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Pessimistic, optimistic, and minimax regret approaches for linear programs under uncertainty

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Abstract Uncertain data appearing as parameters in linear programs can be categorized variously. This paper deals with merely probability, belief (necessity), plausibility (possibility), and random set information of uncertainties. However, most theoretical approaches and models limit themselves to the analysis involving merely one kind of uncertainty within a problem. Moreover, none of the approaches concerns itself with the fact that random set, belief (necessity), and plausibility (possibility) convey the same information. This paper presents comprehensive methods for handling linear programs with mixed uncertainties which also preserve all details about uncertain data. We handle mixed uncertainties as sets of probabilities which lead to optimistic, pessimistic, and minimax regret in optimization criteria.

Keywords Optimization with mixed uncertainties · Minimax regret · Random sets · Belief · Plausibility · Necessity · Possibility

1 Introduction

Linear optimization problems in many real circumstances are not deterministic, since information received is not always precise. Uncertainties, based on lack of information to obtain deterministic data, used in this paper are probability, belief, plausibility, necessity, possibility, and random set. Each of these uncertainties could be represented

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as a set of probabilities. We apply this knowledge in our models to overcome the limitation of modeling approaches presented in literature, e.g. Jamison (1998), Tanaka et al. (1985), Vasant (2003) and Wang and Zhu (2002), which do not deal with all different interpretations of uncertainty.

The novel element of this paper is the incorporation of the total range of uncertainty available in the presence of partial, incomplete information in optimization problems. That is, from partial, incomplete information which is in the form of a plausibility distribution, belief distribution or random set, we obtain the full range of associated uncertainty in the form of pairs of distribution and we optimize with these *pairs* of distributions. Thus, the novel part of our approach this—optimization over the complete range of uncertainty. Secondly, we develop an algorithm to solve problems over the full range of uncertainty and at the same time improve the classical min–max regret algorithm.

Linear programming problems in which coefficients are obtained from partial or incomplete information lead naturally to upper and lower distributions that are often Demster–Shaffer beliefs/plausibilities, possibilities/necessities, and probability intervals. Even though a random set can be transformed as an over/under approximation of a probability interval, in general (see Lemmer and Kyburg 1991), it was presented in Thipwiwatpotjana and Lodwick that we can use a relationship between a probability interval and random sets to get the upper/lower distribution of a given probability interval through random sets. Hence, these are pairs of upper/lower functions which each of which may be transformed into random sets. Since random sets are the most general theory which includes these, if coefficients arise from several of these, a transformation to the most general form (random sets) is required. For example, suppose that all that is known that a coefficient is that the support lies in the interval [2,3], another coefficient's partial information is obtained from knowing that its distribution lies in a trapezoidal number 2/2.5/3/3.5 such that all possible cumulative distributions are bounded by the possibility distribution and necessity distribution of this trapezoidal fuzzy number. This can occur in refinery models where percent sulfur content of various crude oils coming from a variety of sources in the first case is only known to be between 2 and 3 % (say crude from Nigeria is 2 % and from West Texas is 3 % and are mixed in the same pool). In the second case, the probability distribution as derived from some partial chemical analysis to lie in the fuzzy triangular number. Regardless, when one partial information of a variety of types, one must use the most general theory which is often random sets.

Moreover, the uncertainty proposed in this paper cannot be solved via known stochastic optimization or possibilistic methods, alone. In stochastic and possibilistic methods, only one distribution is analyzed. To be sure, the optimistic and pessimistic can be obtained via stochastic or possibilistic optimization methods. What we propose is to solve the problem in which the min/max is a way of capturing the whole range of uncertainty in a rational way not just the optimistic and pessimistic optimization. This is different from what has been done.

We provide definitions of uncertainty interpretations in the next section. Then, we present the problem statement and literature review to emphasize that there are no approaches attempting to deal with more than one or two interpretations of uncertainty in the same problem. Section 5 proposes optimistic, pessimistic, and minimax regret approaches to handle a problem that has more than one or two uncertainty types. An application and conclusion follow.

2 Definitions and notation

We provide the mathematical definitions of uncertainty information appearing in this paper, except the definition of probability. We assume that readers have some basic background in probability theory.

Definition 1 A **belief** measure is a function $Bel : \mathcal{P}(U) \to [0, 1]$ that satisfies the following properties;

- $Bel(\emptyset) = 0$, $Bel(U) = 1$, and
- super-additive property: for $A_1, \ldots, A_t \subseteq U$,

$$Bel(A_1 \cup \ldots \cup A_t) \geq \sum_{1 \leq j \leq t} Bel(A_j) - \sum_{1 \leq j < k \leq t} Bel(A_j \cap A_k) + \cdots + (-1)^{t+1} Bel(A_1 \cap \cdots \cap A_t).$$

Definition 2 A **plausibility** measure is a function $Pl : \mathcal{P}(U) \to [0, 1]$ that satisfies the following properties;

- $Pl(\emptyset) = 0$, $Pl(U) = 1$, and
- sub-additive property: for $A_1, \ldots, A_t \subseteq U$,

$$Pl(A_1 \cup \cdots \cup A_t) \leq \sum_{1 \leq j \leq t} Pl(A_j) - \sum_{1 \leq j < k \leq t} Pl(A_j \cap A_k) + \cdots + (-1)^{t+1} Pl(A_1 \cap \cdots \cap A_t).$$

The definitions above mean that belief and plausibility measures lose the additive property of probability measures. Thus, we have

$$Bel(A) + Bel(A^c) \leq 1,$$
$$Pl(A) + Pl(A^c) \geq 1,$$

for any $A, A^c \subseteq U$, where A^c is the complement set of the set A. The dual relationship between a belief measure and a plausibility measure is that $Bel(A) = 1 - Pl(A^c)$, for every $A \in \mathcal{P}(U)$. The full details on belief and plausibility measures can be found in many resources, e.g., Dempster (1967), Klir and Yuan (1995), and Shafer (1976).

Definition 3 A mapping $m : \mathcal{P}(U) \to [0, 1]$, such that $\sum_{A \in \mathcal{F}} m(A) = 1$, generates a **random set** $(\mathcal{F}, m)$, where $\mathcal{F} = \{A \in \mathcal{P}(U) \ : \ m(A) > 0\}$. We call the mapping m a **probability basic assignment function**.

It is shown in Dempster (1967) and Klir and Yuan (1995) that random sets have a very close relationship with belief and plausibility measures, i.e., for any $A \in \mathcal{P}(U)$,

$$Bel(A) = \sum_{B \in \mathcal{F} \mid B \subseteq A} m(B) \quad \text{and} \quad Pl(A) = \sum_{B \in \mathcal{F} \mid A \cap B \neq \emptyset} m(B). \tag{1}$$

Moreover, if $\mathcal{F} = \{A_1, \ldots, A_l\}$ such that $A_1 \subset A_2 \subset \cdots \subset A_l$, then the corresponding belief and plausibility measures are called ***necessity*** (Nec) and ***possibility*** (Pos) measures, because the nestedness property of $\mathcal{F}$ yields

$$\begin{aligned} Nec(A \cap B) &= \min\{Nec(A),\ Nec(B)\}, \quad \text{and} \\ Pos(A \cup B) &= \max\{Pos(A),\ Pos(B)\}, \end{aligned}$$

for any $A, B \in \mathcal{P}(U)$. We present examples on belief, plausibility, and random set uncertainties in Sect. 3 to enrich our intuition about the information that can be represented by these uncertainties.

The set of all probability measures generated by a random set $(\mathcal{F}, m)$ has the form

$$\mathcal{M}_{RS} = \left\{ Pr \mid Pr(A) = \sum_{i=1}^{l} m(A_i) Pr^i(A),\ \text{for all } A \in \mathcal{P}(U) \right\},$$

where 'Pr' refers to a probability measure and

$$Pr^i \in \left\{ Pr^i : \mathcal{P}(U) \to [0, 1] \mid Pr^i(A) = 1, \quad \text{whenever} \quad A^i \subseteq A \right\}.$$

It was proved in Thipwiwatpotjana (2010) that

$$\inf_{Pr \in \mathcal{M}_{RS}} Pr(A) = Bel(A) \quad \text{and} \quad \sup_{Pr \in \mathcal{M}_{RS}} Pr(A) = Pl(A).$$

Therefore, it is important to point out here the fact that belief (necessity), plausibility (possibility), and random set provide the same information and can be represented by a set of probabilities. For this reason, we should not use only possibility information to obtain a solution when we work on linear program with possibilistic parameters. Instead, we should allow the dual necessity information in the model as well. Our paper has the aim to resolve this issue.

