①

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problem 2.16

ⓐ we choose $D+1$ distinct points in $\mathbb{R}$: $x_0, x_1, \dots, x_D$

let $$X = \begin{pmatrix} 1 & x_0^1 & \dots & x_0^D \\ 1 & x_1^1 & & x_1^D \\ \vdots & \vdots & & \vdots \\ 1 & x_D^1 & & x_D^D \end{pmatrix}_{D+1 \times D+1}$$

$\det(X) \neq 0$ as x_k are all different

dichotomy $y = (y_0, \dots, y_D)^T \in \{-1, 1\}^{D+1}$

let $c = (c_0, \dots, c_D)^T = X^{-1} y$

$Xc = y$

$$h_c(x_k) = \operatorname{sign} \sum_{i=0}^{D} c_i x_k^i = y_k$$

for all $k = 0, \dots, D$

H shatters $x_0, \dots, x_D$ so $m_H(D+1) = 2^{D+1}$ and $d_{vc}(H) \geq D+1$

ⓑ we choose $D+2$ points in $\mathbb{R}$, $x_0, \dots, x_{D+1}$

$D+2$ vectors $(x_k^0, x_k^1, \dots, x_k^D)$ $(k = 0, 1, \dots, D+1)$ are $D+2$ vectors in $D+1$ dimentions

they are linearly dependent $\rightarrow$ There exist index l and $D+1$ coefficients a_k (not all equal to zeros)

$$(x_l^0, x_l^1, \dots, x_l^D) = \sum_{k \neq l} a_k (x_k^0, x_k^1, \dots, x_k^D)$$

dichotomy $y \Rightarrow y_k = \operatorname{sign}(a_k)$)

$c = (c_0, \dots, c_D)^T$

$$(x_l^0, x_l^1, \dots, x_l^D) \begin{pmatrix} c_0 \\ \vdots \\ c_D \end{pmatrix} = \sum_{k \neq l} a_k (x_k^0, x_k^1, \dots, x_k^D) \begin{pmatrix} c_0 \\ \vdots \\ c_D \end{pmatrix} = \sum_{k \neq l} \sum_{i=0} c_i a_k x_k^i$$

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Date / / No

$$(x_l^0, x_l^1, \dots x_l^D)\begin{pmatrix} c_0 \\ \vdots \\ c_D \end{pmatrix} = \sum_{k \neq l} a_k (x_k^0, \dots x_k^D)\begin{pmatrix} c_0 \\ \vdots \\ c_D \end{pmatrix}$$

$$= \sum_{k \neq l} \sum_{i=0}^{D} c_i a_k x_k^i$$

lets assume There exists $c \in \mathbb{R}^{D+1}$ so that

$$y_k = h_c(x_k) = \text{sign}\left(\sum_{i=0}^{D} c_i x_k^i\right)$$

for any k so that $a_k \neq 0$ and

$$y_l = h_c(x_l) = \text{sign}\left(\sum_{i=0}^{D} c_i x_l^i\right)$$

in this case we would get

$$\text{sign}(a_k) = y_k = \text{sign}\left(\sum_{i=0}^{D} c_i x_k^i\right)$$
$$\Rightarrow \sum_{i=0}^{D} c_i a_k x_k^i > 0$$

for any k so that $a_k \neq 0$ so we also get

$$\sum_{k \neq l} \sum_{i=0}^{D} c_i a_k x_k^i = (x_l^0, x_l^1, \dots x_l^D)\begin{pmatrix} c_0 \\ \vdots \\ c_D \end{pmatrix} = \sum_{i=0}^{D} c_i x_l^i > 0$$

$$y_l = \text{sign} \sum_{i=0}^{D} c_i x_l^i = +1$$

However we had $y_l = -1 \Rightarrow$ we have dichotomy $y \in \mathbb{R}^{D+2}$ that cannot be generated by H This implies that

