GUC
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CSEN1022: Machine Learning

Probability Theory Review

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Some parts are adapted from Prof. Rong Jin's slides

Definition of Probability

- ***Experiment***: toss a coin twice

- ***Sample space***: possible outcomes of an experiment
  - S = {HH, HT, TH, TT}

- ***Event***: a subset of possible outcomes
  - A={HH}, B={HT, TH}

- ***Probability of an event***: a number assigned to an event Pr(A)
  - Axiom 1: $\Pr(A) \geq 0$
  - Axiom 2: $\Pr(S) = 1$
  - Axiom 3: For every sequence of disjoint events

$$\Pr(\bigcup_i A_i) = \sum_i \Pr(A_i)$$

  - Example: Pr(A) = n(A)/N: frequentist statistics

    (If we repeat an experiment N times, and denote by n(A) the number of times we observe A, then Pr(A) = n(A)/N)

Joint Probability

- For events A and B, **joint probability** Pr(AB) (or Pr(A, B)) stands for the probability that both events happen

- Example: What is the probability that the first toss is H and the second toss is H?

  Pr(1^{st} is H and 2^{nd} is H)=Pr(1^{st} is H)Pr(2^{nd} is H)=0.5x0.5=0.25

- Example: A={HH}, B={HT, TH}, what is the joint probability Pr(AB)?

  Answer: 0

Independence

- Two events ***A and B are independent*** if

  $\Pr(AB) = \Pr(A)\Pr(B)$

- A set of events $\{A_i\}$ is independent if

$$\Pr(\bigcap_i A_i) = \prod_i \Pr(A_i)$$

- Example: Drug test

	Women	Men
Success	200	1800
Failure	1800	200

A = {Patient is a Woman}

B = {Drug fails}

Is event A independent of event B?

- Pr(AB)=1800/4000, Pr(A)=2000/4000, Pr(B)=2000/4000

  ∵ Pr(AB) ≠ Pr(A)Pr(B) → A and B are dependent

Conditioning

- If A and B are events with Pr(A) > 0, the ***conditional probability of B given A*** is

$$\Pr(B \mid A) = \frac{\Pr(AB)}{\Pr(A)}$$

- Example: Drug test

	Women	Men
Success	200	1800
Failure	1800	200

A = {Patient is a Woman}

B = {Drug fails}

Pr(B|A) = ?

Pr(A|B) = ?

- Pr(B|A)=Pr(AB)/Pr(A)=(1800/4000)/(2000/4000)=0.9
- Pr(A|B)=Pr(AB)/Pr(B)=(1800/4000)/(2000/4000)=0.9

- Given A is independent from B, what is the relationship between Pr(A|B) and Pr(A)?

  Pr(A|B) = Pr(A)

Bayes' Rule

- Given two events A and B and suppose that Pr(A) > 0, then

$$\Pr(B \mid A) = \frac{\Pr(AB)}{\Pr(A)} = \frac{\Pr(A \mid B)\Pr(B)}{\Pr(A)}$$

- Example:

$\Pr(R) = 0.8$

$\Pr(W\|R)$	R	$\neg R$
W	0.7	0.4
$\neg W$	0.3	0.6

R: It is a rainy day

W: The grass is wet

$\Pr(R|W)$ = ?

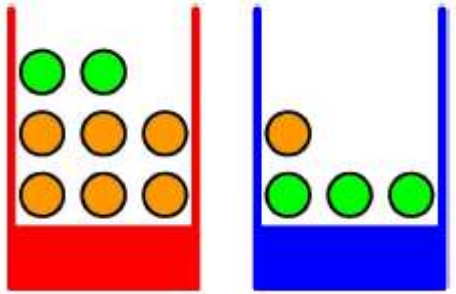
$$\Pr(R \mid W) = \frac{\Pr(W \mid R)\Pr(R)}{\Pr(W)} = \frac{0.7 \times 0.8}{\Pr(W)}$$

$$\Pr(W) = \Pr(WR) + \Pr(W\neg R) = \Pr(W \mid R)\Pr(R) + \Pr(W \mid \neg R)\Pr(\neg R)$$

$$= 0.7 \times 0.8 + 0.4 \times 0.2 = 0.64$$

Bayes' Rule

- Example:
  Two boxes: one red and one blue
  Two kinds of fruit: apples and oranges
  Task: randomly pick one of the boxes
  and then select a fruit



