Inference in FOPL

Lecture 5

February 20, 2018

Semantic Inference

- Recall that semantic inference is the process of identifying valid arguments of a logic (and hence tautologies).
- In PL, we considered three options:
  1. Truth table construction.
  2. Tableau methods (Wang's algorithm).
  3. Any sound and complete method of syntactic inference.
- We need to consider each of these options for FOPL.

Truth Tables (I)

- Can we use truth tables?
  - Yes, but in very restricted cases.
  - Even in those cases, the method will be extremely inefficient.
- Recall that, in PL, a row in the truth table corresponds to a situation defined by the truth assignment function A.
- In FOPL, each row would correspond to a structure-variable assignment pair (M, s).
- The number of such pairs is infinite. We cannot construct the entire table in the standard way.

Truth Tables (II)

- But there are two lucky coincidences:
  1. If all WFFs are closed, then we need not worry about variable assignments (any one will do).
  2. Under certain conditions, it is sufficient to consider one particular structure. This is called the **Herbrand structure.**
- An Herbrand structure is a structure where the universe is the set of all ground terms in the logic. The function $\mathcal{I}_{\mathcal{F}}$ maps each ground term onto itself.
- If $\mathcal{F}$ consists of only constants, truth tables could be used.
- If there is at least one non-constant function symbol, the truth table will be infinite.

Truth Tables (III)

- Suppose that we only have constants in $\mathcal{F}$. Suppose that there are n constants: $c_1, c_2, \ldots, c_n$.
- Construct a truth table by first constructing columns for all ground atomic WFFs.
- For each predicate symbol with arity r, there are n^r ground atomic WFFs. **(How many rows will the truth table have? What does the number of rows depend on?)**
- A WFF of the form $\forall x(\phi)$ is rewritten as $\bigwedge_{i=1}^{n} \phi\{c_i/x\}$.
- A WFF of the form $\exists x(\phi)$ is rewritten as $\bigvee_{i=1}^{n} \phi\{c_i/x\}$.
- If we add one non-constant function symbol, the number of ground atomic WFFs become infinite.

A Tableau Method for FOPL

- Again, the idea here is to try to construct a falsifying structure.
- Extend Wang's algorithm for FOPL; call it "FWang".
  - This is *not* an official name!
- The algorithm FWang operates on two sets T and F.
- It returns "True" if there is no structure that satisfies T and falsifies F.
- It returns "False" otherwise.
- The algorithm may be recursively defined as follows. Note that this is very similar to Wang's algorithm, except for five cases.

FWang: The Algorithm

FWang(T, F)

1. If $\mathbf{T} \in \mathcal{F}$ or $\mathbf{F} \in \mathcal{T}$ or $\mathcal{T} \cap \mathcal{F} \neq \{\}$, return "True";

2. If $T \cup F$ is a set of ground WFFs, return "False";

3. If $\neg\phi \in T$, return FWang$(T - \{\neg\phi\},\ F \cup \{\phi\})$;

4. If $\neg\phi \in F$, return FWang$(T \cup \{\phi\},\ F - \{\neg\phi\})$;

5. If $(\phi \wedge \psi) \in T$, return FWang$((T - \{\phi \wedge \psi\}) \cup \{\phi, \psi\},\ F)$;

6. If $(\phi \wedge \psi) \in F$,
return FWang$(T,\ (F - \{\phi \wedge \psi\}) \cup \{\phi\})$
and FWang$(T,\ (F - \{\phi \wedge \psi\}) \cup \{\psi\})$;

FWang: The Algorithm

7. If $(\phi \vee \psi) \in T$,
return FWang$((T - \{\phi \vee \psi\}) \cup \{\phi\},\ F)$
and FWang$(T - \{\phi \vee \psi\}) \cup \{\psi\},\ F)$;

