A Dynamic Model of Network Formation: Network Participation Game and Network Sharing Game

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***Abstract*— We present a dynamic model of network formation for network participation and sharing problems based on a non-cooperative game theory framework. In both scenarios, we operate within the context of a social network where the network's structure reflects the interactions among participants. In the network participation problem, users have to decide whether to participate in a cooperative activity even if it involves incurring a cost. Their choice depends on a simple criterion: participation becomes attractive if the anticipated benefits, closely tied to the number of their friends also partaking, outweigh the incurred cost. Similarly, in a network-sharing problem, individuals have to decide whether to share their personal resources with their friends or not. Sharing occurs only when a mutual benefit arises, and both parties stand to gain from the exchange. We cast these problems as non-cooperative games and provide a comprehensive characterization of the Nash equilibria in these settings. Furthermore, we introduce Log-Linear Learning (LLL) as a potential decision strategy for the participants and analyze the long-term dynamics of this approach within the framework. To empirically validate our research findings, we perform extensive simulations on random networks. These simulations provide compelling evidence that within our proposed framework, user engagement in both network participation and sharing dilemmas closely aligns with the well-established concepts of k-core and (r, s)-core within network structures.**

I. Introduction

Social and economic networks serve as the backbone of our society. The interaction topologies of these networks play a fundamental role in a diverse set of phenomena such as the adoption of new technologies and conventions, the generation of personalized recommendations, variations in stock markets, and the diffusion of information diffusion in a population (see [1], [2], and [3]).

The exchange of information and resources within a social network depends on the willingness of users to share. These resources may take the form of information, such as job openings, stock prices, or product reviews, or they can be physical resources like sensor suites for weather data in agricultural planning or location-sharing for accurate traffic congestion updates on cellphones. Consequently, when individuals choose not to contribute their available resources, even if they have a large number of connections in the underlying social network, it can hinder the progression of a specific phenomenon. Thus, within the framework of a given social network, an important challenge is to identify users who are inclined to engage in cooperative activities ([4], [5], [6], and [7]).

We propose a game theoretic framework for network formation within network participation and sharing games. In participation games, users face the choice of joining a cooperative activity, which involves incurring certain costs but also reaping associated benefits. Similarly, in sharing games, each player possesses a set of personal resources and must decide whether to share these resources with their friends/neighbors in a social network. When mutual resource sharing occurs among friends with different resources, it can lead to shared benefits for all parties involved. Various models for user engagement have been proposed in the literature on social network analysis, behavioral psychology, and economics, based on empirical evidence and experimental data. An important aspect of these models is the concept of reciprocity as discussed in [8], [9], [10], and [11]. The principle of reciprocity dictates that users only participate in any activity as long as their expected benefits outweigh the cost of participation.

Based on reciprocity, the following model has been widely adopted for the network participation problem. A user decides to participate in a network activity if a minimum number, say k, of his friends are also participating and he opts out of the activity if the number of participating friends drops below k. This model leads to the concept of k-core of the network as a measure of user engagement (see for example [4], [11], [12], [13]). Here, the k-core is a graph theoretic notion and refers to the maximal sub-graph of a graph in which each vertex has at least k neighbors. Similarly, in the network sharing problem as presented in [7], there exists a total of r resources in the network and each user is randomly assigned s of these resources, where s is less than r. Then, the users agree to share their resources if and only if they can get access to all the r resources through their friends. This model leads to the notions of (r, s)-core.

The analysis that establishes k-core and (r, s)-core as a measure of user engagement in [4] and [7], and the other related literature, assumes that all the players are initially participating in the cooperative/sharing activity. Then, the players who do not satisfy the k-neighbor criteria, or who do not get access to all the r resources from their participating friends, decide to opt out. This user attrition can lead to a cascade of users leaving the activity in steps and the iterative process converges to the k-core or (r, s)-core of the network, in which all the users satisfy the desired criteria. However, this approach cannot explain the formation of a participation/sharing network from initial conditions in

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which the number of initial participants is significantly less than the k-core or (r, s)-core of the network.

There also exists an extensive body of literature on network formation in which the primary research question is to figure out the type of networks that emerge when a group of myopic and self-interested agents has to decide whether to participate in a cooperative activity or not (see e.g., [14], [15], [16], [17]). However, the focus in these works is on edge formation such that users can select a specific subset of neighbors with whom they wish to collaborate. This problem is different from the user participation problem described above in which all the neighbors of a participating user benefit.