Let B represent the box ($B = r$ or $B = b$)
Let F represent the fruit ($F = a$ or $F = o$)

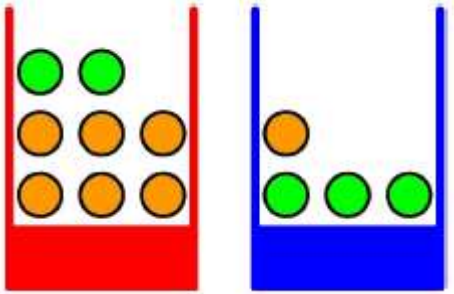
Let the probability of picking the red box be 40% and the blue box be 60%

$$p(B = r) = 4/10$$
$$p(B = b) = 6/10$$

Bayes' Rule

- Conditional Probabilities

$$
\begin{aligned}
p(F = a|B = r) &= 1/4 \\
p(F = o|B = r) &= 3/4 \\
p(F = a|B = b) &= 3/4 \\
p(F = o|B = b) &= 1/4.
\end{aligned}
$$



- What is the probability of picking an apple? ($p(F = a)$)

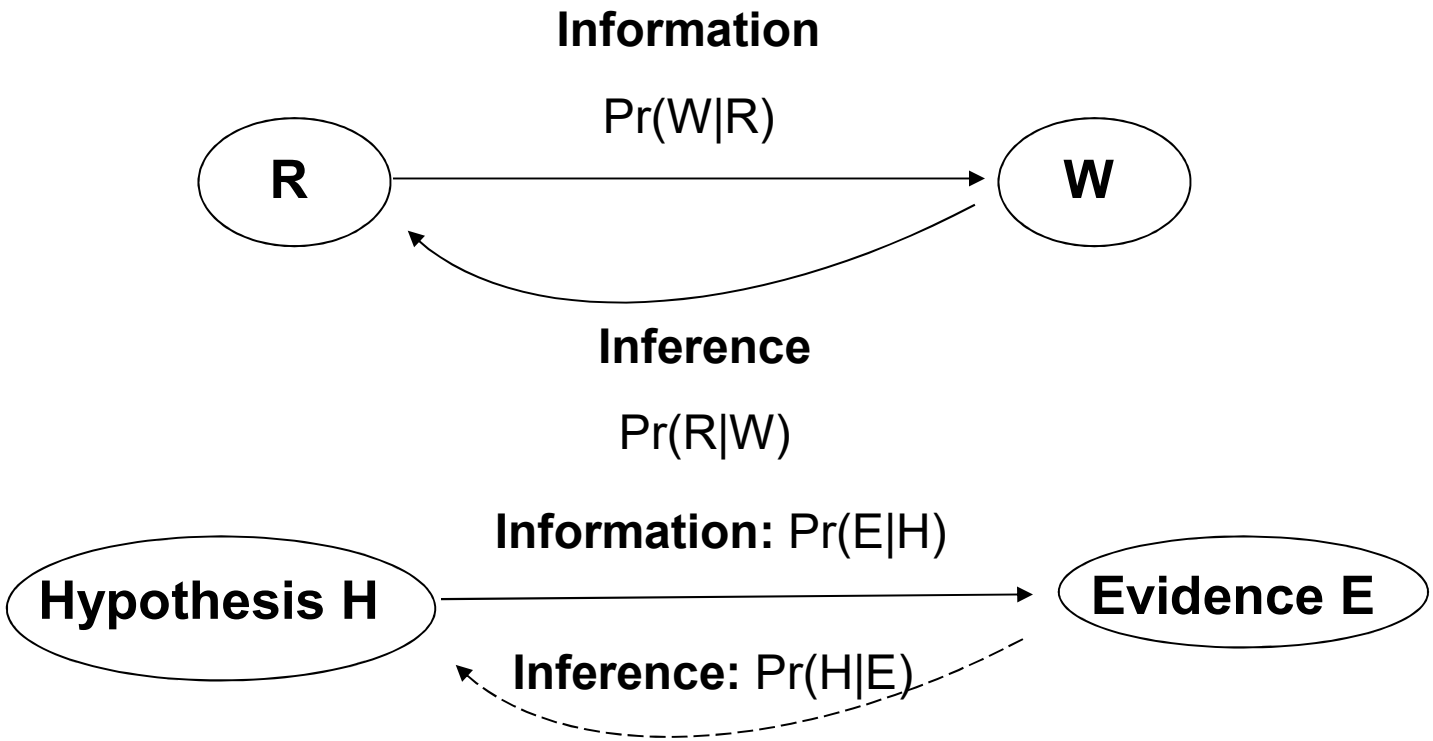
$$
\begin{aligned}
p(F = a) &= p(F = a|B = r)p(B = r) + p(F = a|B = b)p(B = b) \\
&= \frac{1}{4} \times \frac{4}{10} + \frac{3}{4} \times \frac{6}{10} = \frac{11}{20}
\end{aligned}
$$