8. If $(\phi \vee \psi) \in F$, return FWang$(T,\ (F - \{\phi \vee \psi\}) \cup \{\phi, \psi\})$;

9. If $(\phi \Rightarrow \psi) \in T$,
return FWang$((T - \{\phi \Rightarrow \psi\}) \cup \{\psi\},\ F)$
and FWang$(T - \{\phi \Rightarrow \psi\},\ F \cup \{\phi\})$;

10. If $(\phi \Rightarrow \psi) \in F$, return FWang$(T \cup \{\phi\},\ (F - \{\phi \Rightarrow \psi\}) \cup \{\psi\})$;

FWang: The Algorithm

11. If $(\phi \Leftrightarrow \psi) \in T$,

return FWang$((T - \{\phi \Leftrightarrow \psi\}) \cup \{\psi, \phi\},\ F)$

and FWang$(T - \{\phi \Leftrightarrow \psi\},\ F \cup \{\phi, \psi\})$;

12. If $(\phi \Leftrightarrow \psi) \in F$,

return FWang$(T \cup \{\phi\},\ (F - \{\phi \Leftrightarrow \psi\}) \cup \{\psi\})$

and FWang$(T \cup \{\psi\},\ (F - \{\phi \Leftrightarrow \psi\}) \cup \{\phi\})$;

FWang: The Algorithm

13. If $\forall x(\phi) \in T$,

return FWang($T \cup \{\phi\{t/x\}\},\ F$);

//where t is an arbitrary term, preferably one that appears in T or F

14. If $\forall x(\phi) \in F$,

return FWang($T,\ (F - \{\forall x(\phi)\}) \cup \{\phi\{c/x\}\}$);

//where c is new constant that occurs in neither T nor F

//c is referred to as a **Skolem constant**

FWang: The Algorithm

15. If $\exists x(\phi) \in T$,

return FWang$((T - \{\exists x(\phi)\}) \cup \{\phi\{c/x\}\},\ F)$;

//where c is a Skolem constant

16. If $\exists x(\phi) \in F$,

return FWang$(T,\ F \cup \{\phi\{t/x\}\})$;

//where t is an arbitrary term, preferably one that appears in T or F

Examples

- Using the FWang algorithm, determine whether the following are valid arguments.
  - $\models \exists x(P(a) \Rightarrow Q(x)) \Rightarrow (P(a) \Rightarrow \exists z Q(z))$.
  - $\models \exists x(P(a) \Rightarrow Q(x)) \Rightarrow (P(a) \Rightarrow \forall z Q(z))$
- **Do it yourself.**

Soundness and Completeness

- FWang is **sound**:
  - If FWang$(\mathcal{P}, \{\phi\})$ = "True", then $\mathcal{P} \models \phi$.
- FWang algorithm is **complete**:
  - If $\mathcal{P} \models \phi$, then FWang$(\mathcal{P}, \{\phi\})$ = "True".
- But there is a catch.

Another Example

- Using the FWang algorithm, determine whether the following is a valid argument.
  - $\models (\forall x P(x) \Rightarrow \exists y Q(y)) \Rightarrow (P(f(a)) \Rightarrow \exists z Q(z))$.
- **Do it yourself.**
- Clearly, as is, FWang may never terminate. (It is not an "algorithm" after all!)

Possible Fix

- Replace step 2 by the following:

  2. Return "False" if

  (a) T is a set of ground atoms, or of formulas of the form $\forall x(\phi)$ where all terms in T and F have already been used to instantiate x; and

  (b) F is a set of ground atoms, or of formulas of the form $\exists x(\phi)$ where all terms in T and F have already been used to instantiate x.

Proviso

- What about the following valid argument?
  - $\forall x P(x) \models \exists z P(z)$.
- **Do it yourself.**
- We may need to introduce an initial constant to get things moving.

Yet Another Example

- But what about this argument?
  - $\models ((\forall x \exists y P(x, y)) \Rightarrow P(a, b))$.
- **Do it yourself.**
- This is a hopeless situation: FWang is *not* a decision procedure for FOPL.
- In fact, it was proved that the set of valid FOPL formulas is undecidable (only r.e.).

