State Space Representation

By

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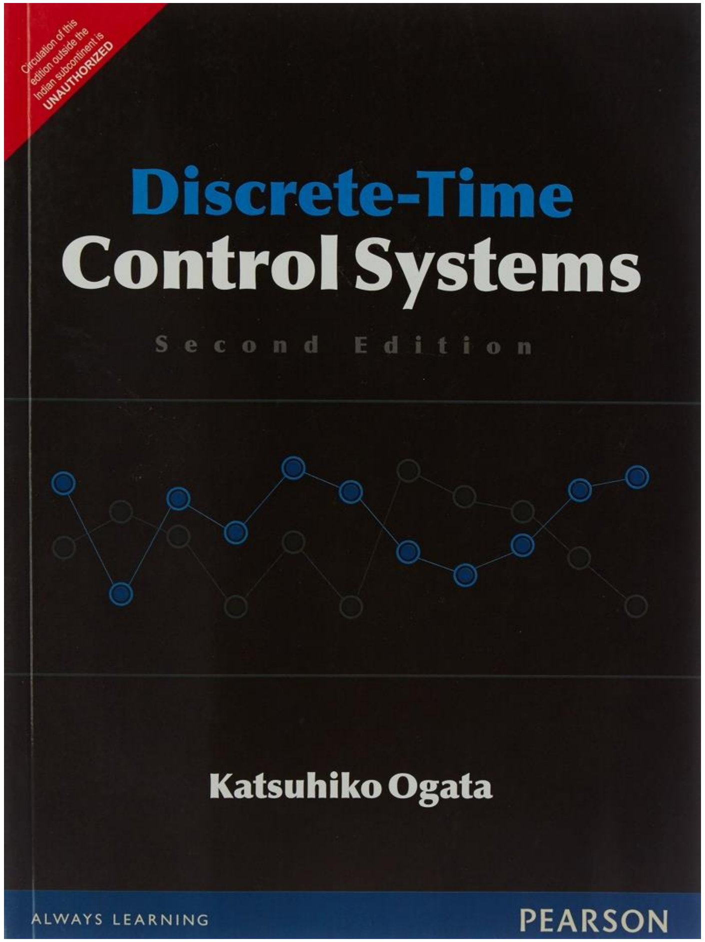

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Introduction

Conventional Control Theory V.S Modern Control Theory

1. The conventional control theory is completely based on the frequency domain approach while the modern control system theory is based on time domain approach.

2. In the conventional theory of control system we have linear and time invariant single input single output (SISO) systems only but with the help of theory of modern control system we can easily do the analysis of even non linear and time variant multiple inputs multiple outputs (MIMO) systems also.

3. In the modern theory of control system the stability analysis and time response analysis can be done by both graphical and analytically method very easily.

- **Conventional control theory is applicable to:**
  - SISO systems.
  - Linear.
  - Time Invariant.

- **Modern control theory is applicable to:**
  - MIMO systems.
  - Linear or Nonlinear Systems.
  - Time Invariant or Time Varying.

Introduction (Cont.)

Linear State Space Models

- There are many alternative model formats that can be used for linear dynamic systems.
- In simple SISO problems, any representation is probably as good as any other.
- However, as we move to more complex problems (especially multivariable problems), it is desirable to use special model formats.
- One of the most flexible and useful structures is the ***State Space Model.***
- State space analysis of control system is based on the modern theory which is applicable to all types of systems like:
  - single input single output systems (SISO),
  - multiple inputs and multiple outputs systems (MIMO),
  - linear and non linear systems,
  - time varying and time invariant systems.

State Space Representation

State Space Analysis

- Let us consider few basic terms related to state space analysis of modern theory of control systems.
- **State in State Space Analysis:** It refers to smallest set of variables whose knowledge at $t = t0$ together with the knowledge of input for $t \geq t0$ gives the complete knowledge of the behavior of the system at any time $t \geq t0$.
- **State Variables in State Space analysis:** It refers to the smallest set of variables which help us to determine the state of the dynamic system.
  - State variables are defined by x1(t), x2(t)……..Xn(t).
- **State Vector:** Suppose there is a requirement of n state variables in order to describe the complete behavior of the given system, then these n state variables are considered to be n components of a vector x(t). Such a vector is known as state vector.
- **State Space:** It refers to the n dimensional space which has x1 axis, x2 axis ………xn axis.

State Space Equations

- **Importance of state space representation of a system (A system description in a time domain Modern method of system description):**

- **describe a large category of systems**, such as linear and nonlinear systems, time-invariant and time-varying systems, systems with time delays, systems with nonzero initial conditions, and others.