Our objective in this work is to generalize the setups of the network participation and sharing problems by proposing a dynamic network formation approach that can lead to a participation/sharing network from any initial condition. In particular, we formulate these problems as non-cooperative games in which users are modeled as self-interested rational players with a utility function. The players decide between participating in a network activity or opting out by maximizing their local utility functions. We provide a complete characterization of the set of Nash equilibria for both of these games. Then, we propose a dynamic model for players' decisions in which players select their actions using Log-Linear Learning as defined in [18] and [19].

Log-linear learning (LLL) is a version of noisy best-response dynamics and assumes that the players have bounded rational. In bounded rationality, players select best response actions with high probability but select suboptimal actions with a small but non-zero probability. In our case, bounded rationality implies that players who do not have k friends participating or players who do not get access to the full set of r resources will still take a risk and decide to participate in the network for some time. However, they will opt out after some time if the reciprocity criteria are not satisfied after some random time. This behavior based on bounded rationality has been explored in a variety of theoretical and empirical settings as in [20], [21], [22], and [23]. After setting up the game, we prove for the network participation game that all the best response paths are acyclic and stochastically stable joint action profiles under LLL will belong to the set of Nash equilibria. Finally, we validate our results through extensive simulations. Thus, our proposed decision model for the players explains network formation from any given initial condition.

II. Setup

We consider a social network of n players represented by a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, n\}$ is the set of players and $\mathcal{E}$ is the edge set. An edge $(i, j) \in \mathcal{E}$ implies that players i and j can interact and benefit from each other. Given a player i and a set $S \subset V$,

$$d(i, S) = |\{j \in S \mid (i, j) \in \mathcal{E}\}|.$$

Similarly, for any two sets S_1 and S_2,

$$d(S_1, S_2) = |\{(i, j) \in \mathcal{E} \mid i \in S_1 \text{ and } j \in S_2\}|.$$

Let $\mathcal{N}(i)$ be the neighborhood set of player i, i.e.,

$$\mathcal{N}(i) = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\},$$

and $\mathcal{N}[i] = \mathcal{N}(i) \cup \{i\}$ be the closed neighborhood of i. Each player has a set of actions A_i and $\mathcal{A} = A_1, A_2, \ldots, A_n$ is the set of joint action profiles. Each element σ in $\mathcal{A}$ is an n-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ such that $\sigma_i \in A_i$ is an action of player i. We use the notation $\sigma = (\sigma_i, \sigma_{-i})$ to represent actions from the perspective of player i where σ_{-i} is an $(n-1)$-tuple representing the actions of all the players other than i. Similarly, for any set $S \subset V$, $\sigma = (\sigma_S, \sigma_{V \setminus S})$ is a decomposition of action profile between players in the sets S and $V \setminus S$ respectively.

A player has a utility function defined over the set of joint action profiles, i.e., $U_i : \mathcal{A} \to \mathbb{R}$. Given an action profile σ_{-i}, player i prefers action σ_i over σ_i' if and only if $U_i(\sigma_i, \sigma_{-i}) > U_i(\sigma_i', \sigma_{-i})$. The set of best responses of player i to σ_{-i} is

$$B_i(\sigma_{-i}) = \{\sigma_i^* \in A_i \mid U_i(\sigma_i^*, \sigma_{-i}) \geq U_i(\sigma_i, \sigma_{-i}) \; \forall \, \sigma_i \in A_i\}.$$

Let

$$d_H(\sigma, \sigma') = |\{j \in V \mid \sigma_j \neq \sigma_j'\}|$$

be the number of players whose actions are different in profiles σ and σ'. A sequence of joint action profiles $\mathcal{P} = (\sigma^0, \sigma^1, \cdots, \sigma^{l-1})$, $\sigma^p \in \mathcal{A}$ for all $p \in \{0, 1, \cdots, l-1\}$, is a best-response path if $d_H(\sigma^p, \sigma^{p+1}) = 1$ for each consecutive pair of profiles and σ_i^{p+1} belongs to $B_i(\sigma_{-i}^p)$ for the updating player i. Thus, at each step of a best response path, only one player updates his action and the updated action is a best response to the actions of other players. For the path $\mathcal{P}$, we define

$$\mathcal{P}(\sigma^p, \sigma^{p+1}) = \{i \in \mathcal{V} \mid \sigma_i^p \neq \sigma_i^{p+1}\},$$

i.e., $\mathcal{P}(\sigma^p, \sigma^{p+1})$ is the index of the player updating his action from σ^p to σ^{p+1}.