The focus of this research is to be able to solve a linear program with these uncertain data within the same problem. To this end, we denote any uncertain parameters in a linear program with a hat notation above those parameters, regardless the interpretations of those uncertainties. The meaning of them will be specified. For instance, we use the notation $\widehat{u}$ to represent that $\widehat{u}$ has a belief information (or other interpretations of uncertainty) over the set of realizations $U = \{u_1, u_2, \ldots, u_n\}$ and we restrict ourselves to a finite set of realizations. Moreover, we use bold letters to present any vector notations in this paper.

3 Problem statement and motivation

A linear programming problem is well-defined when all parameters are deterministic. It is commonly known that when some parameters are probabilistic, (while the rest are deterministic), we use a stochastic program. A stochastic programming problem deals with probabilistic parameters by interpreting probabilities back into deterministic mathematical relations/operations, then presenting these relations/operations as a well-defined optimization problem.

However, not all data can be represented by probability. Some may explicitly stand out as a form of random sets, possibility, or others. We will assume that we will use the information at hand and try to find a decision for our optimization problem with just these data. To see what this means we consider the following toy examples describing uncertain information in forms of belief, plausibility, and random set interpretations of uncertainties.

Example 1 Belief information.
Suppose that there is an art competition in a high school of 1,000 students. There are 5 groups of students (group I, II, III, IV, and V) competing in this event. Groups I, II, III, IV, and V have 5, 7, 10, 4, and 6 members, respectively. In a situation that we need to predict which group will win with the highest number of votes ahead of time, the certain information we have at this moment is that we know for sure that 5, 7, 10, 4, and 6 out of 1,000 who will vote for I, II, III, IV and V, respectively. However, we do not know the decision of the rest, so we cannot provide the proportion for the chance of winning this competition. This information is considered to be a belief information with $U = \{\text{I, II, III, IV, V}\}$, $Bel(\{\text{I}\}) = 5/1000$, $Bel(\{\text{II}\}) = 7/1000$, $Bel(\{\text{III}\}) = 10/1000$, $Bel(\{\text{IV}\}) = 4/1000$, $Bel(\{\text{V}\}) = 6/1000$, and $Bel(U) = 1$. For all other nonempty subsets S's of U, $Bel(S) = 0$. We also be able to obtain random set and plausibility from this information in the table below, using the relationship in system (1). The rest of nonempty subsets S's of U would also have nonzero plausibility values.

Set S	$Bel(S)$	$m(S)$	$Pl(S)$
{I}	5/1000	5/1000	973/1000
{II}	7/1000	7/1000	975/1000
{III}	10/1000	10/1000	978/1000
{IV}	4/1000	4/1000	972/1000
{V}	6/1000	6/1000	974/1000
U	1	968/1000	1

Example 2 Possibility information.
A pair of 2 weeks contact lenses might not be able to be worn comfortably for 2 weeks. Some people may need to change to a new pair within 10 days, but it does not mean that they would change to the new pair every 10 days. The hidden reasons for changing a new pair within 10 days could be, for example, defects may be found when first opening a contact lenses package, or contact lenses could be accidentally dropped during a cleaning process at the end of the fifth day. This means that one might need to change to a new pair without using the defected lenses, or one would start a new pair in the morning of the sixth day. Suppose a survey of 1,000 customers

says that 450 people change to new pairs within 12 days, 250 of customers wear new pairs within 13 days, and the rest have new pairs within 2 weeks. Here, $U = \{0, 1, 2, \ldots, 14\}$ and 'within 12 days', 'within 13 days', and 'within 2 weeks' refer to $\{0, 1, 2 \ldots, 12\}$, $\{0, 1, 2, \ldots, 13\}$, and U, respectively. We can see that these sets are nested. We do not know the number of people wearing a pair of contact lenses for the exact 12 days, therefore we can not provide the value of $Pr(\{12\})$. The information we have is that those who wear new contact lenses within 12 days, 13 days, or 2 weeks could actually wear new lenses every 12 days. This means that the possibility of using a pair of contact lenses for 12 days is $\frac{450}{1000} + \frac{250}{1000} + \frac{300}{1000} = 1$, based on the survey information. Similarly, the possibility of using a pair of contact lenses for 13 days is 0.55, while the possibility of using a pair of contact lenses for the maximum allowance time is 0.30. This information would help managing the stock of contact lenses of a traveler who plans a trip to a destination that has contact lens at more expensive price or not at all.

Example 3 Random set information.
This example is about a survey for a governor election of a state. Suppose that a big company, who plans to be a sponsor for five candidates, works on the survey 3 months before the election. It is too early to tell who has the best chance to win.

Set S of candidate	Number of responses in favor	$m(S)$
$\{a\}$	500	0.05
$\{a, b, c, d, e\}$	500	0.05
$\{b, c\}$	2,000	0.2
$\{a, b\}$	3,000	0.3
$\{c, d, e\}$	4,000	0.4

The survey of 10,000 people has the following results. The candidates 'a' and 'b' are from party A, while 'c','d' and 'e' are from party B. Three (four) out of ten of the survey are A's (B's) supporters. There is a group of 2,000 people undecided about who to choose between candidate 'b' and 'c'. There are 500 of the survey favor 'a'. The other 500 people do not have any preferences about any individual or subsets of the candidates. This information can be represented in the table above.

The company has 5 million dollars for supporting candidates. The plan is to sponsor all candidates but the support amount that each candidate receives may vary due to the chances each may have of winning the election. How does the company divide up its funds based on its survey information so that it could get the best return based on the analysis that the winning candidate can help boost the company's usual profit by some multiplicity of the fund received by the candidate.

The survey information in the previous example may be considered to be a random set. We should not assume a uniform distribution in each set. This is because when we know that there are 2,000 persons who favor either candidate 'b' or 'c', we cannot assume by using a uniform distribution as a priori that 1,000 persons will choose 'b' and the other 1,000 will choose 'c'. Assuming the uniform distribution might not provide the best solution for our optimization problem. We provide an example that support this statement is in Sect. 6.

Our generic linear programming problem with uncertainty is

$$\begin{array}{ll} \min & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{s.t.} & \widehat{\mathbf{A}}\mathbf{x} \geq \widehat{\mathbf{b}} \\ & \mathbf{x} \geq \mathbf{0}, \end{array} \tag{2}$$

where some elements in $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{b}}$ represent uncertain information that can be a mixture of all uncertainties defined in Sect. 2. The system (1) indicates that any information presented by a random set can also be presented by the corresponding belief measure or plausibility measure, as can be inferred from the three examples above. This means that whenever we have one representation of information (random set, belief, or plausibility), we also have the other two representations of the same information. However, researchers neither take advantage of nor concern themselves with the fact that when working on an optimization problem that contains possibility information, they also have at the same time necessity, belief/plausibility, as well as random set information. Possibility optimization as found in the literature, typically appears without the full suite (its dual) of uncertainty.

The next section is devoted to the review of uncertain linear programs which show that we do not have a reasonable method to deal with different types of uncertainties (probability, random set, possibility, necessity, plausibility, and belief) within the same problem.

