Classification using Logistic Regression

This lecture covers

- **Logistic regression hypothesis**
- **Decision Boundary**
- **Cost function**
  (why we need a new one)
- **Simplified Cost function & Gradient Descent**
- **Advanced Optimization Algorithms**
- **Multiclass classification**

Logistic regression
Hypothesis Representation

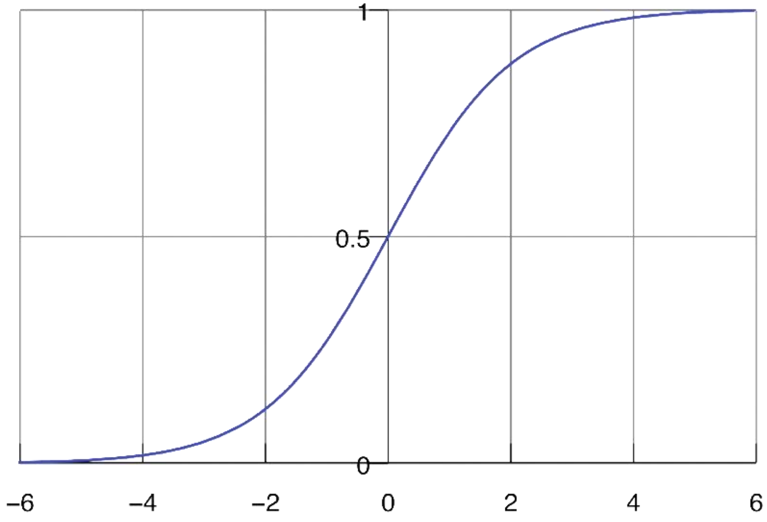
Classification Problems

- ***Classification***
  - malignant or benign cancer
  - Spam or Ham
  - Human face or no human face
  - Positive Sentiment?
- **Binary Decision Task (in most simple case)**
  - **Want** $0 \leq h_\theta(x) \leq 1$
  - **Data point belongs to class if close to 1**
  - **Doesn't belong to class if close to 0**

Logistic Function (Sigmoid Function)

$$g(z) = \frac{1}{1+e^{-z}}$$

- **maps $\mathbb{R}$ into interval** [0;1]
- 0 **asymptote for** $x \to -\infty$
- 1 **asymptote for** $x \to \infty$



Sigmoid Function (S-shape)

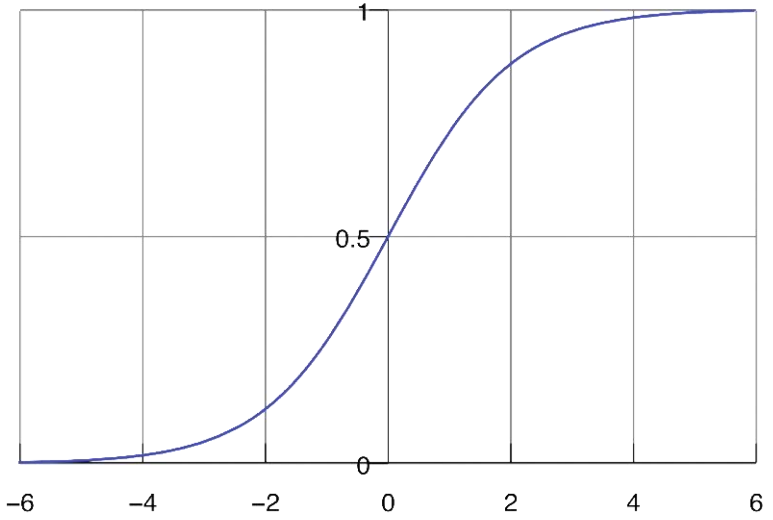
Logistic Function

- **Hypothesis** $$h_\theta(x) = g(\theta^T x) = \frac{1}{1+e^{-\theta^T x}}$$
- **Interpretation**
$$h_\theta(x) = p(y=1|x,\theta)$$
- **Because probabilites should sum to 1, define**
$$p(y=0|x,\theta) := 1 - p(y=1|x,\theta)$$
- **If $h_\theta(x) = 0.7$ interpret as 70% chance data point belongs to class**
- **If $h_\theta(x) \geq 0.5$ classify as *positive sentiment, malignant tumor, ...***

