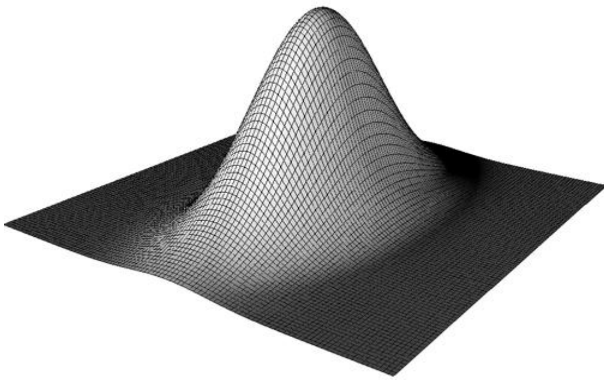
Computer Vision and Pattern Recognition

L18. Contours, Curves and Splines



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Learning Outcomes

After attending this lecture you should be able to:

- Contrast the explicit, parametric, and implict forms of a line
- Describing how piecewise cubic curves can be used to model contours
- Define different orders of curve continuity
- Explain the behaviour of the Hermite and Bezier curve families
- Apply the Catmull-Rom and B-spline approaches to modelling splines
- Define the components of the Frenet frame

Contours

Contours are often used to model the shape of an object

Fitting a contour to an edge map is a common way to find objects

Contours are **curves** – either **open** or **closed**

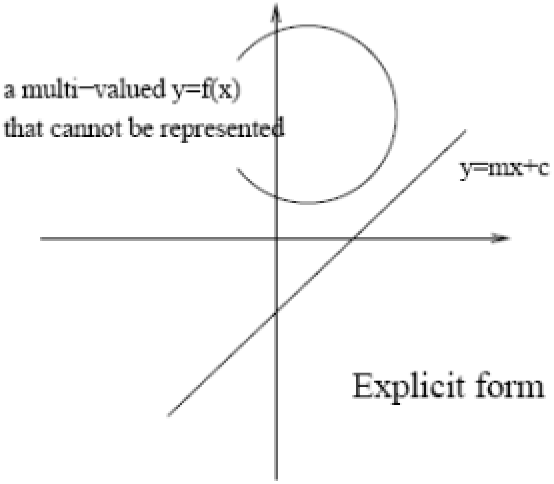
Representations of a line

A straight line can be expressed in three forms:-

1. Explicit *i.e. y=f(x)*

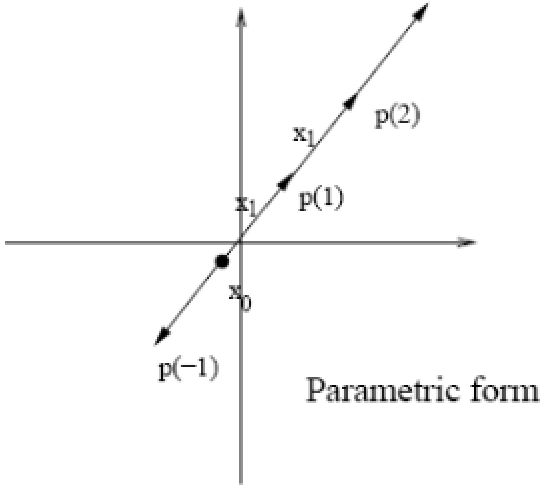
$$y = mx + c$$

(which is restrictive in the lines we can represent)



2. Parametric *(line drawn as 's' is varied)*

$$\underline{p}(s) = \underline{x_0} + s\underline{x_1}$$



To simplify presentation in this talk vectors are single underlined and matrices are double-underlined. Numbers (scalars) have no underlining.

Representations of a line

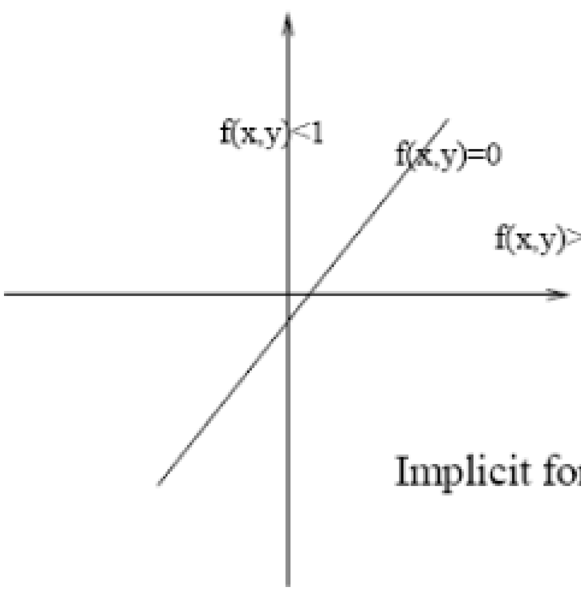
3. Implicit *i.e. f(x,y)=0*

e.g. can derived from the parametric form of a line

$$\underline{p}(s) = \underline{x_0} + s\underline{x_1} \quad \Rightarrow \quad \begin{aligned} x &= x_0 + su \\ y &= y_0 + sv \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} \frac{x - x_0}{u} &= s = \frac{y - y_0}{v} \\ (x - x_0)v &= (y - y_0)u \\ (x - x_0)v - (y - y_0)u &= 0 \end{aligned}$$

