First-Order Predicate Logic

Lecture 4

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The Puzzle of the Politicians [a]

A certain convention numbered 100 politicians. Each politician was either crooked or honest. We are given the following two facts.

1. At least one of the politicians was honest.
2. Given any two of the politicians, at least one of the two was crooked.

Can it be determined from these two facts how many of the politicians were honest and how many were crooked?

Can you do it using propositional logic?

[a] This puzzle is due to Ray Smullyan.

A Simpler Example

- Consider the argument

$$\frac{\begin{array}{l}\text{All men are mortal}\\ \text{Socrates is a man}\end{array}}{\text{Socrates is mortal}}$$

- Is it valid?
- Yes, but why?

More . . .

- Consider another argument

$$\frac{\begin{array}{l}\text{All men are mortal}\\ \text{Some men are wise}\end{array}}{\text{Some mortals are wise}}$$

- Is it valid?
- Yes, but why?

Even More . . .

- Consider trying to represent the knowledge that all GUC students are smart.
- In propositional logic:

  $$A \wedge B \wedge C \wedge \cdots$$

  where
  - A is "Merna is smart"
  - B is "Marc is smart"
  - C is "Ahmed is smart"
  - etc.
- At least two problems:
  - Does not work with infinite domains.
  - Very inefficient representation.
  - Misses important generalizations.

What Do We Need?

- We need a more expressive language.
- One that allows us to go deeper than the statement level.
- One that allows us to explicitly represent
  - Expressions like "all" and "some", called **quantifiers**.
  - Properties like "GUC student" or "smart".
  - Variables standing for arbitrary individuals.
- **(First-order) Predicate Logic** is a language that gives us all that (and more).

Vocabulary of FOPL

- The vocabulary of FOPL is the union of the following sets.
  1. A countably infinite set $\mathcal{V}$ of **variables**. Variables are typically x, y, z, etc. (possibly subscripted).
  2. A countable set $\mathcal{F}$ of **function symbols**. A function $r : \mathcal{F} \longrightarrow \mathbb{N}$ assigns to each function symbol an *arity*. Conventionally, $r(f) = n$ is alternatively indicated by f^n or f/n. A **constant** is a function symbol with zero-arity.
  3. A countable set $\mathcal{P}$ of **predicate symbols**. A function $r : \mathcal{P} \longrightarrow \mathbb{N}$ assigns to each predicate symbol an *arity*. Conventionally, $r(p) = n$ is alternatively indicated by p^n or p/n. Two distinguished zero-ary predicate symbols **T** and **F** are in $\mathcal{P}$.
  4. A set of Syncategorematic Symbols $\{\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow, \forall, \exists, (,), ,\}$

FOPL Expressions

- The sets $\mathcal{V}, \mathcal{F}$, and $\mathcal{P}$ must be disjoint and contain none of the syncategorematic symbols.
- Terminal symbols are used to construct expressions of FOPL.
- FOPL expressions come in two types: **terms** and **formulas**.

FOPL Terms

- The set $\mathcal{T}$ of FOPL terms is the union of the following sets
  1. $\mathcal{V}$
  2. $\{f(t_1, t_2, \ldots, t_n) \mid f \in \mathcal{F}, r(f) = n$, and $t_1, t_2, \ldots, t_n$ are terms $\}$
- That is, a term is a variable, a constant, or a **functional term** defined recursively in terms of function symbols and terms.
  - Constants are displayed without the empty pair of parentheses.
- **Example**
  - Let $\mathcal{V} = \{x_1, x_2, \ldots\}$,
    $\mathcal{F} = \{John^0, Mary^0, FatherOf^1, ChildOf^2\}$.
  - The following are all terms: x_1, $John$, $FatherOf(x_3)$, $ChildOf(John, Mary)$, $FatherOf(ChildOf(x_2, Mary))$, etc.

FOPL Atomic Formulas

- An **atomic formula** of FOPL is of the form $P(t_1, t_2, \ldots, t_n)$, where
  - $P \in \mathcal{P}$,
  - $r(P) = n$, and
  - $t_1, t_2, \ldots, t_n \in \mathcal{T}$.
- Note that this subsumes **T** and **F**.

