

1 GIBBS

Vectors :

$$\vec{r}_1 = -294.32\hat{i} + 4265.1\hat{j} + 5986.7\hat{k} \text{ Km}$$

$$\vec{r}_2 = -1365.5\hat{i} + 3637.6\hat{j} + 6346.8\hat{k} \text{ Km}$$

$$\vec{r}_3 = -2940.3\hat{i} + 2473.7\hat{j} + 6555.8\hat{k} \text{ Km}$$

Magnitude of the vector:

$$r_1 = \sqrt{-294.32^2 + 4265.1^2 + 5986.7^2} = 7356.513 \text{ Km}$$

$$r_2 = \sqrt{-1365.5^2 + 3637.6^2 + 6346.8^2} = 7441.68 \text{ Km}$$

$$r_3 = \sqrt{-2940.3^2 + 2473.7^2 + 6555.8^2} = 7598.886 \text{ Km}$$

Step2: Find \vec{C}_{12} , \vec{C}_{23} and \vec{C}_{31}

$$\vec{C}_{12} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -294.32 & 4265.1 & 5986.7 \\ -1365.5 & 3637.6 & 6346.8 \end{vmatrix} = 5292516.76i + -6306848.67j + 4753375.62k(\text{Km}^2)$$

$$\vec{C}_{23} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1365.5 & 3637.6 & 6346.8 \\ -2940.3 & 2473.7 & 6555.8 \end{vmatrix} = 8147298.92i + -9709551.14j + 7317797.93k(\text{Km}^2)$$

$$\vec{C}_{31} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2940.3 & 2473.7 & 6555.8 \\ -294.32 & 4265.1 & 5986.7 \end{vmatrix} = -13151842.79i + 15673190.95j + -11812614.15k(\text{Km}^2)$$

Step3: The magnitude of \vec{C}_{23} :

$$C_{23} = \sqrt{8147298.92^2 + -9709551.14^2 + 7317797.93^2} = 14635710.764(\text{Km}^2)$$

The unit vector of \vec{C}_{23} :

$$\hat{C}_{23} = \left(\frac{8147298.92i + -9709551.14j + 7317797.93k}{14635710.764} \right) = 0.557i + -0.663j + 0.5k$$

The unit vector of \vec{r}_1 :

$$\hat{u}_{r_1} = \left(\frac{-294.32i + 4265.1j + 5986.7k}{7356.513} \right) = -0.04i + 0.58j + 0.814k$$

Therefore :

$$\hat{u}_{r_1} \cdot \hat{C}_{23} = -6.118058187509767e - 06$$

This is close enough to or equal zero for our purposes.

The three vectors r_1 , r_2 , and r_3 are coplanar.

Step4 :

$$\begin{aligned}
\vec{N} &= r_1 \vec{C}_{23} + r_2 \vec{C}_{31} + r_3 \vec{C}_{12} \\
&= 7356.51(8147298.92i + -9709551.14j + 7317797.93k) \\
&\quad + 7441.68(-13151842.79i + 15673190.95j + -11812614.15k) \\
&\quad + 7598.89(5292516.76i + -6306848.67j + 4753375.62k) \\
\vec{N} &= (2281140568.33i + -2718596493.52j + 2048144274.9k)(\text{Km}^3) \\
N &= \sqrt{2281140568.33^2 + -2718596493.52^2 + 2048144274.9^2} \\
&= 4097470458.447(\text{Km}^3)
\end{aligned}$$

$$\begin{aligned}
\vec{D} &= \vec{C}_{12} + \vec{C}_{23} + \vec{C}_{31} \\
&= (5292516.76i + -6306848.67j + 4753375.62k) \\
&\quad + (-13151842.79i + 15673190.95j + -11812614.15k) \\
&\quad + (8147298.92i + -9709551.14j + 7317797.93k) \\
\vec{D} &= (287972.89i + -343208.86j + 258559.4k)(\text{Km}^2) \\
D &= \sqrt{287972.89^2 + -343208.86^2 + 258559.4^2} \\
&= 517275.237(\text{Km}^2)
\end{aligned}$$

$$\begin{aligned}
\vec{S} &= (\vec{r}_1)(r_2 - r_3) + (\vec{r}_2)(r_3 - r_1) + (\vec{r}_3)(r_1 - r_2) \\
&= (-294.32i + 4265.1j + 5986.7k)(7441.68 - 7598.886) \\
&= (-1365.5i + 3637.6j + 6346.8k)(7598.886 - 7356.513) \\
&= (-2940.3i + 2473.7j + 6555.8k)(7356.513 - 7441.68) \\
\vec{S} &= (-34275.77i + 478.57j + 38810.21k)(\text{Km}^2)
\end{aligned}$$

Step5 :

$$\vec{v}_2 = \sqrt{\left(\frac{\mu}{ND}\right)} \left(\frac{\vec{D} \times \vec{r}_2}{r_2} + \vec{S} \right)$$

$$\begin{aligned}
\vec{v}_2 &= \sqrt{\frac{398,600}{(4097470458.45)(517275.24)}} \\
&\quad \times \left[\frac{\begin{vmatrix} \text{i} & \text{j} & \text{k} \\ -294.32 & 4265.1 & 5986.7 \\ -1365.5 & 3637.6 & 6346.8 \end{vmatrix}}{7441.68} + (-34275.77i + 478.57j + 38810.21k) \right] \\
&= (-6.22i + -4.01j + 1.6k) \text{ (Km/s)}
\end{aligned}$$