4 Review of uncertain linear programs

In this section, we review the strategies to obtain a meaningful solution to a linear programming problem with uncertainty. The well-known approach to remodel problem (2) when all uncertainties have probability interpretation is the stochastic programming (see for instance, Birge and Louveaux 1997; Kall and Wallace 1994). We consider in particular a two-staged expected recourse model, which is one area of the stochastic programming. The first stage is the decision we make before knowing the realizations of uncertainties. Then, the recourse will be the next action,which is used to adjust the decision based on the consequences taken in the first stage. Recourse variables that are used to adjust the first stage decisions generally come with penalties. Therefore, problem (2) is transformed into a two-stages expected recourse model (3), where $\mathbf{x}$ is the first stage decision vector, $\mathbf{y}(\boldsymbol{\xi})$ is the second stage (recourse) decision vector, with respect to vector $\boldsymbol{\xi}$ of all realizations with corresponding probability, and $\mathbf{q}$ is penalty vector respected to $\mathbf{y}(\boldsymbol{\xi})$.

$$\min \mathbf{c}^{\mathrm{T}}\mathbf{x} + E_{\boldsymbol{\xi}}[\mathbf{q}^{\mathrm{T}}\mathbf{y}(\boldsymbol{\xi})]. \tag{3}$$

The term $E_{\boldsymbol{\xi}}[\mathbf{q}^{\mathrm{T}}\mathbf{y}(\boldsymbol{\xi})]$ is the expected value of $\mathbf{q}^{\mathrm{T}}\mathbf{y}(\boldsymbol{\xi})$, with respect to the random vector $\boldsymbol{\xi}$, while $\mathbf{y}(\boldsymbol{\xi}) = \max\left\{\widehat{\mathbf{b}} - \widehat{\mathbf{A}}\mathbf{x},\ 0\right\}$. The uncertain notation $\widehat{\mathbf{b}} - \widehat{\mathbf{A}}\mathbf{x}$ will spread into all scenarios with a corresponding probability.

The other approach that provides a reasonable solution to the problem (2), when uncertainties are presented as deterministic scenarios $\xi^1, \xi^2, \ldots, \xi^N$, is the minimax regret model. These scenarios have no probability attached. The result to this approach is a solution that ensures the smallest difference between a true objective value, (if we know the real scenario), and the value given by this solution. The minimax regret model is given by:

$$\begin{aligned}
&\min \;\; R \\
&\text{where} \;\; R(\mathbf{x}) = \max_k \left\{\mathbf{c}^{\mathrm{T}}\mathbf{x} - z(\xi^k),\; k = 1, 2, \ldots, N\right\}, \\
&\text{and} \quad z(\xi^k) = \min_{\mathbf{x}} \mathbf{c}^{\mathrm{T}}\mathbf{x}, \quad \text{s.t. ;} \;\; \mathbf{A}^k\mathbf{x} \geq \mathbf{b}^k,\;\; \mathbf{x} \geq \mathbf{0},\;\; k = 1, \ldots, N.
\end{aligned} \tag{4}$$

Note that the terms $\mathbf{A}^k$ and $\mathbf{b}^k$ are the matrix and the right-hand-side vector corresponding to the scenario ξ^k. The minimax regret approach can be found in many books that have the robust optimization topic, e.g. in Morrison and Greenberg (2008).

Moreover, articles on modeling uncertainty information in a linear program involved with probability uncertainty, interval uncertainty, or possibilistic uncertainty in the problem. The only mixture of uncertainty in a same problem that are found in the literature is the mixture of possibilistic and probabilistic uncertainties in different constraints of the problem.

Averbakh and Lebedev (2005) proved that finding a minimax regret solution of a linear programming problem with interval objective function oceﬃcients is strongly NP-hard. Jamison (1998) proposes the idea of an expected average model to handle the problem with possibilistic uncertainty, which coincides with the expected recourse approach. An expected average of a possibilistic uncertainty $\widehat{u}$ was originally invented by Yager (1981), with the use of possibility distribution and α-level set of $\widehat{u}$, (please look in Klir and Yuan (1995) for the mathematical definitions and basic properties of a possibility distribution and α-level sets). Let U_α be an α-level set of a possibilistic uncertainty $\widehat{u}$ that has realizations $u_1, u_2, \ldots, u_n$. Then, The mean value of U_α is defined by $\Theta(U_\alpha) = \frac{1}{|U_\alpha|}\sum_{u_i \in U_\alpha} u_i$, which expresses the average value of $\widehat{u}$ at the satisfaction level α. An expected average is the mean of $\Theta(U_\alpha)$ for all α-levels, which is defined by

$$EA\,[\,\widehat{u}\,] = \int_0^1 \Theta(U_\alpha)\, d\alpha.$$

The model by Jamison (1998) is as follows, when $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{b}}$ are possibilistic uncertainties:

$$\min \quad \mathbf{c}^{\mathrm{T}}\mathbf{x} + EA\left[\mathbf{q}^{\mathrm{T}}\left(\max\left\{\widehat{\mathbf{b}} - \widehat{\mathbf{A}}\mathbf{x},\; \mathbf{0}\right\}\right)\right].$$

Yager (1981) also proved that an α-level set of $\widehat{u} + \widehat{v}$ is $U_\alpha \oplus V_\alpha$, where $U_\alpha \oplus V_\alpha$ is the set consisting of the sum of each element of U_α with all the elements of V_α. An acceptance level of uncertainty is another treatment for a linear program with possibilistic uncertainty, (see for examples in Dubois and Prade 1980; Ramik and Rimanek

1985; Tanaka et al. 1985). The solution based on this approach is an undecided satisfaction level, which is sometimes difficult to use. Moreover, the combination of an expected recourse and an expected average model can be used in a linear program with uncertainty when possibility and probability are only uncertainty in the problem and they need to be in the different constraints. However, the semantic of the objective value obtained by this method is unclear. Lodwick and Jamison (2006) presents an interval expected value approach (5), when possibility and probability are in the same constraint.

$$\left.\begin{array}{llcll} \multicolumn{2}{l}{\text{Original problem}} & & \multicolumn{2}{l}{\text{Interval expected value problem}} \\ \min & \mathbf{c}^{\mathrm{T}}\mathbf{x} & & \min & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{s.t.} & \widehat{\mathbf{A}}\mathbf{x} \geq \widehat{\mathbf{b}} & \Rightarrow & \text{s.t.} & \mathcal{E}[\widehat{\mathbf{A}}]\mathbf{x} \geq \mathcal{E}[\widehat{\mathbf{b}}] \\ & \mathbf{x} \geq \mathbf{0} & & & \mathbf{x} \geq \mathbf{0}. \end{array}\right\} \tag{5}$$

An interval expected value model becomes an interval linear programming problem. $\mathcal{E}[\widehat{\mathbf{A}}]$ in (5) is an interval expected value matrix generated by $\widehat{\mathbf{A}}$, while $\mathcal{E}[\widehat{\mathbf{b}}]$ is an interval expected value vector generated by $\widehat{\mathbf{b}}$. An interval expected value in Lodwick and Jamison (2006) is defined through the possibility and necessity measures. Given a set of realizations of possibilistic uncertainty $\widehat{u}$ as $\{u_1, u_2, \ldots, u_n\}$, where $u_1 \leq u_2 \leq \cdots \leq u_n$, then the corresponding possibility measure generates the upper cumulative probability $\overline{F}$ as $\overline{F}(u_k) = Pos\left(\{u_1, u_2, \ldots, u_k\}\right)$. Moreover, the corresponding necessity measure, obtained by the property $Nec(A) = 1 - Pos(A^{\mathrm{c}}); \forall\, A \subseteq \{u_1, u_2, \ldots, u_n\}$, generates the lower cumulative probability $\underline{F}$ as $\underline{F}(u_k) = Nec\left(\{u_1, u_2, \ldots, u_k\}\right)$. $\overline{F}$ and $\underline{F}$ are cumulative probability distributions that have $\overline{f}$ and $\underline{f}$ as their probability mass functions, respectively. An interval expected value is

$$\mathcal{E}[\,\widehat{u}\,] = \left[\,\sum_{k=1}^{n} u_k \underline{f}(u_k),\ \sum_{k=1}^{n} \overline{f}(u_k)\right].$$

It turns out that

$$\underline{f}(u_k) = Nec\left(\{u_k, u_{k+1}, \ldots, u_n\}\right) - Nec\left(\{u_{k+1}, u_{k+2}, \ldots, u_n\}\right) \quad \text{and} \tag{6}$$

$$\overline{f}(u_k) = Nec\left(\{u_1, u_2, \ldots, u_k\}\right) - Nec\left(\{u_1, u_2, \ldots, u_{k-1}\}\right). \tag{7}$$

Using the nondecreasing order of u_i, $i = 1, 2 \ldots, n$, Nguyen (2006) proves that $\underline{f}$ and $\overline{f}$ provide the smallest and the largest expected values of $\widehat{u}$, respectively.