- Due to the fact that state equations are a set of first-order differential equations, they can be **easily programmed and simulated on both digital and analog computers.**

- Formulating and subsequently in investigating a great variety of **properties in system theory**, such as stability, controllability, and observability.

State Space Equations

- **A state of a system:**

  - past, present, and future state of the system.

  - represents by state variables.

- **A system** could be described by a finite number of state variables.

State Space Equations

- **State Variables**:

the minimum number of system variables such that if we know

(a) their values at a certain moment $t0$,

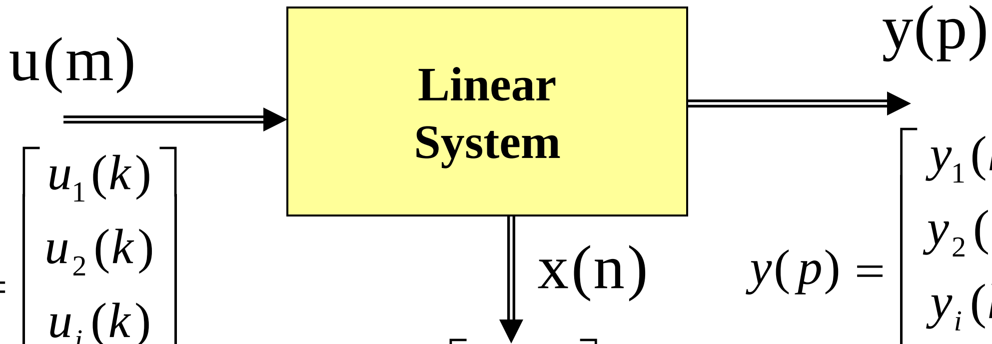
(b) the input function applied to the system for $t >= t0$,

(c) the mathematical model which relates the input, the state variables, and the system itself,

→then the determination of the system's states for $t > t0$ is guaranteed.

The state variables are generally variables representing the state of energy storage elements in a system.

Linear State Space Model



$$u(m) = \begin{bmatrix} u_1(k) \\ u_2(k) \\ u_i(k) \\ u_m(k) \end{bmatrix}$$

Input vector

$$x(n) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_i(k) \\ x_n(k) \end{bmatrix}$$

State vector

$$y(p) = \begin{bmatrix} y_1(k) \\ y_2(k) \\ y_i(k) \\ y_p(k) \end{bmatrix}$$

Output vector

- MIMO : m>1, P>1
- MISO:M>1,P=1, C,D are column vectors.
- SIMO:M=1,P>1, B,D are row vectors
- SISO:M=P=1 B is row vector, c column vector, D is a scalar value.

- ❑ **For linear, time-invariant systems, a discrete-time *state-space model* :**

$$x(k+1) = Gx(k) + Hu(k)$$
$$y = Cx(k) + Du(k)$$

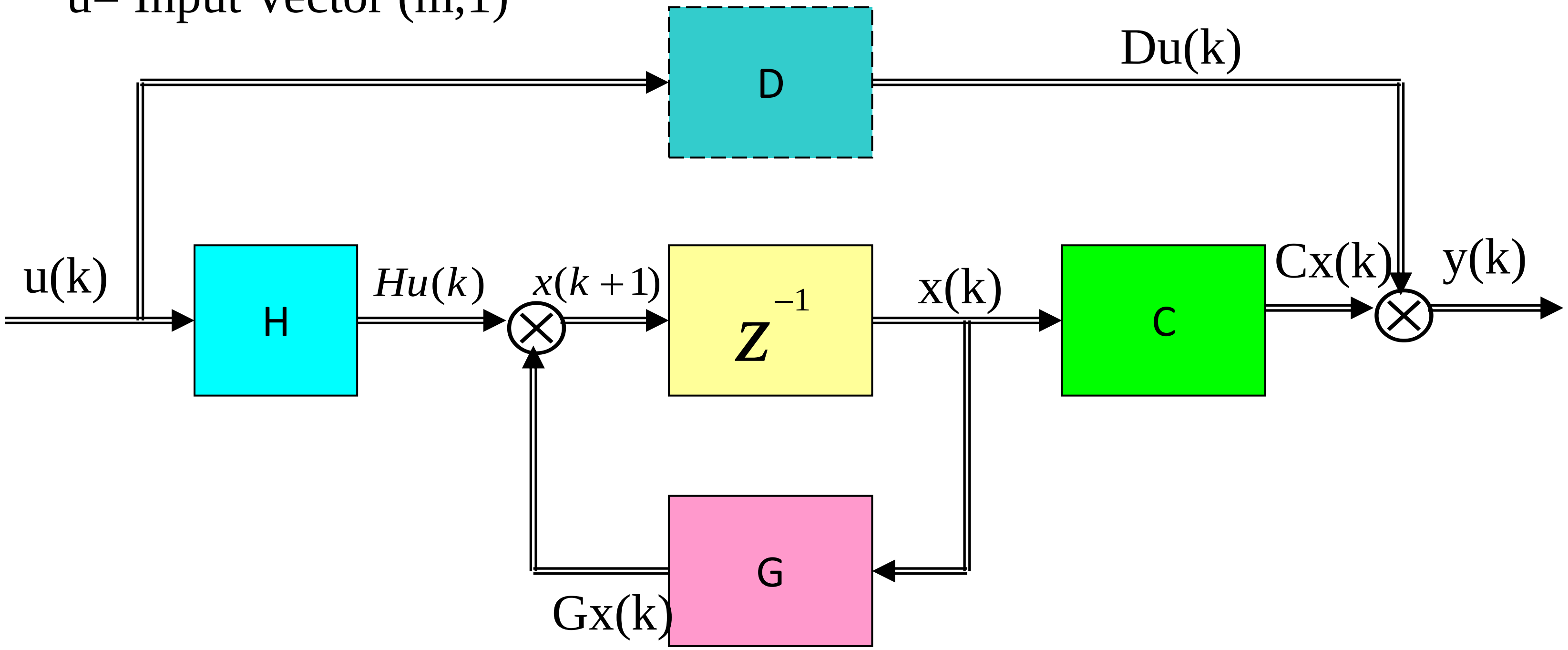
State Space representation of a linear system