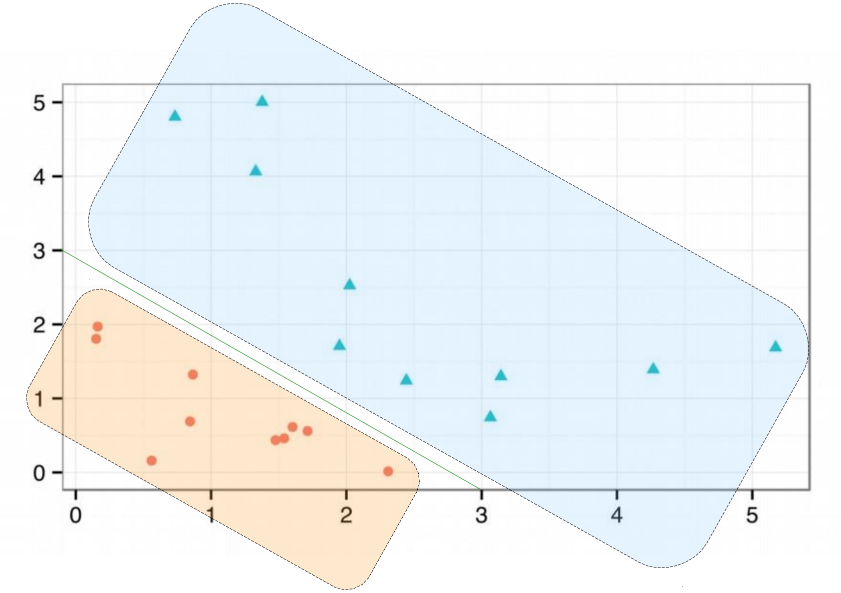
Logistic regression
Decision boundary

$$h_\theta(x) = \frac{1}{1+e^{-\theta^T x}}$$

- **If** $h_\theta(x) \geq 0.5$
  or equivalently $\theta^T x \geq 0$
  predict y = 1
- **If** $h_\theta(x) < 0.5$
  or equivalently $\theta^T x < 0$
  predict y = 0



Example



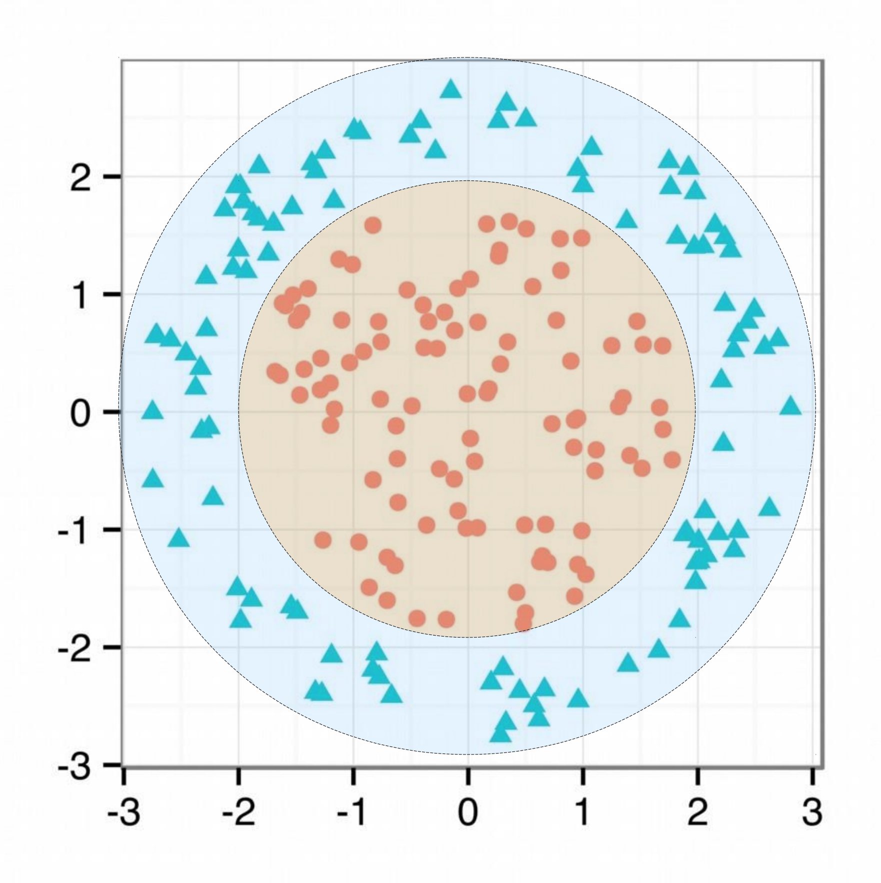
- **If** $h_\theta(x) = g(\theta_0 + \theta_1 x_1 + \theta_2 x_2)$