Bayes' Rule

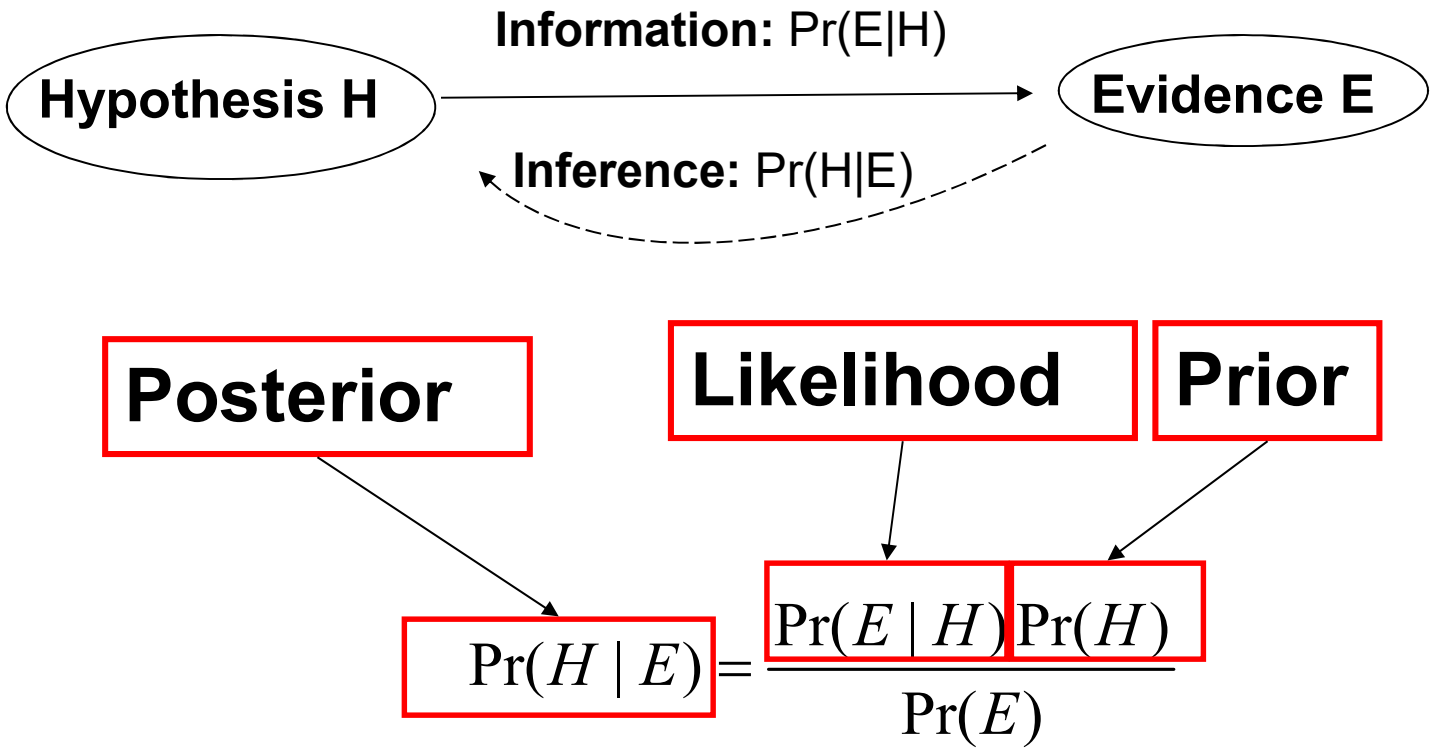
	R	¬R
W	0.7	0.4
¬W	0.3	0.6

R: It rains

W: The grass is wet



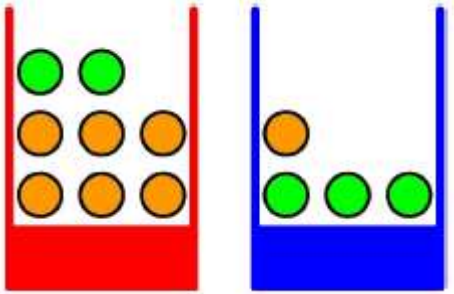
Bayes' Rule



- Prior Probability: Probability available ***before*** observing the evidence
- Posterior Probability: Probability obtained ***after*** observing the evidence

