RELEVANCE LOGIC

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Abstract

In this paper, we give a quick overview on Relevance Logic. We talk about its history and the motivation behind it. We then talk about a couple of important syntactical and semantical approaches to achieve the goal of relevance logic.

There are of course much more than that about relevance logic but unfortunately for the sake of brevity, we are not going to talk about them. These topics are like consistency and decidability results, quantification, etc.

Chapter 1

Introduction

Relevance logic, or relevant logic in Britain and Australia, is a non-classical logic where the antecedent and the consequent of an implication are relevantly related. Relevance logic was devised with the intention of avoiding the paradoxes of weak (material) and strict implication.

1.1 Origin of Relevance Logic

The idea that the premises and the conclusion must be relevant is not new. It actually dates back to the Greek philosophers in their discussion about the nature of conditionals. The Greek philosopher and physician Sextus Empiricus mentioned material implication. He said "those who introduce connection or coherence assert that it is a valid hypothetical whenever the opposite of its consequent contradicts its antecedent". In other words, he said some people claim that the only case that violates the implication $p \to q$ is $p \wedge \neg q$.

In 1975, Nuel Belnap and Alan Ross Anderson mentioned In "Entailment: The logic of Relevance and Necessity" that the earliest bedrock for relevance logic was that of Moh and Church in 1950 and 1951. However, in 1992, they published the second volume of their work stating in the preface that they missed the work of Orlov; a russian philosopher who published the paper "The logic of Compatibility of propositions" in 1928

1.2 Paradoxes of material Implication

Implication in classical logic does not capture the intuitive understanding we know about implication. Meaning that $A \to B$ holds only if there is a relation between A and B. That was the motivation behind relevance logic. The absence of this notion in classical logic actually led to some paradoxes. We here talk about some of the most common ones.

If the antecedent is false then it implies anything.

$$\neg p \to (p \to q)$$

e.g. It is the case that it is not winter now. If it is winter now, then $2 + 2 = 5$.

if the consequent is true then anything implies it.

$$p \to (q \to p)$$

e.g. It is the case that it is 2019. If it is 9:00pm now then it is 2019.

From these 2 paradoxes, we can say that for any three arbitrary propositions p , q and r, either p implies q or q implies r

$$(p \to q) \vee (q \to r)$$

This is valid since if q is true then anything implies it. If not then it implies anything. Nothing says that r and p have to be different propositions so we can end up with $(p \to q) \vee (q \to p)$. This means that for any two arbitrary propositions, one must imply the other (even if they contradict each other)

e.g. If it is 5:00pm now then it is 6:00pm now or if it is 6:00pm now then it is 5:00pm now.

1.3 Modal Logic and Strict Implication

As we can see, the previous formulae are valid though they are fallacies and sometimes paradoxical. C. I. Lewis wanted to avoid these paradoxes so he came up with modal logic and strict implication. In our first paradox, it is the case that p is false so $p \to q$ holds. In modal logic, this only holds if we can not imagine a world (not just ours) where p is true and q is false. Unfortunately we still get some paradoxes. This typically happens when the antecedent is impossible (we can not imagine it in any world)

$$(p \wedge \neg p) \to q$$

or when the consequent is necessarily true. e.g.

$$p \to (q \vee \neg q)$$

and

$$p \to (q \to q)$$

Chapter 2

Relevant Entailment

A couple of conditions were devised to enforce relevant implication. We are going to discuss two of the most prominent ones; Variable Sharing and Derivational Utility.

2.1 Variable Sharing Principle

Many philosophers and logicians argue that one of the reasons behind these paradoxes is that the antecedent and the consequent are not related.

For the antecedent and the consequent to be related, relevance logicians apply the Variable Sharing Principle. For any formula $A \to B$, the formula A and the formula B must share at least one propositional variable or otherwise $A \to B$ can not be proven. This can eliminate the paradoxes of strict implication that we mentioned since each of them has only p in the antecedent and only q in the consequent.

2.1.1 Criticism of Variable Sharing

Some paradoxes like $p \to (q \to p)$ still hold though they satisfy variable sharing. This made Anderson and Belnap say that variable sharing is a necessary condition for the relevant entailment but not sufficient.

Being necessary can be a bit controversial. Let's take this example. Someone says "I am hungry", another one replies "There are some fruits on the table". The two sentences do not seem to share a common variable though they seem relevant to each other.