  and $\theta = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

- **Prediction y = 1 whenever**

$$\begin{array}{rccc} & \theta^T x & \geq & 0 \\ \Leftrightarrow & -3 + x_1 + x_2 & \geq & 0 \\ \Leftrightarrow & x_1 + x_2 & \geq & 3 \end{array}$$

Example



- **If**

$$h_\theta(x) = g(\theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_1^2 + \theta_4 x_2^2$$

and

$$\theta = \begin{bmatrix} -2 & 0 & 0 & 1 & 1 \end{bmatrix}^T$$

- **Prediction y = 1 whenever**

$$\mathrm{x}_1^2 + x_2^2 \geq 2$$

Logistic regression
Cost Function

Training and cost function

- **Training data wih m datapoints, n features**

$$\{(x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \ldots, (x^{(m)}, y^{(m)})\}$$

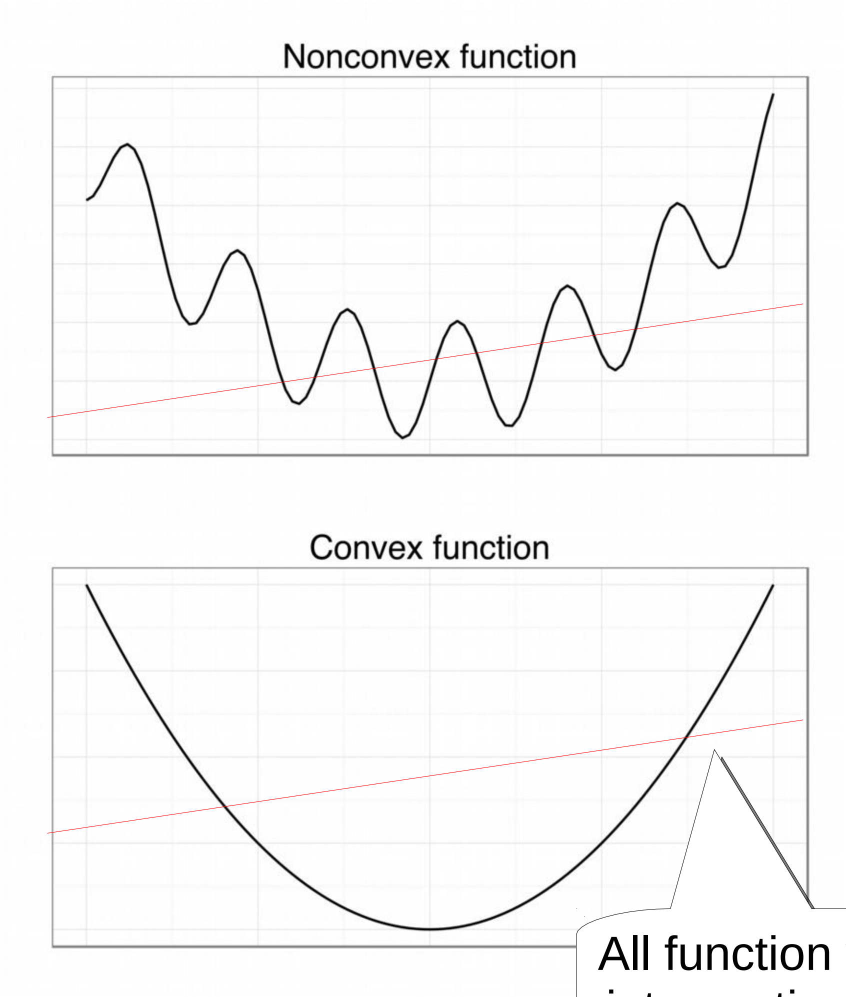
where

$$x \in \mathbb{R}^{n+1} \text{ with } x_0 := 1, y \in \{0, 1\}$$

- **Average cost**

$$J(\theta) = \frac{1}{m} \sum_{i=1}^{m} Cost(h_\theta(x^{(i)}), y^{(i)})$$

Reusing Linear Regression cost



All function values below intersection with *any* line

- **Cost from linear regression**

$$Cost(h_\theta(x), y) := \frac{1}{2}\left(h_\theta(x) - y\right)^2$$

with logistic regression hypothesis

$$h_\theta(x) = \frac{1}{1+e^{-\theta^T x}}$$

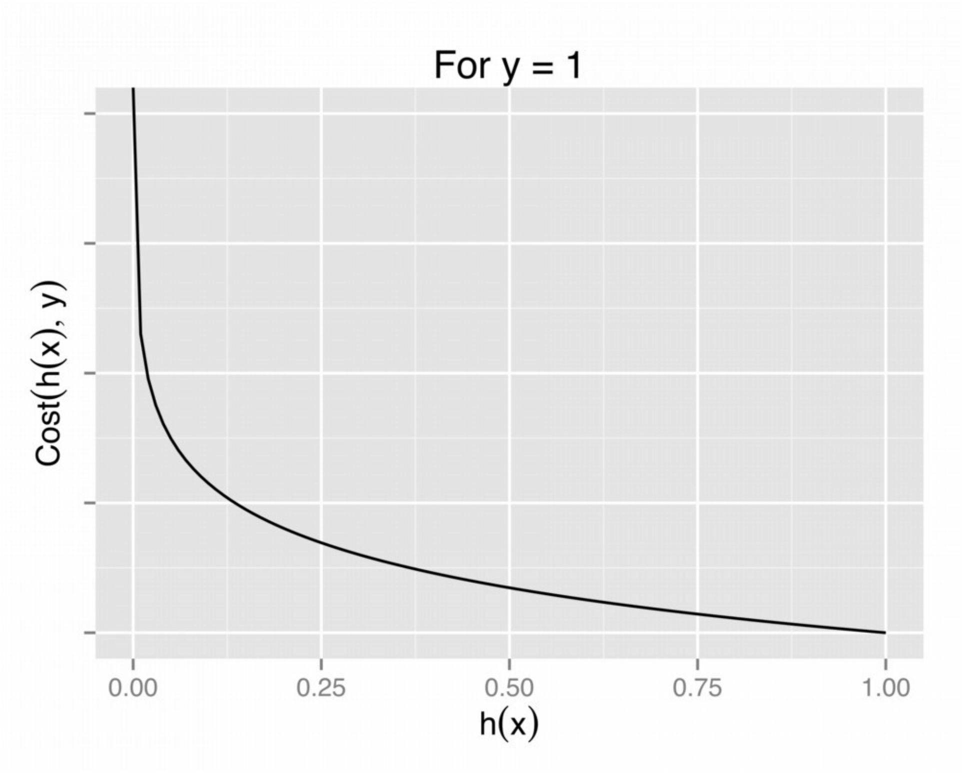
leads to non-convex average cost

$$J(\theta) = \frac{1}{m}\sum_{i=1}^{m} Cost(h_\theta(x^{(i)}), y^{(i)})$$