Given an action profile, players update their actions based on some strategic rule. A variety of learning rules have been proposed and analyzed in the literature. We assume that players' action selection strategy can be modeled by Log-Linear Learning (LLL), which is a version of noisy best response dynamics (see e.g., [18] and [24]). Log-linear learning is an asynchronous learning rule in which, at each decision time, only one player updates his action. Below are the steps:

1) Log-Linear Learning (LLL) : In LLL, actions are selected at discrete time steps, $k = 1, 2, \cdots$. Let $\sigma = (\sigma_i, \sigma_{-i})$ be the current action profile at some time step k. Then, for the next step,

- One player is randomly selected with any distribution that assigns non-zero selection probability to all the players. Say player i is selected.
- All the other players repeat their actions.
- Player i selects an action σ_i' with probability

$$p_i^{\text{LLL}}(\sigma_{-i}, T) = \frac{e^{-\frac{1}{T}\left(U_i(\sigma_i^*, \sigma_{-i}) - U_i(\sigma_i', \sigma_{-i})\right)}}{Z_i(\sigma_{-i})}, \qquad (1)$$

where $Z_i(\sigma_{-i}) = \sum_{\hat{\sigma}_i \in A_i} e^{-\frac{1}{T}(U_i(\sigma_i^*, \sigma_{-i}) - U_i(\hat{\sigma}_i, \sigma_{-i}))}$ and $\sigma_i^* = \underset{\hat{\sigma}_i \in A_i}{\operatorname{argmax}}\ U_i(\hat{\sigma}_i, \sigma_{-i})$.

In (1), the parameter T represents noise in the decision-making, and as $T \to 0$, $p_i^{\mathrm{LLL}}(\sigma_{-i}, T)$ converges to the probability in which all the weight is assigned to the best response of i given σ_{-i}. Log-linear learning induces a Markov chain on the $\mathcal{A}$, and the induced Markov chain is ergodic and reversible with a unique stationary distribution μ_T^{LLL}. An action profile σ is *stochastically stable* if and only if $\lim_{T \to 0} \mu_T^{\mathrm{LLL}}(\sigma) > 0$,

III. Network Participation Game-k neighbor setup

Consider a social network represented by a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \cdots, n\}$ is the set of players and $\mathcal{E}$ is the set of connections among the players. Suppose, the players are presented with an opportunity to participate in some collaborative activity. Each agent will have to pay some cost to participate in that activity. Let, the cost of participation be k. Each player also anticipates receiving some benefit from participating in this activity and the benefit depends on the number of neighbors of the player who also decide to participate. To formulate this problem as a game, each player has two actions in his action set:

$$A_i = \begin{cases} 1 & \text{participate} \\ 0 & \text{do not participate} \end{cases}$$

Given an action profile σ in $\mathcal{A}$. We define two sets

$$V_\sigma = \{j \in \mathcal{V} \mid \sigma_j = 1\},$$

i.e., V_σ is the set of players participating in the network in action profile σ. From a player's perspective, we define

$$\mathcal{N}_i^p(\sigma) = \{j \in \mathcal{N}(i) \mid \sigma_j = 1\}.$$

where $\mathcal{N}_i^p(\sigma)$ is the set of neighbors of i that are already participating in the network. As before $\mathcal{N}_i^p[\sigma] = \mathcal{N}_i^p(\sigma) \cup \{i\}$ be the *closed neighborhood of participating players of* i.

A key challenge in formulating the network participation problem as a non-cooperative game is to propose a utility function that should support the empirical evidence and experimental observations regarding players' participation decisions, and should also result in a game setup that can be analyzed in detail. The utility function of a player that we propose in this work is

$$U_i(\sigma_i, \sigma_{-i}) = \sigma_i \left(\frac{1}{\mathcal{N}(i)} (\mathcal{N}_i^p(\sigma) - k) + \frac{\alpha}{\mathcal{N}(i)} \mathbf{1}_i(\sigma) \right). \tag{2}$$

The utility of a player is zero for not participating in the cooperative activity. For $\sigma_i = 1$, the utility has two components. The first term of the utility of a player depends on the number of his neighbors participating in the network, which is represented by $\mathcal{N}_i^p(\sigma)$. A player receives a positive utility if $\mathcal{N}_i^p(\sigma)$ is greater than k and a negative utility if $\mathcal{N}_i^p(\sigma) < k$. The second term is an indicator function

$$\mathbf{1}_i(\sigma) = \begin{cases} 1 & |\mathcal{N}_i^p(\sigma)| = k, \\ 0 & \text{otherwise}, \end{cases}$$

and $\alpha \in (0, 1)$. The objective of the second term is to break the tie when the number of participating neighbors of a player is exactly equal to k.