Given:

$$x(k+1) = Gx(k) + Hu(k)$$
$$y = Cx(k) + Du(k)$$

G= State (System) Matrix(n,n)
H= Input Matrix (n,m)
x= State Vector (n,1)
u= Input Vector (m,1)

C= Output Matrix (r,n)
D= Direct Transmission Matrix (r,m)
y= Output Vector (r,1)



Block diagram of state space

State-Space Canonical Forms

- Canonical forms are the standard forms of state space models.
- Each of these canonical form has specific advantages which makes it convenient for use in particular design technique.
- The most interesting canonical forms are the following: –
  - Controllable Canonical form
  - Observable Canonical form
  - Diagonal Canonical form (Jordan Canonical Form)
- It is interesting to note that the dynamics properties of system remain unchanged whichever the type of representation is used.

State-Space Canonical Forms

We have seen that transform domain analysis of a digital control system yields a transfer function of the following form.

$$G(z) = \frac{Y(z)}{U(z)} = \frac{\beta_0 z^m + \beta_1 z^{m-1} + \ldots + \beta_m}{z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n} \quad m \leq n$$

❑ Controllable Canonical Form:

- The controllable canonical form arranges the coefficients of the transfer function denominator across one row of the A matrix:
- **In state space form, we have:** $x(k+1) = Ax(k) + Bu(k)$
- where: $y = Cx(k) + Du(k)$

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \ldots & -\alpha_1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Note

The controllable canonical from is useful for the pole placement controller design technique.

$$C = \left[\beta_n - \alpha_n\beta_0 \;\; \beta_{n-1} - \alpha_{n-1}\beta_0 \;\; \ldots \;\; \beta_1 - \alpha_1\beta_0\right] \qquad D = \beta_0$$

State-Space Canonical Forms

- **Observable Canonical Form:**

  The observable canonical form arranges the coefficients of the transfer function denominator across one column of the A matrix

$$G(z) = \frac{Y(z)}{U(z)} = \frac{\beta_0 z^m + \beta_1 z^{m-1} + \ldots + \beta_m}{z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n} \quad m \le n$$

$$A = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & -\alpha_n \\ 1 & 0 & 0 & \ldots & 0 & -\alpha_{n-1} \\ 0 & 1 & 0 & \ldots & 0 & -\alpha_{n-2} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \\ 0 & 0 & 0 & \ldots & 1 & -\alpha_1 \end{bmatrix} \quad B = \begin{bmatrix} \beta_n - \alpha_n \beta_0 \\ \beta_{n-1} - \alpha_{n-1}\beta_0 \\ \vdots \\ \beta_1 - \alpha_1 \beta_0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & \ldots & 1 \end{bmatrix} \quad D = \beta_0$$

Duality

In previous two sections we observed that the system matrix A in observable canonical form is transpose of the system matrix in controllable canonical form. Similarly, control matrix B in observable canonical form is transpose of output matrix C in controllable canonical form. So also output matrix C in observable canonical form is transpose of control matrix B in controllable canonical form.