FOPL Well-Formed Formulas

- The set of FOPL WFFs is the smallest set containing all of the following forms.
  - Atomic formulas.
  - $\neg\phi$, provided that ϕ is a WFF.
  - $(\phi \wedge \psi)$, provided that ϕ and ψ are WFFs.
  - $(\phi \vee \psi)$, provided that ϕ and ψ are WFFs.
  - $(\phi \Rightarrow \psi)$, provided that ϕ and ψ are WFFs.
  - $(\phi \Leftrightarrow \psi)$, provided that ϕ and ψ are WFFs.
  - $\forall x(\phi)$, where $x \in \mathcal{V}$ and ϕ is a WFF.
  - $\exists x(\phi)$, where $x \in \mathcal{V}$ and ϕ is a WFF.

Note

1. For convenience, grouping parentheses may be omitted, or replaced by square brackets.

2. The language, $\mathcal{L}_{\mathcal{T},\mathcal{P}}$, of a particular FOPL is the set of all WFFs formed according to the above rules, with $\mathcal{T}$ as the set of terms and $\mathcal{P}$ the set of predicate symbols.

3. Note that $\mathcal{T}$ and $\mathcal{P}$ are what distinguish one language of FOPL from another (i.e., these are the user-defined components of the language).

Quantified Formulas

- A **quantified formula** is a formula of the form $\forall x(\phi)$ or $\exists x(\phi)$.
- $\forall$ is a **universal quantifier**, and $\forall x(\phi)$ is a **universally-quantified** formula.
- $\exists$ is an **existential quantifier**, and $\exists x(\phi)$ is an **existentially-quantified** formula.
- In either case, ϕ is called the **scope** of the quantifier.

Free and Bound Variables

- In a quantified formula $Qx(\phi)$, the quantifier Q is said to **bind** the variable x.
- An **occurrence** of a variable x is **bound in a formula ϕ** if a sub-formula of ϕ in which that occurrence of x exists is the scope of a quantifier, in ϕ, that binds x.
  - An occurrence of a variable is one that appears in argument position. Occurrences can be identified by their left-to-right order of appearance in a WFF.
- An **occurrence** of a variable x is **free in a formula ϕ** if it is not bound in ϕ.
- Note that all occurrences of a variable x in a formula $Qx(\phi)$ are bound.

Example

- Identify all bound and free occurrences of x in $(\forall y[\exists x[\exists z[P(x, z)]] \Rightarrow Q(y, x)] \wedge R(x))$.
- **Do it yourself.**

More Terminology

- A WFF with free occurrences of variables is an **open** formula.
- A WFF with no free occurrences of variables is a **closed** formula.
- A WFF (term) with no variables is a **ground** WFF (term).

A Useful Definition

Definition. Let f be a function, $x \in Domain(f)$, and $y \in Range(f)$. $f[y/x]$ denotes a function such that $f[y/x](x) = y$ and $f[y/x](z) = f(z)$ for $z \neq x$.

Substitutions

- A **substitution** is a function $\sigma : \mathcal{V} \longrightarrow \mathcal{T}$.
- Two conventions:
  1. Instead of the standard ordered-pair notation, substitutions are usually displayed as $\{t_1/v_1, t_2/v_2 \ldots\}$, $\{v_1 \rightarrow t_1, v_2 \rightarrow t_2 \ldots\}$, or $\{v_1 := t_1, v_2 := t_2 \ldots\}$.
  2. Instead of listing all pairs of a substitution, it is conventional to drop pairs of the form x/x.
- If E is an FOPL expression and σ a substitution, then $E\sigma$ denotes the result of applying σ on E.

Substitution Application I: Terms

- Let σ be a substitution.
  1. If $x \in \mathcal{V}$, then $x\sigma = \sigma(x)$.
  2. If $t^0 \in \mathcal{T}$ is a constant, then $t\sigma = t$.
  3. If $f^n(t_1, t_2, \ldots, t_n) \in \mathcal{T}$ is a functional term $(n > 0)$, then $f^n(t_1, t_2, \ldots, t_n)\sigma = f^n(t_1\sigma, t_2\sigma, \ldots, t_n\sigma)$.