The utility function is normalized by $\mathcal{N}(i)$, which results in a utility function that is asymmetric about the real axis depending on the relative values of k and $\mathcal{N}(i)$. This asymmetry helps in modeling the phenomenon that for players with the number of neighbors significantly higher than k, the cost of joining the network when $\mathcal{N}_i^p(\sigma)$ is less than k is small as compared to the players with a small neighborhood set. The implication of this relative difference in cost is that players with large number of neighbors will be more willing to take risks and the probability of selecting noisy actions in LLL will be higher as compared to the players with a small number of neighbors. Thus, players who are well connected in the network are more likely to join early as compared to players with a small number of neighbors.

A. Analysis

Next, we analyze the network participation game for which the player utilities are defined in (2), and players use noisy best response, which is modeled by LLL, as their decision strategy. To analyze network formation behavior for this setup, we will use Nash equilibrium and stochastic stability as our solution concept.

Proposition 1: For the network participation game with utility function defined in (2), an action profile σ^* is a Nash equilibrium if either of the following conditions is satisfied.

1) $\sigma_i^* = 0$ for all $i \in \mathcal{V}$.
2) For all i in V_{σ^*}, $d(i, V_{\sigma^*}) \geq k$ and for all j in $V \setminus V_{\sigma^*}$, $d(j, V_{\sigma^*}) < k$.

Proof: To prove the first part of the proposition, suppose the joint action profile is $\sigma = (0, 0 \ldots, 0)$. Then, the utility of all the players is zero. At each iteration, a random player, say i is selected to update his action under LLL. Then,

$$U_i(0, \sigma_{-i}) = 0 \text{ and } U_i(1, \sigma_{-i}) = -k/|\mathcal{N}(i)|.$$

Thus, each player prefers no participation over participation if the current joint profile is $(0, \ldots, 0)$.

For condition 2), the joint action profile σ is represented as $\sigma = (\sigma_{V_{\sigma^*}}, \sigma_{V \setminus V_{\sigma^*}})$. For each player in V_{σ^*}, the condition $d(i, V_{\sigma^*}) \geq k$ implies that player i has at least k neighbors who are participating in the activity. Therefore, $U_i(1, \sigma_{-i}^*) > U_i(0, \sigma_{-i}^*)$. The strict inequality is enforced by the second term in the utility action that adds a value of α to the utility function when the number of participating neighbors is exactly equal to k. Similarly, for all players $j \in V \setminus V_{\sigma^*}$, the condition $d(j, V_{\sigma^*}) < k$ implies that the number of participating neighbors is less than k and therefore $U_i(0, \sigma_{-i}^*) < U_i(1, \sigma_{-i}^*)$. ∎

Proposition 2: For the network participation game with utility function defined in (2), all best response paths are acyclic.

Proof: Suppose the above statement is not true and there exists a best response path $\mathcal{P} = (\sigma^1, \sigma^2, \cdots, \sigma^l, \sigma^{l+1}$ that is a cycle, i.e., $\sigma^1 = \sigma^{l+1}$. For a path to be a cycle, every player that transitions from 0 to 1 must transition back to 1 and every player that transitions from 1 to 0 must transition back from 0 to 1. Thus, we can only have a cycle with an even number of transitions, which implies that l must be an even number with $\sigma^{l+1} = \sigma^1$.

Let G be a global function on the set of joint action profiles and is defined as

$$G(\sigma) = \sum_{i=1}^{n} \hat{U}_i(\sigma_i, \sigma_{-i}),$$

where

$$\hat{U}_i(\sigma_i, \sigma_{-i}) = \mathcal{N}(i)U_i(\sigma_i, \sigma_{-i}) - \alpha \mathbf{1}_i(\sigma)\sigma_i. \tag{3}$$

If the path $\mathcal{P}$ is a cycle, then the following equality must hold

$$\sum_{q=1}^{l} \left[G(\sigma^{q+1}) - G(\sigma^q)\right] = 0.$$

Since every transition must be reversed in a cyclic path. Suppose

$$\mathcal{P}(\sigma^{q_1}, \sigma^{q_1+1}) = \mathcal{P}(\sigma^{q_2}, \sigma^{q_2+1}) = i,$$

i.e., player i updates the actions in transitions $(\sigma^{q_1}, \sigma^{q_1+1})$ and $(\sigma^{q_2}, \sigma^{q_2+1})$. Without loss of generality, we assume that i transitions from 0 to 1 in σ^{q_1} to σ^{q_1+1} transition and from 1 to 0 in some later transition σ^{q_2} to σ^{q_2}, where $1 < q_1 < q_2 < l$. Then,