- **Convex *J* easier to optimize (no local optima)**

Logistic Regression Cost function

$$Cost(h_\theta(x), y) = \begin{cases} -log(h_\theta(x)) & \text{if } y = 1 \\ -log(1 - h_\theta(x)) & \text{if } y = 0 \end{cases}$$

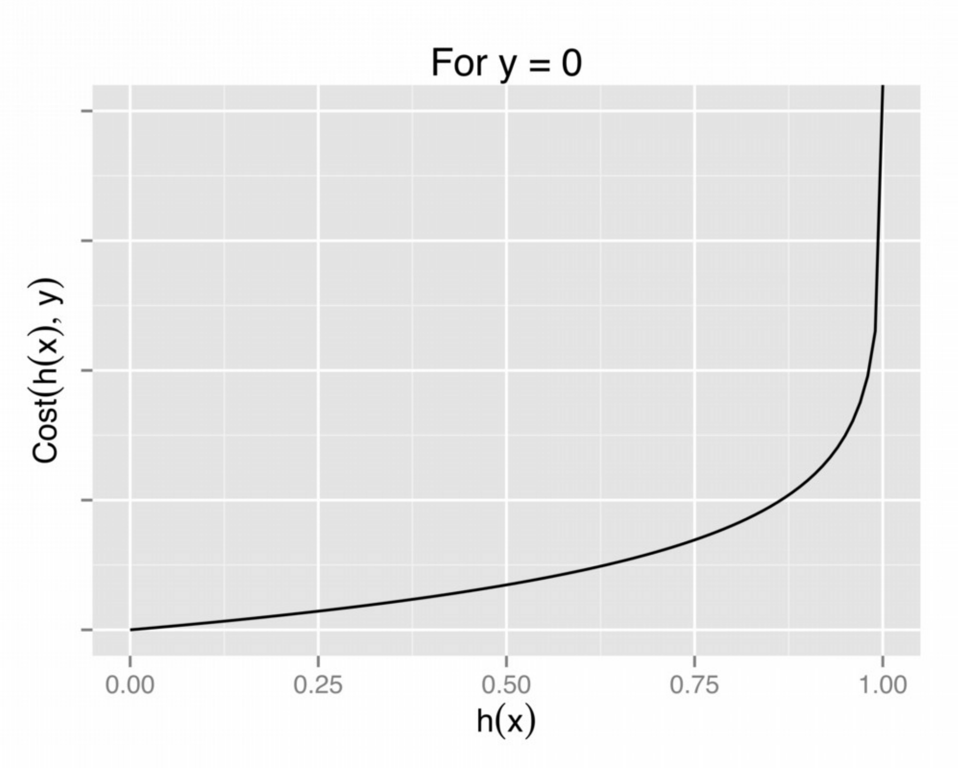


- **If** y = 1 **and** h(x) = 1**,** Cost = 0
- **But for** $h(x) \rightarrow 0$

  $Cost \rightarrow \infty$
- **Corresponds to intuition: if prediction is** h(x) = 0 **but actual value was** y = 1**, learning algorithm will be penalized by large cost**

Logistic Regression Cost function

$$Cost(h_\theta(x), y) = \begin{cases} -log(h_\theta(x)) & \text{if } y = 1 \\ -log(1 - h_\theta(x)) & \text{if } y = 0 \end{cases}$$