All approaches presented in this section are limited to merely probabilistic or possibilistic uncertainties. A solution obtained by an interval expected value model is in a form of an interval, which is not a user friendly solution. We also loose necessity information, which comes automatically by the dual relationship, when applying the expected average in a possibilistic model. To date, there is no approach that provides a semantical and reasonable solution for a linear programming problem with more than two interpretations of uncertainties. Thus, we propose in the next section three expected recourse based models that treat probability, possibility, necessity, belief, plausibility, and random set with in the same linear programs with uncertainty. These approach are (I) optimistic expected recourse model, (II) pessimistic expected recourse model, and (III) minimax regret problem of all expected recourse models. There is also research dealing with fuzzy optimization by many authors. One comprehensive review source of fuzzy optimization is that of Untiedt (2006). However, we do not include fuzzy information in this paper. Our approaches cannot handle purely fuzzy (transitional) data, since the semantic and axiomatic representation of fuzzy sets and probability-base uncertainty are distinct.

5 Comprehensive approaches for uncertainty linear programs

The approaches presented in this section have the aim of carrying the information of uncertainty along with all calculations. It considers to be the '*new*' approaches since no researcher deals with mixed uncertain linear programs in the matter presented here before. To fulfill this purpose, we shall take the advantage of the knowledge that belief (necessity) and plausibility (possibility) have the dual relationship between each other, and all uncertainties considered in this research can be represented as a set of probability measures. Possibilistic information generates its dual necessity, and vise versa. Similarly, if the information with belief interpretation generates the dual plausibility interpretation, and vise versa. All four interpretations of uncertainties (possibility, necessity, belief, and plausibility) can actually be considered as random set information because of the relationship (1). We can construct a particular probabilities $\underline{f}$ and $\overline{f}$ that provide the smallest and the largest expected values of an uncertainty with random set information by using (6) and (7).

5.1 Optimistic and pessimistic expected recourse problems

Given data as one of our uncertainty types that is neither deterministic nor probabilistic, and without knowing further information, we do not know which probability is the one we should use. The optimistic expected recourse problem is the minimum of the expected recourse values over a set M of probability vector measures, when uncertainties are converted into the set M. We will provide the clarification on the meaning and construction of the set M shortly. The mathematical representation of the optimistic expected recourse problem is

$$\min_{\mathbf{f}\in M}\ \min_{\mathbf{x},\,\mathbf{y}} \left(\mathbf{c}^{\mathrm{T}}\mathbf{x} + E_{\boldsymbol{\xi}_{\mathbf{f}}}\left[\mathbf{q}^{\mathrm{T}}\mathbf{y}(\boldsymbol{\xi})\right]\right). \tag{8}$$

The term $E_{\boldsymbol{\xi}_{\mathbf{f}}}[\mathbf{q}^{\mathrm{T}}\mathbf{y}(\boldsymbol{\xi})]$ is the expected value of $\mathbf{q}^{\mathrm{T}}\mathbf{y}(\boldsymbol{\xi})$, with respect to the random vector $\boldsymbol{\xi}$ whose probability is $\mathbf{f} \in M$. The pessimistic expected recourse problem is defined with the opposite semantic, maximum, of the expected recourse values over the set M, and presented as

$$\max_{\mathbf{f}\in M}\ \min_{\mathbf{x},\,\mathbf{y}} \left(\mathbf{c}^{\mathrm{T}}\mathbf{x} + E_{\boldsymbol{\xi}_{\mathbf{f}}}\left[\mathbf{q}^{\mathrm{T}}\mathbf{y}(\boldsymbol{\xi})\right]\right). \tag{9}$$

Let's us clarify the definition of set M in the context of the problem (2). Suppose that the uncertain constraint system $\widehat{\mathbf{A}}\mathbf{x} \geq \widehat{\mathbf{b}}$ has m inequality constraints. Each constraint has K_i scenarios, $i = 1, 2, \ldots, m$. For example, if the i^{th} constraint is $\widehat{2}x_1 + 3x_2 \geq \widehat{5}$, where $\widehat{2}$ represents the set of realizations $\{1, 1.5, 2, 2.5, 3\}$ with a possibility distribution, and $\widehat{5}$ describes a random set of the set of realizations $\{4, 5, 6\}$, then this constraint has a total of $K_i = 15$ scenarios. With the pairwise independence assumption, let $\boldsymbol{f_i} = \left[f_i^1, f_i^2, \ldots, f_i^{K_i}\right]$ be a joint probability vector for the ith constraint, i.e., f_i^k is a joint probability for the kth scenario, $k = 1, 2, \ldots, K_i$, where $\sum_{k=1}^{K_i} f_i^k = 1$. From the constraint example, $f_i^k \quad (k = 1, 2, \ldots, 15)$ is derived from a probability in the set $\mathcal{M}_{\widehat{2}}$ of probability measures generated by the possibility $\widehat{2}$ and a probability in the set $\mathcal{M}_{\widehat{5}}$ of probability measures generated by the random set $\widehat{5}$. An element in the set M is a vector of probability, $\mathbf{f} = \left[\boldsymbol{f_1}, \boldsymbol{f_2}, \ldots, \boldsymbol{f_m}\right]^{\mathrm{T}}$. The vector $\mathbf{y}$, as the recourse action of scenarios for every constraints, can be expressed in the similar pattern as $\mathbf{f}$, i.e., $\mathbf{y} = \left[\boldsymbol{y_1}, \boldsymbol{y_2}, \ldots, \boldsymbol{y_m}\right]^{\mathrm{T}}$, where $\boldsymbol{y_i} = \left[y_i^1, y_i^2, \ldots, y_i^{K_i}\right]$, for each i. The term $E_{\boldsymbol{\xi}_{\mathbf{f}}}\left[\mathbf{q}^{\mathrm{T}}\mathbf{y}(\boldsymbol{\xi})\right]$ can expand to $q_1 \sum_{k=1}^{K_1} f_1^k y_1^k + q_2 \sum_{k=1}^{K_2} f_2^k y_2^k + \cdots + q_m \sum_{k=1}^{K_m} f_m^k y_m^k$.

The problems (8) and (9) seem to be nonlinear optimization problems because of the multiplication between an unknown vector $\mathbf{f}$ and a variable $\mathbf{y}$ to obtain an expected value. By assuming the feasibility of all scenario constraints, we show in Theorem 1 that the simplification of the problems (8) and (9) is a linear program.

Theorem 1 *Assume that all uncertainties are pairwise independent from each other. Given*

$$z_{\mathbf{f}}(\mathbf{x}, \mathbf{y}) = \mathbf{c}^T\mathbf{x} \; + \; q_1 \sum_{k=1}^{K_1} f_1^k y_1^k + q_2 \sum_{k=1}^{K_2} f_2^k y_2^k + \cdots + q_m \sum_{k=1}^{K_m} f_m^k y_m^k,$$

suppose that $\min_{\mathbf{x},\mathbf{y}} \; z_{\mathbf{f}}(\mathbf{x}, \mathbf{y})$ *is bounded for each* $\mathbf{f} \in M$. *Then, there exist constructible* $\underline{\mathbf{f}}$ *and* $\overline{\mathbf{f}} \in M$ *such that*

$$\min_{\mathbf{f}\in M} \min_{\mathbf{x},\mathbf{y}} z_{\mathbf{f}}(\mathbf{x}, \mathbf{y}) = \min_{\mathbf{x},\mathbf{y}} \mathbf{c}^T\mathbf{x} \; + \; q_1 \sum_{k=1}^{K_1} \underline{f}_1^k y_1^k + \cdots + q_m \sum_{k=1}^{K_m} \underline{f}_m^k y_m^k, \tag{10}$$

$$\max_{\mathbf{f}\in M} \min_{\mathbf{x},\mathbf{y}} z_{\mathbf{f}}(\mathbf{x}, \mathbf{y}) = \min_{\mathbf{x},\mathbf{y}} \mathbf{c}^T\mathbf{x} \; + \; q_1 \sum_{k=1}^{K_1} \overline{f}_1^k y_1^k + \cdots + q_m \sum_{k=1}^{K_m} \overline{f}_m^k y_m^k. \tag{11}$$