$$\begin{aligned} G(\sigma^{q_1+1}) - G(\sigma^{q_1}) &= [\hat{U}_i(1, \sigma^{q_1}_{-i}) - \hat{U}_i(0, \sigma^{q_1}_{-i})] + \\ &\sum_{j \in \mathcal{N}_i^p(\sigma^{q_1})} \left[\hat{U}_j(\sigma^{q_1}_j, 1, \sigma^{q_1}_{-\{i,j\}}) - \hat{U}_j(\sigma^{q_1}_j, 0, \sigma^{q_1}_{-\{i,j\}})\right] + \\ &\sum_{j \notin \mathcal{N}_i^p[\sigma^{q_1}]} \left[\hat{U}_j(\sigma^{q_1}) - \hat{U}_j(\sigma^{q_1})\right]. \end{aligned}$$

Here $\hat{U}_j(\sigma^{q_1}_j, 1, \sigma^{q_1}_{-\{i,j\}})$ is the updated utility of player j when $\sigma^{q_1}_j$ is the action of j, $\sigma^{q_1}_i = 1$ is the action of player i and $\sigma^{q_1}_{-\{i,j\}}$ is the joint action profile of actions of players other than i and j.

The last term in the above equation is equal to zero since the utility of the players that are not in the participating neighborhood of player i remains the same from σ^{q_1} to σ^{q_1+1}. Player i transitions from 0 to 1 as his best response if the $|\mathcal{N}_i^p(\sigma^{q_1})| \geq k$. Thus,

$$[\hat{U}_i(1, \sigma^{q_1}_{-i}) - \hat{U}_i(0, \sigma^{q_1}_{-i})] = |\mathcal{N}_i^p(\sigma^{q_1})| - k.$$

The second term in the expression for $G(\sigma^{q_1}) - G(\sigma^{q_1+1})$ is the impact of player i's participation on the utility of his participating neighbors and is equal to $\mathcal{N}_i^p(\sigma^{q_1})$ since the utility of each participating neighbor of i is increased by a factor of $1/\mathcal{N}(j)$. Thus,

$$G(\sigma^{q_1}) - G(\sigma^{q_1+1}) = 2\mathcal{N}_i^p(\sigma^{q_1}) - k.$$

When player i transitions from 1 to 0 at some later step of the cycle, say $\sigma^{q_2} = (1, \sigma^{q_2}_{-i})$ and $\sigma^{q_2+1} = (0, \sigma^{q_2}_{-i})$, then

$$G(\sigma^{q_2}) - G(\sigma^{q_2+1}) = 2\mathcal{N}_i^p(\sigma^{q_2}) - k.$$

Since a transition from 1 to 0 is a best response only if the number of participating neighbors is less than k, we get $\mathcal{N}^p(\sigma^{q_2}) < k \leq \mathcal{N}^p(\sigma^{q_1})$. Therefore,

$$[G(\sigma^{q_1}) - G(\sigma^{q_1+1})] - [G(\sigma^{q_2}) - G(\sigma^{q_2+1})] > 0$$

Thus, given a best response cycle, we can divide the entire path into pair of transitions of individual players from 1 to 0 and then from 0 to 1. The corresponding difference in the global function $G(\cdot)$ for these transition pairs is always greater than zero. Therefore,

$$\sum_{p=1}^{l-1} \left[G(\sigma^{p+1}) - G(\sigma^p)\right] > 0$$

for any best response cycle, which is a contradiction. Thus, a best response path cannot be a cycle for our network participation game with the utility function defined in (2). ■

Proposition 3: Consider the network participation game with the utility function defined in (2). If all the players adhere to Log-Linear Learning for decision-making, then the stochastically stable joint action profiles belong to the set of Nash equilibria.

Proof: The proof is a direct consequence of Props. 1 and 2. Prop. 1 implies that all the Nash equilibria of the game are strict, i.e. if $\sigma^* \in \mathcal{A}$ is a Nash equilibrium then $U_i(\sigma_i^*, \sigma_{-i}^*)$ is strictly greater than $U_i(\sigma_i, \sigma_{-i}^*)$ for any $\sigma_i \in A_i$. Prop 2 establishes that the best response paths are acyclic. For a finite number of players with a finite number of actions, $|\mathcal{A}|$ is finite and therefore every best response path should have a finite length. Thus, every best response path has to terminate and it cannot terminate to any profile other than a Nash equilibrium. Since LLL is a noisy best response dynamics, the Markov chain induced by LLL follows best response paths with high probability and Nash equilibria are the only absorbing states of the Markov chain if $T = 0$. Thus, in the limiting case of $T \to 0$, the stochastically stable profiles for which $\mu_T^{\text{LLL}} > 0$ will belong to the set of Nash equilibria, which concludes the proof. ■

IV. Network Participation Game: Assorted Resources Setup

We consider a generalization of the network participation game as presented in [7]. In the generalized setup, each player has a set of personal resources, which can be certain physical sensors, some information, or some expertise for performing certain tasks. A player can share his resources with his immediate neighbors in the network. Let $\mathcal{R} = \{0, 1, \ldots, r-1\}$ be the set of resources available in the network. Each node is assigned a subset of s resources where $s \leq r$. This setup can be represented as a labeled graph in which each node in the graph is assigned a label, where the label of anode is the set of resources assigned to that player.