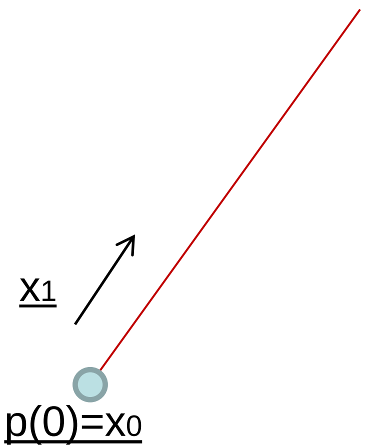


Implicit form

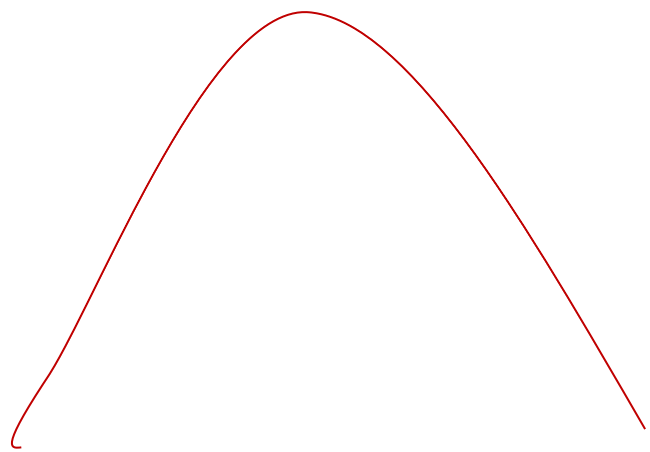
Parametric curve

Adding higher order terms increases the number of ‘turns’ on curve

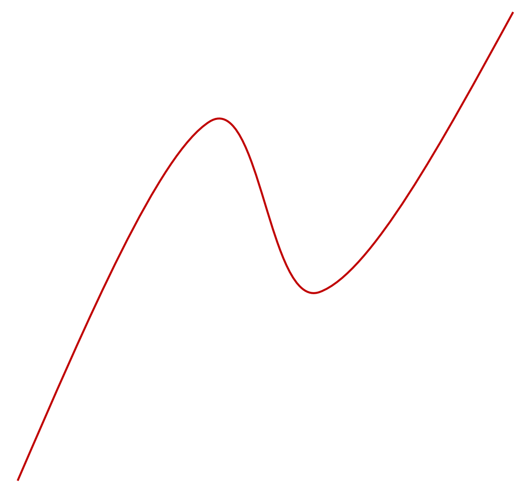
$$\underline{p}(s) = \underline{x}_0 + s\underline{x}_1 \cdot$$



Line (order 1)



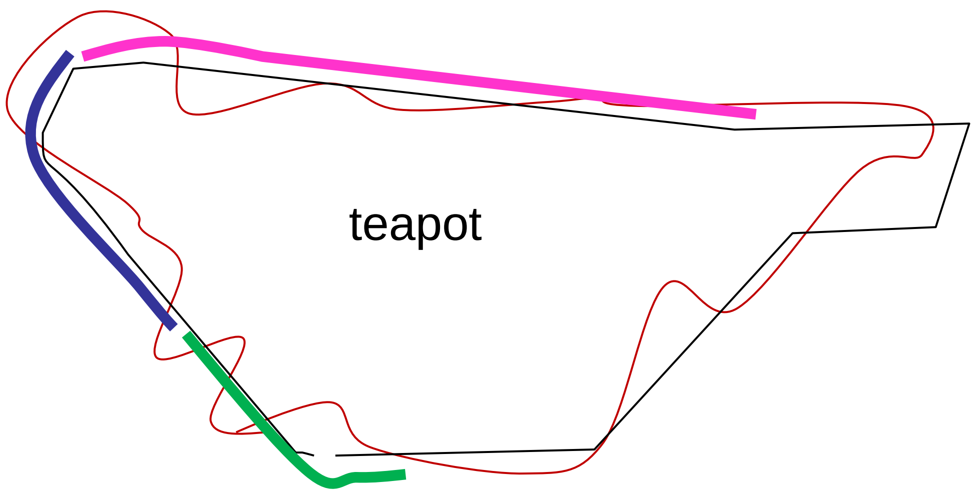
Quadratic (order 2)



Cubic (order 3)

Piecewise modelling

It is difficult to model complex shapes using a single high order poly



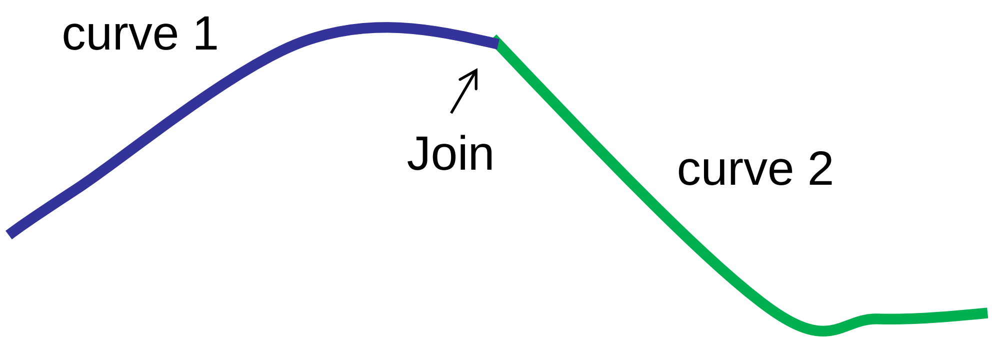
Therefore we model complex shape in **piecewise** form using **several cubic curves.**

$$\underline{p}(s) = \underline{x_0} + s\underline{x_1} + s^2\underline{x_2} + s^3\underline{x_3}$$

Curve Continuity

Ideally we would "like the **piecewise curves** to be **continuous**"