Substitution Application II: WFFs

- Let σ be a substitution.
  1. $\mathbf{T}\sigma = \mathbf{T}$.
  2. $\mathbf{F}\sigma = \mathbf{F}$.
  3. If $P^n(t_1, t_2, \ldots, t_n)$ is an atomic WFF $(n > 0)$, then $P^n(t_1, t_2, \ldots, t_n)\sigma = P^n(t_1\sigma, t_2\sigma, \ldots, t_n\sigma)$.
  4. If $\neg\phi$ is a WFF, then $\neg\phi \quad \sigma = \neg \quad \phi\sigma$.
  5. If $(\phi * \psi)$ is a WFF with $* \in \{\wedge, \vee, \Rightarrow, \Leftrightarrow\}$, then $(\phi * \psi)\sigma = (\phi\sigma * \psi\sigma)$

Substitution Application III: Quantified Formulas

- Let σ be a substitution and let Q be $\forall$ or $\exists$.
- Consider formulas of the form $Qx(\phi)$.
  1. If there is a variable y such that $x/y \in \sigma$, then $Qx(\phi)\sigma = Qz(\phi\{z/x\})\sigma$, where $z \neq x$ and z does not occur free in ϕ. This variable renaming is required here to avoid **variable capture**.
  2. Otherwise, $Qx(\phi)\sigma = Qx(\phi\tau)$, where $\tau = \sigma[x/x]$.
- Substitution application is valid only for free occurrences of variables. It should not result in binding previously free occurrences.

Semantics of FOPL

- Semantics of an FOPL language $\mathcal{L}_{\mathcal{T},\mathcal{P}}$ is based on the notion of a **structure**.
- An $\mathcal{L}_{\mathcal{T},\mathcal{P}}$-structure M is a (user-defined) triple $(\mathcal{U}, \mathcal{I}_{\mathcal{F}}, \mathcal{I}_{\mathcal{P}})$:
  1. $\mathcal{U}$, the **universe (of discourse)** (or the **domain (of discourse)**) is a non-empty set of entities.
     – The study of what exactly those entities are is what is called **ontology**. But, for now, think of ordinary physical objects.
  2. $\mathcal{I}_{\mathcal{F}} : \mathcal{F} \longrightarrow \bigcup_{i \in \mathbb{N}} \mathcal{U}^{\mathcal{U}^i}$ is a function that maps each n-ary function symbol in $\mathcal{F}$ to an n-ary *function* on $\mathcal{U}$.
  3. $\mathcal{I}_{\mathcal{P}} : \mathcal{P} \longrightarrow \bigcup_{i \in \mathbb{N}} 2^{\mathcal{U}^i}$ is a function that maps each n-ary predicate symbol in $\mathcal{P}$ to an n-ary *relation* on $\mathcal{U}$.
     – A unary relation is typically referred to as a **property**.

Interpretations, Structures, and Variable Assignments

- Semantics of FOPL expressions is defined by an interpretation function $[\![\cdot]\!]^{M,s}$, where
  1. M is an $\mathcal{L}_{\mathcal{T},\mathcal{P}}$-structure.
  2. $s : \mathcal{V} \longrightarrow \mathcal{U}$ is a **variable assignment** that assigns to each variable an entity in the universe of discourse.

Semantics of Terms

- For every $x \in \mathcal{V}$, $[\![x]\!]^{M,s} = s(x)$.
- For every constant symbol $f^0 \in \mathcal{F}$, $[\![f^0]\!]^{M,s} = \mathcal{I}_{\mathcal{F}}(f^0)$.
- For every functional term $f^n(t_1, t_2, \ldots, t_n) \in \mathcal{T}$,
  $[\![f^n(t_1, t_2, \ldots, t_n)]\!]^{M,s} = \mathcal{I}_{\mathcal{F}}(f^n)([\![t_1]\!]^{M,s}, [\![t_2]\!]^{M,s}, \ldots, [\![t_n]\!]^{M,s})$.

