simulated separately works as it is results in close values to the theoretical. In this case it is computationally more expensive to simulate them separately as one could easily include the homogeneous part of $\lambda_1 = 3$ and include it in the non-homogeneous simulation. However there are many properties of Poisson properties that are displayed when separating them and is the reason why this method was chosen for this project.