Z - Transform

Definition:

$$X(z) = \sum_{n=0}^{\infty} X(nT)\, Z^{-n}$$

where T is the sampling time, $n = 0, 1, 2, \ldots$

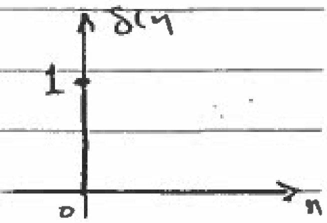
* for sequence

$$X(z) = \sum_{n=0}^{\infty} X(n)\, Z^{-n}$$

where $T = 1\,sec$

* Z-Transform for some elementary functions

1] impulse function ($\delta(t)$ or $\delta(n)$)

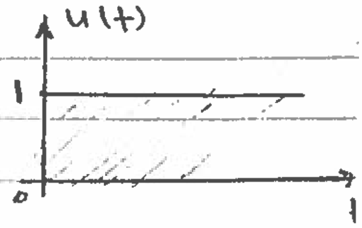


$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$\delta(z) = \sum_{n=0}^{\infty} \delta(n)\, Z^{-n} = 1$$

$$\boxed{\therefore\ \delta(n) \xrightarrow{Z.T} 1}$$

2] unit step function



$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$u(z) = \sum_{n=0}^{\infty} u(nT)\, Z^{-n} = \sum_{n=0}^{\infty} 1 \cdot Z^{-n} = 1 + Z^{-1} + Z^{-2} + \cdots$$

$$= \underbrace{1 + \tfrac{1}{z} + \left(\tfrac{1}{z}\right)^2 + \cdots}_{\text{geometric series}}$$

$$u(z) = \frac{1}{1 - \text{base}} = \frac{1}{1 - \frac{1}{z}} = \frac{z}{z - 1}$$

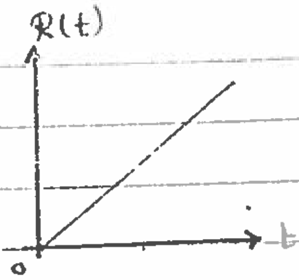
$\text{base} = \frac{1}{z}$

$$\boxed{u(t) \xrightarrow{Z.T} \frac{z}{z-1}}$$

* Properties of Z-Transform

Property	$x[n]$ or $x(t)$	$X(z)$
- Multiply by "Const"	$a\,x(t)$	$a\,X(z)$
- linearity	$x(t) + h(t)$	$X(z) + H(z)$
- Multiply by t	$t\,x(t)$	$-z\underbrace{T}_{\text{Sampling}} \frac{d}{dz} X(z)$
- Multiply by n	$n\,x(n)$	$-z \frac{d}{dz} X(z)$
- Multiply by a^n	$\underbrace{a}_{\text{Const}}{}^{n}\,x(n)$	$X(z/a)$
- Multiply by exp	$e^{-at} x(t)$	$X(z e^{aT})$
	$e^{-an} x(n)$	$X(z e^{a})$
- Shifting	$x(t - NT)$	$z^{-N} X(z)$ shift
	$x(n - N)$	$z^{-N} X(z)$
	$N = 0, 1, 2, 3, \ldots$	
	$x(t + NT)$	$z^{N} X(z) - z^{N} x(0) - z^{N-1} x(T) \ldots$
	$x(n + N)$	$z^{N} X(z) - z^{N} x(0) - z^{N-1} x(1) \ldots$

3] Ramp Function



$$R(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$R(t) = t\,u(t)$$

but $u(t) \xrightarrow{Z.T} \frac{Z}{Z-1}$ AND $t\,u(t) \longrightarrow -ZT \frac{d}{dZ} \frac{Z}{Z-1}$

$$= \cancel{-}ZT \cdot \frac{\cancel{Z}-1-\cancel{Z}}{(Z-1)^2}$$

$$\therefore t\,u(t) \xrightarrow{Z.T} \frac{ZT}{(Z-1)^2}$$

$$\boxed{\therefore R(t) \xrightarrow{Z.T} \frac{ZT}{(Z-1)^2}}$$

4] Polynomial function

$$x(n) = \begin{cases} a^n & n = 0, 1, 2, \ldots \\ 0 & n < 0 \end{cases}$$

$$x(n) = a^n u(n)$$

but $u(n) = \frac{Z}{Z-1}$ AND $a^n u(n) \longrightarrow \frac{Z/a}{Z/a - 1} = \frac{Z}{Z-a}$

$$\boxed{\therefore x(n) \xrightarrow{Z.T} \frac{Z}{Z-a}}$$

5] exponential function

$$x(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$x(t) = e^{-at} u(t)$$

but

$$u(t) = \frac{z}{z-1} \quad \text{AND} \quad e^{at} u(t) \longrightarrow \frac{ze^{aT}}{ze^{aT}-1} \cdot \frac{e^{-aT}}{e^{-aT}}$$

$$\boxed{x(t) \xrightarrow{z \cdot T} \frac{z}{z-e^{-aT}}}$$

6] Sinusoidal function

$$x(t) = \begin{cases} \sin wt & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Note:

$$e^{jwT} = \cos wT + j \sin wT$$

$$e^{-jwT} = \cos wT - j \sin wT$$

then

$$\cos wT = \frac{1}{2}\left(e^{jwT} + e^{-jwT}\right)$$

$$\sin wT = \frac{1}{2j}\left(e^{jwT} - e^{-jwT}\right)$$

$$x(t) = \frac{1}{2}\left(e^{jwT} - e^{-jwT}\right)$$

$$X(z) = \frac{1}{2}\left(\frac{z}{z-e^{jwT}} - \frac{z}{z-e^{-jwT}}\right)$$

$$= \frac{z}{2}\left(\frac{\cancel{z} - e^{-jwT} - \cancel{z} + e^{jwT}}{z^2 - z\underbrace{\left(e^{jwT} + e^{-jwT}\right)} + 1}\right)$$

$$\boxed{x(t) \xrightarrow{z \cdot T} \frac{z \sin wT}{z^2 - 2z \cos wT + 1}}$$

Assignment 1

* Obtain Z-Transform of

① $x_1(t) = t\, e^{-2t}$, Assum sampling time $T = 0.2\,sec$

Sol. $t \xrightarrow{Z.T} \dfrac{TZ}{(Z-1)^2}$

$$e^{-2t}\, t \xrightarrow{Z.T} \frac{T(Z e^{2T})}{(Z e^{2T} - 1)^2}$$ but $T = 0.2\,sec$

$$x_1(t) = \frac{0.298\,Z}{(1.49Z - 1)^2}$$

② $x_2(n) = e^{-3n} \sin wn$

Sol: $\sin wn \xrightarrow{Z.T} \dfrac{Z \sin w}{Z^2 - 2Z \cos w + 1}$

$$e^{-3n} \sin wn \xrightarrow{Z.T} \frac{(Ze^{3}) \sin w}{(Ze^{3})^2 - 2(Ze^{3}) \cos w + 1}$$

③ $X_3(n) = 5^n e^{-2n}$

Sol. $e^{-2n} \xrightarrow{Z.T} \dfrac{Z}{Z - e^{-2}}$

$$5^n e^{-2n} \xrightarrow{Z.T} \frac{Z/5}{Z/5 - e^{-2}}$$

④ $x_4(t) = 5\,u(t-2T)$

Sol: $u(t) \xrightarrow{Z.T} \dfrac{Z}{Z-1}$

$5\,u(t-2T) \xrightarrow{Z.T} 5\,Z^{-2} \cdot \dfrac{Z}{Z-1} \longrightarrow \dfrac{5}{Z(Z-1)}$

⑤ $x_5(n) = 7^n \cdot n$

Sol: $n \xrightarrow{Z.T} \dfrac{Z}{(Z-1)^2}$

$7^n \cdot n \xrightarrow{Z.T} \dfrac{(Z/7)}{((Z/7)-1)^2}$

⑥ $x_6(n) = 3\,\delta(n) - 2\,\delta(n-1)$

Sol: $\delta(n) \xrightarrow{Z.T} 1$

$3\delta(n) - 2\delta(n-1) \xrightarrow{Z.T} 3 - 2Z^{-1}$

* Inverse Z-Transform

$X(Z)$	$x(n)$
$\dfrac{Z}{Z-1}$	$u(n)$
$\dfrac{Z}{Z-a}$	a^n
$\dfrac{Z}{Z-e^{-aT}}$	e^{-at}
$\dfrac{Z}{Z-e^{-a}}$	e^{-an}
$\dfrac{Z\sin\omega T}{Z^2-2Z\cos\omega T+1}$	$\sin\omega T$
$\dfrac{Z(Z-\cos\omega T)}{Z^2-2Z\cos\omega T+1}$	$\cos\omega T$

I] Partial fraction Method :