Syntactic Inference

- Recall
  - An **inference rule** is a rule that licences the **derivation** of WFFs of a certain form from a (possibly empty) set of WFFs of certain forms.
  - **Syntactic inference** is the process of identifying correct derivations, based on some set of inference rules.
- We are going to extend the system of natural deduction presented for PL by four new rules—an introduction and an elimination rule for each quantifier.

Universal Elimination

$$\frac{\forall x(\phi)}{\phi\{t/x\}}$$

- t is an arbitrary term.
- Rationale: If it's true for all entities in the domain, then it's true for any particular entity in the domain.

Example

Prove that

$$\{\forall x(P(x) \Rightarrow Q(x)), P(a)\} \vdash Q(a)$$

where a is a constant.

Example

Prove that

$$\{\forall x(P(x) \Rightarrow Q(x)), P(a)\} \vdash Q(a)$$

where a is a constant.

1.$\forall x(P(x) \Rightarrow Q(x))$ (hypothesis)

Example

Prove that

$$\{\forall x(P(x) \Rightarrow Q(x)), P(a)\} \vdash Q(a)$$

where a is a constant.

1.$\forall x(P(x) \Rightarrow Q(x))$ (hypothesis)

2.$P(a)$ (hypothesis)

Example

Prove that

$$\{\forall x(P(x) \Rightarrow Q(x)), P(a)\} \vdash Q(a)$$

where a is a constant.

1.$\forall x(P(x) \Rightarrow Q(x))$	(hypothesis)
2.$P(a)$	(hypothesis)
3.$P(a) \Rightarrow Q(a)$	(1, $\forall$-elim)

Example

Prove that

$$\{\forall x(P(x) \Rightarrow Q(x)), P(a)\} \vdash Q(a)$$

where a is a constant.

1.$\forall x(P(x) \Rightarrow Q(x))$	(hypothesis)
2.$P(a)$	(hypothesis)
3.$P(a) \Rightarrow Q(a)$	(1, $\forall$-elim)
4.$Q(a)$	(2, 3, $\Rightarrow$-elim)

Another Example

Something's wrong with the following derivation. Can you tell what it is?

1.$\forall x(\exists y P(x, y))$ (hypothesis)

2.$\exists y P(y, y)$ (1, $\forall$-elim)

Another Example

Something's wrong with the following derivation. Can you tell what it is?

1.$\forall x(\exists y P(x, y))$ (hypothesis)

2.$\exists y P(y, y)$ (1, $\forall$-elim) $\Longleftarrow$ variable capture

Another Example

Something's wrong with the following derivation. Can you tell what it is?

1.$\forall x(\exists y P(x, y))$ (hypothesis)

2.$\exists y P(y, y)$ (1, $\forall$-elim) $\Longleftarrow$ variable capture

The first occurrence of y should have been free, but it is now captured by the existential quantifier.

Existential Introduction

$$\frac{\phi\{t/x\}}{\exists x(\phi)}$$

- t is an arbitrary term.
- Rationale: If it's true for a particular entity in the domain, then there is an entity in the domain for which it's true.

Example

Prove that

$$\{\forall x P(x)\} \vdash \exists z P(z)$$

Example

Prove that

$$\{\forall x P(x)\} \vdash \exists z P(z)$$

$\mathbf{1}. \forall x P(x)$ (hypothesis)

Example

Prove that

$$\{\forall x P(x)\} \vdash \exists z P(z)$$

1.$\forall x P(x)$ (hypothesis)

2.$P(c)$ (1, $\forall$-elim)

Example

Prove that

$$\{\forall x P(x)\} \vdash \exists z P(z)$$

1.$\forall x P(x)$ (hypothesis)

2.$P(c)$ (1, $\forall$-elim)

3.$\exists z P(z)$ (2, $\exists$-intro)

Another Example

Something's wrong with the following derivation. Can you tell what it is?