Semantics of Atomic WFFs

- $[\![\mathbf{T}]\!]^{M,s} = \top$.
- $[\![\mathbf{F}]\!]^{M,s} = \bot$.
- For $n > 0$, $[\![P^n(t_1, t_2, \ldots, t_n)]\!]^{M,s} = \top$ iff $([\![t_1]\!]^{M,s}, [\![t_2]\!]^{M,s}, \ldots, [\![t_n]\!]^{M,s}) \in \mathcal{I}_{\mathcal{P}}(P^n)$.

Semantics of Compound WFFs

- For $\phi, \psi \in \mathcal{L}_{\mathcal{T},\mathcal{P}}$
  1. $[\![\neg\phi]\!]^{M,s} = \top$ iff $[\![\phi]\!]^{M,s} = \bot$.
  2. $[\![(\phi \wedge \psi)]\!]^{M,s} = \top$ iff $[\![\phi]\!]^{M,s} = [\![\psi]\!]^{M,s} = \top$.
  3. $[\![(\phi \vee \psi)]\!]^{M,s} = \bot$ iff $[\![\phi]\!]^{M,s} = [\![\psi]\!]^{M,s} = \bot$.
  4. $[\![(\phi \Rightarrow \psi)]\!]^{M,s} = \bot$ iff $[\![\phi]\!]^{M,s} = \top$ and $[\![\psi]\!]^{M,s} = \bot$.
  5. $[\![(\phi \Leftrightarrow \psi)]\!]^{M,s} = \top$ iff $[\![\phi]\!]^{M,s} = [\![\psi]\!]^{M,s}$.

Semantics of Quantified WFFs

- $[\![\forall x(\phi)]\!]^{M,s} = \top$ iff $[\![\phi]\!]^{M,s[a/x]} = \top$ for every $a \in \mathcal{U}$.
- $[\![\exists x(\phi)]\!]^{M,s} = \top$ iff $[\![\phi]\!]^{M,s[a/x]} = \top$ for some $a \in \mathcal{U}$.

The Extended Light-Switch World (I)

- The extended light-switch world consists of two light switches connected in parallel to a light source.
- Suppose that one switch is up, one is down, and the light is on.
- We define a simplistic FOPL $\mathcal{L}_{\mathcal{T},\mathcal{P}}$ to discuss the extended light-switch world:
  - $\mathcal{F} = \{S1^0, S2^0, b^0\}$
  - $\mathcal{P} = \{\mathsf{Switch}^1, \mathsf{Bulb}^1, \mathsf{Up}^1, \mathsf{Down}^1, \mathsf{On}^1, \mathsf{Off}^1\}$
- Also, consider a set of variables $\mathcal{V} = \{x, y, z_1, z_2, \ldots\}$.
- We now need to define a structure M and a variable assignment s.

The Extended Light-Switch World (II)

- $M = (\mathcal{U}, \mathcal{I}_{\mathcal{F}}, \mathcal{I}_{\mathcal{P}})$, where
  - $\mathcal{U}$ is a set containing the two light switches and the light bulb. Since we cannot put the two switches and the bulb on the slide, we will only consider a representation: $\mathcal{U} = \{\mathbf{switch1}, \mathbf{switch2}, \mathbf{bulb}\}$.
  - $\mathcal{I}_{\mathcal{F}} = \{(S1, \mathbf{switch1}), (S2, \mathbf{switch2}), (b, \mathbf{bulb})\}$.
  - $\mathcal{I}_{\mathcal{P}}(\mathsf{Switch}) = \{\mathbf{switch1}, \mathbf{switch2}\}$.
  - $\mathcal{I}_{\mathcal{P}}(\mathsf{Bulb}) = \{\mathbf{bulb}\}$.
  - $\mathcal{I}_{\mathcal{P}}(\mathsf{Up}) = \{\mathbf{switch1}\}$.
  - $\mathcal{I}_{\mathcal{P}}(\mathsf{Down}) = \{\mathbf{switch2}\}$.
  - $\mathcal{I}_{\mathcal{P}}(\mathsf{On}) = \{\mathbf{bulb}\}$.
  - $\mathcal{I}_{\mathcal{P}}(\mathsf{Off}) = \{\}$.