Proof From $\widehat{\mathbf{y}} = \max\left\{\widehat{\mathbf{b}} - \widehat{\mathbf{A}}\mathbf{x}, \; \mathbf{0}\right\}$, define $\widehat{\theta}_i = \widehat{b}_i - \widehat{a}_{i1}x_1 - \widehat{a}_{i2}x_2 - \cdots - \widehat{a}_{in}x_n$, for $i = 1, 2, \ldots, m$. Then, we can obtain two probability masses $\underline{\boldsymbol{g}}_{\boldsymbol{i}}$ and $\overline{\boldsymbol{g}}_{\boldsymbol{i}}$ that provide the smallest and the largest expected values of $\widehat{b}_i$ by using the constructions in (6) and (7) when $\widehat{b}_i$ has belief, plausibility, necessity, possibility, or random set information. By the same fashion, we can obtain $\underline{\boldsymbol{h}}_{\boldsymbol{ij}}$ and $\overline{\boldsymbol{h}}_{\boldsymbol{ij}}$ that provide the smallest and the largest expected values of $-\widehat{a}_{ij}, \forall \, j = 1, 2, \ldots, n$. Using the pairwise independence assumption and the linear property of an expected value function, the smallest expected value of $\widehat{\theta}_i$ is

$$E_{\widehat{b_i}_{\underline{g}_i}}[\widehat{b_i}] + E_{\widehat{a}_{i1}\underline{h}_{i1}}[-\widehat{a}_{i1}]x_1 + E_{\widehat{a}_{i2}\underline{h}_{i2}}[-\widehat{a}_{i2}]x_2 + \cdots + E_{\widehat{a}_{in}\underline{h}_{in}}[-\widehat{a}_{in}]x_n = E_{\widehat{\theta}_{i\underline{f}_i}}[\widehat{\theta}_i],$$

where $\underline{\boldsymbol{f}}_{\boldsymbol{i}}$ is the joint probability of $\underline{\boldsymbol{h}}_{\boldsymbol{i1}}, \underline{\boldsymbol{h}}_{\boldsymbol{i2}}, \ldots, \underline{\boldsymbol{h}}_{\boldsymbol{in}}$, and $\underline{\boldsymbol{g}}_{\boldsymbol{i}}$. Since $\underline{\boldsymbol{g}}_{\boldsymbol{i}}$ and $\underline{\boldsymbol{h}}_{\boldsymbol{ij}}$, are obtained from (6), this means that the realizations of each uncertainty are in nondecreasing order, with respect to their corresponding probabilities. Let β_i be the number of scenarios of $\widehat{b_i}$ and α_{ij} be the number of scenarios of $-\widehat{a}_{ij}$, $\forall\, j = 1, 2, \ldots, n$. Then, there are total of $K_i = \Pi_{j=1}^{n} \beta_i \cdot \alpha_{ij}$ scenarios of $\widehat{\theta_i}$, each represented as $\theta_i^{k,k_1,k_2,\ldots,k_n} = b_i^k - a_{i1}^{k_1} x_1 - a_{i2}^{k_2} x_2 - \cdots - a_{in}^{k_n} x_n$, where $b_i^1 \le b_i^2 \le \cdots \le b_i^k \le \cdots \le b_i^{\beta_i}$, and $-a_{ij}^1 \le -a_{ij}^2 \le \cdots \le -a_{ij}^{k_j} \le \cdots \le -a_{ij}^{\alpha_{ij}}$. The smallest expected value of $\widehat{\theta_i}$ in the form of joint probability is

$$E_{\widehat{\theta}_{i\underline{f}_i}}[\widehat{\theta}_i] = \sum_{j=1}^{n} \sum_{k_j=1}^{\alpha_{ij}} \left[\Pi_{j=1}^{n} \underline{h}_{ij}^{k_j} \left(\underline{g}_i^1 \theta_i^{1,k_1,k_2,\ldots,k_n} + \cdots + \underline{g}_i^{\beta_i} \theta_i^{\beta_i,k_1,k_2,\ldots,k_n} \right) \right]. \tag{12}$$

For any fixed scenarios indexes $k_1, k_2, \ldots, k_n$, it is clear that

$$\theta_i^{1,k_1,k_2,\ldots,k_n} \le \theta_i^{2,k_1,k_2,\ldots,k_n} \le \cdots \le \theta_i^{\beta_i,k_1,k_2,\ldots,k_n}.$$

Therefore, by setting $y_i^{k,k_1,k_2,\ldots,k_n} = \begin{cases} 0; & \text{if } \theta_i^{k,k_1,k_2,\ldots,k_n} < 0 \\ \theta_i^{k,k_1,k_2,\ldots,k_n}; & \text{otherwise,} \end{cases}$ the smallest expected value of $\widehat{\mathbf{y}}_{\mathbf{i}}$ is obtained from Eq. (12) by substituting y_i instead of θ_i, i.e.,

$$E_{\widehat{\mathbf{y}}_{i\underline{f}_i}}[\widehat{\boldsymbol{y}}_i] = \sum_{j=1}^{n} \sum_{k_j=1}^{\alpha_{ij}} \left[\Pi_{j=1}^{n} \underline{h}_{ij}^{k_j} \left(\underline{g}_i^1 y_i^{1,k_1,k_2,\ldots,k_n} + \cdots + \underline{g}_i^{\beta_i} y_i^{\beta_i,k_1,k_2,\ldots,k_n} \right) \right]. \tag{13}$$

We obtain the largest expected value of $\widehat{\boldsymbol{y}}_{\boldsymbol{i}}$ as $E_{\widehat{\mathbf{y}}_{i\overline{f}_i}}[\widehat{\boldsymbol{y}}_i]$ by using a similar argument, where $\overline{\boldsymbol{f}}_{\boldsymbol{i}}$ is the joint probability of $\overline{\boldsymbol{h}}_{\boldsymbol{i1}}, \overline{\boldsymbol{h}}_{\boldsymbol{i2}}, \ldots, \overline{\boldsymbol{h}}_{\boldsymbol{in}}$, and $\overline{\boldsymbol{g}}_{\boldsymbol{i}}$. Thus, for all joint probability vectors $\boldsymbol{f}_{\boldsymbol{i}}$ of constraint i,

$$E_{\widehat{\mathbf{y}}_{i\underline{f}_i}}[\widehat{\boldsymbol{y}}_i] \;\le\; E_{\widehat{\mathbf{y}}_{i f_i}}[\widehat{\boldsymbol{y}}_i] \;\le\; E_{\widehat{\mathbf{y}}_{i\overline{f}_i}}[\widehat{\boldsymbol{y}}_i], \tag{14}$$

Let $\left(\underline{\mathbf{x}}, \underline{\mathbf{y}}\right)$ and $(\overline{\mathbf{x}}, \overline{\mathbf{y}})$ be optimal solutions to the problems on the right-hand-side of (10) and (11), respectively. For a given $\mathbf{f} = [\boldsymbol{f_1}, \boldsymbol{f_2}, \ldots, \boldsymbol{f_m}]^{\mathrm{T}} \in M$ and its corresponding optimal solution $(\mathbf{x}^*, \mathbf{y}^*)$, we have

$$\begin{aligned} \min_{f \in M} z_f(\mathbf{x}, \mathbf{y}) &= \min_{\mathbf{x},\mathbf{y}} \; \mathbf{c}^{\mathrm{T}}\mathbf{x} + \sum_{i=1}^{m} q_i E_{\widehat{\mathbf{y}}_{\mathbf{i}\underline{f}_i}}[\widehat{\boldsymbol{y}}_i] = \mathbf{c}^{\mathrm{T}}\underline{\mathbf{x}} + \sum_{i=1}^{m} q_i E_{\widehat{\underline{\mathbf{y}}}_{\mathbf{i}\underline{f}_\mathbf{i}}}[\widehat{\underline{\boldsymbol{y}}_i}] \\ &\le \mathbf{c}^{\mathrm{T}}\mathbf{x}^* + \sum_{i=1}^{m} q_i E_{\widehat{\mathbf{y}^*}_{\mathbf{i}\underline{f}_\mathbf{i}}}[\widehat{\boldsymbol{y}^*}_{\mathbf{i}}] \quad \left((\underline{\mathbf{x}}, \underline{\mathbf{y}}) \text{ is optimal with respect to } \underline{\boldsymbol{f}}_{\boldsymbol{i}} \right) \end{aligned}$$