But there are many ways to define continuity



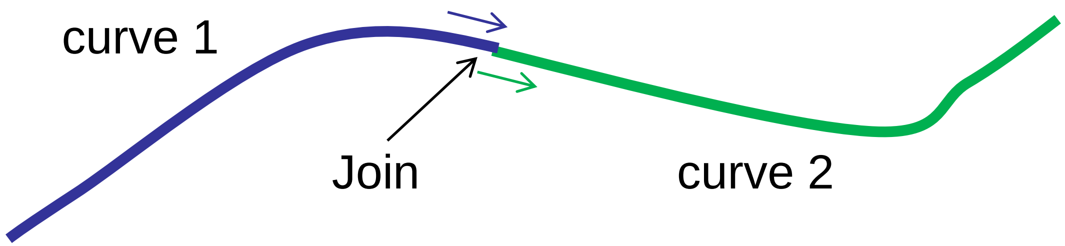
This curve is **C^0 continuous** – the ends meet but there is a kink in the curve as the tangents are not equal at the join.

It is continuous but only to the 0^{th} derivative of the curve p(s)

Curve Continuity

These curves are C^1 continuous because their tangents are equal.

Under definition of C^n continuity, C^1 implies C^0 *(and so on...)*



It is possible to define curves of higher order continuity than C1 but this is rarely useful.

Piecewise curves are useful in Computer graphics and Computer vision

Parametric curve

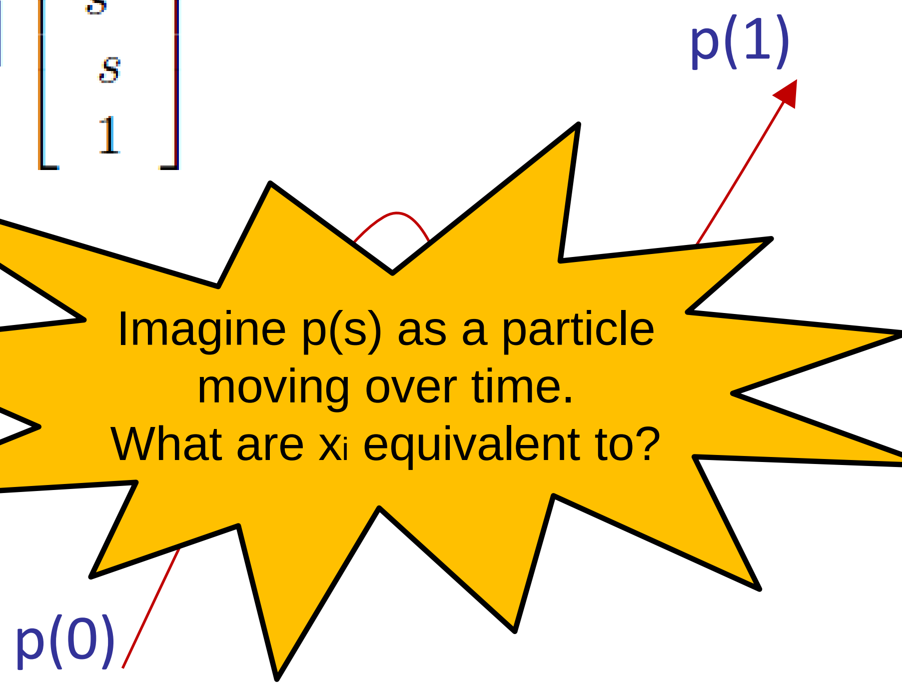
We can rewrite this in a matrix form. For a **cubic** curve:

$$\underline{p}(s) = \underline{x}_0 + s\underline{x}_1 + s^2\underline{x}_2 + s^3\underline{x}_3$$

$$\underline{p}(s) \quad = \quad \begin{bmatrix} \underline{x}_3 & \underline{x}_2 & \underline{x}_1 & \underline{x}_0 \end{bmatrix} \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix}$$

$$\underline{p}(s) \quad = \quad \underline{\underline{C}}\,\underline{Q}(s)$$

By convention, s = [0,1]



Imagine p(s) as a particle moving over time.
What are x_i equivalent to?

Simplifying control

We can generalise from:

$$\underline{p}(s) = \begin{bmatrix} \underline{x_3} & \underline{x_2} & \underline{x_1} & \underline{x_0} \end{bmatrix} \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix}$$

Inserting a matrix “M” called the **blending matrix**

$$\underline{p}(s) = \begin{bmatrix} \underline{x_3} & \underline{x_2} & \underline{x_1} & \underline{x_0} \end{bmatrix} \begin{bmatrix} & & & \\ & \text{“M”} & & \\ & \text{(4x4)} & & \\ & & & \end{bmatrix} \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix}$$

$$\boxed{\underline{p}(s) = \underline{\underline{G}}\,\underline{\underline{M}}\,\underline{Q}(s)}$$

We call **G** the **geometry matrix** as it defines the shape of the curve

M changes the meaning of G to something more intuitive to control

Hermite Curve

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The Hermite curve has the form:

$$\underline{p}(s) = \begin{bmatrix} \underline{p}(0) & \underline{p}(1) & \underline{p}'(0) & \underline{p}'(1) \end{bmatrix} \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix}$$