- **If** y = 0 **and** h(x) = 0**,** Cost = 0
- **But for** $h(x) \rightarrow 1$

  $Cost \rightarrow \infty$

Logistic regression
Simplified Cost Function &
Gradient Descent

Simplified Cost Function (17

- **Original cost of single training example**

$$Cost(h_\theta(x), y) = \begin{cases} -log(h_\theta(x)) & \text{if } y = 1 \\ -log(1 - h_\theta(x)) & \text{if } y = 0 \end{cases}$$

- **Because we always have *y = 0* or *y = 1* we can simplify the cost function definition to**

$$Cost(h_\theta(x), y) \quad = \quad -y\ log(h_\theta(x)) - (1 - y) log(1 - h_\theta(x))$$

- **To convince yourself, use the simplified cost function to calculate**

$$\begin{aligned} Cost(h_\theta(x), 1) &= -log(h_\theta(x)) \\ Cost(h_\theta(x), 0) &= -log(1 - h_\theta(x)) \end{aligned}$$

Simplified Cost Function (18

- **Cost function for training set**

$$J(\theta) = \frac{1}{m}\sum_{i=1}^{m} Cost(h_\theta(x^{(i)}), y^{(i)})$$

$$= -\frac{1}{m}\left(\sum_{i=1}^{m} y^{(i)}\ log(h_\theta(x^{(i)})) + (1-y^{(i)})log(1-h_\theta(x^{(i)}))\right)$$

- **Find parameter argument** θ' **that minimizes** *J***:** $argmin\ \ J(\theta)$
- **To make predictions given new** *x* **output**

$$h_{\theta'}(x) = \frac{1}{1+e^{-\theta^T x}}$$

$$= p(y=1|x,\theta')$$

Gradient Descent for logistic regression

- **Gradient Descent to minimize logistic regression cost function**

$$J(\theta) = -\frac{1}{m}\left(\sum_{i=1}^{m} y^{(i)}\ log(h_\theta(x^{(i)})) + (1-y^{(i)})log(1-h_\theta(x^{(i)}))\right)$$

with identical algorithm as for linear regression

```
while not converged:
    for all $j$:
        $tmp_j := \theta_j - \alpha \frac{1}{m} \sum_{i=1}^{m} \left(h_\theta(x^{(i)}) - y^{(i)}\right) x_j^{(i)}$
    $\theta := \begin{bmatrix} tmp_0 \\ \vdots \\ tmp_n \end{bmatrix}$
```

Beyond Gradient Descent
- Advanced Optimization

Advanced Optimization Algorithms

- **Given functions to compute**
  - $J(\theta)$
  - $\frac{\partial}{\partial\theta_j}J(\theta)$

  an optimization algorithm will compute $\underset{\theta}{argmin}\ J(\theta)$

Optimization Algorithms

- *(Gradient Descent)*
- **Conjugate Gradient**
- **BFGS & L-BFGS**

Advantages

- **Often faster convergence**
- **No learning rate to choose**

Disadvantages

- **Complex**

Preimplemented Alorithms

- **Advanced optimization algorithms exist already in Machine Learning packages for important languages**
  - **Octave/Matlab**
  - **R**
  - **Java**
  - **Rapidminer – under the hood**