The Extended Light-Switch World (III)

- Define s as follows:
  - $s(x) = \mathbf{switch1}$.
  - $s(y) = \mathbf{switch2}$.
  - $s(z_i) = \mathbf{bulb}$.
- Now, determine whether each of the following WFFs is true.
  1. $\exists x[\mathsf{Switch}(x) \wedge \mathsf{Up}(x)]$.
  2. $\forall x[\mathsf{Switch}(x) \Rightarrow (\mathsf{Up}(x) \Leftrightarrow \neg\mathsf{Down}(x))]$.
  3. $\forall x[(\mathsf{Bulb}(x) \wedge \mathsf{On}(x)) \Rightarrow \exists y[\mathsf{Switch}(y) \wedge \mathsf{Up}(y)]]$.

Satisfiability and Validity

- A structure M **satisfies** a WFF ϕ with variable assignment s if $[\![\phi]\!]^{M,s} = \top$. This is also denoted by $M, s \models \phi$.

- A WFF ϕ is **satisfiable** if there are M and s such that $M, s \models \phi$.

- A WFF ϕ is **valid in** M if $M, s \models \phi$ for every assignment s. This is denoted by $M \models \phi$.

- A WFF ϕ is **valid** if $M \models \phi$ for every structure M. This is denoted by $\models \phi$.

- The above terms may be extended to sets of WFFs. For example, a set Γ of WFFs is valid ($\models \Gamma$) if every WFF in Γ is valid.

Models

- A structure M is a **model** of a set of formulas Γ if $M \models \Gamma$.
- Note that, if Γ does not include any open WFFs, then M is a model of Γ if M satisfies Γ with some assignment s. **(Why?)**
- **Define logical implication for FOPL.**
- **In the extended light-switch world, for which of the WFFs discussed is the structure M a model?**

Some Important Equivalences

- $\neg\forall x(P(x)) \equiv \exists x(\neg P(x))$
- $\neg\exists x(P(x)) \equiv \forall x(\neg P(x))$
- $\forall x(P(x) \wedge Q(x)) \equiv (\forall x(P(x)) \wedge \forall x(Q(x)))$
- $\exists x(P(x) \vee Q(x)) \equiv (\exists x(P(x)) \vee \exists x(Q(x)))$

Note!

- $\forall x(P(x) \vee Q(x)) \not\equiv (\forall x(P(x)) \vee \forall x(Q(x)))$
- $\exists x(P(x) \wedge Q(x)) \not\equiv (\exists x(P(x)) \wedge \exists x(Q(x)))$

Example

- Domain is the set $\mathbb{Z}$ of integers.
- $P(x)$: x is even.
- $Q(x)$: x is odd

More FOPL Equivalences

1. $\forall x(\phi \Rightarrow \psi) \equiv (\phi \Rightarrow \forall x(\psi))$ (if x does not occur free in ϕ)
2. $\forall x(\phi \Rightarrow \psi) \equiv (\exists x(\phi) \Rightarrow \psi)$ (if x does not occur free in ψ)
3. $\exists x(\phi \Rightarrow \psi) \equiv (\phi \Rightarrow \exists x(\psi))$ (if x does not occur free in ϕ)
4. $\exists x(\phi \Rightarrow \psi) \equiv (\forall x(\phi) \Rightarrow \psi)$ (if x does not occur free in ψ)
5. $\forall x(\phi) \equiv \forall y(\phi)\{y/x\}$
6. $\exists x(\phi) \equiv \exists y(\phi)\{y/x\}$
7. $\forall x(\forall y(\phi)) \equiv \forall y(\forall x(\phi))$
8. $\exists x(\exists y(\phi)) \equiv \exists y(\exists x(\phi))$