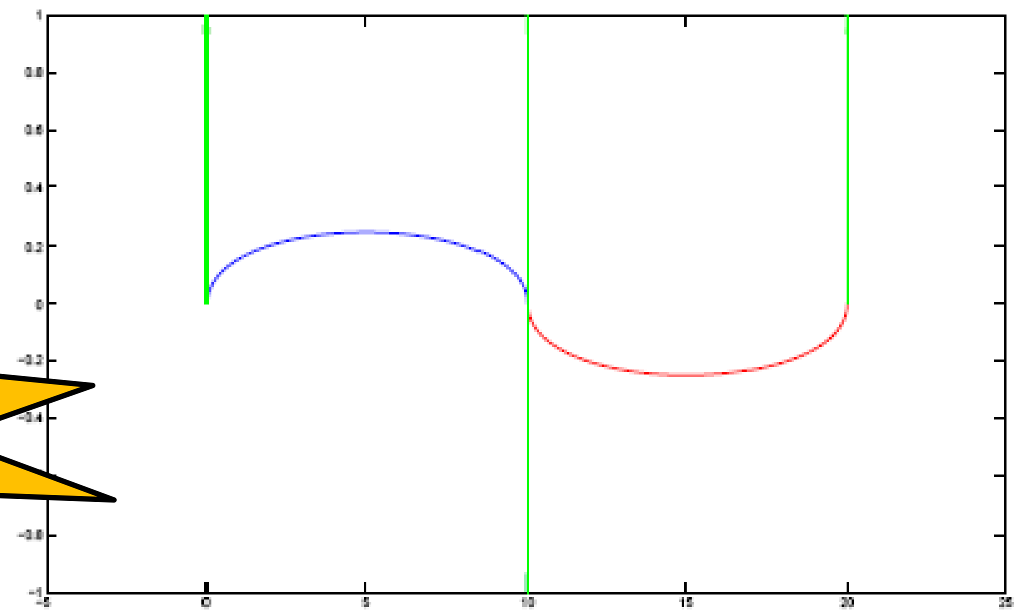
start point

end point

start tangent

end tangent

How would you create a C_1 continuous piecewise curve?



Tangent to a parametric curve

How would you compute the tangent to a parametric curve p(s) ?

$$\underline{p}(s) = \underline{\underline{GM}} \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix}$$

Answer:

$$\underline{p'}(s) = \underline{\underline{GM}} \begin{bmatrix} 3s^2 \\ 2s \\ 1 \\ 0 \end{bmatrix}$$

Hermite Curve - Derivation

The position and tangent of a cubic curve at 's' is:

$$\left[\ \underline{p}(s) \quad \underline{p}'(s)\ \right] = \underline{\underline{GM}} \begin{bmatrix} s^3 & 3s^2 \\ s^2 & 2s \\ s & 1 \\ 1 & 0 \end{bmatrix}$$

Writing the position and tangent at s=0, s=1 as per G in last frame:

$$\left[\ \underline{p}(0) \quad \underline{p}(1) \quad \underline{p}'(0) \quad \underline{p}'(1)\ \right] = \left[\ \underline{p}(0) \quad \underline{p}(1) \quad \underline{p}'(0) \quad \underline{p}'(1)\ \right] \underline{\underline{M}} \begin{bmatrix} s^3 & s^3 & 3s^2 & 3s^2 \\ s^2 & s^2 & 2s & 2s \\ s & s & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\left[\ \underline{p}(0) \quad \underline{p}(1) \quad \underline{p}'(0) \quad \underline{p}'(1)\ \right] = \left[\ \underline{p}(0) \quad \underline{p}(1) \quad \underline{p}'(0) \quad \underline{p}'(1)\ \right] \underline{\underline{M}} \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Hermite Curve - Derivation

$$\begin{bmatrix} \underline{p}(0) & \underline{p}(1) & \underline{p}'(0) & \underline{p}'(1) \end{bmatrix} = \begin{bmatrix} \underline{p}(0) & \underline{p}(1) & \underline{p}'(0) & \underline{p}'(1) \end{bmatrix} \underline{\underline{M}} \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Cancelling the left hand side (G) we get:

$$\underline{\underline{I}} = \underline{\underline{M}} \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{M}} = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}^{-1}$$

Bezier Curve

The Bezier curve approximates its control points (in general)

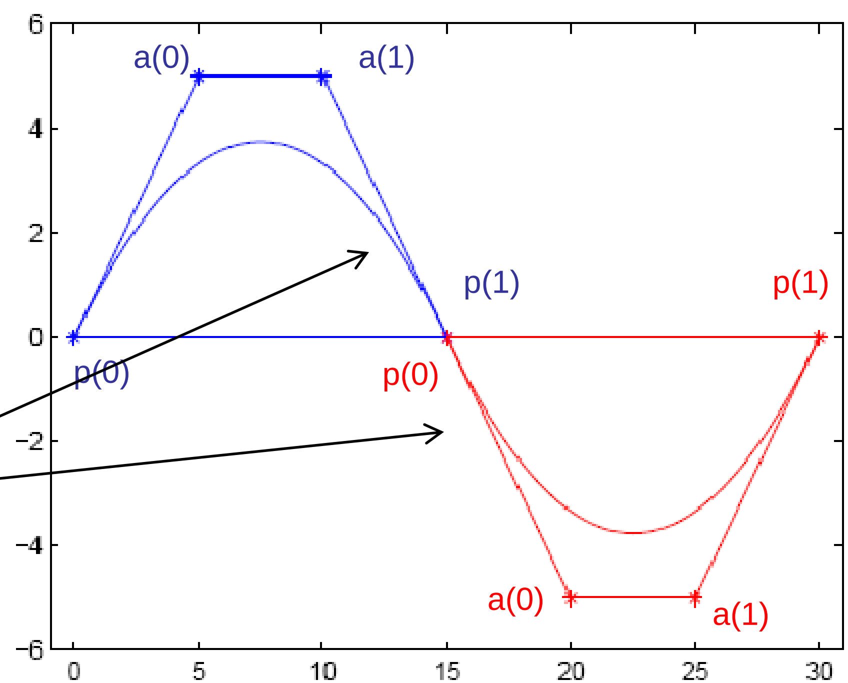
$$\underline{p}(s) = \left[\ \underline{p}(0) \quad \underline{a}_0 \quad \underline{a}_1 \quad \underline{p}(1)\ \right] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix}$$

start point

approximated point 0

approximated point 1

end point



If the start/end points are coincident then = curve is C0

If the approximating points a(1) on curve 1 and a(0) on curve 2 are co-linear and equidistant from the join = curve is C1