$$
\begin{aligned}
&\leq \mathbf{c}^{\mathrm{T}}\mathbf{x}^{*} + \sum_{i=1}^{m} q_i E_{\widehat{\mathbf{y}^{*}}_{\mathbf{i}_{f_\mathbf{i}}}} [\widehat{\boldsymbol{y}^{*}}_{\boldsymbol{i}}] \quad \text{(because of (14))} \\
&\leq \mathbf{c}^{\mathrm{T}}\overline{\mathbf{x}} + \sum_{i=1}^{m} q_i E_{\widehat{\overline{\mathbf{y}}}_{i_{f_i}}} [\widehat{\overline{\boldsymbol{y}}}_{\mathbf{i}}] \quad \big((\mathbf{x}^{*}, \mathbf{y}^{*}) \text{ is optimal with respect to } \boldsymbol{f}_{\boldsymbol{i}}\big) \\
&\leq \mathbf{c}^{\mathrm{T}}\overline{\mathbf{x}} + \sum_{i=1}^{m} q_i E_{\widehat{\overline{\mathbf{y}}}_{\mathbf{i}_{\overline{f}_i}}} [\widehat{\overline{\boldsymbol{y}}}_{\mathbf{i}}] \quad \text{(because of (14))} \\
&= \min_{\mathbf{x},\mathbf{y}} \ \mathbf{c}^{\mathrm{T}}\mathbf{x} + \sum_{i=1}^{m} q_i E_{\widehat{\mathbf{y}}_{i_{\overline{f}_i}}} [\widehat{\boldsymbol{y}}_{\boldsymbol{i}}] \\
&= \max_{\boldsymbol{f}\in M} \ z_{\boldsymbol{f}}(\mathbf{x}, \mathbf{y}).
\end{aligned}
$$

□

The other approach that provides a meaningful solution to the problem (2) with probabilistic uncertainties is the minimax expected regret model, which we develop next.

5.2 Minimax expected regret problem

The minimax expected regret problem is the problem to minimize the maximum regret over all unknown probability vectors $\mathbf{f} \in M$ of

$$
z_{\mathbf{f}}(\mathbf{x}, \mathbf{y}) = \mathbf{c}^{\mathrm{T}}\mathbf{x} + q_1 \sum_{k=1}^{K_1} f_1^k y_1^k + q_2 \sum_{k=1}^{K_2} f_2^k y_2^k + \cdots + q_m \sum_{k=1}^{K_m} f_m^k y_m^k. \tag{15}
$$

Therefore, the best of the worst regret over $\mathbf{f} \in M$ is

$$
\min_{(\mathbf{x},\mathbf{y})} \max_{\mathbf{f}\in M} \left(z_{\mathbf{f}}(\mathbf{x}, \mathbf{y}) - \min_{(\mathbf{x},\mathbf{y})} z_{\mathbf{f}}(\mathbf{x}, \mathbf{y}) \right) \tag{16}
$$

The reason to call model (16) minimax expected regret is because the scenarios for (16) are $\mathbf{f} \in M$. An iterative method to handle a minimax expected regret problem is a relaxation procedure which can be found in many literature, e.g., Inuiguchi and Sawaka (1995), Mausser and Laguna (1998), and Aissi et al. (2009), for instance.

Algorithm 1 *Relaxation procedure.*

1. *Initialization. Choose* $\mathbf{f}^{(1)} = \underline{\mathbf{f}}$. *Solve* $\min_{(\mathbf{x},\mathbf{y})} z_{\underline{\mathbf{f}}}(\mathbf{x}, \mathbf{y})$ *and obtain an optimal solution* $(\mathbf{w}^{(1)}, \mathbf{v}^{(1)})$. *Set* $j = 1$.
2. *Solve the following current relaxed problem to obtain an optimal solution* $\big(R^{(j)},\ (\mathbf{x}^{(j)}, \mathbf{y}^{(j)})\big)$.

$$\begin{aligned}
&\min_{R,\ (\mathbf{x},\mathbf{y})} && R \\
&s.t. && R \geq 0 \\
& && R \geq z_{\mathbf{f}^{(i)}}(\mathbf{x},\mathbf{y}) - z_{\mathbf{f}^{(i)}}(\mathbf{w}^{(i)},\mathbf{v}^{(i)}), \quad i = 1, 2, \ldots, j.
\end{aligned}$$

3. *Obtain an optimal solution* $\left(\mathbf{f}^{(j+1)},\ (\mathbf{w}^{(j+1)}, \mathbf{v}^{(j+1)})\right)$ *where its optimal solution* $Z^{(j)}$ *is*

$$Z^{(j)} = \max_{\mathbf{f}\in M,\ (\mathbf{x},\mathbf{y})} \left(z_{\mathbf{f}}(\mathbf{x}^{(j)}, \mathbf{y}^{(j)}) - z_{\mathbf{f}}(\mathbf{x},\mathbf{y}) \right). \tag{17}$$

4. *If* $Z^{(j)} \leq R^{(j)}$, *terminate the procedure. An optimal solution to the minimax expected regret model (16) is* $\left(\mathbf{x}^{(j)}, \mathbf{y}^{(j)}\right)$. *Otherwise, set* $j = j + 1$ *then return to step* 2. □

Inuiguchi and Sawaka (1995, 1996) manage step 3 in the relaxation procedure by using complementary slackness and branch-and-bound method when the objective coefficients are intervals. Mausser and Laguna (1998, 1999) create binary variables to handle the complementary slackness, yielding a mixed integer program for solving step 3 with interval objective coefficients. However, M containing joint probability vectors is more complicated than interval information. Therefore, these techniques are not suitable. We provide another procedure to handle this particular set M in step 3. It is fortunate that we can still apply the properties (6) and (7) of random set information to our new procedure.

The set M contains infinitely many elements, since each individual uncertain parameter, e.g. $\widehat{b} = \left\{b^1, b^2, b^3, b^4\right\}$ with some corresponding uncertainty interpretation, provides probabilities in the set

$$\mathcal{M}_{\widehat{b}} = \left\{ Pr \mid Pr(A) \in [Bel(A), Pl(A)],\ \text{ where } A \subseteq \left\{b^1, b^2, b^3, b^4\right\} \right\}.$$

However, we can reduce the problem (17) in step 3 to a problem where $\mathbf{f}$ belongs to a set $\overline{M}$ of finite elements, by the help of (6) and (7). The details are as follows. We first expand (17) to

$$\begin{aligned}
Z^{(j)} = \max_{\mathbf{f}\in M,\ (\mathbf{x},\mathbf{y})} \Bigg(& \mathbf{c}^{\mathrm{T}}(\mathbf{x}^{(j)} - \mathbf{x}) + q_1 \sum_{k=1}^{K_1} f_1^k (y_1^{(j)^k} - y_1^k) \\
& + \cdots + q_m \sum_{k=1}^{K_m} f_m^k (y_m^{(j)^k} - y_m^k) \Bigg).
\end{aligned}$$

If the term $(\mathbf{y}^{(j)} - \mathbf{y})$ is known, we would know the order of each subelement of this term and be able to apply (6) or (7) to find an appropriate $\mathbf{f}$ for $(\mathbf{y}^{(j)} - \mathbf{y})$. However, $(\mathbf{x}, \mathbf{y})$ is an unknown variable vector, we will not be able to know the term $(\mathbf{y}^{(j)} - \mathbf{y})$ in advance. Therefore, we will not be able to know a probability $\mathbf{f}$ that provides the largest expected value respected to this $(\mathbf{y}^{(j)} - \mathbf{y})$.