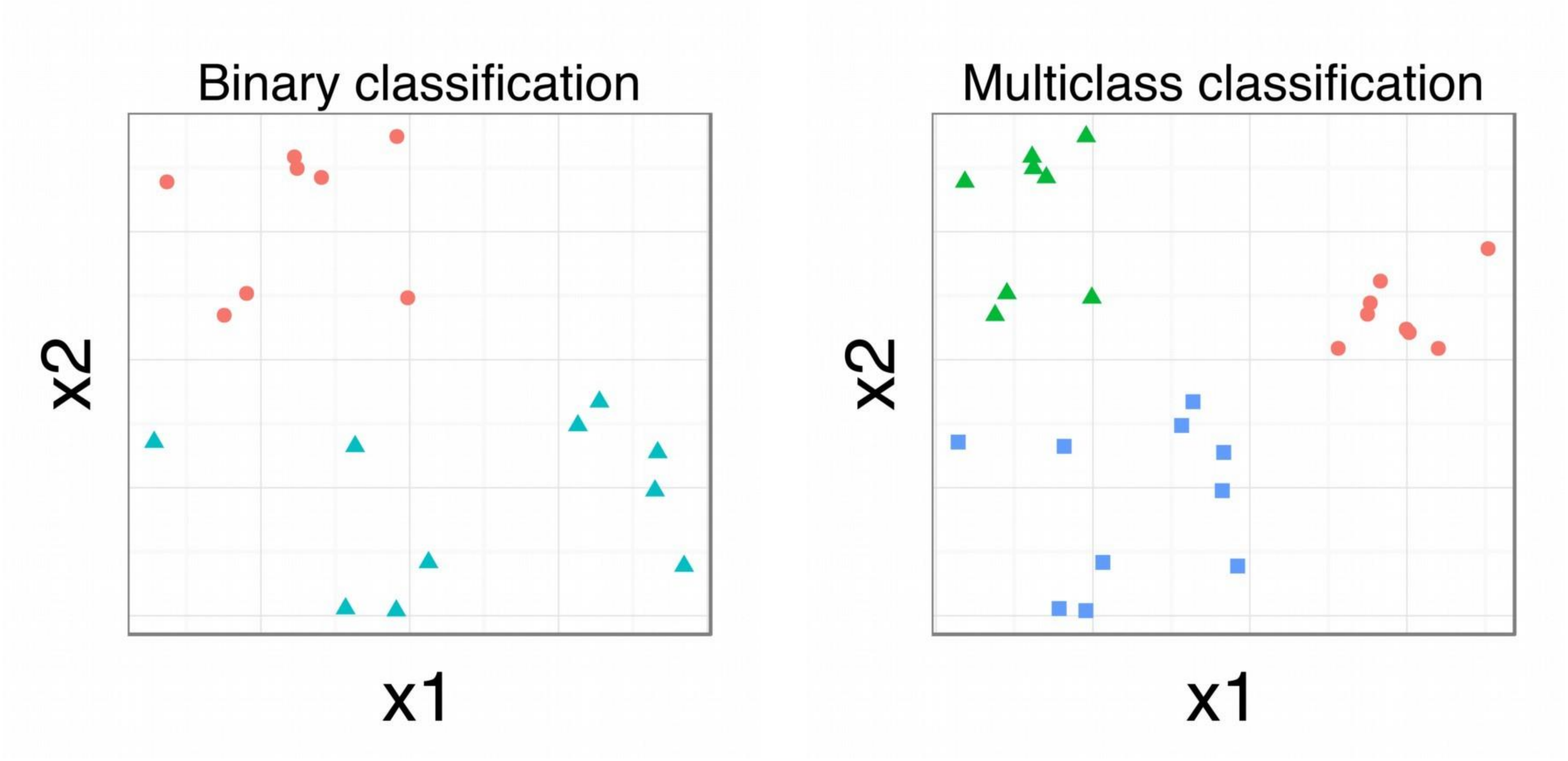
Multiclass Classification (by cheap trickery)