Ex: find the inverse Z-Transform of $X(Z) = \dfrac{Z}{(Z-1)(Z+2)}$

sol:

$$X(Z) = \frac{Z}{(Z-1)(Z+2)} \Rightarrow \frac{X(Z)}{Z} = \frac{1}{(Z-1)(Z+2)}$$

$$\frac{X(Z)}{Z} = \frac{A}{Z-1} + \frac{B}{Z+2}$$

$$A = \lim_{Z\to 1} \frac{1}{Z+2} = 1/3$$

$$B = \lim_{Z\to -2} \frac{1}{Z-1} = -1/3$$

$$\frac{X(Z)}{Z} = \frac{1/3}{Z-1} + \frac{-1/3}{Z+2}$$

$$X(Z) = 1/3\,\frac{Z}{Z-1} - 1/3\,\frac{Z}{Z-(-2)}$$

$$x(n) = \frac{1}{3}\, u(n) - \frac{1}{3}\, (-2)^n$$

Ex: find the inverse Z-Transform of $X(Z) = \dfrac{10Z+5}{(Z-1)(Z+2)}$

Sol: $X(Z) = \dfrac{10Z+5}{(Z-1)(Z+2)} = \dfrac{A}{Z-1} + \dfrac{B}{Z+2}$

$$A = \lim_{Z\to 1} \frac{10Z+5}{Z+2} = 5$$

$$B = \lim_{Z\to -2} \frac{10Z+5}{Z-1} = 5$$

$$X(Z) = \frac{5}{Z-1} + \frac{5}{Z+2} * Z Z^{-1}$$

$$X(Z) = 5Z^{-1} \frac{Z}{Z-1} + 5Z^{-1} \frac{Z}{Z-(-2)}$$

$$X(n) = 5\, u(n-1) + 5\, (-2)^{n-1}$$

Ex: find the inverse Z-Transform of $X(Z) = \dfrac{1}{(Z-1)^2(Z+2)}$

Sol: $X(Z) = \dfrac{1}{(Z-1)^2(Z+2)} = \dfrac{A}{(Z-1)^2} + \dfrac{B}{Z-1} + \dfrac{C}{Z+2}$

$$A = \lim_{Z\to 1} \frac{1}{Z+2} = \frac{1}{3}$$

$$B = \lim_{Z\to 1} \frac{d}{dz} \frac{1}{Z+2} = \lim_{Z\to 1} \frac{-1}{(Z+2)^2} = -\frac{1}{9}$$

$$C = \lim_{Z\to -2} \frac{1}{(Z-1)^2} = \frac{1}{9}$$

$$X(Z) = \frac{1/3}{(Z-1)^2} + \frac{-1/9}{(Z-1)} + \frac{1/9}{(Z+2)} \cdot Z \cdot Z^{-1}$$

$$X(Z) = \frac{1}{3} Z^{-1} \frac{Z}{(Z-1)^2} - \frac{1}{9} Z^{-1} \frac{Z}{Z-1} + \frac{1}{9} Z^{-1} \frac{Z}{Z-(-2)}$$

$$X(n) = \frac{1}{3}(n-1) - \frac{1}{9}\, u(n-1) + \frac{1}{9}\, (-2)^{n-1}$$

$\underline{Ex_4}$: Find the inverse Z-Transform of $X(Z) = \dfrac{Z^2+Z+2}{(Z-1)(Z^2-Z+1)}$

$\underline{Sol}$: $X(Z) = \underbrace{\dfrac{Z^2+Z+2}{(Z-1)(Z^2-Z+1)}} = \dfrac{A}{(Z-1)} + \dfrac{BZ+C}{Z^2-Z+1}$

$$A = \lim_{Z\to 1} \frac{Z^2+Z+2}{Z^2-Z+1} = 4$$

$\underline{but}$ $\dfrac{Z^2+Z+2}{(Z-1)(Z^2-Z+1)} = \dfrac{A}{Z-1} + \dfrac{BZ+C}{Z^2-Z+1}$

$$\therefore Z^2+Z+2 = A(Z^2-Z+1) + (BZ+C)(Z-1)$$
$$= AZ^2 - AZ + A + BZ^2 - BZ + CZ - C$$
$$= (A+B)Z^2 + (C-A-B)Z + (A-C)$$

* Compare coefficient of both sides:

Coeff Z^2: $A+B=1 \Rightarrow A=4$ then $B=-3$

Coeff Z^0: $A-C=2 \Rightarrow A=4$ then $C=2$

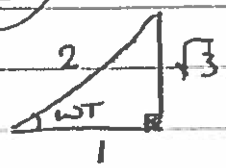
$$\therefore X(Z) = \frac{4}{Z-1} + \frac{-3Z+2}{Z^2-Z+1} \qquad Z\,Z^{-1}$$

$$= 4Z^{-1}\frac{Z}{Z-1} + Z^{-1}\underbrace{\frac{Z(-3Z+2)}{Z^2-Z+1}}$$

$$\cos \omega t \quad \underset{Z^{-1}}{\overset{Z.T}{\rightleftarrows}} \quad \frac{Z(Z-\cos\omega T)}{\underbrace{Z^2-2Z\cos\omega T+1}}$$

$2\cos\omega T = 1 \quad \therefore \cos\omega T = 1/2$

$\sin\omega T = \sqrt{3}/2$



$$X(Z) = 4Z^{-1}\frac{Z}{Z-1} - 3Z^{-1}\frac{Z(Z-2/3)}{Z^2-Z+1}$$

$$= 4Z^{-1}\frac{Z}{Z-1} - 3Z^{-1}\frac{Z(Z-1/2-1/6)}{Z^2-Z+1}$$

note

$2/3 = 4/6 = 1/6 + 3/6$

$= 1/6 + 1/2$

$$X(Z) = 4Z^{-1}\frac{Z}{Z-1} - 3Z^{-1}\frac{Z(Z-\frac{1}{2})}{Z^2-Z+1} - 3Z^{-1}\frac{Z(-\frac{1}{6})}{Z^2-Z+1}$$

$$= 4Z^{-1}\frac{Z}{Z-1} - 3Z^{-1}\frac{Z(Z-\frac{1}{2})}{Z^2-Z+1} + \frac{1}{2}Z^{-1}\frac{Z}{Z^2-Z+1} \quad \frac{\sqrt{3}}{\sqrt{3}}$$

$$\boxed{\sin \omega t \xrightarrow{Z \cdot T} \frac{Z \sin \omega T}{Z^2 - 2Z\cos \omega T + 1}}$$

$$= 4Z^{-1}\frac{Z}{Z-1} - 3Z^{-1}\frac{Z(Z-\frac{1}{2})}{Z^2-Z+1} + \frac{\sqrt{3}}{2\sqrt{3}}Z^{-1}\frac{Z}{Z^2-Z+1}$$

$$\boxed{X([illegible]) = 4u(t-T) - 3\cos\omega(t-T) + \frac{1}{\sqrt{3}}\sin\omega(t-T)}$$

or $\boxed{X(n) = 4u(n-1) + 3\cos\omega(n-1) + \frac{1}{\sqrt{3}}\sin\omega(n-1)}$

2] Direct long Division Method

$$X(Z) = \sum_{n=0}^{\infty} X(n)\, Z^{-n}$$

$$\boxed{X(Z) = X(0) + X(1)Z^{-1} + X(2)Z^{-2} + \ldots} \quad \text{--- (I)}$$

Ex: find $X(n)$ for $n = 0, 1, 2, 3, 4$ when $X(Z)$ is given by: [illegible]

$$X(Z) = \frac{Z^2 + 2Z}{Z^3 + Z^2 + Z + 1} \times \frac{Z^{-3}}{Z^{-3}}$$

First, rewrite $X(Z)$ as a ratio of polynomial in Z^{-1} as follow:

$$X(Z) = \frac{Z^{-1} + 2Z^{-2}}{1 + Z^{-1} + Z^{-2} + Z^{-3}}$$

$X(z)$

$$z^{-1} + z^{-2} - 2z^{-3} + z^{-5}$$

$$1 + z^{-1} + z^{-2} + z^{-3} \,\Big)\, \cancel{z^{-1}} + 2z^{-2}$$

$$-$$

$$\ominus \cancel{z^{-1}} \ominus z^{-2} \ominus z^{-3} \ominus z^{-4}$$

$$\cancel{z^{-2}} - z^{-3} - z^{-4}$$

$$\ominus \cancel{z^{-2}} \ominus z^{-3} \ominus z^{-4} \ominus z^{-5}$$

$$-2\cancel{z^{-3}} - 2\cancel{z^{-4}} - z^{-5}$$

$$\oplus 2\cancel{z^{-3}} \oplus 2\cancel{z^{-4}} \oplus 2z^{-5} \oplus 2z^{-6}$$

$$z^{-5} + 2z^{-6}$$

$$\therefore X(z) = z^{-1} + z^{-2} - 2z^{-3} + z^{-5} + \dots \quad ---\ \text{II}$$

Compare I & II

$x(0) = 0$

$x(1) = 1$

$x(2) = 1$

$x(3) = -2$

$x(4) = 0$

*) Z-Transform method for Solving Difference Equations:

≠ Consider the linear difference equation

$$\boxed{y(n) + a_1 y(n-1) + \cdots + a_n y(n-N) = b_0 u(n) + b_1 u(n-1) + \cdots + b_n u(n-N)}$$

where $u(n)$ and $y(n)$ are the systems input and output at nth iteration

~~this~~ if $\mathcal{Z}[y(n)] = Y(z)$

then $x(n+1), x(n+2) \cdots$ and $x(n-1), x(n-2) \cdots$

can be expressed in terms of $X(z)$ and initial conditions

function	Z-Transform
$\vdots$	
$x(n+2)$	$z^2 X(z) - z^2 x(0) - z x(1)$
$x(n+1)$	$z X(z) - z x(0)$
$x(n)$	$X(z)$
$x(n-1)$	$z^{-1} X(z)$
$x(n-2)$	$z^{-2} X(z)$
$\vdots$	

shifting

Ex: Solve the following difference equ using Z-Transform

$$y(n) + y(n-1) = u(n)$$

Sol:

$$y(n) + y(n-1) = u(n)$$

Take Z-Transform

$$Y(z) + z^{-1} Y(z) = \frac{z}{z-1}$$

$$Y(z)\left[1 + z^{-1}\right] = \frac{z}{z-1}$$

$$Y(z)\left[\frac{z+1}{z}\right] = \frac{z}{z-1}$$

$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z+1)}$$

$$\frac{Y(z)}{z} = \frac{A}{z-1} + \frac{B}{z+1}$$

$$A = \lim_{z \to 1} \frac{z}{z+1} = 1/2$$

$$B = \lim_{z \to -1} \frac{z}{z-1} = 1/2$$

$$\frac{Y(z)}{z} = \frac{1/2}{z-1} + \frac{1/2}{z-(-1)}$$

$$Y(z) = 1/2 \, \frac{z}{z-1} + 1/2 \, \frac{z}{z-(-1)}$$

$$\boxed{\therefore y(n) = 1/2 \, u(n) + 1/2 \, (-1)^n}$$

Ex: Solve the following difference eqn $y(n+1)+2y(n)=0$
where $y(0)=0.5$

sol:

$$y(n+1)+2y(n)=0$$

* Take Z-Transform

$$Z\,Y(Z)-Z\,Y(0)+2\,Y(Z)=0$$

$$Y(Z)\,[Z+2]-0.5\,Z=0$$

$$Y(Z)=\frac{+0.5\,Z}{Z-(-2)}$$

* Take Z-inverse

$$y(n)=\tfrac{1}{2}\,(-2)^n$$

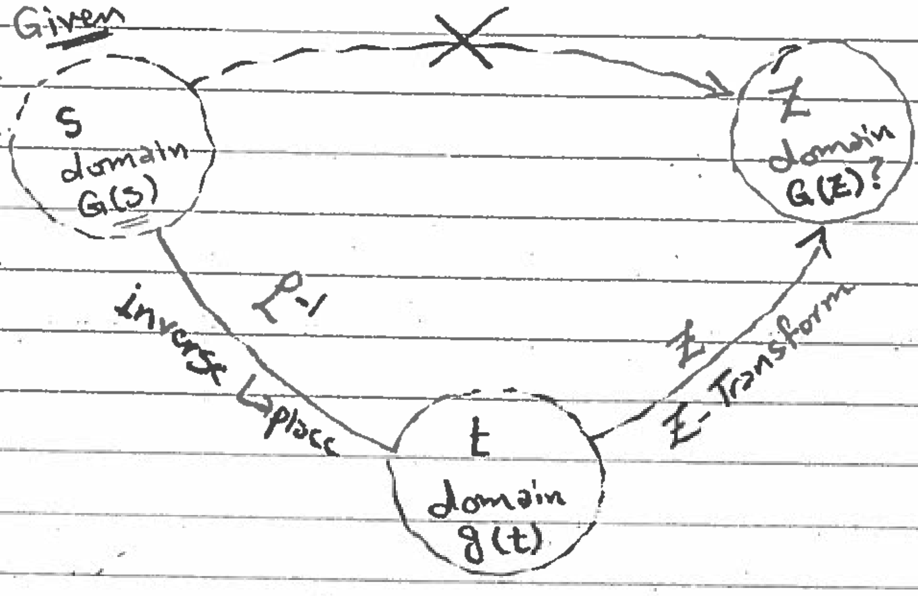
Digital Equivalent Transfer Function of Analog system

Given:

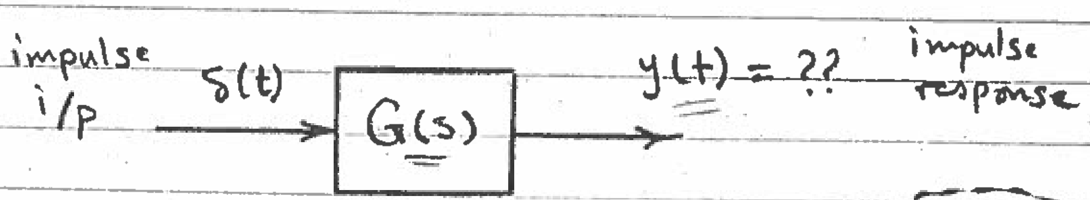
Analog T.F $G(s)$.

Required:

Digital Equivalent T.F $G(z)$.



I] impulse Invariant Method: required $G(z) = ??$



$$y(t) = \delta(t) * g(t) = g(t)$$ impulse response

$$g(t) = \mathcal{L}^{-1}\, G(s) \quad \text{--- (1)}$$

$$\therefore\; G(z) = \mathcal{Z}\, g(t) \quad \text{--- (2)}$$

$$\boxed{G(z) = \mathcal{Z}\, \mathcal{L}^{-1}\, G(s)}$$

Z-Transform   Laplace inverse

Table of Z-Transform

$X(t)$ or $x(n)$	$X(s)$	$X(z)$
$\delta(t)$	1	1
$u(t)$	$\frac{1}{s}$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
$e^{\pm at}$	$\frac{1}{s \mp a}$	$\frac{z}{z-e^{\pm aT}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z \sin \omega T}{z^2 - 2z\cos\omega T + 1}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$\frac{z(z-\cos\omega T)}{z^2 - 2z\cos\omega T + 1}$

Ex: find the impulse response and the digital Transfer function for the following system: $G(s) = \frac{1}{s+1}$

Sol: impulse response:

$$\delta(t) \longrightarrow \boxed{\frac{1}{s+1}} \longrightarrow g(t) = ??$$

$$g(t) = \mathcal{L}^{-1} G(s) = \mathcal{L}^{-1} \frac{1}{s+1} \qquad \therefore g(t) = e^{-t}$$

digital Transfer function:

$$G(z) = \mathcal{Z}\, g(t) = \mathcal{Z}\, e^{-t} \qquad \therefore G(z) = \frac{z}{z-e^{-T}}$$

Ex 2: Find the impulse response and the digital Transfer function for the following system:

$$G(s) = \frac{1}{s(s+2)}$$

Sol: impulse response:

$$g(t) = \mathcal{L}^{-1} G(s) = \mathcal{L}^{-1} \frac{1}{s(s+2)} = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s+2}\right\}$$

$$A = \lim_{s \to 0} \frac{1}{s+2} = 1/2$$

$$B = \lim_{s \to -2} \frac{1}{s} = -1/2$$

$$\therefore g(t) = \mathcal{L}^{-1}\left\{1/2 \cdot \frac{1}{s} - 1/2 \frac{1}{s+2}\right\}$$

$$\therefore g(t) = 1/2\, u(t) - 1/2\, e^{-2t}$$

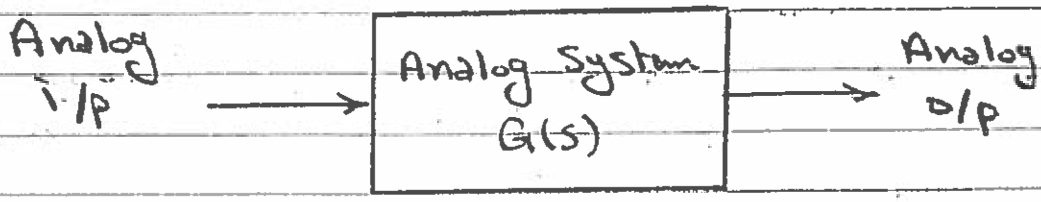
digital Transfer function

$$G(z) = \mathcal{Z}\, g(t) = \mathcal{Z}\left\{1/2\, u(t) - 1/2\, e^{-2t}\right\}$$

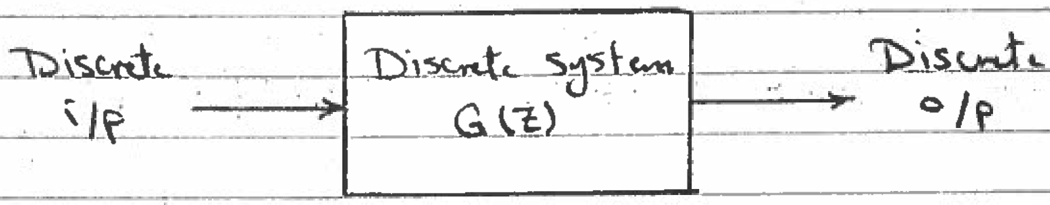
$$G(z) = 1/2 \frac{z}{z-1} - 1/2 \frac{z}{z-e^{-2T}}$$