Note the relationship between the observable and controllable forms:

$$
\begin{aligned}
A_{obs} &= A_{cont}^T \\
B_{obs} &= C_{cont}^T \\
C_{obs} &= B_{cont}^T \\
D_{obs} &= D_{cont}
\end{aligned}
$$

State-Space Canonical Forms

❑ Jordan Canonical Form

Assume that $z = \lambda_i, \quad i = 1, 2, \ldots, n$ are the distinct poles of the given transfer function (3). Then partial fraction expansion of the transfer function yields

$$\begin{aligned}
\frac{Y(z)}{U(z)} &= \beta_0 + \frac{\bar{\beta}_1 z^{n-1} + \bar{\beta}_2 z^{n-2} + \ldots + \bar{\beta}_n}{z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n} \\
&= \beta_0 + \frac{\bar{\beta}_1 z^{n-1} + \bar{\beta}_2 z^{n-2} + \ldots + \bar{\beta}_n}{(z-\lambda_1)(z-\lambda_2)\ldots(z-\lambda_n)} \\
&= \beta_0 + \frac{r_1}{z-\lambda_1} + \frac{r_2}{z-\lambda_2} + \ldots + \frac{r_n}{z-\lambda_n}
\end{aligned}$$

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} r_1 & r_2 & \ldots & r_n \end{bmatrix} \quad D = \beta_0$$

State-Space Canonical Forms

Example: Consider the following discrete transfer function.

$$G(z) = \frac{0.17z + 0.04}{z^2 - 1.1z + 0.24}$$

Find out the state variable model in 3 different canonical forms.

Solution:

The state variable model in controllable canonical form can directly be derived from the transfer function, where the A, B, C and D matrices are as follows:

$$A = \begin{bmatrix} 0 & 1 \\ -0.24 & 1.1 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ C = \begin{bmatrix} 0.04 & 0.17 \end{bmatrix}, \ D = 0$$

The matrices in state model corresponding to observable canonical form are obtained as,

$$A = \begin{bmatrix} 0 & -0.24 \\ 1 & 1.1 \end{bmatrix}, \ B = \begin{bmatrix} 0.04 \\ 0.17 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \ D = 0$$

State-Space Canonical Forms

To find out the state model in Jordan canonical form, we need to fact expand the transfer function using partial fraction, as

$$G(z) = \frac{0.17z + 0.04}{z^2 - 1.1z + 0.24} = \frac{0.352}{z - 0.8} + \frac{-0.182}{z - 0.3}$$

Thus the A, B, C and D matrices will be:

$$A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.352 & -0.182 \end{bmatrix}, \quad D = 0$$

State-Space Model to Transfer Function

Consider a discrete state variable model

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{1}$$

Taking the Z-transform on both sides of Eqn. (1), we get

$$\begin{aligned} zX(z) - zx_0 &= AX(z) + BU(z) \\ Y(z) &= CX(z) + DU(z) \end{aligned}$$

where x_0 is the initial state of the system.

$$\begin{aligned} \Rightarrow \qquad & (zI - A)X(z) = zx_0 + BU(z) \\ \text{or,} \quad & X(z) = (zI - A)^{-1}zx_0 + (zI - A)^{-1}BU(z) \end{aligned}$$

To find out the transfer function, we assume that the initial conditions are zero, i.e., $x_0 = 0$, thus

$$Y(z) = \left(C(zI - A)^{-1}B + D\right)U(z)$$

Therefore, the transfer function becomes

$$G(z) = \frac{Y(z)}{U(z)} = C(zI - A)^{-1}B + D \tag{2}$$

Construction rules for root locus

Given:
$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Find the pulse transfer function y(z)/u(z)=?

Solution:
$$G(z) = \frac{Y(z)}{U(z)} = C(zI - A)^{-1}B + D$$

$$[zI \; - \; A] = \mathrm{z}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} z & -1 \\ 2 & z+3 \end{bmatrix}$$

$$[zI \; - \; A]^{-1} = \frac{1}{|zI \; - \; A|} \,\mathrm{adj}\, [zI \; - \; A]$$

$$= \frac{1}{z^2+3z+2}\begin{bmatrix} z+3 & 1 \\ -2 & z \end{bmatrix}$$

$$\text{T.F} \quad = \frac{1}{z^2+3z+2} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} z+3 & 1 \\ -2 & z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{T.F} \quad = \frac{1}{z^2+3z+2} \begin{bmatrix} -2 & -Z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\boxed{\text{T.F} \quad = \frac{-z}{z^2+3z+2}}$$