In this game setup with assorted resources, $\mathcal{V} = \{1, 2, \ldots, n\}$ is the set of players and each player can interact with a subset of other players. Each player has a set of actions:

$$A_i = \begin{cases} 1 & \text{participate} \\ 0 & \text{do not participate} \end{cases}$$

A player should participate in the network only if they can get access to the full set of resources from the neighboring players who have decided to participate in the network. Given an action profile $\sigma = (\sigma_i, \sigma_{-i})$ in $\mathcal{A}$, where σ_i is the action of player i and σ_{-i} is the joint action profile of all the players other than i, we define

$$L_i(\sigma_{-i}) = \bigcup_{j \in \mathcal{N}_i^p[\sigma]} l(j).$$

where $\mathcal{N}_i^p[\sigma]$ is the closed neighborhood of i and comprises i and all the neighbors of i that are participating in the sharing network in profile σ_{-i} Thus, $L_i(\sigma_{-i})$ is the set of resources that i can access either directly or through its immediate neighbors who are participating. We propose the following utility function for each player

$$U_i(\sigma_i, \sigma_{-i}) = \frac{\sigma_i}{r}\left(|L_i(\sigma_{-i})| - r] + \alpha \mathbf{1}(\sigma)\right), \qquad (4)$$

where

$$\mathbf{1}_i(\sigma) = \begin{cases} 1 & |L_i(\sigma_{-i})| = r, \\ 0 & \text{otherwise}, \end{cases}$$

Notice that we have redefined $\mathbf{1}_i$ for notational convenience and it will be obvious from the situation which definition is applicable.

Player i receives zero utility in the case of no participation, i.e., $\sigma_i = 0$. If the player decides to participate, the resulting utility, as defined in (4), is negative if the number of resources that i can access in the closed neighborhood of participating players, $\mathcal{N}_i^p(\sigma)$, is less than r. However, if i has access to all the r resources, then $U_i$is equal to α/r for some positive α.

A. Analysis

For the network sharing game with assorted resources, we have set up a non-cooperative game by defining a utility function in (4). For players' decision strategy, we consider LLL in which players select their best response to the actions of other players with high probability and with a small but non-zero probability, which depends on the noise parameter T, players select the non-optimal action.

For analyzing network behavior, we again consider Nash equilibrium as our primary solution concept. The analysis approach will be similar to the approach for k neighbor setup.

Proposition 4: For the network sharing game with utility function defined in (4), an action profile σ^* is a Nash equilibrium if either of the following conditions is satisfied.

1) $\sigma_i^* = 0$ for all $i \in V$.
2) For all i in V_{σ^*}, $|L_i(\sigma_{-i}^*)| = r$ and for all j in $V \setminus V_{\sigma^*}$, $|L_i(\sigma_{-i}^*)| < r$, where $L_i(\sigma_{-i}^*)$ is the set of resources that player i can access in the closed neighborhood of participating players.

Proof: We can verify the above statements by applying the definition of Nash equilibrium. Consider condition 1) and let $\sigma^* = (0, 0 \ldots, 0)$, i.e., no player is participating in the sharing network. Suppose player i is randomly selected to update the action. Then, i's utilities for both actions will be the following.

$$U_i(0, \sigma_{-i}^*) = 0 \text{ and } U_i(1, \sigma_{-i}^*) = (s - r)/r.$$

Thus, i will prefer no participation over participation if σ^* is all zero.

Let $\sigma^* = (\sigma_{V_{\sigma^*}}^*, \sigma_{V \setminus V_{\sigma^*}}^*)$ be an action profile that satisfies condition 2). Suppose player i is randomly selected to update the action under LLL. Then, if i belongs to V_{σ^*}, condition 2) implies that $L_i(\sigma_{-i}^*) = r$, i.e., i has access to all the r resources in the closed neighborhood of participating players. Therefore,

$$U_i(1, \sigma_{-i}^*) = \alpha/r \text{ and } U_i(0, \sigma_{-i}^*) = 0,$$

and player i will prefer to keep participating in the sharing network over not participating. Suppose $i \in V \setminus V_{\sigma^*}$ and condition 2) implies that $\sigma_i^* = 0$ and $L_i(\sigma_{-i}^*)| < r$. When i will receive an opportunity to update his action, his utility for both the actions will be

$$U_i(0, \sigma_{-i}^*) = 0 \text{ and } U_i(1, \sigma_{-i}^*) = (s - r).$$

Therefore, player i's best response will be $\sigma_i = 0$.