Multiclass classification problems

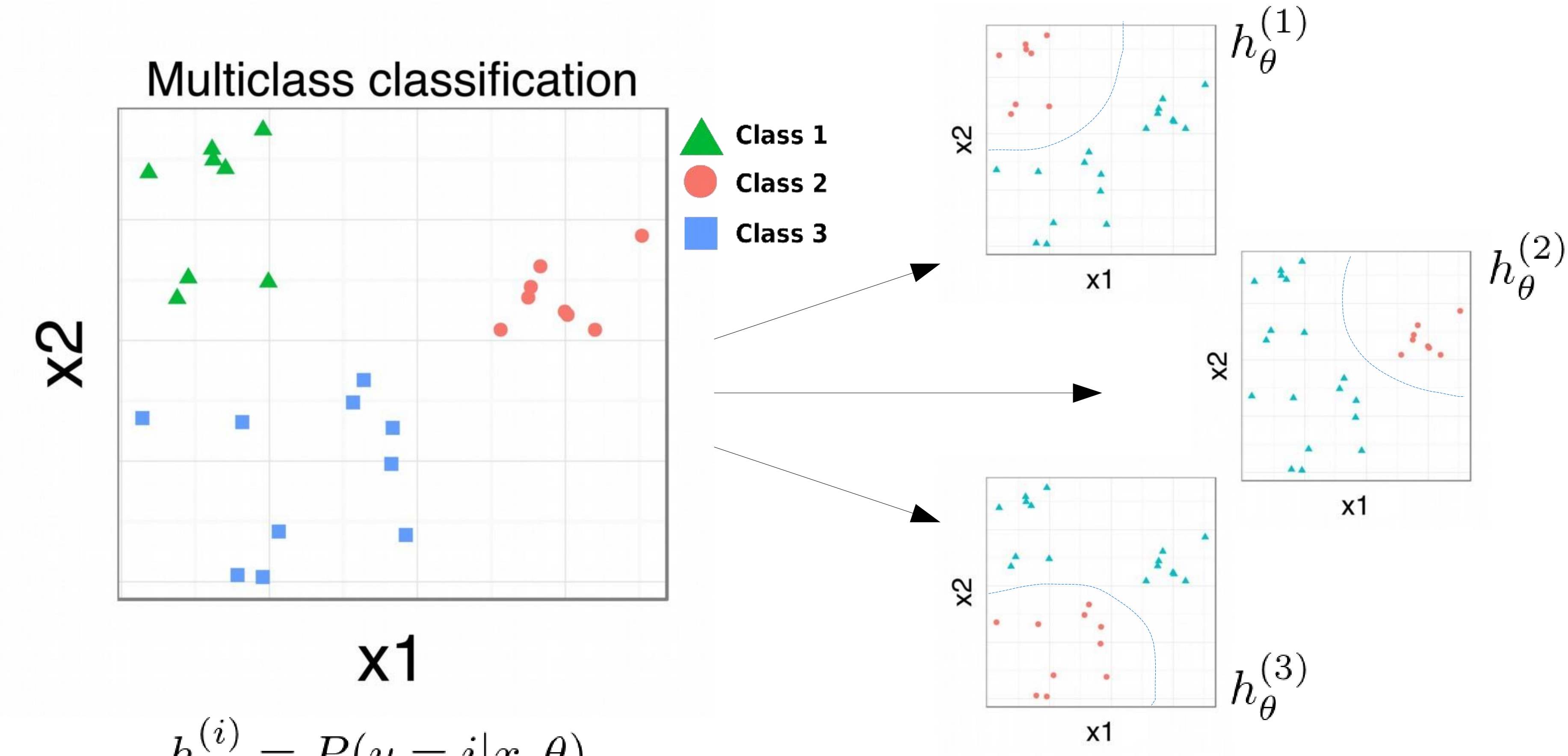
- **Classes of Emails: Work, Friends, Invoices, Job Offers**
- **Medical diagnosis: Not ill, Asthma, Lung Cancer**
- **Weather: Sunny, Cloudy, Rain, Snow**

- **Number classes as 1, 2, 3, ...**

Binary vs. Multiclass Classification



One versus all (one-vs-rest)



$$h_\theta^{(i)} = P(y = i|x, \theta)$$

$$\text{where } i \in \{1, 2, 3\}$$

- **Train logistic regression classifier** $h_\theta^{(i)}(x)$
  for each class *i* **to predict probability of** *y* = *i*
- **On new** *x* **predict class** *i* **which satisfies**

$$\underset{i}{argmax}\ h_\theta^{(i)}(x)$$

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