Bayes' Rule

- $p(B)$: Prior probability: Probability available before observing the identity of the fruit



- $p(B|F)$: Posterior probability: Probability obtained after observing the picked fruit

- Since $p(B = r)$ is 0.4, we are more likely to pick the blue box

- However, once we observe that the picked fruit is an orange we find that it's more likely that we picked from the red box (posterior probability)

$$p(B = r|F = o) = \frac{p(F = o|B = r)p(B = r)}{p(F = o)} = \frac{3}{4} \times \frac{4}{10} \times \frac{20}{9} = \frac{2}{3}$$

Bayes' Rule: More Complicated

- Suppose that $B_1, B_2, \ldots B_k$ form a partition of S:

$$B_i \bigcap B_j = \varnothing; \ \bigcup_i B_i = S$$

Suppose that Pr(B_i) > 0 and Pr(A) > 0. Then

$$\Pr\left(B_i \middle| A\right) = \frac{\Pr\left(A \middle| B_i\right)\Pr\left(B_i\right)}{\Pr\left(A\right)}$$

$$= \frac{\Pr\left(A \middle| B_i\right)\Pr\left(B_i\right)}{\sum_{j=1}^{k} \Pr\left(AB_j\right)}$$

$$= \frac{\Pr\left(A \middle| B_i\right)\Pr\left(B_i\right)}{\sum_{j=1}^{k} \Pr\left(B_j\right)\Pr\left(A \middle| B_j\right)}$$

Random Variable and Distribution

- A ***random variable*** X is a numerical outcome of a random experiment

- The ***distribution*** of a random variable is the collection of possible outcomes along with their probabilities:

  – Discrete case: $\Pr(X = x) = p_\theta(x)$

  – Continuous case: $\Pr(a \le X \le b) = \int_a^b p_\theta(x)dx$

  θ represents the parameter(s) of the distribution

Random Variable: Example and Distribution

- Let S be the set of all sequences of three rolls of a die. Let X be the sum of the number of dots on the three rolls

- What are the possible values of X?

  Answer: 3, 4, 5, 6, …, 18

- Pr(X = 5) = ?

  To get X = 5: (1,1,3), (1,3,1), (3,1,1),
  (1,2,2), (2,1,2), (2,2,1)

  Pr(X = 5) = $6/6^3$

Expectation

- A random variable $X \sim \Pr(X = x)$. Then, its expectation is

$$E[X] = \sum_x x \Pr(X = x)$$

  - In an empirical sample, $x_1, x_2, \ldots, x_N$, the sample mean is

$$E[X] = \frac{1}{N} \sum_{i=1}^{N} x_i$$

- Continuous case: $$E[X] = \int_{-\infty}^{\infty} x p_\theta(x) dx$$

- Expectation of sum of random variables

$$E[X_1 + X_2] = E[X_1] + E[X_2]$$

- Expectation of product of random variables

$$E[X_1 X_2] = \sum_{x_1} \sum_{x_2} x_1 x_2 \Pr(X_1 = x_1, X_2 = x_2)$$

If X_1 and X_2 are independent, $\Pr(X_1 = x_1, X_2 = x_2) = \Pr(X_1 = x_1) \Pr(X_2 = x_2)$

$$\therefore E[X_1 X_2] = \sum_{x_1} x_1 \Pr(X_1 = x_1) \sum_{x_2} x_2 \Pr(X_2 = x_2) = E[X_1] E[X_2]$$

Expectation: Example

- Let S be the set of all sequences of three rolls of a die. Let X be the sum of the number of dots on the three rolls.

- What is $E[X]$?

  Answer: $X=X_1+X_2+X_3$

$$E[X]= E[X_1] + E[X_2] +E[X_3]$$

$$E[X_i]=\sum_{x_i} x_i \Pr(X_i = x_i)$$

$$=1\times\frac{1}{6}+2\times\frac{1}{6}+3\times\frac{1}{6}+4\times\frac{1}{6}+5\times\frac{1}{6}+6\times\frac{1}{6}=\frac{21}{6}$$

$$\therefore E[X]=\frac{21}{6}+\frac{21}{6}+\frac{21}{6}=10.5$$

Expectation: Example

- Let S be the set of all sequences of three rolls of a die. Let X be the product of the number of dots on the three rolls

- What is $E[X]$?