Bezier Curve - Explanation

Multiply out the blending matrix:

$$\underline{p}(s) = \begin{bmatrix} \underline{p}(0) & \underline{a_0} & \underline{a_1} & \underline{p}(1) \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix}$$

start point

approximated point 0

approximated point 1

end point

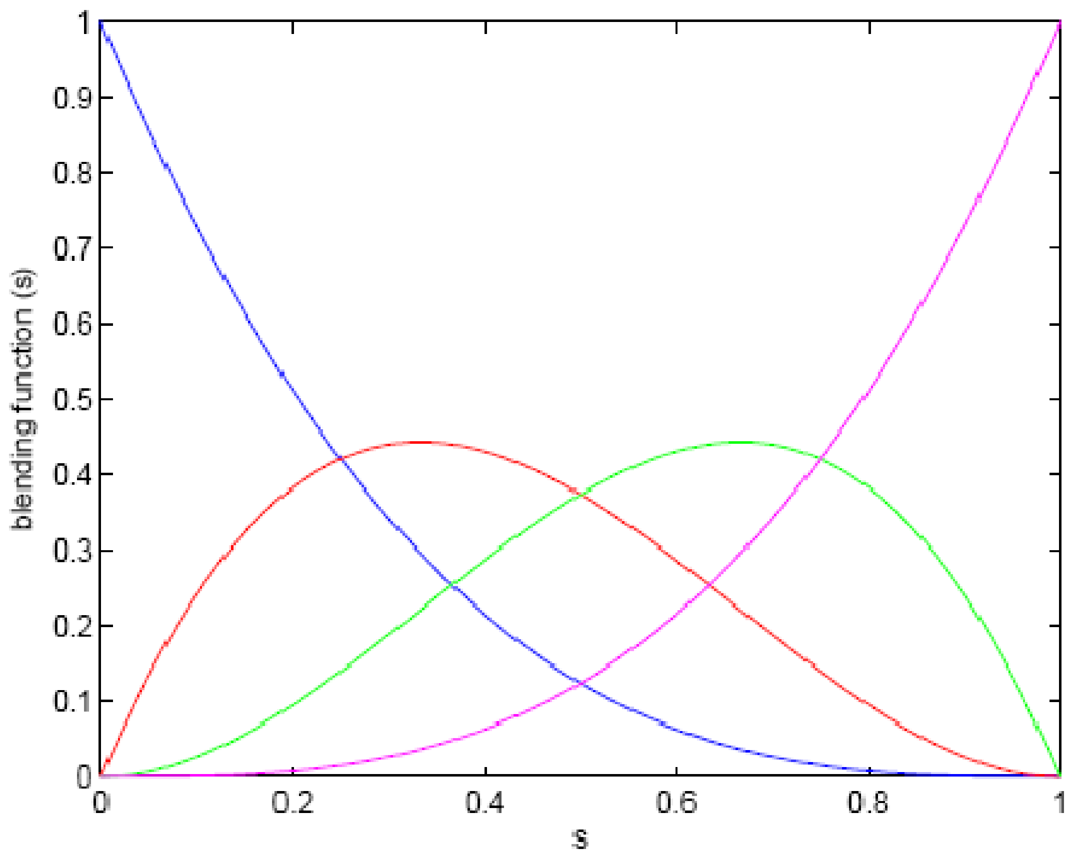
$$\underline{p}(s) = \underline{G}\,\underline{M}\,\underline{\underline{Q}}(s)$$

$$\underline{M}\,\underline{\underline{Q}}(s) \quad = \quad \begin{bmatrix} -s^3 + 3s^2 - 3s + 1 \\ 3s^3 - 6s^2 + 3s \\ -3s^2 + 3s^2 \\ s^3 \end{bmatrix}$$

Bezier Curve - Explanation

Multiply out the blending matrix:

$$\underline{p}(s) = \underline{\underline{GM}}\,\underline{Q}(s) \quad = \quad \begin{bmatrix} \underline{p}(0) & \underline{a}_0 & \underline{a}_1 & \underline{p}(1) \end{bmatrix} \begin{bmatrix} -s^3 + 3s^2 - 3s + 1 \\ 3s^3 - 6s^2 + 3s \\ -3s^2 + 3s^2 \\ s^3 \end{bmatrix}$$



Catmull-Rom Spline

The Catmull-Rom <u>spline</u> interpolates all its control points with C_1

$$\underline{p}(s) = \begin{bmatrix} \underline{a} & \underline{p}(0) & \underline{p}(1) & \underline{b} \end{bmatrix} \frac{1}{2} \begin{bmatrix} -1 & 2 & -1 & 0 \\ 3 & -5 & 0 & 2 \\ -3 & 4 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix}$$

$$\underline{p}(s) = \underline{\underline{G_j}}\,\underline{\underline{M}}\,\underline{\underline{Q}}(s)$$