Note that

For Analog system



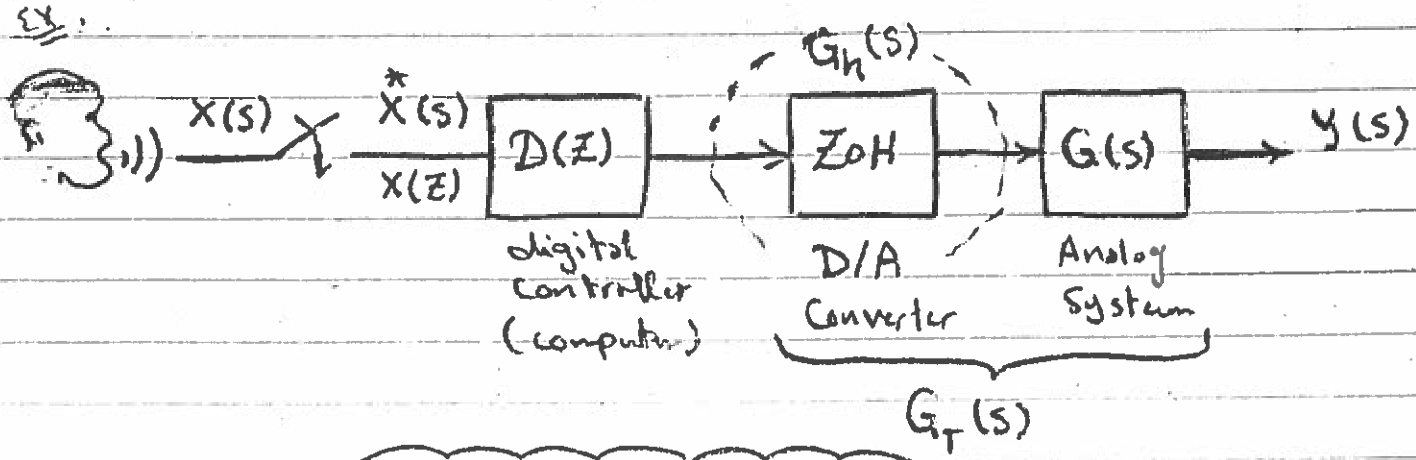
For Discrete system



Note that

Every Anlog T.F should be preconnected by Zero order Hold (ZoH) between the Computer (digital Controller) and the analog T.F.

Ex.:

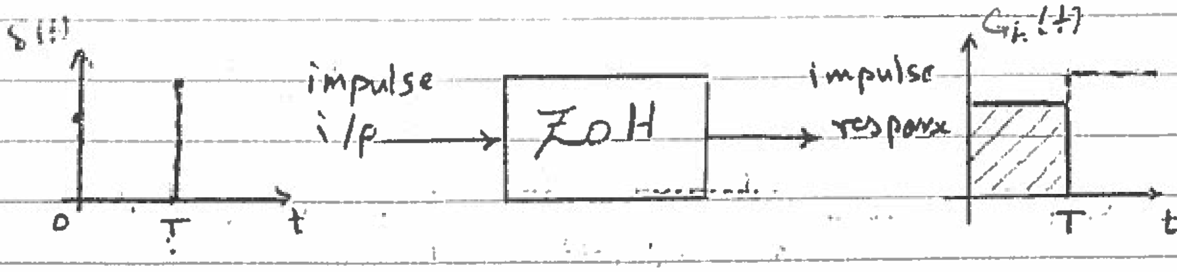


$$\boxed{G_T(s) = G_h(s) \cdot G(s)}$$

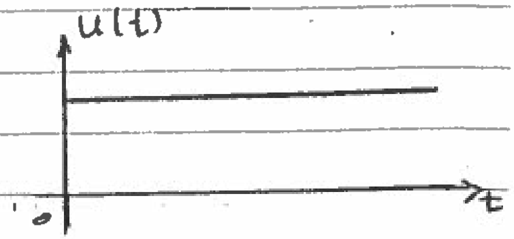
Given

Required

Zero order Hold (ZOH)

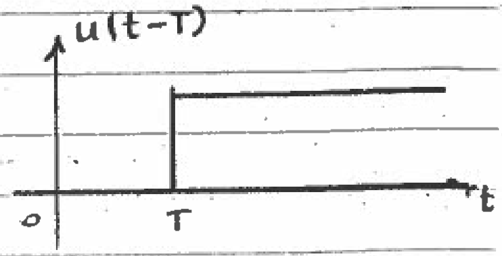


$$G_h(t) = u(t) - u(t-T)$$

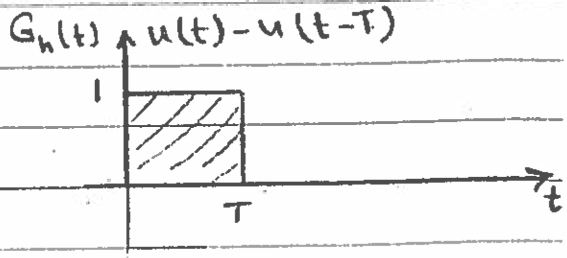


* Take Laplace Transform

$$G_h(s) = \frac{1}{s} - \frac{e^{-sT}}{s}$$



$$\therefore G_h(s) = \frac{1 - e^{-sT}}{s}$$



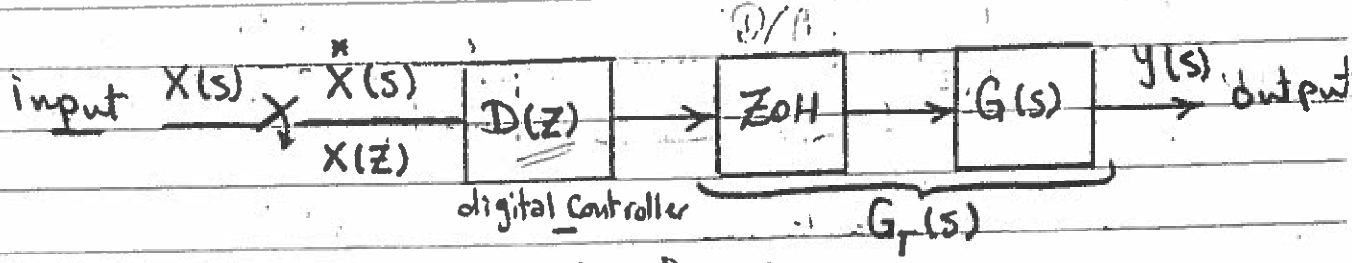
Note

$$X(t) \xrightarrow{L.T} X(s)$$

$$X(t-T) \xrightarrow{L.T} e^{-sT} X(s)$$

* Pulse Transfer function

① open loop System



Required: digital Transfer function

$$T.F = \frac{y(z)}{X(z)} = D(z)\, G_T(z)$$

but:

$$G_T(z) = \mathcal{Z}\, G_T(s) = \mathcal{Z}\; G_h(s) \cdot G(s)$$

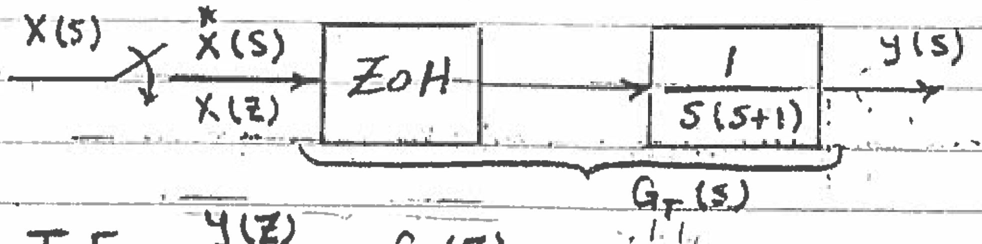
$$= \mathcal{Z}\; \frac{1-e^{-sT}}{s} \cdot G(s)$$

but $z = e^{sT}$

$$G_T(z) = \mathcal{Z}\; \frac{(1-z^{-1})}{s} G(s)$$

$$\therefore G_T(z) = (1-z^{-1})\; \mathcal{Z}\; \frac{G(s)}{s}$$

Ex: find the digital Transfer Function for the following open Loop system. Assum sampling time = 1 sec.



Sol

$$T.F = \frac{y(Z)}{X(Z)} = G_T(Z)$$

$$G_T(Z) = (1 - Z^{-1})\, \mathcal{Z}\, \frac{G(S)}{S}$$

$$= \frac{Z-1}{Z}\, \mathcal{Z}\, \frac{1}{S^2(S+1)} = \frac{Z-1}{Z}\, \mathcal{Z}\left\{\frac{A}{S^2} + \frac{B}{S} + \frac{C}{S+1}\right\}$$

$$A = \lim_{S\to 0} \frac{1}{S+1} = 1$$

$$B = \lim_{S\to 0} \frac{d}{dS}\left(\frac{1}{S+1}\right) = \lim_{S\to 0} \frac{-1}{(S+1)^2} = -1$$