Consider any constraint i that has β_i scenarios of $\widehat{b_i}$ and α_{il} scenarios of $\widehat{a}_{il}$, $l = 1, 2, \ldots, n$. If the order of these scenarios are unknown, there will be $\beta_i!$ ways that

$\widehat{b_i}$ could be ordered. Thus, there are $\beta_i!$ probabilities as a candidate that provides the highest expected value of this unknown $\widehat{b_i}$. Hence, there are $t_1 = \beta_i! \times \alpha_{i1}! \times \alpha_{i2}! \times \cdots \times \alpha_{in}!$ joint probabilities $\boldsymbol{f}_i$, which one of them will generates the largest value of $\sum_{k=1}^{K_i} f_i^k(y_i^{(j)^k} - y_i^k)$, for each $i = 1, 2, \ldots, m$. It follows that one of $t = \Pi_{i=1}^m t_i$ probabilities $\mathbf{f}$ would provide an optimal for (17). We conclude that the size of M in the problem (17) could be reduced to the set $\overline{M}$ of the size t, where elements in $\overline{M}$ generate from all possible ordering of uncertainties, i.e.,

$$Z^{(j)} = \max_{\mathbf{f}\in\overline{M},\, (\mathbf{x},\mathbf{y})} \left(z_{\mathbf{f}}(\mathbf{x}^{(j)}, \mathbf{y}^{(j)}) - z_{\mathbf{f}}(\mathbf{x}, \mathbf{y}) \right). \tag{18}$$

Given a probability $\mathbf{f}$, we have the sequence of probabilities generated upon the order of $\mathbf{y}^{(j)} - \mathbf{y}$ through the following relationship.

$$\left.\begin{aligned}
&Z_{\mathbf{f}}^{(j)}\\
&= \max_{(\mathbf{x},\mathbf{y})} \left(z_{\mathbf{f}}(\mathbf{x}^{(j)}, \mathbf{y}^{(j)}) - z_{\mathbf{f}}(\mathbf{x}, \mathbf{y})\right)\\
&= \max_{(\mathbf{x},\mathbf{y})} \left(\mathbf{c}^{\mathrm{T}}(\mathbf{x}^{(j)} - \mathbf{x}) + q_1 \sum_{k=1}^{K_1} f_1^k(y_1^{(j)^k} - y_1^k) + \cdots + q_m \sum_{k=1}^{K_m} f_m^k(y_m^{(j)^k} - y_m^k) \right)\\
&= \mathbf{c}^{\mathrm{T}}(\mathbf{x}^{(j)} - \mathbf{x}_{\mathbf{f}}) \;+\; q_1 \sum_{k=1}^{K_1} f_1^k(y_1^{(j)^k} - y_{1_{\mathbf{f}}}^k) + \cdots + q_m \sum_{k=1}^{K_m} f_m^k(y_m^{(j)^k} - y_{m_{\mathbf{f}}}^k)\\
&\le \mathbf{c}^{\mathrm{T}}(\mathbf{x}^{(j)} - \mathbf{x}_{\mathbf{f}}) \;+\; q_1 \sum_{k=1}^{K_1} f_1^{(r)^k}(y_1^{(j)^k} - y_{1_{\mathbf{f}}}^k) + \cdots + q_m \sum_{k=1}^{K_m} f_m^{(r)^k}(y_m^{(j)^k} - y_{m_{\mathbf{f}}}^k)\\
&\text{where} \boldsymbol{f}_i^{(r)}, \text{ corresponding to the order of } (y_i^{(j)^k} - y_{i_{\mathbf{f}}}^k), \text{ is obtained}\\
&\text{by applying the system (7)}\\
&\le \mathbf{c}^{\mathrm{T}}(\mathbf{x}^{(j)} - \mathbf{x}_{\mathbf{f}^{(r)}}) \;+\; q_1 \sum_{k=1}^{K_1} f_1^{(r)^k}(y_1^{(j)^k} - y_{1_{\mathbf{f}^{(r)}}}^k) + \cdots +\\
&q_m \sum_{k=1}^{K_m} f_m^{(r)^k}(y_m^{(j)^k} - y_{m_{\mathbf{f}^{(r)}}}^k)\\
&= \max_{(\mathbf{x},\mathbf{y})} \left(z_{\mathbf{f}^{(r)}}(\mathbf{x}^{(j)}, \mathbf{y}^{(j)}) - z_{\mathbf{f}^{(r)}}(\mathbf{x}, \mathbf{y})\right)\\
&= Z_{\mathbf{f}^{(r)}}^{(j)}.
\end{aligned}\right\} \tag{19}$$

We continue to find a new $\mathbf{f}^{(r)}$ using the above relation (19) until the new probability $\mathbf{f}^{(r)*}$ could apply to the each suborder of $(y_i^{(j)^k} - y_{i_{\mathbf{f}^{(r)*}}}^k)$. It is clear that $Z_{\mathbf{f}^{(r)*}}^{(j)} \le Z^{(j)}$. However, if $Z_{\mathbf{f}^{(r)*}}^{(j)} > R^{(k)}$, then we quit step 3, (even if we did not compute $Z^{(j)}$), and return to step 2 in the relaxation procedure with $j = j + 1$ to find an updated optimal solution $\left(R^{(j)}, (\mathbf{x}^{(j)}, \mathbf{y}^{(j)})\right)$. Otherwise, select $\mathbf{f}$ that has not been used in this iteration of step 3, and reprocess the system (19) until we find $\mathbf{f}^{(r)*}$ such that $Z_{\mathbf{f}^{(r)*}}^{(j)} > R^{(k)}$ or obtain an optimal solution to the minimax regret problem. Hence, we relax step 3 and obtain a modify version of the relaxation procedure in Algorithm 2.

Algorithm 2 *Modify relaxation procedure.*

1. *Initialization. Choose* $\mathbf{f}^{(1)} = \underline{\mathbf{f}}$*. Solve* $\min_{(\mathbf{x},\mathbf{y})} z_{\underline{\mathbf{f}}}(\mathbf{x},\mathbf{y})$*, and obtain an optimal solution* $(\mathbf{w}^{(1)}, \mathbf{v}^{(1)})$*. Set* $j = 1$.
2. *Solve the following current relaxed problem to obtain an optimal solution* $\left(R^{(j)},\ (\mathbf{x}^{(j)}, \mathbf{y}^{(j)})\right)$.

$$\begin{aligned}
\min_{R,\ (\mathbf{x},\mathbf{y})} \quad & R \\
s.t. \quad & R \geq 0 \\
& R \geq z_{\mathbf{f}^{(i)}}(\mathbf{x},\mathbf{y}) - z_{\mathbf{f}^{(i)}}(\mathbf{w}^{(i)}, \mathbf{v}^{(i)}), \quad i = 1, 2, \ldots, j.
\end{aligned}$$

3. *Start with* $\mathbf{f}^{(j)}$ *and work on the system (19) to find* $\mathbf{f}^{(r)*}$*. Calculate* $Z^{(j)}_{\mathbf{f}^{(r)*}}$ *and its optimal solution* $(\mathbf{w}^{(j+1)}, \mathbf{v}^{(j+1)})$.
4. *If* $Z^{(j)}_{\mathbf{f}^{(r)*}} > R^{(j)}$*, set* $\mathbf{f}^{(r)*}$ *as* $\mathbf{f}^{(j+1)}$*. Set* $j = j+1$*, then return to step* 2.
5. *If* $Z^{(j)}_{\mathbf{f}^{(r)*}} \leq R^{(j)}$*, select* $\mathbf{f}$ *that has not been used in this iteration of step 3, and reprocess the system (19) until we find* $\mathbf{f}^{(r)*}$ *such that* $Z^{(j)}_{\mathbf{f}^{(r)*}} > R^{(k)}$*, then continue step 4. Otherwise,* $\overline{M} = \emptyset$ *and we terminate the procedure. An optimal solution to the minimax expected regret model (16) is* $\left(\mathbf{x}^{(j)}, \mathbf{y}^{(j)}\right)$. □

Since there are finite number of joint probabilities $\boldsymbol{f}_i$, this algorithm terminates in a finite number of iterations. The intension for Algorithm 2 is to point out the usefulness of the ordering in a random set (see page 11). We would not be able to improve the worse case computation iteration, as it was proved in Averbakh and Lebedev (2005) that the minimax regret approach provides a strongly NP-hard problem. However, by using the ordering property, we should be able to speed up the calculation of Eq. (17), since the ordering property leads us to a probability that provides the most improved direction as explained in the system (19).

An example application of an uncertain linear program in the next section provides the optimistic and pessimistic solutions based on Theorem 1, and the minimax expected regret solution based on the technique presented in this section.