Calculate the orbital elements

we have \vec{r} and \vec{v}

$$\vec{r} = (-1365.5)\hat{i} + (3637.6)\hat{j} + (6346.8)\hat{k} \text{ (Km)}$$

$$\vec{v} = (-6.22)\hat{i} + (-4.01)\hat{j} + (1.6)\hat{k} \text{ (Km/s)}$$

Step1

$$r = \sqrt{\vec{r} \cdot \vec{r}} = \sqrt{((-1365.5))^2 + ((3637.6))^2 + ((6346.8))^2} = 7441.68 \text{ (Km)}$$

Step2

$$v = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{((-6.22))^2 + ((-4.01))^2 + ((1.6))^2} = 7.57 \text{ (Km/s)}$$

Step3

$$v_r = \frac{\vec{v} \cdot \vec{r}}{r} = \frac{((-6.22)) \cdot ((-1365.5)) + ((-4.01)) \cdot ((3637.6)) + ((1.6)) \cdot ((6346.8))}{7441.68} = 0.54 \text{ (Km/s)}$$

Since $v_r > 0$, the satellite is flying away from perigee.

Step4

$$\vec{h} = \vec{r} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (-1365.5) & (3637.6) & (6346.8) \\ (-6.22) & (-4.01) & (1.6) \end{vmatrix} = 31280.88\hat{i} + -37277.19\hat{j} + 28095.03\hat{k} \text{ (Km}^2\text{/s)}$$

Step5

$$h = \sqrt{\vec{h} \cdot \vec{h}} = \sqrt{(31280.88)^2 + (-37277.19)^2 + (28095.03)^2} = 56190.86 \text{ (Km}^2\text{/s)}$$

Step6

$$i = \cos^{-1} \frac{h_z}{h} = \cos^{-1} \left(\frac{28095.03}{56190.86} \right) \Rightarrow 60.0^\circ$$

Since $0 > i < 90$, this is prograde orbit.

Step7

$$\vec{N} = \vec{K} \times \vec{h} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ 31280.88 & -37277.19 & 28095.03 \end{vmatrix} = 37277.19\hat{i} + 31280.88\hat{j} + -0.0\hat{k} \text{ (Km}^2\text{/s)}$$

Step8

$$N = \sqrt{\vec{N} \cdot \vec{N}} = \sqrt{(37277.19)^2 + (31280.88)^2 + (-0.0)^2} = 48662.95 \text{ (Km}^2\text{/s)}$$

Step9

$$\Omega = \cos^{-1} \frac{N_x}{N} = \cos^{-1} \left(\frac{31280.88}{48662.95} \right) \Rightarrow 40.0^\circ$$

Step10

$$\vec{e} = \frac{1}{\mu} \left[\left(v^2 - \frac{\mu}{r} \right) \vec{r} - r v_r \vec{v} \right]$$

$$\begin{aligned} \vec{e} &= \frac{1}{398600} \left[\left(7.57^2 - \frac{398600}{7441.68} \right) \right] ((-1365.5)\hat{i} + (3637.6)\hat{j} + (6346.8)\hat{k}) \\ &\quad - 7441.68 * 0.54 ((-6.22)\hat{i} + (-4.01)\hat{j} + (1.6)\hat{k}) \\ \vec{e} &= (0.05\hat{i} + 0.07\hat{j} + 0.04\hat{k}) \end{aligned}$$

Step11

$$e = \sqrt{\vec{e} \cdot \vec{e}} = \sqrt{(0.05)^2 + (0.07)^2 + (0.04)^2} = 0.1$$

Clearly, the orbit is an ellipse.

Step12

We know that $e_z > 0$; therefore ω must lie in the first quadrant

$$\omega = \cos^{-1} \frac{\vec{N} \cdot \vec{e}}{N e} = \cos^{-1} \left[\frac{(37277.19)(0.05) + (31280.88)(0.07) + (-0.0)(0.04)}{(48662.95)(0.1)} \right] \Rightarrow 30.07^\circ$$

Step13

We know that $v_r > 0$, which means $0^\circ \geq \theta < 180^\circ$. Therefore,

$$\theta = \cos^{-1} \frac{\vec{e} \cdot \vec{r}}{e r} = \cos^{-1} \left[\frac{(0.05)(-1365.5) + (0.07)(3637.6) + (0.04)(6346.8)}{(0.1)(7441.68)} \right] \Rightarrow 49.93^\circ$$

Having found the six orbital elements, we can go on to compute other parameters

The perigee and apogee radii are

$$\begin{aligned} r_p &= \frac{h^2}{\mu} \frac{1}{1 + e \cos(0^\circ)} = \left(\frac{56190.86^2}{398600} \right) \left(\frac{1}{1 + 0.1 * 1.0} \right) = 7200.46 \text{ (Km)} \\ r_a &= \frac{h^2}{\mu} \frac{1}{1 + e \cos(180^\circ)} = \left(\frac{56190.86^2}{398600} \right) \left(\frac{1}{1 + 0.1 * -1.0} \right) = 8802.41 \text{ (Km)} \end{aligned}$$

From these it follows that the semimajor axis of the ellipse is

$$a = \frac{1}{2} (r_p + r_a) = 8001.44 \text{ (Km)}$$