1. $P(a) \Rightarrow \forall x Q(x, a)$ (hypothesis)

2. $\exists x(P(x) \Rightarrow \forall x Q(x, x))$ (1, $\exists$-intro)

Another Example

Something's wrong with the following derivation. Can you tell what it is?

1.$P(a) \Rightarrow \forall x Q(x,a)$ (hypothesis)

2.$\exists x(P(x) \Rightarrow \forall x Q(x,x))$ (1, $\exists$-intro) $\Longleftarrow 2\{a/x\} \neq 1$

Existential Elimination

$$\frac{\exists x(\phi)}{\phi\{c/x\}}$$

- c is a Skolem constant.
- Restrictions:
  - c has not been previously used in the derivation.
  - c does not occur in the conclusion.
- Rationale:
  - We temporarily introduce a name for the individual of which $\exists x(\phi)$ holds.
  - The name must not refer to anyone that we already know.

Example

The following are legitimate steps in a derivation.

1.$\forall x(P(x) \Rightarrow Q(x))$ (hypothesis)

2.$\exists y P(y)$ (hypothesis)

3.$P(a)$ (2, $\exists$-elim)

4.$P(a) \Rightarrow Q(a)$ (1, $\forall$-elim)

5.$Q(a)$ (3, 4, $\Rightarrow$-elim)

- **Note!** The above is not a proof of the validity of $\{\forall x(P(x) \Rightarrow Q(x)), \exists y P(y)\} \vdash Q(a)$.
- Recall that a Skolem constant cannot occur in the conclusion.

Universal Introduction

$$\frac{\phi\{t/x\}}{\forall x(\phi)}$$

- Restriction:
  - t does not occur (free, if a variable) in the hypotheses or the conclusion.
  - any Skolem constant in ϕ was introduced into the derivation strictly before t.
- **How could t have been introduced into the derivation?**

Universal Introduction: Rationale

- If it's true for an arbitrary entity, then it is true for all entities.
- We use this rule all the time, whenever we need to prove that some property is true of all elements in a set.
- The restrictions ensure the arbitrariness of x and prevent the danger of switching quantifier order.

Example

Prove that

$$\{\forall x(P(x) \Rightarrow Q(x)), \forall x(P(x))\} \vdash \forall x(Q(x))$$

Example

Prove that

$$\{\forall x(P(x) \Rightarrow Q(x)), \forall x(P(x))\} \vdash \forall x(Q(x))$$

1.$\forall x(P(x) \Rightarrow Q(x))$ (hypothesis)

Example

Prove that

$$\{\forall x(P(x) \Rightarrow Q(x)), \forall x(P(x))\} \vdash \forall x(Q(x))$$

1.$\forall x(P(x) \Rightarrow Q(x))$ (hypothesis)

2.$\forall x(P(x))$ (hypothesis)

Example

Prove that

$$\{\forall x(P(x) \Rightarrow Q(x)), \forall x(P(x))\} \vdash \forall x(Q(x))$$

1.$\forall x(P(x) \Rightarrow Q(x))$ (hypothesis)

2.$\forall x(P(x))$ (hypothesis)

3.$P(x) \Rightarrow Q(x)$ (1, $\forall$-elim)

Example

Prove that

$$\{\forall x(P(x) \Rightarrow Q(x)), \forall x(P(x))\} \vdash \forall x(Q(x))$$

1.$\forall x(P(x) \Rightarrow Q(x))$ (hypothesis)

2.$\forall x(P(x))$ (hypothesis)

3.$P(x) \Rightarrow Q(x)$ (1, ∀-elim)

4.$P(x)$ (2, ∀-elim)

Example

Prove that

$$\{\forall x(P(x) \Rightarrow Q(x)), \forall x(P(x))\} \vdash \forall x(Q(x))$$

1.$\forall x(P(x) \Rightarrow Q(x))$ (hypothesis)

2.$\forall x(P(x))$ (hypothesis)

