Propositional Modal Logic

Lecture 8

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Trouble with $\Rightarrow$

- Standard semantics of material implication would deem as true the sentence
  If pigs can fly, then horses can sing
- The intuition behind such a judgement is that if the impossible would be true, then anything would.
- The "impossible" here is taken to be anything that does not happen to be true.
- But, while we might be ready to swallow such an argument for the above example, we would not find it very impressive in other cases:
  If I do not have a beard, than 2+2=5

The Paradoxes of Implication

- In standard PL, the following WFFs are valid. **(Right?)**
  1. $P \Rightarrow (Q \Rightarrow P)$
  2. $\neg P \Rightarrow (P \Rightarrow Q)$
- Now, it is easy to show that the following is also valid.
  - $(P \Rightarrow Q) \vee (Q \Rightarrow P)$
- That is, given any pair of propositions, one of them implies the other.
- At this point, we must admit that, whatever the merits of material implication are, it does not capture our intuitive notion of a conditional statement.

Strict Implication

- Of course, the problem with material implication is the possible lack of a "link" between the antecedent and the consequent.
- Logical implication, however, necessitates such a "link".
- Recall that logical implication is the relation denoted by $P \models Q$
- This, however, is a semantic relation. A syntactic version was first introduced by MacColl: $P \multimap\!\ni Q$
- This is referred to as **strict implication**. (MacColl used the symbol ":"; "$\multimap\!\ni$" was later introduced by C. I. Lewis.)
- MacColl accepts $(P \multimap\!\ni Q) \models (\neg P \vee Q)$.
- However, he denies $(P \multimap\!\ni Q) \equiv (\neg P \vee Q)$.
  - MacColl's example: $P =$ "He persists in his extravagancy" and $Q =$ "He will be ruined".

Necessity

- What is the semantics of $\multimap$?
- It seems clear that, whatever we do, we must maintain that

$$\models \phi \multimap \psi \quad \text{iff} \quad \phi \models \psi$$

- This raises the following question: given the relation between logical implication and logical validity, since $\multimap$ corresponds to logical implication, what sort of notion corresponds to logical validity?
- This is the notion of **necessity**.

Necessary and Contingent Propositions

- The following statements are both true.
  1. Someone has blue eyes.
  2. If someone has blue eyes, then they have eyes.
- However, only (2) is necessarily true; we cannot conceive of a state of affairs in which it is not.
- (1) is only contingent; it just happens to be true in the current/factual state of affairs.
- This is exactly the relation between material and strict implication: material implication is only contingent, strict implication is necessary.
- That is, assuming we have a necessity operator,

$$P \multimap Q \equiv \text{necessary}(P \Rightarrow Q)$$

Modal Logic

- To start with, you can think of **modal logic** as the logic of necessity; that is, the logic of the operator "necessary" we assumed in the previous slide.
- However, as we shall see shortly, modal logic is much more powerful: it could be used to represent notions of tense, knowledge, belief, moral obligation, ability, and others.
- Since modal operators are necessarily non-truth-functional, you may also think of modal logic as a logic of (some) non-truth-functional (unary) operators.

Propositional Modal Logic

- The vocabulary of propositional modal logic (**PML**) is exactly that of PL in addition to two dual unary operators: $\Box$ and $\Diamond$.
- WFFs of PML are all WFFs of PL in addition to
  1. $\Box\phi$, and
  2. $\Diamond\phi$

  where ϕ is a PML WFF.
- Intuitively, $\Box\phi$ means that "it is necessary that ϕ".
- $\Box$ and $\Diamond$ are related by the equivalence $\Box\phi \equiv \neg \Diamond (\neg\phi)$.
- Thus, $\Diamond\phi$ may be paraphrased as "it is possible that ϕ".

Semantics

- So far, we have only alluded to intuition when it comes to the meaning of $\Box$ and $\Diamond$.
- Recall that, intuitively, $\Box\phi$ is taken to be true if ϕ is true in every state of affairs we can conceive of.
- In PL, a state of affairs corresponds to a truth assignment $\mathcal{A}$.
- Unlike in PL, where we evaluate a proposition with respect to some particular $\mathcal{A}$, in PML we seem to need to evaluate a proposition with respect to each $\mathcal{A}$ which is somehow conceivable.
- How do we give a precise account of the notion of "conceivable states of affairs"?
- The idea is formally quite simple, but ontologically problematic.

Possible Worlds

- The technical term for "conceivable state of affairs" is **"possible world"**.
- There are different ways to think of what a possible world is, but you may consider it to be an alternative world which might be quite similar or quite different from our actual world with respect to the truth values of propositions.
- That is, you might want to think of a possible world as just a set of propositions.
- In AI, possible worlds are often characterized syntactically by maximal, consistent sets of atomic WFFs. These are typically referred to as **possible models**.
- **(Can you count the number of possible models describable by some PML?)**

The Modal Game

- Before considering a formal treatment of possible-worlds semantics, let's play Hughes and Cresswell's so-called modal game.
- Initial Setting:
  1. Settle on some language of PML.
  2. Assign to each player a set of players that they can "see" (that is, that they should monitor).
     – Note: a player does not necessarily see themselves.
  3. Give each player a piece of paper with some propositional variables written on it. These represent the true atoms with respect to each player.

The Modal Game (Cont'd)

- The Game:
  1. Write a WFF on a board that everyone can see.
  2. If the WFF is a propositional variable, then only those players with this variable on their paper should raise their hand.
  3. If the WFF is PL WFF whose sub-formulas have been written on the board before, then only players whose papers make this WFF true should raise their hand. **(How?)**
  4. If the WFF is of the form $\Box\phi$ where ϕ has been written on the board before, then for each player, if all players they can see have raised their hands when ϕ was written on the board, they should raise their hand.