Note
$e^{-1} = 0.36$

$$C = \lim_{S\to -1} \frac{1}{S^2} = 1$$

$$\therefore G_T(Z) = \frac{Z-1}{Z}\, \mathcal{Z}\left(\overset{t}{\frac{1}{S^2}} - \overset{u}{\frac{1}{S}} + \overset{e^{-t}}{\frac{1}{S+1}}\right)$$

$$= \frac{Z-1}{Z}\left(\frac{TZ}{(Z-1)^2} - \frac{Z}{Z-1} + \frac{Z}{Z-e^{-T}}\right)$$

$$G_T(z) = \frac{1}{Z-1} - 1 + \frac{Z-1}{Z-0.36}$$

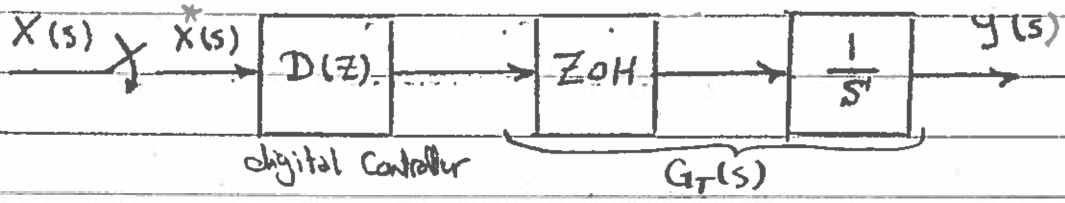
$$= \frac{Z - 0.36 + (Z-1)^2 - (Z-1)(Z-0.36)}{(Z-1)(Z-0.36)}$$

$$= \frac{Z - 0.36 + Z^2 - 2Z + 1 - Z^2 + 1.36Z - 0.36}{Z^2 - 1.36Z + 0.36}$$

$$T.F = G_T(Z) = \frac{0.36Z + 0.28}{Z^2 - 1.36Z + 0.36}$$

Ex: Find the digital Transfer function for the following open Loop system, where $D(z) = \frac{z-1}{z}$, then find the step response, Assume Sampling time $T_s = 1\,sec$

Sol:



digital Transfer function:

$$T.f = \frac{y(z)}{x(z)} = D(z) \cdot G_T(z)$$

but:

$$G_T(z) = (1 - z^{-1})\, \mathcal{Z}\, \frac{G(s)}{s}$$

$$\therefore G_T(z) = \frac{z-1}{z}\, \mathcal{Z}\, \frac{1}{s^2} \sim t$$

$$G_T(z) = \frac{\cancel{z-1}}{\cancel{z}} \cdot \frac{T\cancel{z}}{(z-1)^{\cancel{2}}} \qquad T_s = 1\,sec$$

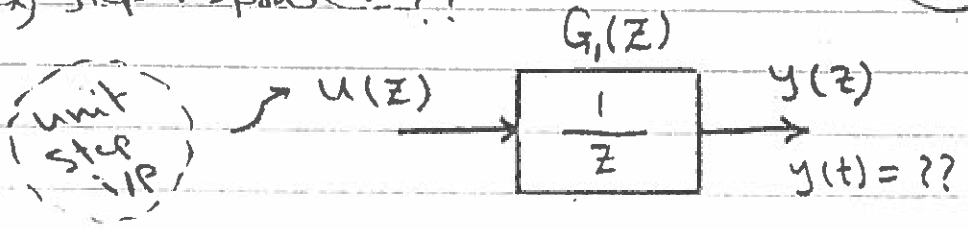
then

$$G_T(z) = \frac{1}{z-1}$$

$$\therefore T.f = \frac{z-1}{z} \cdot \frac{1}{z-1} \qquad \therefore \text{digital Transfer function}$$

$$T.f = \frac{1}{z}$$

*) Step response = ??



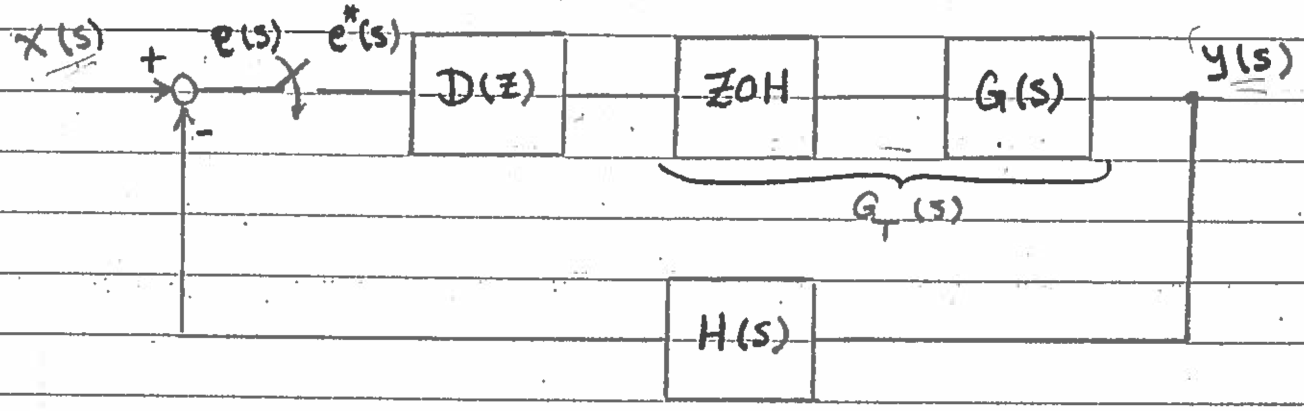
$$y(z) = u(z) \cdot G_1(z) = \frac{z}{z-1} \cdot \frac{1}{z} \Rightarrow y(z) = \frac{1}{z-1}$$

then: Step response $y(t) = Z^{-1}\, y(z)$

$$y(t) = Z^{-1} \frac{1}{z-1} \cdot z \cdot z^{-1}$$

$$y(t) = u(t - T)$$

② Closed Loop System:



*) digital Transfer function:

$$T.F = \frac{Y(z)}{X(z)} = \frac{D(z)\, G_T(z)}{1 + D(z)\, G_T H(z)}$$

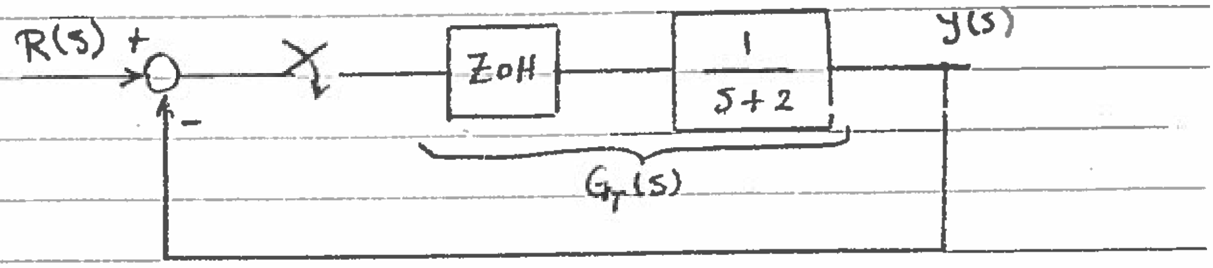
where

$$G_T(z) = (1 - z^{-1})\, Z\, \frac{G(s)}{s}$$

AND

$$G_T H(z) = (1 - z^{-1})\, Z\, \frac{G(s) \cdot H(s)}{s}$$

Ex: find the digital Transfer Function for the following closed Loop system, Assume sampling time $T_s = 0.5$ sec, then find step response



Sol:

digital Transfer Function

$$T.F = \frac{y(z)}{R(z)} = \frac{G_T(z)}{1+G_T(z)}$$

but

$$G_T(Z) = (1-Z^{-1})\, \mathcal{Z}\, \frac{G(S)}{S}$$

$$= \frac{Z-1}{Z}\, \mathcal{Z}\, \frac{1}{S(S+2)}$$

$$= \frac{Z-1}{Z}\, \mathcal{Z}\left(\frac{A}{S} + \frac{B}{S+2}\right)$$

$$A = \lim_{S\to 0} \frac{1}{S+2} = 1/2$$

$$B = \lim_{S\to -2} \frac{1}{S} = -1/2$$

$u \qquad e^{-2t}$

$$\therefore G_T(Z) = \frac{Z-1}{Z}\, \mathcal{Z}\left(1/2\, \frac{1}{S} - 1/2\, \frac{1}{S+2}\right)$$

$$= \frac{Z-1}{Z}\left(1/2\, \frac{Z}{Z-1} - 1/2\, \frac{Z}{Z-e^{-2T}}\right) \quad \text{but } T_s = \tfrac{1}{2}$$

$$= 1/2\left(1 - \frac{Z-1}{Z-0.36}\right)$$

$$= 1/2\left(\frac{Z-0.36-Z+1}{Z-0.36}\right)$$

then

$$G_T(z) = \frac{0.32}{z - 0.36}$$

but

$$T.F = \frac{G_T(z)}{1 + G_T(z)} \qquad \frac{\frac{0.32}{z-0.36}}{1 + \frac{0.32}{z-0.36}} = \frac{0.32}{z - 0.36 + 0.32}$$

then

the digital Transfer function

$$\boxed{T.F = \frac{0.32}{z - 0.04}}$$

* unit step response:

$$\begin{array}{ccc} & G_1(z) & \\ \xrightarrow[u(t)]{u(z)} & \boxed{\dfrac{0.32}{z-0.04}} & \xrightarrow[y(t) = ??]{y(z)} \end{array}$$

$$y(z) = u(z) \cdot G(z)$$

$$y(z) = \frac{z}{z-1} \cdot \frac{0.32}{z-0.04} = \frac{0.32\,z}{(z-1)(z-0.04)}$$

$$\frac{y(z)}{z} = \frac{0.32}{(z-1)(z-0.04)} = \frac{A}{z-1} + \frac{B}{z-0.04}$$