  Answer: $X=X_1X_2X_3$

  Since the three rolls are independent, then

$$E[X]=E[X_1]E[X_2]E[X_3]$$

$$= \left(\frac{21}{6}\right)^3$$

Variance

- The variance of a random variable measures how much variability there is in the random variable

$$\begin{aligned} Var(X) &= E((X - E[X])^2) \\ &= E(X^2 + E[X]^2 - 2XE[X]) \\ &= E(X^2 - E[X]^2) \\ &= E[X^2] - E[X]^2 \end{aligned}$$

- Population variance: $\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$, where μ is the sample mean
- Example

  Let X represent randomly sampled integer numbers in the range (1, 100):

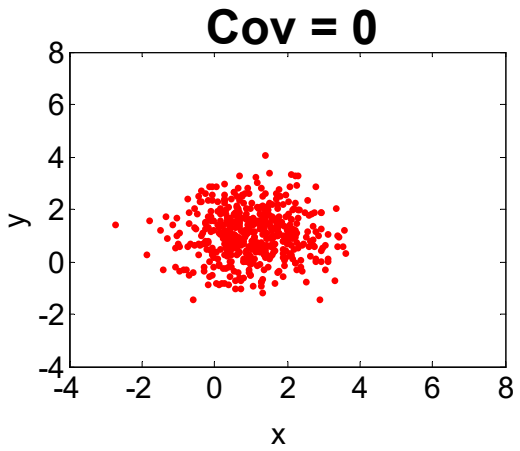
  - {2, 50, 9, 4, 23, 65, 99} → $\sigma^2 = \frac{1}{7}\sum_{i=1}^{7}(x_i - 36)^2 = 1154.85$
  - {33, 34, 35, 36, 37, 38, 39} → $\sigma^2 = \frac{1}{7}\sum_{i=1}^{7}(x_i - 36)^2 = 4$

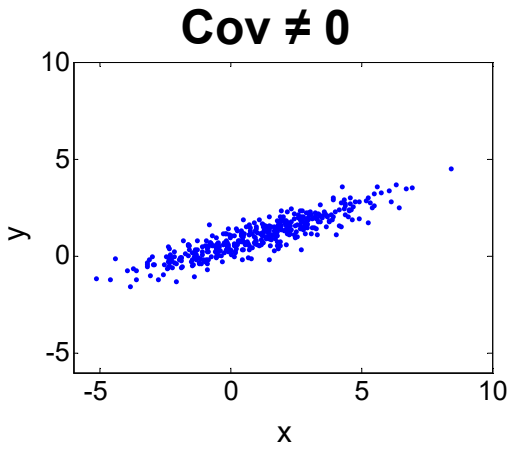
Covariance

- The covariance of two random variables measures the extent to which the two variables vary together

$$
\begin{aligned}
Cov[X,Y] &= E[(X - E[X])(Y - E[Y])] \\
&= E[XY - XE[Y] - YE[X] + E[X]E[Y]] \\
&= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] \\
&= E[XY] - E[X]E[Y]
\end{aligned}
$$

- Sample covariance $C_{XY} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu_X)(y_i - \mu_Y)$
- Example





Bernoulli Distribution

- The outcome of an experiment can either be success (1) or failure (0)

- Pr(X = 1) = θ, Pr(X = 0) = 1- θ, or

$$p_\theta(x) = \theta^x (1-\theta)^{1-x}$$

- $E[X] = \theta$, $Var(X) = \theta(1-\theta)$

Gaussian Distribution

- $X \sim N(\mu, \sigma)$

$$p_\theta(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

$$\Pr(a \le X \le b) = \int_a^b p_\theta(x)dx = \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

- $E[X] = \mu$, Var$(X) = \sigma^2$

- Mean: μ, Standard deviation: σ

- If $X_1 \sim N(\mu_1, \sigma_1)$ and $X_2 \sim N(\mu_2, \sigma_2)$, $X = X_1 + X_2$?

  Answer: X is Gaussian as well with mean $\mu_1 + \mu_2$ and variance

$$\sigma^2 = \sigma_1^2 + \sigma_2^2$$

Plots of Gaussian Distribution