$$m_H(D+2) < 2^{D+2}$$

$$d_{vc}(H) \leq D+1$$

Dar kareem

problem 2.24

(a) $\bar{g}(x) = E_D[g(x)]$

$$= E_D\left[\frac{y_1 - y_2}{x_1 - x_2}x + \frac{x_1 y_2 - x_2 y_1}{x_1 - x_2}\right]$$

$$= \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}\frac{x_1^2 - x_2^2}{x_1 - x_2}\,dx_1\,dx_2 \cdot x + \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}\frac{x_1 x_2^2 - x_2 x_1^2}{x_1 - x_2}\,dx_1\,dx_2$$

$$= \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}(x_1 + x_2)\,dx_1\,dx_2 \cdot x - \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}x_1 x_2\,dx_1\,dx_2$$

$$= \frac{1}{4}\times 0 - \frac{1}{4}\times 0 = 0$$

(d) $E_{out} = E_x\left[(g(x) - f(x))^2\right] = E_x\left[(ax + b - x^2)^2\right]$

$$= E_x[x^4] - 2aE_x[x^3] + (a^2 - 2b)E_x[x^2] + 2abE_x[x] + b^2$$

$$= \frac{1}{2}\int_{-1}^{1}x^4\,dx - 2a\frac{1}{2}\int_{-1}^{1}x^3\,dx + (a^2 - 2b)\frac{1}{2}\int_{-1}^{1}x^2\,dx + 2ab\frac{1}{2}\int_{-1}^{1}x\,dx + b^2$$

$$= \frac{1}{5} + \frac{(a^2 - 2b)}{3} + b^2$$

we Take expectation with respect to D to get the test performance and we replace a and b by $(x_1 + x_2)$ and $(-x_1 x_2)$ respectively, we get

$$E_D[E_{out}] = \frac{1}{5} + \frac{1}{3}E_D\left[(x_1 + x_2)^2 + 2x_1 x_2\right] + E_D[x_1^2 x_2^2]$$

$$= \frac{1}{5} + \frac{1}{3}\times\frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}(x_1^2 + x_2^2 + 4x_1 x_2)\,dx_1\,dx_2 + \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}x_1^2 x_2^2\,dx_1\,dx_2$$

$$= \frac{1}{5} + \frac{1}{3}\times\frac{1}{4}\times\frac{8}{3} + \frac{1}{4}\times\frac{4}{9} = \frac{8}{15}$$

$$\text{bias}(x) = (\bar{g}(x) - f(x))^2 = f(x)^2 = x^4$$

$$\text{bias} = E_x[x^4] = \frac{1}{2}\int_{-1}^{1}x^4\,dx = \frac{1}{5}$$

(4)

Date / / No

$$\mathrm{var}(x) = E_D\left[\left(g(x) - \bar{g}(x)\right)^2\right] = E_D\left[a^2x^2 + 2abx + b^2\right]$$

$$= E_D[a^2] \times x^2 + 2\,E_D[ab] \times x + E_D[b^2]$$

$$= E_D\left[(x_1 + x_2)^2\right] x^2 - 2E_D\left[(x_1 + x_2)\,x_1 x_2\right] x + E_D\left[x_1^2 x_2^2\right]$$

$$= \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}\left(x_1^2 + 2x_1x_2 + x_2^2\right) dx_1\,dx_2\; x^2 - \frac{2}{4}\int_{-1}^{1}\int_{-1}^{1}\left(x_1^2 x_2 + x_1 x_2^2\right) dx_1\,dx_2\; x$$

$$+ \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} x_1^2 x_2^2\, dx_1\,dx_2$$

$$= \frac{1}{4}\left(\frac{4}{3} + 0 + \frac{4}{3}\right) x^2 - 0 \times x + \frac{1}{4} \times \frac{4}{9} = \frac{2}{3}x^2 + \frac{1}{9}$$

$$\mathrm{Var} = E_x\left[\frac{2}{3}x^2 + \frac{1}{9}\right] = \frac{2}{3} \times \frac{1}{2}\int_{-1}^{1} x^2\, dx + \frac{1}{9} = \frac{1}{3}$$

It is close to the numerically obtaind ones