$$A = \lim_{z \to 1} \frac{0.32}{z-0.04} = 1/3$$

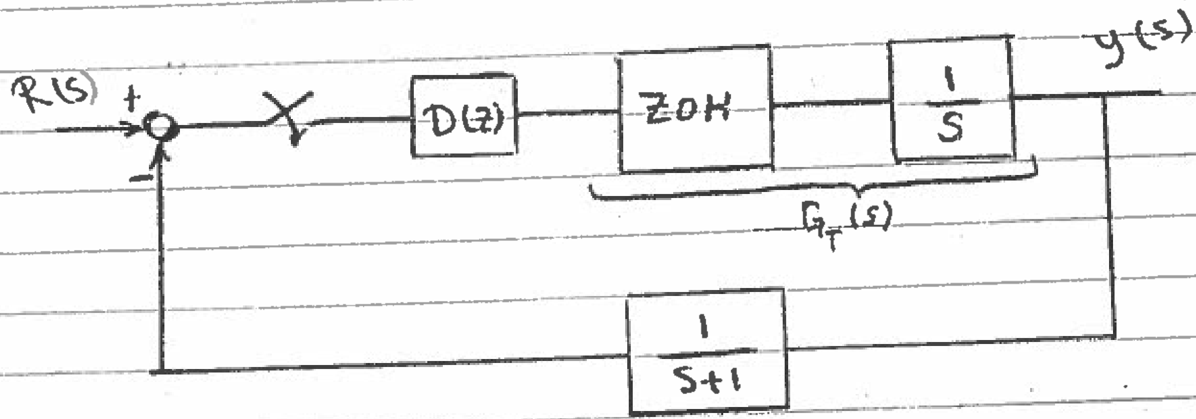
$$B = \lim_{z \to 0.04} \frac{0.32}{z-1} = -1/3$$

$$y(z) = 1/3 \, \frac{z}{z-1} - 1/3 \, \frac{z}{z-0.04}$$

∴ step response

$$\boxed{y(n) = 1/3 \, u(n) - 1/3 \, (0.04)^n}$$

Ex: find the digital Transfer function for the following Closed loop system, where $D(z) = \frac{z-1}{z}$, Assume $T_s = 1$ sec



Sol:

digital Transfer function

$$T.F = \frac{y(z)}{R(z)} = \frac{D(z) \cdot G_T(z)}{1 + D(z)\, G_T H(z)}$$

$$G_T(z) = (1 - z^{-1})\, \mathcal{Z}\, \frac{G(s)}{s}$$

$$= \frac{z-1}{z}\, \mathcal{Z}\, \frac{1}{s^2} \to t$$

$$= \frac{z-1}{z} \cdot \frac{Tz}{(z-1)^2}$$

$$\therefore G_T(z) = \frac{1}{z-1} \quad \text{then} \quad D(z) \cdot G_T(z) = \frac{1}{z}$$

$$G_T H(z) = (1 - z^{-1})\, \mathcal{Z}\, \frac{G(s) \cdot H(s)}{s}$$

$$= \frac{z-1}{z}\, \mathcal{Z}\, \frac{1}{s^2(s+1)}$$

$$= \frac{z-1}{z}\, \mathcal{Z}\left(\frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}\right)$$

$$A = \lim_{s\to 0} \frac{1}{s+1} = 1$$

$$B = \lim_{s\to 0} \frac{d}{ds}\left(\frac{1}{s+1}\right) = \lim_{s\to 0} \frac{-1}{(s+1)^2} = -1$$

$$C = \lim_{s\to -1} \frac{1}{s^2} = 1$$

$$\therefore G_1H(z) = \frac{z-1}{z} \mathcal{Z}\left(\overset{t}{\frac{1}{s^2}} - \overset{u}{\frac{1}{s}} + \overset{e^{-t}}{\frac{1}{s+1}}\right)$$

$$= \frac{z-1}{\cancel{z}}\left(\frac{T\cancel{z}}{(z-1)^2} - \frac{\cancel{z}}{z-1} + \frac{\cancel{z}}{z-e^{-T}}\right)$$

but $T = 1\,\text{sec}$

$$G_T H(z) = \left(\frac{1}{z-1} - 1 + \frac{z-1}{z-0.36}\right)$$

then

$$D(z).G_T H(z) = \frac{z-1}{z}\left(\frac{1}{z-1} - 1 + \frac{z-1}{z-0.36}\right)$$

$$= \left(\frac{1}{z} - \frac{z-1}{z} + \frac{(z-1)^2}{z(z-0.36)}\right)$$

$$= \frac{z-0.36-(z-1)(z-0.36)+(z-1)^2}{z(z-0.36)}$$

$$= \frac{z-0.36-z^2+1.36z-0.36+z^2-2z+1}{z(z-0.36)}$$

$$D(z).G_T H(z) = \frac{0.36z+0.28}{z(z-0.36)}$$

but

$$T.F = \frac{D(z).G_T(z)}{1+D(z)G_T H(z)}$$

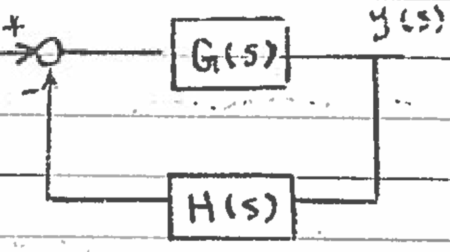
$$T.F = \frac{1/z}{1+\frac{0.36z+0.28}{z(z-0.36)}} = \frac{z-0.36}{z(z-0.36)+0.36z+0.28}$$

$$T.F = \frac{z - 0.36}{z^2 - \cancel{0.36 z} + \cancel{0.36 z} + 0.28}$$

$$T.F = \frac{z - 0.36}{z^2 + 0.28}$$

* Stability analysis for Discrete time control

* for Analog system

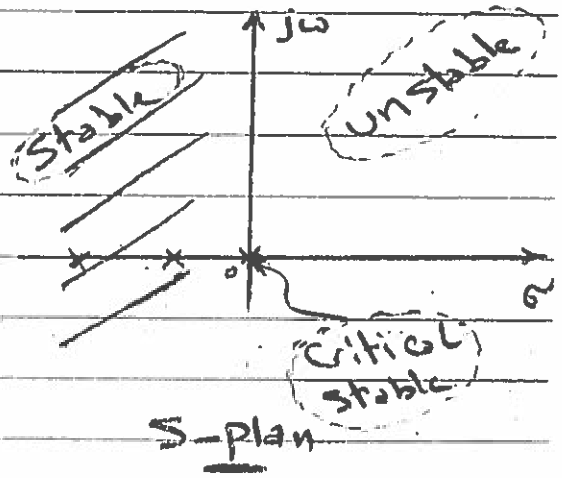


$$T.F = \frac{y(s)}{R(s)} = \frac{G(s)}{1+GH(s)} = \frac{q(s)}{P(s)}$$

zeros → $q(s)$; Poles → $P(s)$

then, the stability of the system is given by the c/cs equ:

$$P(s) = 1 + GH(s) = 0$$



but

$$S = \sigma + j\omega \quad \text{---} (I)$$

for stable $\sigma < 0$

* Mapping from S-plan to Z-plan:

$$Z = e^{sT} \quad \text{---} (II)$$

* from (I) in (II)

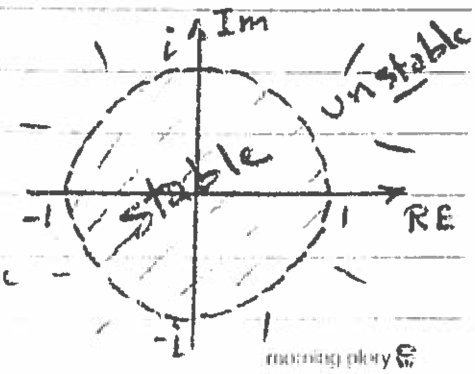
$$Z = e^{(\sigma + j\omega)T}$$
$$= e^{\sigma T} \cdot e^{j\omega T} = |Z| \angle \omega T \quad \text{---} (III)$$
$$= |Z| \angle Z$$

*) the boundary between stable region & unstable region given by:

$$\sigma = 0 \quad \text{---} (IV)$$

* from (IV) in (III);

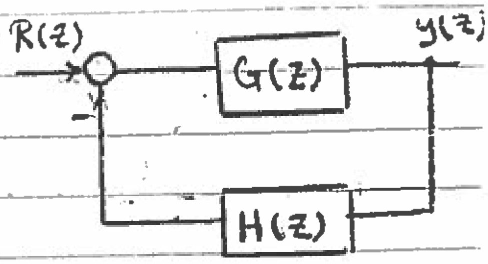
$$Z = |1| \angle \omega T$$



* for stable: $\sigma < 0$ then system stable inside the unit circle

* Stability analysis of closed loop system in z-plan:

* for the following closed loop system the transfer function can be given by:



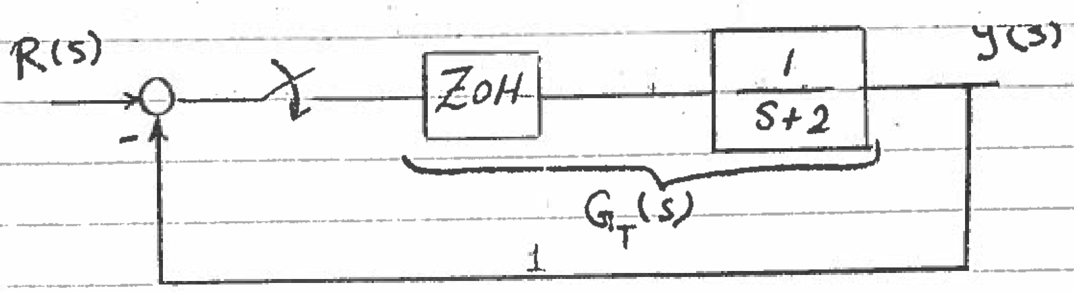
$$T.F = \frac{G(z)}{1+GH(z)} = \frac{q(z)}{P(z)}$$

* The stability of the system defined by the c/cs eqn:

$$P(z) = 1 + GH(z) = 0$$

- if all poles lie within the unit circle in z-plan then $\longrightarrow$ system is stable.
- if any pole lies outside the unit circle in z-plan then $\longrightarrow$ system is unstable.
- if a simple pole lies at $z=1$ or complex conjugate on unit circle then $\longrightarrow$ system is criticle stable.