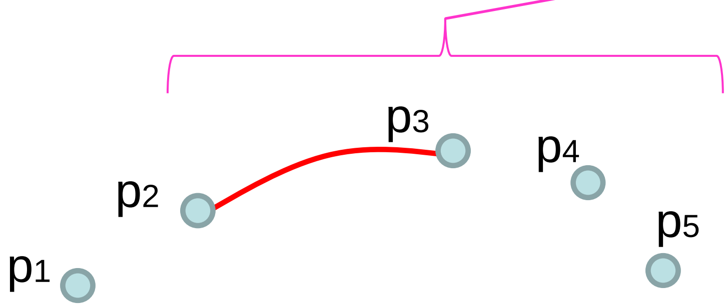
$$\underline{\underline{G_1}} = \begin{bmatrix} \underline{p_1} & \underline{p_1} & \underline{p_2} & \underline{p_3} \end{bmatrix}$$

$$\underline{\underline{G_2}} = \begin{bmatrix} \underline{p_1} & \underline{p_2} & \underline{p_3} & \underline{p_4} \end{bmatrix}$$

$$\underline{\underline{G_3}} = \begin{bmatrix} \underline{p_2} & \underline{p_3} & \underline{p_4} & \underline{p_5} \end{bmatrix}$$

$$\underline{\underline{G_4}} = \begin{bmatrix} \underline{p_3} & \underline{p_4} & \underline{p_5} & \underline{p_6} \end{bmatrix}$$

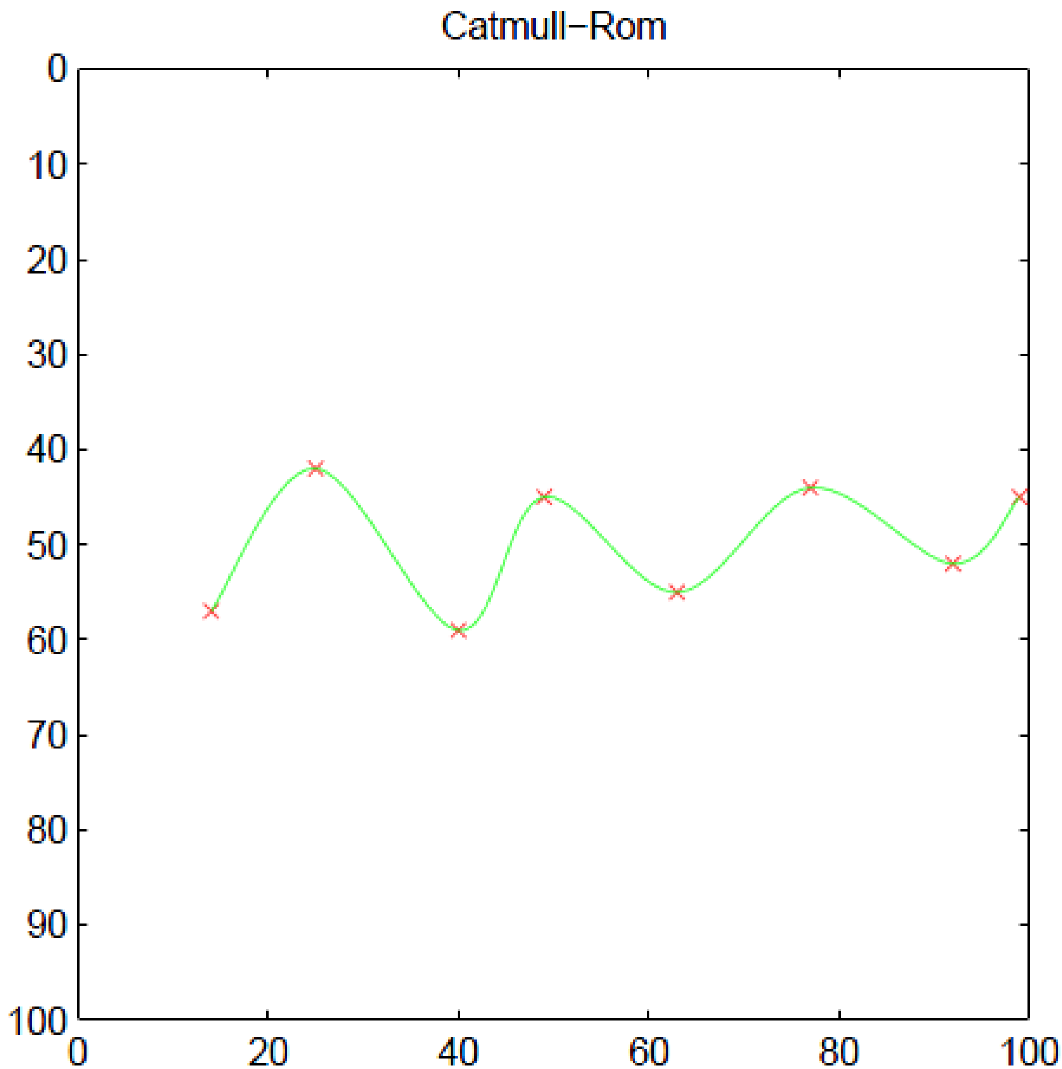
$$\underline{\underline{G_5}} = \begin{bmatrix} \underline{p_4} & \underline{p_5} & \underline{p_6} & \underline{p_6} \end{bmatrix}$$