6 Application to optimistic, pessimistic and minimax regret approaches

The following example is an uncertainty version of the problem adopted from a farming example in Birge and Louveaux (1997) and has been solved using GAMS. A farmer raises wheat, corn, and sugar beets on 500 acres of land. The planting cost per acre of wheat, corn, and sugar beet are \$150, \$230, and \$260, respectively. He knows that at least 200 tons of wheat and 240 tons of corn are needed for cattle feed. These amounts can be raised on the farm or bought from a market. Any production in excess of the feeding requirement will be sold. Selling prices are \$170 and \$150 per ton of wheat and corn, respectively. The purchase price is higher, \$238 per ton of wheat and \$210 per ton of corn. Another profitable crop is sugar beet, which sells at \$36 per ton. However, the state commission imposes a quota on sugar beet production of 6,000 tons. Any amount in excess of the quota can be sold only at \$10 per ton. The mean yield on the farmer's land is 2.5, 3, and 20 tons per acre for wheat, corn, and sugar beets, respectively.

The problem can be formulated as the model (20)

$$\left.\begin{array}{ll} \min & 150x_1 + 230x_2 + 260x_3 + 238w_1 - 170u_1 + 210w_2 \\ & -150u_2 - 36u_3 - 10u_4 \\ \text{s.t.} & x_1 + x_2 + x_3 \leq 500 \\ & 2.5x_1 + w_1 - u_1 \geq 200 \\ & 3x_2 + w_2 - u_2 \geq 240 \\ & 20x_3 - u_3 - u_4 \geq 0 \\ & u_3 \leq 6000 \\ & x_1, x_2, x_3, u_1, u_2, u_3, u_4, w_1, w_2 \geq 0, \quad \text{where} \end{array}\right\} \tag{20}$$

x_1 = acres of land devoted to wheat,
x_2 = acres of land devoted to corn,
x_3 = acres of land devoted to sugar beets,
w_1 = tons of wheat purchased,
w_2 = tons of corn purchased,
u_1 = tons of wheat sold,
u_2 = tons of corn sold,
u_3 = tons of sugar beets sold at \$36, and
u_4 = tons of sugar beets sold at \$10.

Suppose now that the farmer collects the weather data for the past 360 years of his town from the local government sources. The farmer finds out that there are 120 years that have bad weather, but he could not tell what the pattern of bad weather was from the data. There are 180 years in which no bad things happen; no monster storms, no devastating hail, no outbreak of insects. From the weather data, the farmer can determine the planting schedule during these 180 years. However, since there are no data on yields, the farmer does not know for sure that these 180 years have above average or on average yields. Moreover, the weather data show that there are 60 years of chaotic conditions; mixed good and bad seasons for farming. The farmer cannot tell which of these 60 years provide below average, average, or above average crops yields. For this information, the farmer has a random set with respect to the uncertainty $\hat{\mathbf{v}}$:

$$\hat{\mathbf{v}} = \left\{\mathbf{v}_1 = [2, 2.4, 16]^{\mathrm{T}}, \mathbf{v}_2 = [2.5, 3, 20]^{\mathrm{T}}, \mathbf{v}_3 = [3, 3.6, 24]^{\mathrm{T}}\right\}$$

such that

$$m(\{\mathbf{v}_1\}) = \frac{1}{3}, \qquad m(\{\mathbf{v}_2, \mathbf{v}_3\}) = \frac{1}{2}, \qquad m(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}) = \frac{1}{6}, \tag{21}$$

where $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ refer to the vectors of the below average, average, and above average yields of wheat, corn, and sugar beets. For example, $\mathbf{v}_1 = [2, 2.4, 16]^{\mathrm{T}}$ means that the below average yields for wheat, corn and sugar beets are 2, 2.4, and 16 tons/acre, respectively. We also can see that $\mathbf{v}_1 \leq \mathbf{v}_2 \leq \mathbf{v}_3$. We note that if each of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ has probability $\frac{1}{3}$, the problem (20) becomes a standard two-stage recourse problem as presented in Birge and Louveaux (1997) with the average overall profit of \$ 108,390 when using 170, 80, and 250 acres for wheat, corn, and sugar beets, respectively.

However, if we obtained the random set information (21) instead of an exact probability, we would have

$$\begin{aligned}
Bel(\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}) &= 1, & Pl(\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}) &= 1,\\
Bel(\{\mathbf{v}_1,\mathbf{v}_2\}) &= \tfrac{1}{3}, & Pl(\{\mathbf{v}_1,\mathbf{v}_2\}) &= 1,\\
Bel(\{\mathbf{v}_1,\mathbf{v}_3\}) &= \tfrac{1}{3}, & Pl(\{\mathbf{v}_1,\mathbf{v}_3\}) &= 1,\\
Bel(\{\mathbf{v}_2,\mathbf{v}_3\}) &= \tfrac{1}{2}, & Pl(\{\mathbf{v}_2,\mathbf{v}_3\}) &= \tfrac{2}{3},\\
Bel(\{\mathbf{v}_1\}) &= \tfrac{1}{3}, & Pl(\{\mathbf{v}_1\}) &= \tfrac{1}{2},\\
Bel(\{\mathbf{v}_2\}) &= 0, & Pl(\{\mathbf{v}_2\}) &= \tfrac{2}{3},\\
Bel(\{\mathbf{v}_3\}) &= 0, & Pl(\{\mathbf{v}_3\}) &= \tfrac{2}{3}.
\end{aligned}$$

Appling the optimistic, pessimistic and minimax regret approaches with this information, we provide three meaningful solutions for this random set linear program given in Table 1. We should make clear to the readers about the difference between the pessimistic (optimistic) approach and the pessimistic (optimistic) weather condition. The pessimistic weather condition is the below average scenario. If we know that it happens, we would proceed to solve the corresponding deterministic problem with the below average data. However, the pessimistic approach is used when we do not have the exact probability information on the weather, and would like to know the worst average outcome (profit) due to the unknown probability.

Moreover, suppose we do not use the regret solution, and treat the unclear information equally. That is, 180 years of the mixture between above average and on average yields is treated as 90 years of above average yields and 90 years of on average yields. Similarly, 60 years of chaotic conditions is interpreted as 20 years each for above average, average, and below average yields. Hence, the proportions for the above average, on average, and below average years are $\frac{11}{36}$, $\frac{11}{36}$, and $\frac{14}{36}$, respectively. Using these probabilities to set up an expected recourse model results in an optimal solution in which we obtain a profit of \$113,271.88. The column 'U' in Table 1 shows an optimal solution with respect to this expected recourse model with the uniform probability. If we use this solution as a solution for the minimax expected recourse model instead of the regret solution, the regret becomes \$8,224.65. This means that the uncertain information should not be assumed uniform priori, otherwise we may regret more than what we expect.

7 Conclusion and remarks

We are now able to handle an uncertain linear program with more than one interpretation of uncertainty in the problem which has not been done before. The approaches

Table 1 Optimal solutions based on the pessimistic, optimistic, minimax regret, and uniform expected recourse of the farming problem

	Wheat				Corn				Sugar beets			
	P	R	U	O	P	R	U	O	P	R	U	O
Devoted land (acres)	100	145.98	170	183.33	100	82.32	80	66.67	300	271.70	250	250
Below Average												
Yield (tons)	200	291.95	340	366.67	240	197.57	192	160	4,800	4,347.25	4,000	4,000
Sales (tons)	0	91.95	140	166.67	0	0	0	0	4,800	4,347.25	4,000	4,000
Purchase (tons)	0	0	0	0	0	42.43	48	80	–	–	–	–
Average												
Yield (tons)	250	364.94	425	458.33	300	246.97	240	200	6,000	5,434.06	5,000	5,000
Sales (tons)	50	164.94	225	258.33	60	6.97	0	0	6,000	5,434.06	5,000	5,000
Purchase (tons)	0	0	0	0	0	0	0	40	–	–	–	–
Above Average												
Yield (tons)	300	437.93	510	550	360	296.36	288	240	7,200	6,520.87	6,000	6,000
Sales (tons)	100	237.93	310	350	120	56.36	48	0	6,000	6,000	6,000	6,000
									1, 200*	520.87*		
Purchase (tons)	0	0	0	0	0	0	0	0	–	–	–	–

O := optimistic, P := pessimistic, R := minimax regret, and U := uniform
∗ Sugar beets sold at $10/ton
Pessimistic profit $87,150
Optimistic profit $127,677.78
Profit by uniform probability $113,271.88
Regret value using uniform solution $8,224.65
Minimax regret value $4,656.51

we use for solving this problem were the optimistic, pessimistic and minimax regret approaches by using the fact that each uncertainty in this paper can be represented as a set of probabilities. Therefore, we overcome many of the existing limitation in literature.

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