Ex,: Check the stability for the following closed loop system assume sampling time $T = 0.5$ sec.



Sol:

- the digital Transfer function:

$$T.F = \frac{G_T(z)}{1 + G_T(z)}$$

$$G_T(z) = (1 - z^{-1})\, \mathcal{Z}\, \frac{G(s)}{s}$$

$$= \frac{z-1}{z}\, \mathcal{Z}\, \frac{1}{s(s+2)} = \frac{z-1}{z}\, \mathcal{Z}\left(\frac{A}{s} + \frac{B}{s+2}\right)$$

$$A = \lim_{s \to 0} \frac{1}{s+2} = 1/2$$

$$B = \lim_{s \to -2} \frac{1}{s} = -1/2$$

u $\qquad e^{-2t}$

$$\therefore G_T(z) = \frac{z-1}{z}\, \mathcal{Z}\left(1/2\, \frac{1}{s} - 1/2\, \frac{1}{s+2}\right)$$

$$G_T(z) = \frac{z-1}{z}\left[1/2\, \frac{z}{z-1} - 1/2\, \frac{z}{z - e^{-2T}}\right]$$

but $T = 0.5$ sec

$$\therefore G_T(z) = 1/2\left[1 - \frac{z-1}{z-0.36}\right]$$

$$\therefore G_T(z) = 1/2\left[\frac{z - 0.36 - z + 1}{z - 0.36}\right] \qquad \therefore G_T(z) = \frac{0.32}{z - 0.36}$$

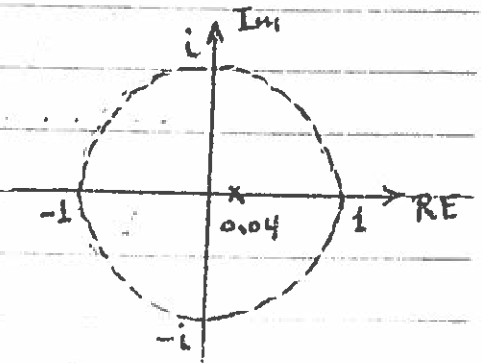
$$T.F = \frac{G_T(z)}{1 + G_T(z)} = \frac{\frac{0.32}{z-0.36}}{1 + \frac{0.32}{z-0.36}} = \frac{0.32}{z - 0.36 + 0.32}$$

$$\therefore T.F = \frac{0.32}{z - 0.04} = \frac{q(z)}{p(z)}$$

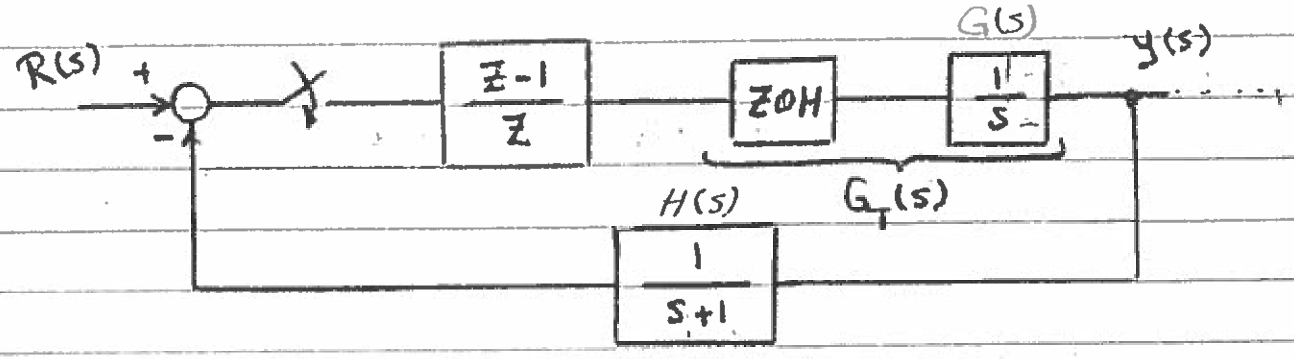
- The stability of the system defined by the C.L.C.'s equ

$$p(z) = z - 0.04 = 0 \qquad \therefore z = 0.04$$

Since all poles lie inside the unit circle then System is stable.



$\underline{Ex_2}$: Check the stability for the following closed loop system, assume sampling time $T = 1$ sec



$\underline{Sol}$: $\underline{\text{the digital Transfer function:}}$

$$T.F = \frac{Y(z)}{R(z)} = \frac{D(z).G_T(z)}{1+D(z)\,G_TH(z)}$$

where

$$\therefore G_T(z) = (1-z^{-1})\, \mathcal{Z}\, \frac{G(s)}{s}$$

And

$$G_TH(z) = (1-z^{-1})\, \mathcal{Z}\, \frac{G(s).H(s)}{s}$$

$$\therefore G_T(z) = \frac{z-1}{z}\, \mathcal{Z}\, \frac{1}{s^2} \to t$$

$$= \frac{z-1}{\cancel{z}} \cdot \frac{T\cancel{z}}{(z-1)^2}$$

but $T = 1$ sec

$$\therefore G_T(z) = \frac{1}{z-1} \Rightarrow D(z).G_T(z) = \frac{z-1}{z} \cdot \frac{1}{z-1}$$

$$\therefore D(z).G_T(z) = 1/z$$

$$G_TH(z) = \frac{z-1}{z}\, \mathcal{Z}\, \frac{1}{s^2(s+1)} = \frac{z-1}{z}\, \mathcal{Z}\left(\frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}\right)$$

$$A = \lim_{s\to 0} \frac{1}{s+1} = 1$$

$$B = \lim_{s\to 0} \frac{d}{ds} \frac{1}{s+1} = \lim_{s\to 0} \frac{-1}{(s+1)^2} = -1$$

$$C = \lim_{s \to -1} \frac{1}{s^2} = 1$$

$$G_T H(z) = \frac{z-1}{z} \mathcal{Z}\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right)$$

t, u, e^{-t}

$$G_T H(z) = \frac{z-1}{z}\left(\frac{Tz}{(z-1)^2} - \frac{z}{(z-1)} + \frac{z}{z-e^{-T}}\right)$$

but $T = 1\,sec$

$$G_T H(z) = \left(\frac{1}{z-1} - 1 + \frac{z-1}{z-0.36}\right)$$

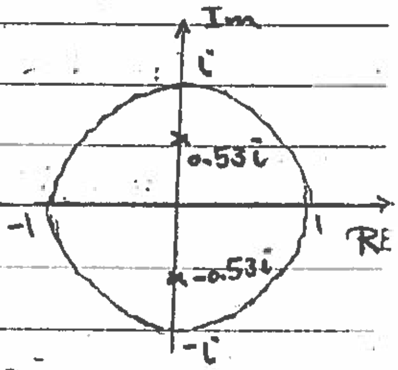
$$\Rightarrow D(z).G_T H(z) = \frac{z-1}{z}\left(\frac{1}{z-1} - 1 + \frac{z-1}{z-0.36}\right)$$

$$= \frac{1}{z} - \frac{z-1}{z} + \frac{(z-1)^2}{z(z-0.36)}$$

$$= \frac{z-0.36-(z-1)(z-0.36)+(z-1)^2}{z(z-0.36)}$$

$$= \frac{z-0.36-z^2+1.36z-0.36+z^2-2z+1}{z(z-0.36)}$$

$$D(z).G_T H(z) = \frac{0.36z+0.28}{z(z-0.36)}$$



$$\therefore T.F = \frac{D(z)\, G_T(z)}{1 + D(z)\, G_T H(z)}$$

$$\therefore T.F = \frac{1/z}{1 + \frac{0.36z+0.28}{z(z-0.36)}} = \frac{z-0.36}{z(z-0.36)+0.36z+0.28}$$

$$= \frac{z-0.36}{z^2 - \cancel{0.36z} + \cancel{0.36z} + 0.28} = \frac{z-0.36}{z^2+0.28} = \frac{q(z)}{P(z)}$$

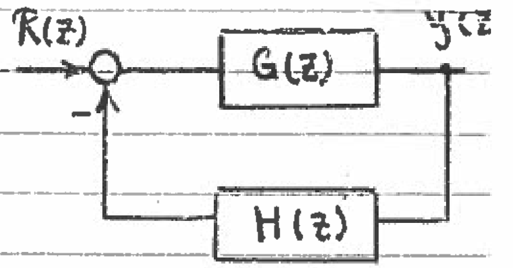
* the stability of the system defined by the cl cs equ.

$$P(z) = z^2 + 0.28 = 0 \Rightarrow z = \pm 0.53i$$

Since all poles lie inside the unit circle then system is stable.

