Differentiation

Definition:

The derivative of the function $f(x)$ with respect to the variable x is :

$$\frac{df}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.

Example :

Find the first derivative of the function $f(x) = x^2$ by using the definition of differentiation.

Solution

$\because f(x) = x^2 \quad , \quad f(x+h) = (x+h)^2$
$= x^2 + 2hx + h^2$

$$\therefore f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2hx + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = \lim_{h \to 0} (2x+h)$$

$$= 2x$$

Review on Basic differential rules

① $y = c$ (Constant) $\rightarrow y' = \frac{dy}{dx} = 0$

② $y = x^n \longrightarrow y' = n\,x^{n-1}$

③ $y = c\,f(x) \longrightarrow y' = c\,f'(x) = c\,\frac{df}{dx}$

where C is a constant

④ $y = f(x) \pm g(x) \longrightarrow y' = f'(x) \pm g'(x)$

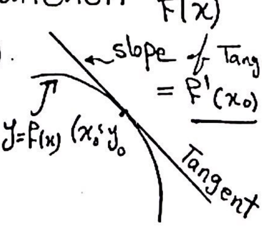
⑤ $y = [f(x)]^n \longrightarrow y' = n\,[f(x)]^{n-1}\,f'(x)$

⑥ $y = \sqrt{f(x)} \longrightarrow y' = \dfrac{f'(x)}{2\sqrt{f(x)}}$

⑦ $y = f(x).g(x) \longrightarrow y' = f(x).g'(x) + g(x)\,f'(x)$

⑧ $y = \dfrac{f(x)}{g(x)} \longrightarrow y' = \dfrac{g(x)\,f'(x) - f(x)\,g'(x)}{g^2(x)}$

- Note: The derivative of the function $f(x)$ at a point x_o denoted by $f'(x_o)$ is the slope of the tangent to a curve of $f(x)$ at x_o



Examples:

Find $\frac{dy}{dx}$ for each of the following:

① $y = x^5 - \frac{4}{x^3} + \sqrt[4]{x^3} + 5x - 16$

$y = x^5 - 4x^{-3} + x^{3/4} + 5x - 16$

$y' = 5x^4 + 12x^{-4} + \frac{3}{4}x^{-1/4} + 5$

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② $y = \frac{6}{\sqrt[3]{(x^2-5)^2}}$

$y = 6(x^2-5)^{-2/3}$

$y' = 6\left(\frac{-2}{3}\right)(x^2-5)^{-5/3}(2x).$

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③ $y = (x^3-1)^5(2+7x^{-4})^7 + \left(\frac{x^2-3}{x^{-4}+2}\right)^{4/3}$

$y' = (x^3-1)^5 \cdot 7(2+7x^{-4})^6 \cdot (-28x^{-5})$

$+ (2+7x^{-4})^7 \cdot 5(x^3-1)^4 \cdot (3x^2)$

$+ \frac{4}{3}\left(\frac{x^2-3}{x^{-4}+2}\right)^{1/3} \cdot \left(\frac{(x^{-4}+2)(2x) - (x^2-3)(-4x^{-5})}{(x^{-4}+2)^2}\right.$

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④ If $y = (x + \sqrt{x^2-1})^n$

Prove that $y' = \dfrac{ny}{\sqrt{x^2-1}}$

— Proof —

$$\frac{\sqrt{x^2-1}}{\sqrt{x^2-1}}$$

$$y' = n(x+\sqrt{x^2-1})^{n-1}\cdot\left(1 + \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2-1}}\right)$$

$$= n(x+\sqrt{x^2-1})^{n-1}\left(\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right)$$

$$= n(x+\sqrt{x^2-1})^{n-1}\cdot\frac{(x+\sqrt{x^2-1})^1}{\sqrt{x^2-1}}$$

$$= \frac{n(x+\sqrt{x^2-1})^n}{\sqrt{x^2-1}}$$

$$= \frac{ny}{\sqrt{x^2-1}}$$

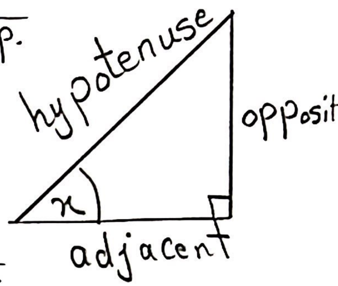
H.W

⑤ Find the slope of the tangent for the Curve $y = \dfrac{x}{x-2}$ at the point $(3,3)$