Finally, we argue that any action profile that does not satisfy conditions 1) or 2) is not a Nash equilibrium. Suppose there exists a non-zero such profile $\hat{\sigma}$ that does not satisfy condition 2) but is a Nash equilibrium. Not satisfying 2) implies that either there exists a player with $\hat{\sigma}_i = 1$ but $|L_i(\hat{\sigma}_{-i})| < r$ or there exists a player with $\hat{\sigma}_i = 0$ but $|L_i(\hat{\sigma}_{-i})| = r$. In both of these scenarios, the best response of that player will be to switch the current action, and hence $\hat{\sigma}$ cannot be Nash equilibrium. ■

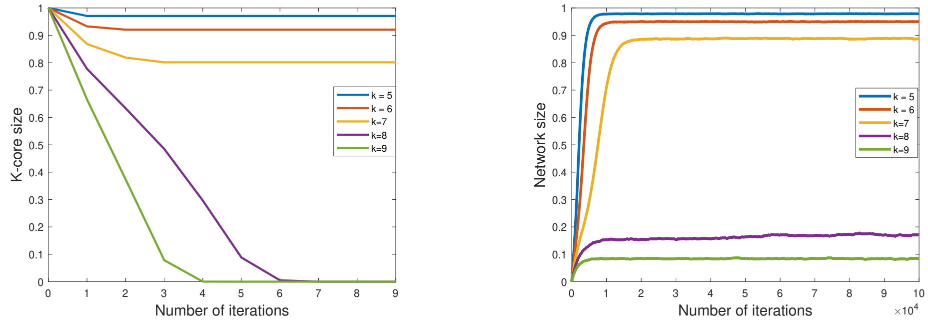
V. Simulation

We performed extensive simulations in Matlab to validate our results. We modeled the underlying social network with Erdös-Rényi (ER) graphs. ER graphs are the most widely used models of random graphs, which are specified by a single parameter p called the edge probability. In an ER graph, an edge exists between any node pair with probability p. Thus, the formation of each edge in an ER graph is completely independent of the network structure and only depends on p.

In our setup, we considered ER graphs with $n = 1000$ players and link formation probability $p = 0.010$. Thus, each player had ten neighbors on average. We generated 100 ER graphs and verified our results on all the randomly generated networks. Then, we computed the average performance over all the hundred networks.

A. Simulation results for the Network Participation Game:

For the network participation game, we selected $T = 0.3$ and $\alpha = 0.5$, where T was the noise parameter in LLL and α was a parameter in the utility function defined in (2). We simulated networks with $k = \{5, 6, 7, 8, 9\}$. The results are presented in Fig. 1. In Fig. 1(a), we present the results of the standard k-core selection algorithm as described in [4]



(a) k-core size

(b) Participation Network size with LLL

Fig. 1. Simulation Results for Network Participation game

in which we start with all the players participating in the network, and then we iteratively remove the players who do not satisfy the k neighbor criteria. The x-axis in this plot corresponds to the number of iterations and the y-axis corresponds to the expected fraction of players that belong to the k-core of the network, averaged over one hundred randomly generated networks. As can be seen from the figure the, k-core of the networks were empty for $k = 8$ and $k = 9$, and a significant fraction of players were included in the k-core for $k < 8$.

In Fig. 1(b), we present the results of our proposed approach for the network participation problem. In this plot, the x-axis represents the number of iterations of LLL and the y-axis represents the network size, which is equal to the fraction of players that are participating in the network. For our results, we started with the initial condition with no participating player. However, under LLL, the fraction of players who decide to participate in the network start to increase and reach a steady state after around twenty thousand iterations. From these results, we can immediately observe a direct relation between the sizes of the k-core and network sizes under our proposed scheme for various values of k.

