**German University in Cairo**
**Department of Computer Science**
**Assoc. Prof. Haythem O. Ismail**

**Introduction to Artificial Intelligence**, Winter Term 2019
**Problem Set 6**

### Exercise 6-1

With respect to the rules of natural deduction introduced in class, prove each of the following.

a) $\{(P \Rightarrow Q) \Rightarrow P, P \Rightarrow Q, \neg R\} \vdash Q \vee R$.

**Solution:**

| | |
|---|---|
| 1.$(P \Rightarrow Q) \Rightarrow P$ | (premise) |
| 2.$P \Rightarrow Q$ | (premise) |
| 3.$P$ | (1,2, $\Rightarrow$-elimination) |
| 4.$Q$ | (2,3,$\Rightarrow$-elimination) |
| 5.$Q \vee R$ | (4, $\vee$-introduction) |

b) $\{\exists x[P(x) \Rightarrow Q(x)], \forall x P(x)\} \vdash \exists x Q(x)$.

**Solution:**

| | |
|---|---|
| 1.$\exists x[P(x) \Rightarrow Q(x)]$ | (premise) |
| 2.$\forall x P(x)$ | (premise) |
| 3.$P(A) \Rightarrow Q(A)$ | (1,$\exists$-elimination) |
| 4.$P(A)$ | (2,$\forall$-elimination) |
| 5.$Q(A)$ | (2,3, $\Rightarrow$-elimination) |
| 6.$\exists x Q(x)$ | (5, $\exists$-introduction) |

c) $\{P(A), \forall x[\exists y[P(y) \vee R(y)] \Rightarrow Q(x)]\} \vdash \exists z[Q(z) \wedge P(z)]$.

**Solution:**

| | |
|---|---|
| 1.$P(A)$ | (premise) |
| 2.$\forall x[\exists y[P(y) \vee R(y)] \Rightarrow Q(x)]$ | (premise) |
| 3.$\exists y[P(y) \vee R(y)] \Rightarrow Q(A)$ | (2, $\forall$-elimination) |
| 4.$P(A) \vee R(A)$ | (1, $\vee$-introduction) |
| 5.$\exists y[P(y) \vee R(y)]$ | (4, $\exists$-introduction) |
| 6.$Q(A)$ | (3, 5, $\Rightarrow$-elimination) |
| 7.$Q(A) \wedge P(A)$ | (1, 3, $\wedge$-introduction) |
| 8.$\exists z[Q(z) \wedge P(z)]$ | (1, 3, $\wedge$-introduction) |

d) $\{\forall x \forall y[P(f(x,y)) \Rightarrow P(f(y,g(x)) \wedge Q(g(y),g(x))], \exists x \exists y[P(f(x,y)]\} \vdash \exists x \exists y[Q(g(g(x)),y)]$.

**Solution:**

| | |
|---|---|
| $1.\exists x \exists y[P(f(x,y))]$ | (premise) |
| $2.\forall x \forall y[P(f(x,y)) \Rightarrow P(f(y,g(x)) \wedge Q(g(y),g(x))]$ | (premise) |
| $3.\exists y[P(f(A,y))]$ | (1, $\exists$-elimination) |
| $4.P(f(A,B)$ | (3, $\exists$-elimination) |
| $5.\forall y[P(f(A,y)) \Rightarrow P(f(y,g(A)) \wedge Q(g(y),g(A))]$ | (1, $\forall$-elimination) |
| $6.P(f(A,B) \Rightarrow P(f(B,g(A)) \wedge Q(g(B),g(A))]$ | (5, $\forall$-elimination) |
| $7.P(f(B,g(A)) \wedge Q(g(B),g(A))]$ | (3,6, $\Rightarrow$-elimination) |
| $8.P(f(B,g(A))$ | (7, $\wedge$-elimination) |
| $9.\forall y[P(f(B,y)) \Rightarrow P(f(y,g(B)) \wedge Q(g(y),g(B))]$ | (2, $\forall$-elimination) |
| $10.P(f(B,g(A)) \Rightarrow P(f(g(A),g(B)) \wedge Q(g(g(A)),g(B))]$ | (9, $\forall$-elimination) |
| $11.P(f(g(A),g(B)) \wedge Q(g(g(A)),g(B))$ | (8,10, $\Rightarrow$-elimination) |
| $12.Q(g(g(A)),g(B))$ | (11, $\wedge$-elimination) |
| $13.\exists y[Q(g(g(A)),y)]$ | (12, $\exists$-introduction) |
| $14.\exists x \exists y[Q(g(g(x)),y)]$ | (13, $\exists$-introduction) |

**Exercise 6-2**

For each of the following pairs of atomic sentences, give a most general unifier, if one exists.

a) $Q(y, G(A,B))$, $Q(G(x,x), y)$.