Use N-1 piecewise curves to interpolate N control points $p_{1 \dots N}$

Catmull-Rom Spline

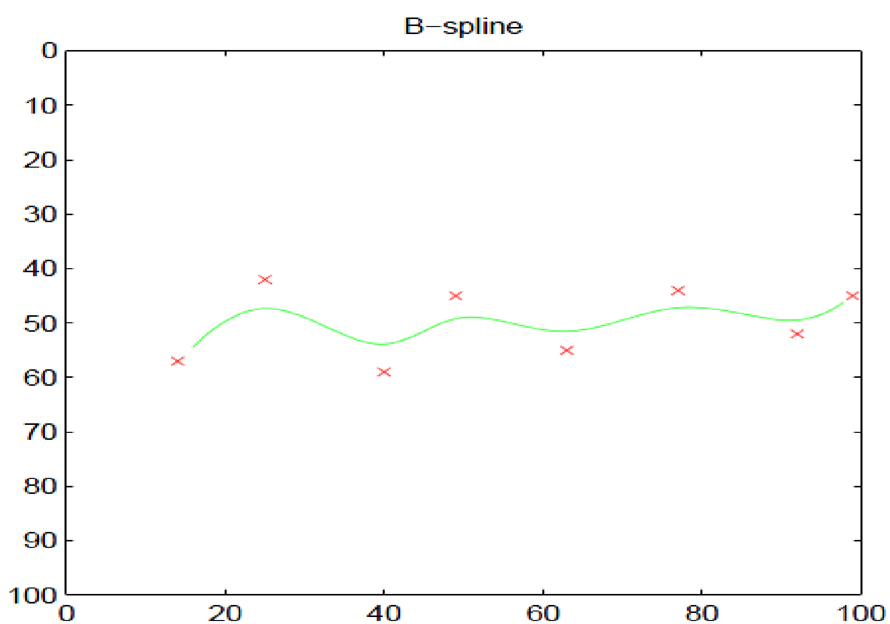
The Catmull-Rom spline is an interpolating curve



β-Spline

The β-spline is similar to Catmull-Rom but approximates all control points with C_1 (this can be useful to smooth noisy control points)

$$\underline{p}(s) = \begin{bmatrix} \underline{a} & \underline{p}(0) & \underline{p}(1) & \underline{b} \end{bmatrix} \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 0 & 4 \\ -3 & 3 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix}$$



Curvature of a parametric curve

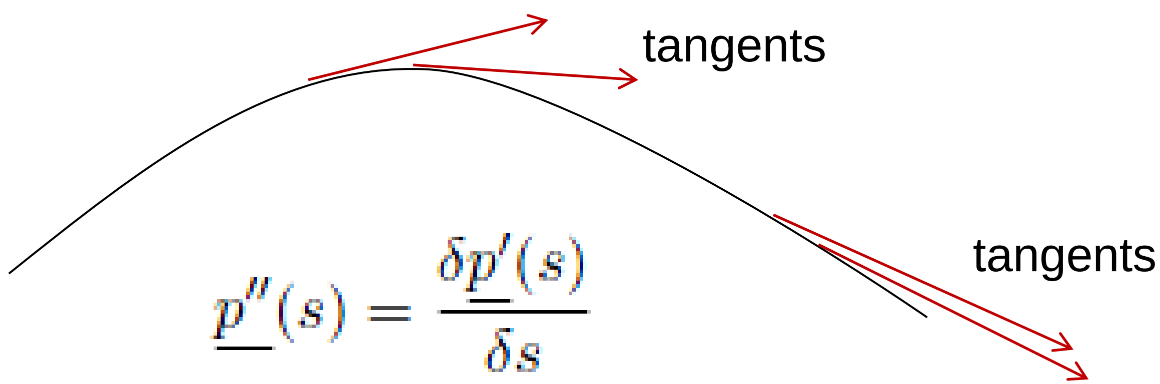
What does the **rate of change of the tangent** give us?

$$\underline{p}(s) = \underline{\underline{GM}} \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix}$$

$$\underline{p}'(s) = \frac{\delta \underline{p}(s)}{\delta s}$$

$$\underline{p}'(s) = \underline{\underline{GM}} \begin{bmatrix} 3s^2 \\ 2s \\ 1 \\ 0 \end{bmatrix}$$

The **magnitude** of the second derivative is the **curvature**



$$\underline{p}''(s) = \frac{\delta \underline{p}'(s)}{\delta s}$$

Frenet Frame

The first and second derivatives of the curve give us two components of the Frenet Frame – the natural reference frame defined at p(s)