3.$P(x) \Rightarrow Q(x)$ (1, $\forall$-elim)

4.$P(x)$ (2, $\forall$-elim)

5.$Q(x)$ (2, 3, $\Rightarrow$-elim)

Example

Prove that

$$\{\forall x(P(x) \Rightarrow Q(x)), \forall x(P(x))\} \vdash \forall x(Q(x))$$

1.$\forall x(P(x) \Rightarrow Q(x))$	(hypothesis)
2.$\forall x(P(x))$	(hypothesis)
3.$P(x) \Rightarrow Q(x)$	(1, $\forall$-elim)
4.$P(x)$	(2, $\forall$-elim)
5.$Q(x)$	(2, 3, $\Rightarrow$-elim)
6.$\forall x(Q(x))$	(5, $\forall$-intro)

Another Example

Something's wrong with the following proof. Can you tell what it is?

1.$P(a)$ (hypothesis)

2.$\forall x(P(x))$ (1, $\forall$-intro)

Another Example

Something's wrong with the following proof. Can you tell what it is?

1.$P(a)$ (hypothesis)

2.$\forall x(P(x))$ (1, $\forall$-intro) $\Longleftarrow$ a occurs free in a hypothesis

Yet Another Example

Something's wrong with the following proof. Can you tell what it is?

1.$\forall x(\exists y(P(x, y)))$ (hypothesis)

2.$\exists y(P(a, y))$ (1, $\forall$-elim)

3.$P(a, c)$ (2, $\exists$-elim)

4.$\forall x(P(x, c))$ (3, $\forall$-intro)

Yet Another Example

Something's wrong with the following proof. Can you tell what it is?

1.$\forall x(\exists y(P(x,y)))$ (hypothesis)

2.$\exists y(P(a,y))$ (1, $\forall$-elim)

3.$P(a,c)$ (2, $\exists$-elim)

4.$\forall x(P(x,c))$ (3, $\forall$-intro) $\Longleftarrow$ $P(x,c)$ contains a Skolem constant introduced after a

One Final Example

Prove that

$$\{\forall x(\exists y(P(x) \Rightarrow Q(x,y))\} \vdash \forall x(P(x) \Rightarrow \exists y(Q(x,y)))$$

One Final Example

Prove that

$$\{\forall x(\exists y(P(x) \Rightarrow Q(x,y))\} \vdash \forall x(P(x) \Rightarrow \exists y(Q(x,y)))$$

1.$\forall x(\exists y(P(x) \Rightarrow Q(x,y))$ (hyp)

One Final Example

Prove that

$$\{\forall x(\exists y(P(x) \Rightarrow Q(x,y))\} \vdash \forall x(P(x) \Rightarrow \exists y(Q(x,y)))$$

1.$\forall x(\exists y(P(x) \Rightarrow Q(x,y))$ (hyp)

2.$\exists y(P(a) \Rightarrow Q(a,y))$ (1, $\forall$-elim)

One Final Example

Prove that

$$\{\forall x(\exists y(P(x) \Rightarrow Q(x,y))\} \vdash \forall x(P(x) \Rightarrow \exists y(Q(x,y)))$$

1.$\forall x(\exists y(P(x) \Rightarrow Q(x,y))$ (hyp)

2.$\exists y(P(a) \Rightarrow Q(a,y))$ (1, $\forall$-elim)

3.$P(a) \Rightarrow Q(a,c)$ (2, $\exists$-elim)

One Final Example

Prove that

$$\{\forall x(\exists y(P(x) \Rightarrow Q(x,y))\} \vdash \forall x(P(x) \Rightarrow \exists y(Q(x,y)))$$

1.$\forall x(\exists y(P(x) \Rightarrow Q(x,y))$ (hyp)

2.$\exists y(P(a) \Rightarrow Q(a,y))$ (1, $\forall$-elim)

3.$P(a) \Rightarrow Q(a,c)$ (2, $\exists$-elim)