*) Jury Stability Test

For the following Closed Loop system the transfer function can be given by:



$$T.F = \frac{G(z)}{1+GH(z)} = \frac{q(z)}{P(z)}$$

* The stability of the system defined by the C/cs equ:

$$P(z) = 1 + GH(z) = 0$$

* Assume that the c/cs equ $P(z)$ is polynomial in z as follows:

$$P(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_{n-1} z + a_n$$

*) Conditions for stability.

1. $|a_n| < a_0$

2. $P(z)\Big|_{z=1} > 0$

3. $P(z)\Big|_{z=-1}$ → for n even $P(-1) > 0$; for n odd $P(-1) < 0$

4. Construct Jury Table

$|b_{n-1}| > |b_0|$

$|C_{n-2}| > |C_0|$

$\vdots$

$|q_2| > |q_0|$ no. of Coefficient equal '3' stop

z^0	z^1	z^2	z^3	$\dots$	z^{n-1}	z^n
a_n	a_{n-1}	a_{n-2}	a_{n-3}	$\dots$	a_1	a_0
a_0	a_1	a_2	a_3	$\dots$	a_{n-1}	a_n
b_{n-1}	b_{n-2}	b_{n-3}	$\dots$	b_1	b_0	
b_0	b_1	b_2	$\dots$	b_{n-2}	b_{n-1}	
c_{n-2}	c_{n-3}	$\dots$	c_1	c_0		
c_0	c_1	$\dots$	c_{n-3}	c_{n-2}		
$\vdots$						
q_2	q_1	q_0				

no. of coefficient equal '3' stop

* iff all conditions satisfied then system is stable.

$$b_{n-1} = \begin{vmatrix} a_n & a_0 \\ a_0 & a_n \end{vmatrix}, \quad b_{n-2} = \begin{vmatrix} a_n & a_1 \\ a_0 & a_{n-1} \end{vmatrix}, \quad \dots, \quad b_0 = \begin{vmatrix} a_n & a_{n-1} \\ a_0 & a_1 \end{vmatrix}$$

$$c_{n-2} = \begin{vmatrix} b_{n-1} & b_0 \\ b_0 & b_{n-1} \end{vmatrix}, \quad c_{n-3} = \begin{vmatrix} b_{n-1} & b_1 \\ b_0 & b_{n-2} \end{vmatrix}, \quad \dots, \quad c_0 = \begin{vmatrix} b_{n-1} & b_{n-2} \\ b_0 & b_1 \end{vmatrix}$$

Ex: Construct the Jury Table and check the stability of the c/cs eqn

$$P(z) = z^4 - 1.2z^3 + 0.07z^2 + 0.3z - 0.08$$

Sol:

* Conditions of stability

1- $|a_n| < a_0$, $|-0.08| < 1$ Satisfy

2- $P(z)\big|_{z=1} = (1)^4 - 1.2(1)^3 + 0.07(1)^2 + 0.3(1) - 0.08 = 0.09 > $ Satisfy

3- $P(z)\big|_{z=-1} = (-1)^4 - 1.2(-1)^3 + 0.07(-1)^2 + 0.3(-1) - 0.08 = 1.89 > 0$

$n = 4$ (even) && $P(z)\big|_{z=-1} > 0$ Satisfy

4- Construct Jury Table:

z^0	z^1	z^2	z^3	z^4
-0.08	0.3	0.07	-1.2	1
1	-1.2	0.07	0.3	-0.08
-0.994	1.176	-0.756	-0.204	
-0.204	-0.756	1.176	-0.994	
0.946	-1.184	0.315		

Coefficients equal '3' stop

$b_3 = \begin{vmatrix} -0.08 & 1 \\ 1 & -0.08 \end{vmatrix}$ $b_3 = -0.994$, $b_2 = \begin{vmatrix} -0.08 & -1.2 \\ 1 & 0.3 \end{vmatrix}$ $\therefore b_2 = 1.176$

$b_1 = \begin{vmatrix} -0.08 & 0.07 \\ 1 & 0.07 \end{vmatrix}$ $b_1 = -0.756$, $b_0 = \begin{vmatrix} -0.08 & 0.3 \\ 1 & -1.2 \end{vmatrix}$ $\therefore b_0 = -0.204$

Check

$|b_3| > |b_0| \Rightarrow 0.994 > 0.204$ Satisfy

$C_2 = \begin{vmatrix} -0.994 & -0.204 \\ -0.204 & -0.994 \end{vmatrix}$ $C_2 = 0.946$

$C_1 = \begin{vmatrix} -0.994 & -0.756 \\ -0.204 & 1.176 \end{vmatrix}$ $C_1 = -1.184$

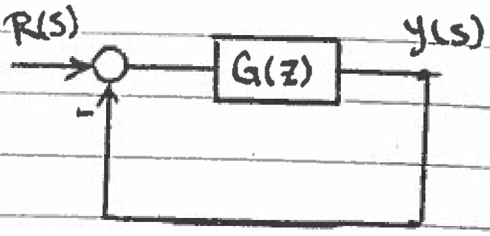
$C_0 = \begin{vmatrix} -0.994 & 1.176 \\ -0.204 & -0.756 \end{vmatrix}$ $C_0 = 0.315$

$|C_2| > |C_0| \Rightarrow 0.946 > 0.315$ Satisfy

Since all Conditions satisfied then system is stable

Ex: For the following unity feed back closed loop system find Range of k for stability. Give that the open loop Transfer function of the system is

$$G(z) = \frac{k(0.3679z + 0.2642)}{(z-0.3679)(z-1)}$$



Sol:

the digital Transfer function:

$$T.F = \frac{G(z)}{1+G(z)} = \frac{q(z)}{P(z)}$$

$\therefore$ the stability of the system is given by the c/cs equation:

$$P(z) = 1 + G(z) = 0$$

$$\therefore P(z) = 1 + \frac{k(0.3679z + 0.2642)}{(z-0.3679)(z-1)} = 0$$

$$P(z) = (z-0.367)(z-1) + k(0.3679z + 0.2642) = 0$$

$$P(z) = z^2 - 1.367z + 0.367 + 0.3679kz + 0.2642k = 0$$

$$P(z) = z^2 + (0.3679k - 1.367)z + 0.2642k + 0.367 = 0$$

Conditions of stability

1- $|a_n| < a_0 \Rightarrow |0.2642k + 0.367| < 1$

$$-1 < 0.264k + 0.367 < 1$$

$$-1.367 < 0.264k < 0.63$$

$$-5.1775 < k < 2.3925 \quad (1)$$

2- $P(z)\big|_{z=1} > 0$

$$P(z)\big|_{z=1} = (1)^2 + (0.3679k - 1.367)(1) + 0.2642k + 0.367 > 0$$

$$= 1 + 0.3679k - 1.367 + 0.2642k + 0.367 > 0$$

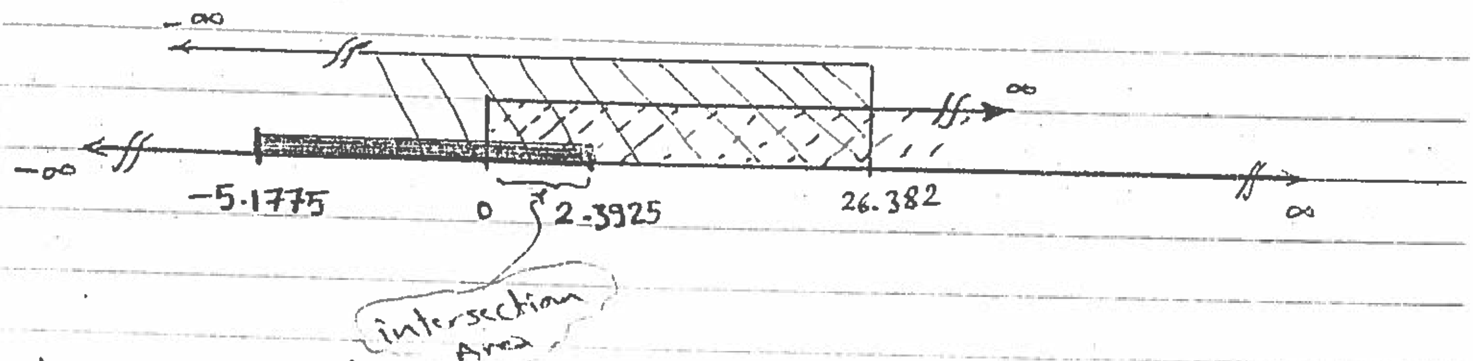
$$k > 0 \quad \text{--- (2)}$$

3- $P(z)\big|_{z=-1} > 0$ for n even

$$\therefore P(z)\big|_{z=-1} = (-1)^2 + (0.3679k - 1.367)(-1) + 0.2642k + 0.367 > 0$$

$$k < 26.382$$

so we select the Range that satisfy the three conditions (the intersection between them).



then : range of k for stability is given by

$$0 < k < 2.3925$$