2.2 Derivational Utility

Since Variable Sharing is not sufficient (and maybe not always necessary), Anderson and Belnap devised a syntactical approach to solve the problem if relevance. They called it Derivational Utility. Derivational Utility condition states that for any conclusion, the hypotheses or assumptions used to derive it must be used. They improve on Fitch style natural deduction to explicitly state the used hypotheses.

Fitch Style

To explain Fitch style, it may be better to do so using an example. let's try to prove the paradox $p \to (q \to p)$ using natural deduction rules and Fitch style in propositional logic.

1	p	hyp
2	$\quad q$	hyp
3	$\quad p$	reit
4	$q \to p$	⇒I, 2, 3
5	$p \to (q \to q)$	⇒I, 1, 4

We begin with our assumption or hypothesis (1) in the first level. We then introduce another hypothesis (2) in the second level (subproof). In this subproof we can not use formula outside it. Meaning that we can not use the first hypothesis p.If we want to use our first hypothesis in the second level, we reiterate it i.e. we just write it one more time inside our subproof. We do so in step (3). We then exit our subproof in step (4) by saying that the assumption (2) implies the conclusion (3). Then we do the same in step (5).

We can apply Derivational Utility to block that proof but first we need to give another example to see how we can imporve Fitch style. Consider $p \to ((p \to q) \to q)$

1	p	hyp, (1)
2	$\quad p \to q$	hyp, (2)
3	$\quad p$	reit, (1)
4	$\quad q$	⇒E, 2, 3, (1, 2)
5	$(p \to q) \to q$	⇒I, 2, 4, (1)
6	$p \to ((p \to q) \to q)$	⇒I, 1, 5

There are three main differences¿ We are going to state these differences first then explain our example.

- for each new hypothesis, we give it a new index (written here in parenthesis).
- each time we add a new line to our proof, we state the hypotheses used in that proof.
- each time we exit a subproof, the consequent of the implication must have the index of the hypothesis that we are discharging and that index should be subtracted from the set of indices .

The first two lines are different hypotheses so we give them two different indices (1) and (2) between parentheses. In step (3), we reiterate hypothesis (1) so we write its index. In step (4), we use arrow elimination rule on step (2) and (3) which are the two hypotheses (1) and (2) so we write both indices between parentheses. In step (5), we exit the subproof and remove the index of the second hypothesis. It is permissible to do that because we used hyp (2) to get the consequent of step (5). In step (6), We exit the other subproof and remove the index of the first hypothesis.

The paradox in the first example is unprovable under the condition of Derivational Utility.

1	$\vert\ p$	hyp, (1)
2	$\vert\quad\vert\ q$	hyp, (2)
3	$\vert\quad\vert\ p$	reit, (1)
4	$\vert\ q \to p$	⇒I, 2, 3, ()
5	$p \to (q \to q)$	⇒I, 1, 4, ()

As stated before, to exit a subproof, the conclusion (consequent) must use the hypothesis (antecedent). This means that to exit the inner subproof, hyp (2) must be used. In step (4), p is the consequent but it is a reiteration of hyp (1) which of course does not use hyp (2) as it does not have the index of hyp (2). Because of that, step (4) is invalid and the proof is blocked.

Unlike Variable Sharing, Anderson and Belnap consider Derivational Utility as a necessary and sufficient condition. Derivational Utility is also syntactical condition whereas Variable Sharing is semantical.

2.2.1 How good is Derivational Utility

Unfortunately, like Variable Sharing, there is a backdoor that can be used to construct a proof for the aforementioned paradox without breaking the condition. All what we need to do is to find a workaround to deduce p using q then we find ourselves having hyp (2) in p and thus we can exit our subproof. What better way to do that but by using $\wedge$ Intro. Here is a proof for the same paradox under Derivational Utility:-

1	p	hyp, (1)
2	$\quad q$	hyp, (2)
3	$\quad p$	reit, (1)
4	$\quad p \wedge q$	$\wedge$I, 2, 3, (1, 2)
5	$\quad p$	$\wedge$E, 4, (1, 2)
6	$q \rightarrow p$	$\Rightarrow$I, 2, 5, (1)
7	$p \rightarrow (q \rightarrow q)$	$\Rightarrow$I, 1, 6

We could overcome the condition by introducing two intermediate steps. Step (3) (p) is a reiteration of hyp (1) thus it was not derived from hyp (2) which we need to exit the subproof. However, We can use $\wedge$ Intro on p and q

like in step (4) then we can eliminate it again. By doing so we could obtain p that was derived using both the hyp (1) and hyp (2). Now the proof can continue normally and exit the two subproofs ending with the same paradox again.