Trigonometric Functions

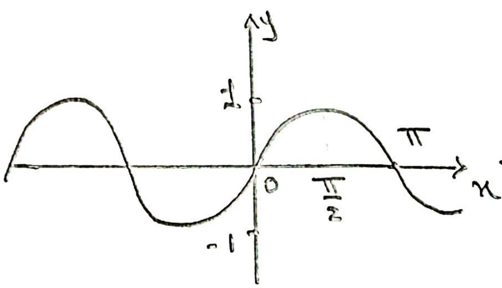
$\sin x = \dfrac{opp.}{hyp.}$, $\operatorname{Cosec} x = \dfrac{1}{\sin x} = \dfrac{hyp.}{opp.}$

$\cos x = \dfrac{adj}{hyp.}$, $\sec x = \dfrac{1}{\cos x} = \dfrac{hyp.}{adj.}$

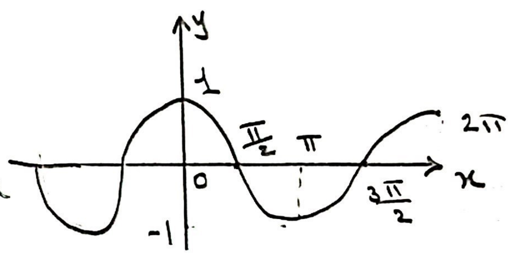


$\tan x = \dfrac{opp.}{adj.}$, $\cot x = \dfrac{1}{\tan x} = \dfrac{adj.}{opp.}$

$\tan x = \dfrac{\sin x}{\cos x}$, $\cot x = \dfrac{\cos x}{\sin x}$



$y = \sin x$
(odd function)



$y = \cos x$
(even function)

Identities for Trigonometric Functions

① $\sin^2 x + \cos^2 x = 1$

② $1 + \tan^2 x = \sec^2 x$

③ $1 + \cot^2 x = \operatorname{Cosec}^2 x$

(4) $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

(5) $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

(6) $\sin(2x) = 2 \sin x \cos x$

(7) $\cos(2x) = \cos^2 x - \sin^2 x$

(8) $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$

(9) $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$

Note

$$\sin^2 x \neq \sin x^2$$

$$\sin^2 x = (\sin x)^2$$

Derivative of Trigonometric Functions

Let $u = u(x)$, (u is a function of x)

① $y = \sin(u) \longrightarrow y' = \cos(u) \cdot u'$

② $y = \cos(u) \longrightarrow y' = -\sin(u) \cdot u'$

③ $y = \tan(u) \longrightarrow y' = \sec^2(u) \cdot u'$

④ $y = \cot(u) \longrightarrow y' = -\csc^2(u) \cdot u'$

⑤ $y = \sec(u) \longrightarrow y' = \sec(u)\tan(u) \cdot u'$

⑥ $y = \csc(u) \longrightarrow y' = -\csc(u)\cot(u) \cdot u'$

Examples

Find y' for the following

① $y = \tan^2(\cos^3 x^2) + \sqrt{x\sec x + \sin\sqrt{x}}$

$$y' = 2\tan(\cos^3 x^2) \cdot \sec^2(\cos^3 x^2) \cdot (3\cos^2 x^2) \cdot (-\sin x^2) \cdot (2x)$$

$$+ \frac{x\sec x\tan x + \sec x \cdot (1) + \cos\sqrt{x} \cdot \left(\frac{1}{2\sqrt{x}}\right)}{2\sqrt{x\sec x + \sin\sqrt{x}}}$$

$$\to (\quad)^{1/2} \qquad \to \left(\sec\left(\sqrt{\cos x}\right)\right)^3$$

(2) $$y = \sqrt{\frac{1-\sin 2x}{1+\cot x}} + \sec^3\left(\sqrt{\cos x}\right)$$

$$y' = \frac{1}{2}\left(\frac{1-\sin 2x}{1+\cot x}\right)^{-1/2} \cdot \left(\frac{(1+\cot x)(-2\cos 2x) - (1-\sin 2x)(-\csc^2 x)}{(1+\cot x)^2}\right)$$

$$+\ 3\left(\sec\sqrt{\cos x}\right)^2 \cdot \sec\sqrt{\cos x}\ \tan\sqrt{\cos x} \cdot \frac{-\sin x}{2\sqrt{\cos x}}$$

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(3) $$y = \sqrt{x^5 - \tan^2 x + x\cos^2 x}$$

$$y' = \frac{5x^4 - 2\tan x \cdot \sec^2 x + x \cdot (2\cos x)(-\sin x) + \cos^2 x \cdot (1)}{2\sqrt{x^5 - \tan^2 x + x\cos^2 x}}$$