1) Tangent

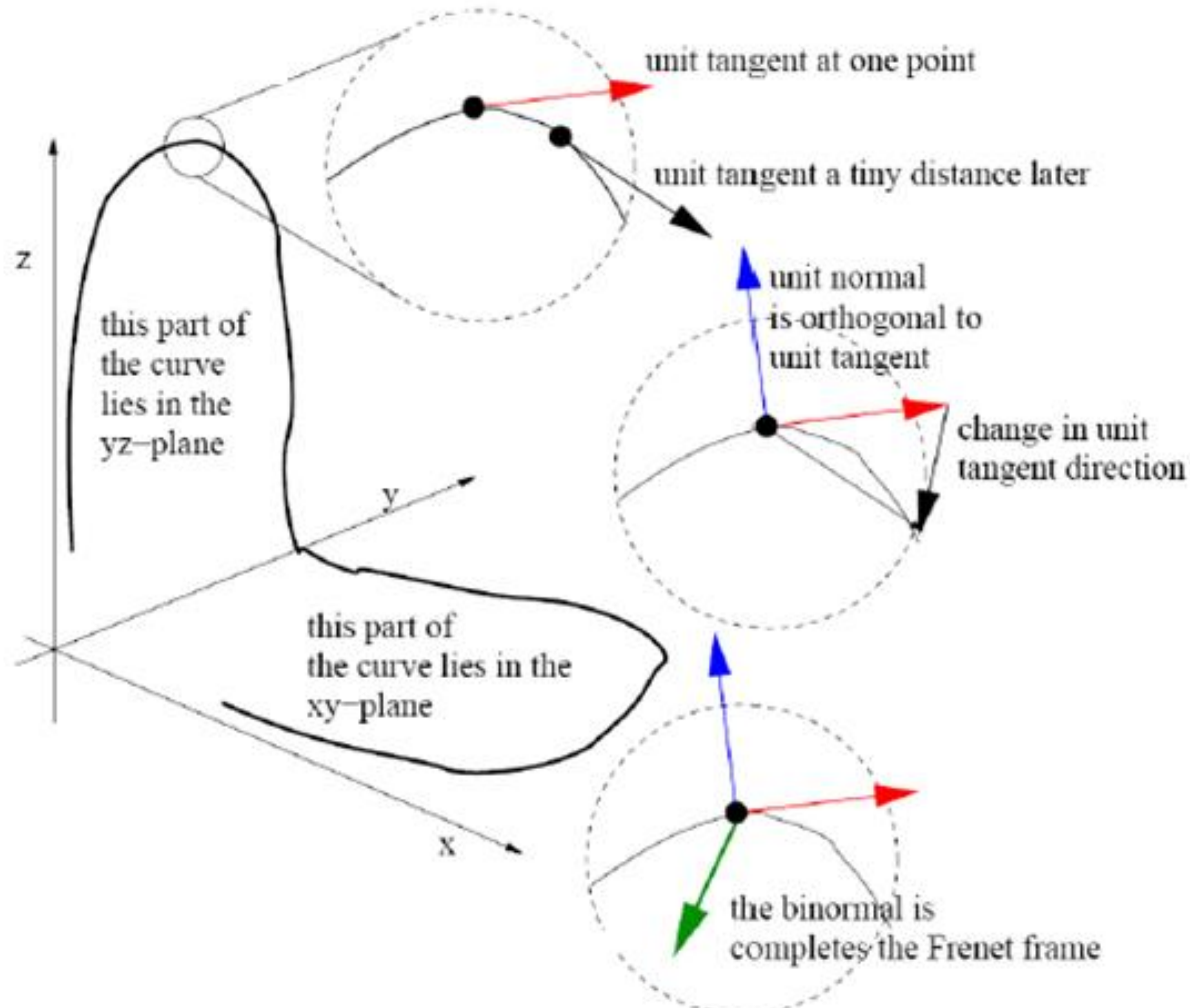
$$\underline{p}'(s) = \frac{\delta \underline{p}(s)}{\delta s}$$

2) Normal

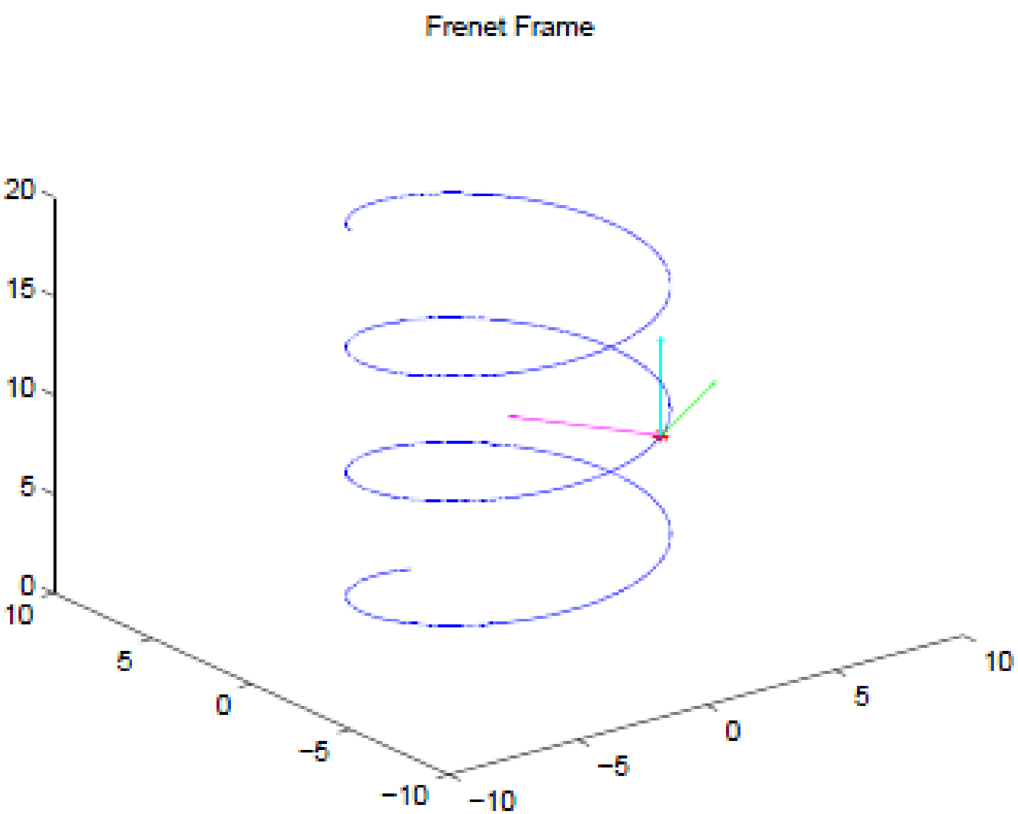
$$\underline{p}''(s) = \frac{\delta \underline{p}'(s)}{\delta s}$$

3) Binormal

$$\underline{b}(s) = \underline{p}'(s) \times \underline{p}''(s)$$



Frenet Frame Demo



$$\underline{p}(\theta) = \begin{bmatrix} -\cos\theta \\ -\sin\theta \\ \theta \end{bmatrix}$$

1) Tangent

$$\underline{p}'(\theta) = \begin{bmatrix} \sin\theta \\ -\cos\theta \\ 1 \end{bmatrix}$$

2) Normal

$$\underline{p}''(\theta) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

3) Binormal

$$\underline{p}'(\theta) \times \underline{p}''(\theta).$$

Frenet Frame

Summary

After attending this lecture you should be able to:

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