Possible Solution

There is a solution to this problem. It puts some restrictions on using $\wedge$ Introduction and elimination and splits $\wedge$ into intensional and extensional ones. Most importantly, it says the following

$$\text{From } A_i \text{ and } B_i \text{ to infer } (A \wedge B)_i$$

This means that for the two formulas A and B to be conjoined, both of them must have the same hyp indices thus ending up with the same set of indices. In our example they had two different hyp indices which were (1) and (2) respectively.

Chapter 3

Semantics

In chapter 3, we investigated the conditions that we can restrain our deduction rules to so that we can accomplish relevance. Here we will discuss more semantical approaches and possible their possible interpretations.

Many different semantics were developed for relevance logics; operational semantics (also known as semilattice semantics), algebraic semantics and ternary relation semantics. The ternary relation semantics is the standard semantics for relevance logics.

3.1 Ternary Relation Semantics

In 1970s, Routley and Meyer developed the ternary relation semanitcs. The idea behind it is similar to Kripke semantics in modal logic. It also uses worlds that are connected by some relation function but that function connects three worlds with each other not just two as in Kripke's. In modal logic, for A to imply B in some world W, it must be the case that in every world

W' that is accessible from W, either A is false or B is true. In Routly-Meyer semantics, we go one step further. $A \to B$ in some world x if and only if A is false in the world y or B is true in the world z where the relation function relates x, y and z (written $Rxyz$).

The strict implication paradox $p \to (q \to q)$ is not valid with Routly-Meyer semantics. $q \to q$ is false if q holds in world y but does not hold in world z. However, so far we still get the other two paradoxes $(p \wedge \neg p) \to q$ and $p \to (q \vee \neg q)$

3.1.1 Interpretation

So, what does it mean to have ternary relations on the set of worlds?

There are a couple of interpretations for this semantics. Three important ones are Information-Based Interpretations, Conditionality Interpretation and The Truthmaker Interpretation.

Information-Based Interpretations

The basic idea behind it goes back to Urquharts semilattice semantics in 1972. It considers each of x, y, z where $Rxyz$ holds as pieces of information. Also $z = x \sqcup y$. This means that the information contained in x and the information contained in y are contained in z. This was revised later and it turned out that it is better to consider $x \sqcup y \sqsubseteq z$.

Another similar interpretation considers x to be a medium that transfers information (information channel) and y and z to be two information sites. $A \to B$ holds at x if and only if x connects the two sites y and z and whenever A is true in y, B is true in z.

The last information-based interpretation that we are going to talk about is the most recent one and was developed by Mares in 2004. It considers x, y and z as situations. Situations are parts of the universe. the more situations we know, the more we know about the universe. Let's consider x to be the situation of the laws of general relativity, y to be a hypothesized situation of a star moving in an ellipse. Using the information from x and the information from y we can infer a situation z stating that there is a very heavy object acting on that star. To formalize it, x contains $A \to B$ if there is a hypothesized situation y that contains A and from it and the other information in x, we can derive a situation z where B holds

3.1.2 Constraints on R

The ternary relation semantics sounds at first a bit odd because it is not as intuitive as Kripke's. However, it turns out to be a powerful semantics. Actually it is a generalization of Kripke semantics. if we set y to be the same world as z, we end up with Kripke semantics. Similarly we can impose other constraints on R to make other rules of inference valid and to obtain different logics (including both relevance and non-relevance logics like modal logic). For instance, if we say that $Rwww$ (as in classical logic) holds for any world w, then $(A \to B), A \vdash B$. This condition by the way is called pseudo-modus ponens

Chapter 4

Relevance Logics

There are many different relevance logics. They are obtained by applying more constraints on the ternary Relation R or in other words, adding more axioms.

The weakest logic is called B. It has the least number of axioms which are 9 and no constraints on R.

There are 12 more axioms that can be used to obtain other logics. All of these axioms are achieved by restricting R more.

The strongest relevance logic is called R. It has 11 out of 12 of these extra axioms. that last axiom is called weakening and it is $A \to (B \to A)$. Yes, it is the axiom that causes the very same paradox $p \to (q \to p)$ hence the name and hence it is not an axiom of the strong logic R

References

Chapter 5, Relevance Logic

Recent Work in Relevant Logic

Relevance Logic, Stanford Encyclopedia of Philosophy