Kripke Structures

- Let $\mathcal{L}_{\mathcal{P}}$ be a PML with a set $\mathcal{P}$ of propositional variables.
- A **Kripke structure** (or a **frame**) $\mathcal{K}$ is a pair $(\mathcal{W}, \mathcal{R})$, where
  - $\mathcal{W}$ is a set possible worlds, and
  - $\mathcal{R} \subseteq \mathcal{W} \times \mathcal{W}$ is the **accessibility relation** on $\mathcal{W}$.
- A **model** is a triple $(\mathcal{W}, \mathcal{R}, \mathcal{A})$, where $(\mathcal{W}, \mathcal{R})$ is a Kripke structure and $\mathcal{A} : \mathcal{P} \longrightarrow (\mathcal{W} \longrightarrow \{\top, \bot\})$ is a function that assigns to each propositional variable a function from possible worlds to truth values.

Semantics of PML

- The interpretation function $[\![\cdot]\!]$ assigns to each $\mathcal{L}_{\mathcal{P}}$ WFF a truth value with respect to some model $\mathcal{M} = (\mathcal{W}, \mathcal{R}, \mathcal{A})$ and some world $w \in \mathcal{W}$.
  - $[\![\mathbf{T}]\!]^{\mathcal{M},w} = \top$.
  - $[\![\mathbf{F}]\!]^{\mathcal{M},w} = \bot$.
  - $[\![\mathbf{P}]\!]^{\mathcal{M},w} = \mathcal{A}(P)(w)$, for every $P \in \mathcal{P}$.

Semantics of PML (Cont'd)

- For $\phi, \psi \in \mathcal{L}_{\mathcal{V}}$
  1. $[\![\neg\phi]\!]^{\mathcal{M},w} = \top$ iff $[\![\phi]\!]^{\mathcal{M},w} = \bot$.
  2. $[\![(\phi \wedge \psi)]\!]^{\mathcal{M},w} = \top$ iff $[\![\phi]\!]^{\mathcal{M},w} = [\![\psi]\!]^{\mathcal{M},w} = \top$.
  3. $[\![(\phi \vee \psi)]\!]^{\mathcal{M},w} = \bot$ iff $[\![\phi]\!]^{\mathcal{M},w} = [\![\psi]\!]^{\mathcal{M},w} = \bot$.
  4. $[\![(\phi \Rightarrow \psi)]\!]^{\mathcal{M},w} = \bot$ iff $[\![\phi]\!]^{\mathcal{M},w} = \top$ and $[\![\psi]\!]^{\mathcal{M},w} = \bot$.
  5. $[\![(\phi \Leftrightarrow \psi)]\!]^{\mathcal{M},w} = \top$ iff $[\![\phi]\!]^{\mathcal{M},w} = [\![\psi]\!]^{\mathcal{M},w}$.
  6. $[\![\Box\phi]\!]^{\mathcal{M},w} = \top$ iff for every $w' \in \mathcal{W}$ such that $w \; \mathcal{R} \; w'$ $[\![\phi]\!]^{\mathcal{M},w'} = \top$.
  7. $[\![\Diamond\phi]\!]^{\mathcal{M},w} = \top$ iff for some $w' \in \mathcal{W}$ such that $w \; \mathcal{R} \; w'$ $[\![\phi]\!]^{\mathcal{M},w'} = \top$.

Validity

- A PML WFF ϕ is **valid in a model** $\mathcal{M} = (\mathcal{W}, \mathcal{R}, \mathcal{A})$ if, for every $w \in \mathcal{W}$, $[\![\phi]\!]^{\mathcal{M},w} = \top$.
- A PML WFF ϕ is **valid in a structure** $(\mathcal{W}, \mathcal{R})$ if, for every model $\mathcal{M} = (\mathcal{W}, \mathcal{R}, \mathcal{A})$, ϕ is valid in $\mathcal{M}$.
- A PML WFF ϕ is **K-valid** if it is valid in every structure.

Examples of K-valid WFFs

1. $\Box(\phi \Rightarrow \psi) \Rightarrow (\Box\phi \Rightarrow \Box\psi)$
   - This WFF is well-known as just **K**.
   - Is the converse of **K** **K**-valid?
2. $\Box(\phi \wedge \psi) \Leftrightarrow (\Box\phi \wedge \Box\psi)$
3. $(\Box\phi \vee \Box\psi) \Rightarrow \Box(\phi \vee \psi)$
4. $\Diamond\phi \Rightarrow (\Box\psi \Rightarrow \Diamond\psi)$
5. $\Diamond(\phi \Rightarrow \psi) \Leftrightarrow (\Box\phi \Rightarrow \Diamond\psi)$

Modal Logics

- Different modal logics can be devised by employing different sets of axioms.
- Some common axioms are the following:

  K: $\Box(\phi \Rightarrow \psi) \Rightarrow (\Box\phi \Rightarrow \Box\psi)$

  T: $\Box\phi \Rightarrow \phi$

  D: $\Box\phi \Rightarrow \Diamond\phi$

  4: $\Box\phi \Rightarrow \Box\Box\phi$

  5: $\Diamond\phi \Rightarrow \Box \Diamond \phi$

  B: $\phi \Rightarrow \Box \Diamond \phi$

- Modal logics are named according to which axioms they employ. For example, *System* **K**, *System* **T**, *System* **T4** (also known as **S4**), etc.

Semantic Interpretation of Axioms

- Not all axioms presented above are **K**-valid.
- We only require them to be valid in a structure $(\mathcal{W}, \mathcal{R})$.
- In a way, the axioms impose restrictions on the structures that may be employed in semantic interpretation.
- In particular, they impose some properties on the accessibility relation $\mathcal{R}$.
- **For each of T, D, 4, 5, and B, describe such restrictions.**