For performance comparison, we compared the action of each player in the k-core formulation and our proposed formulation. Because of the noise in LLL, players have a non-zero probability of switching their actions. However, choosing a suboptimal action is a rare event that happens with very low probability. Therefore, we considered the actions of all the players when LLL reached a steady state. In particular, we considered the actions of all the players in the final thirty thousand iterations of LLL. From Fig. 1(b). we can observe that the state of the network is the steady state for the final thirty thousand iterations. Then we imposed a constraint that if a player decided to participate in the network for over 95% of these thirty thousand iterations, then we consider that player to be a member of the network. Otherwise, we assume that the player's participation was a rare event and we do not consider that player as part of the network. The results of this analysis are presented in Table I. For the cases of $k \in \{5, 6, 7, 8, 9\}$, the expected error is less than 2.5%, which is an extremely tight bound considering

k	% error
5	0.2190
6	0.2820
7	2.2830
8	1.2960
9	0

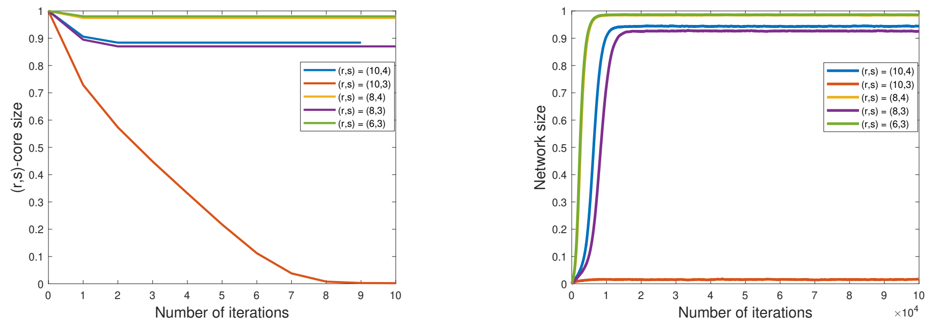
TABLE I
PERFORMANCE COMPARISON FOR NETWORK PARTICIPATION GAME

the stochastic nature of players' decision strategy. Based on these results, we can declare with high confidence that in the network participation game that we presented in Section III where the objective of the players was to maximize their utility defined in (2) using LLL, the global state of network converges to the k-core of the network. Thus, we have addressed the challenge of k-core formation as a non-cooperative game.

B. Simulation results for the Network Sharing Game:

We performed a similar analysis for the Network Sharing Game and the results are presented in Fig. 2. As before, the underlying social network is an ER graph with $n = 1000$ and $p = 0.01$, which results in an expected number of ten neighbors per player. The simulations were performed for (r, s) in $\{(10, 4), (10, 3), (8, 4), (8, 3), (6, 3)\}$. We selected the noise parameter $T = 0.15$ and $\alpha = 0.5$ for (4). The results were averaged over one hundred randomly generated ER networks. In Fig. 2(a), we show the results for the expected fraction of players included in the (r, s) core of ER networks. The only case in which (r, s)-core was empty was $(r, s) = (10, 3)$. For all the other simulated scenarios, the percentage of players that were included in the (r, s)-core was over 80%. In Fig. 2, we present the results for our proposed scheme for the network sharing game. The y-axis represents the size of the network in terms of the expected fraction of players sharing their resources over the network with LLL as their decision strategy. We can immediately observe that the results are very similar.

We compared the network performance under (r, s)-core and the size of the sharing network under our proposed scheme. For this comparison, we again compared the actions



(a) (r, s)-core size

(b) Sharing network size with LLL

Fig. 2. Simulation Results for Network Sharing game

(r, s)	% error
(10,4)	0.533
(10,3)	0.2430
(8,4)	0.1560
(8,3)	0.769
(6,4)	0.08

TABLE II

PERFORMANCE COMPARISON FOR NETWORK SHARING GAME.

of all players in the (r, s)-core and our proposed scheme in the steady state, which is the final thirty thousand iterations as described above. The results of this comparison are presented in Table II and it can be seen that the errors in all the scenarios is less than 1%. Thus, the actions of the players under LLL are the same as in the (r, s)-core setup, which validates our proposed approach for the network sharing game as presented in Section IV.

VI. CONCLUSIONS

We presented a game theoretic model for user participation and sharing problems, which have a central importance in social network analysis. The existing analysis approaches either assumed full participation as an initial condition or considered network formation at an edge level in which users had to decide which connections they would like to maintain with their neighbors. We formulated these problems as non-cooperative games with players having bounded rationality. We proposed utility functions for these games and completely characterized their Nash equilibria. We also analyzed the best response paths for the network participation game and established that LLL dynamics will converge to the set of Nash equilibria. We validated our setups through extensive simulations and showed that under our proposed approach in which players' action selection strategy was modeled as noisy best response dynamics, the participation network evolved from an initial condition of zero user engagement and converged to the k-cores and (r, s)-cores of randomly generated ER networks.

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