  4.$P(a)$ (assumption)

One Final Example

Prove that

$$\{\forall x(\exists y(P(x) \Rightarrow Q(x,y))\} \vdash \forall x(P(x) \Rightarrow \exists y(Q(x,y)))$$

1.$\forall x(\exists y(P(x) \Rightarrow Q(x,y))$ (hyp)

2.$\exists y(P(a) \Rightarrow Q(a,y))$ (1, $\forall$-elim)

3.$P(a) \Rightarrow Q(a,c)$ (2, $\exists$-elim)

4.$P(a)$ (assumption)

5.$Q(a,c)$ (3, 4, $\Rightarrow$-elim)

One Final Example

Prove that

$$\{\forall x(\exists y(P(x) \Rightarrow Q(x,y))\} \vdash \forall x(P(x) \Rightarrow \exists y(Q(x,y)))$$

1.$\forall x(\exists y(P(x) \Rightarrow Q(x,y))$ (hyp)

2.$\exists y(P(a) \Rightarrow Q(a,y))$ (1, $\forall$-elim)

3.$P(a) \Rightarrow Q(a,c)$ (2, $\exists$-elim)

  4.$P(a)$ (assumption)

  5.$Q(a,c)$ (3, 4, $\Rightarrow$-elim)

  6.$\exists y Q(a,y)$ (5, $\exists$-intro)

One Final Example

Prove that

$$\{\forall x(\exists y(P(x) \Rightarrow Q(x,y))\} \vdash \forall x(P(x) \Rightarrow \exists y(Q(x,y)))$$

1.$\forall x(\exists y(P(x) \Rightarrow Q(x,y))$ (hyp)

2.$\exists y(P(a) \Rightarrow Q(a,y))$ (1, $\forall$-elim)

3.$P(a) \Rightarrow Q(a,c)$ (2, $\exists$-elim)

  4.$P(a)$ (assumption)

  5.$Q(a,c)$ (3, 4, $\Rightarrow$-elim)

  6.$\exists y Q(a,y)$ (5, $\exists$-intro)

7.$P(a) \Rightarrow \exists y Q(a,y)$ (4, 6, $\Rightarrow$-intro)

One Final Example

Prove that

$$\{\forall x(\exists y(P(x) \Rightarrow Q(x,y))\} \vdash \forall x(P(x) \Rightarrow \exists y(Q(x,y)))$$

1.$\forall x(\exists y(P(x) \Rightarrow Q(x,y))$	(hyp)
2.$\exists y(P(a) \Rightarrow Q(a,y))$	(1, $\forall$-elim)
3.$P(a) \Rightarrow Q(a,c)$	(2, $\exists$-elim)
4.$P(a)$	(assumption)
5.$Q(a,c)$	(3, 4, $\Rightarrow$-elim)
6.$\exists y Q(a,y)$	(5, $\exists$-intro)
7.$P(a) \Rightarrow \exists y Q(a,y)$	(4, 6, $\Rightarrow$-intro)
8.$\forall x(P(x) \Rightarrow \exists y Q(x,y))$	(7, $\forall$-intro)

Soundness and Completeness

- FOPL is sound. That is, there are inference procedures that would derive only tautologies.
- In his **Completeness Theorem**, Gödel proved that FOPL is complete. That is, there are inference procedures that would derive all tautologies.
- However, Gödel did not construct such a procedure, Robinson did later.
- In his **Incompleteness Theorem**, Gödel proved that a language strong enough to represent any consistent set of axioms of number theory is incomplete.

Decidability

- But Gödel also proved that FOPL is **semi-decidable**. That is, the set of FOPL tautologies is **recursively-enumerable**, but not **recursive**.

- This means that if a given WFF is a tautology, then our sound and complete inference procedure will indeed derive it. However, if the given WFF is not a tautology, our procedure might run forever. (And we wouldn't know if the WFF is not a tautology or if it is just taking too long to derive it.)