**Solution:**

The two statements do not unify.

b) $\mathit{Older}(\mathit{Father}(y), y)$, $\mathit{Older}(\mathit{Father}(x), \mathit{John})$.

**Solution:**

$\mu = \{\mathit{John}/x, \mathit{John}/y\}$

c) $\mathit{Knows}(\mathit{Father}(y), y)$, $\mathit{Knows}(x, x)$.

**Solution:**

The two statements do not unify.

**Exercise 6-3** (From R&N)

Write down logical representations for the following sentences, suitable for use with Generalized Modus Ponens.

a) Horses, cows, and pigs are mammals.

**Solution:**

- $H(x) \Rightarrow M(x)$
- $C(x) \Rightarrow M(x)$
- $P(x) \Rightarrow M(x)$

b) An offspring of a horse is a horse.

**Solution:**

- $H(x) \wedge O(y,x) \Rightarrow H(y)$

c) Bluebird is a horse.

**Solution:**

- $H(BB)$

d) Bluebird is Charlie's parent.

**Solution:**

- $Pt(BB, Ch)$

e) Offspring and parent are inverse relations.

**Solution:**

- $Pt(x,y) \Rightarrow O(y,x)$
- $O(x,y) \Rightarrow Pt(y,x)$

f) Every mammal has a parent.

**Solution:**

- $M(x) \Rightarrow Pt(Prnt(x), x)$
  $Prnt(x)$ is a skolem function. Note that variables in Horn Clause are assumed to be universally quantified.

What is the result of backward-chaining on $\exists x, y\, \mathit{OffSpring}(x,y)$? What is the result of forward-chaining on $Cow(Moo)$?

**Solution:**

Backward Chaining on $\exists x, y\, \mathit{OffSpring}(x,y)$

- Start with $O(x,y)$.
- Standardize apart: $O(u,v)$
- $O(u,v)$ is the goal with the empty substitution.
- Backchain into $Pt(x,y) \Rightarrow O(y,x)$, getting the unifier $\mu = \{y/u, x/v\}$
- Apply $\mu$ on the antecedent to get $Pt(x,y)$. (The application does not change the antecedent.)
- $Pt(x,y)$ is now the new goal, with substitution $\mu$.
- $Pt(x,y)$ unifies with $Pt(BB, Ch)$, the unifier is $\mu' = \{BB/x, Ch/y\}$.
- Return the composition of $\mu$ and $\mu'$: $\mu \circ \mu' = \{BB/x, BB/v, Ch/y, Ch/u\}$

The restriction of the final unifier to $u$ and $v$ is the answer.

**Solution:**

Forward Chaining on $Cow(Moo)$

- Start with $C(Moo)$.
- Forward chain into $C(x) \Rightarrow M(x)$,
  with the unifier $\mu_1 = \{Moo/x\}$
- Apply $\mu_1$ to the consequent to get $M(Moo)$.
- $M(Moo)$ forward chains into $M(x) \Rightarrow Pt(Prnt(x), x)$,
  with the unifier $\mu_2 = \{Moo/x\}$.
- Apply $\mu_2$ to the consequent to get $Pt(Prnt(Moo), Moo)$.
- $Pt(Prnt(Moo), Moo)$ forward chains into $Pt(x, y) \Rightarrow O(y, x)$,
  with the unifier $\mu_3 = \{Prnt(Moo)/x, Moo/y\}$.
- Apply $\mu_3$ to the consequent to get $O(Moo, Pt(Moo))$.
- $O(Moo, Pt(Moo))$ forward chains into $O(x, y) \Rightarrow Pt(y, x)$,
  with the unifier $\mu_4 = \{Moo/x, Prnt(Moo)/y\}$.
- Apply $\mu_4$ to the consequent to get $Pt(Prnt(Moo), Moo)$.
- Since we already have that, we stop.

Note that, had we insisted that $Prnt(x)$ must also be a mammal in sentence (f), forward chaining would have gone on forever.