Ex: Solve the following difference eqn.

$$y(n) + y(n-1) = u(n)$$

Sol:

Step: Take Z-Transform:

$$Y(z) + z^{-1} Y(z) = \frac{z}{z-1}$$

$$Y(z)\left\{1 + \frac{1}{z}\right\} = \frac{z}{z-1}$$

$$Y(z)\left\{\frac{z+1}{z}\right\} = \frac{z}{z-1}$$

$$Y(z) = \frac{z^2}{(z-1)(z+1)}$$

Step 3: Take inverse z-Transform

$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z+1)}$$

$$\frac{Y(z)}{z} = \frac{A}{z-1} + \frac{B}{z+1}$$

$$A = \lim_{z \to 1} \frac{z}{z+1} = \frac{1}{2}$$

$$B = \lim_{z \to -1} \frac{z}{z-1} = \frac{1}{2}$$

$$\frac{Y(z)}{z} = \frac{1}{2}\frac{1}{z-1} + \frac{1}{2}\frac{1}{z-(-1)}$$

$$Y(z) = \frac{1}{2}\frac{z}{z-1} + \frac{1}{2}\frac{z}{z-(-1)}$$

$$\boxed{y(n) = \frac{1}{2}u(n) + \frac{1}{2}(-1)^n}$$

Ex 2 Solve the following difference equation.

$$y(n+1) + 2y(n) = 0 \quad \& \quad y(0) = 0.5.$$

Sol: Step 1: Take z-Transform

$$z\,Y(z) - z\,y(0) + 2\,Y(z) = 0$$

but $y(0) = 0.5$

$$Y(z)\left[z+2\right] = \frac{z}{2}$$

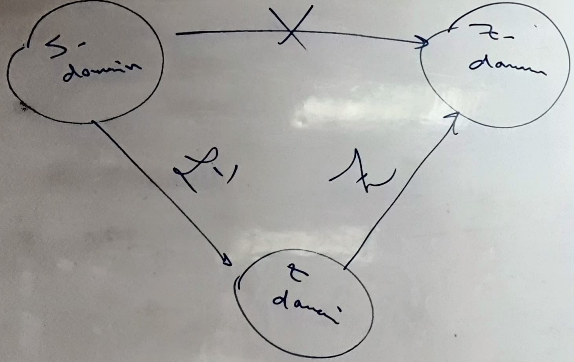
$$Y(z) = \frac{1}{2}\,\frac{z}{z-(-2)}$$

Step 2: take inverse z-Transform

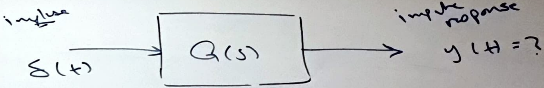
$$y(n) = \frac{1}{2}(-2)^n$$

Digital Equivalent Transfer function of Analog system

Given: $G(s)$ Req $G(z)$



Impulse invariant Method:



$$y(t) = \delta(t) * g(t) = g(t)$$

$$g(t) = \mathcal{L}^{-1} G(s) \quad \text{— ①}$$

$$G(z) = Z\, g(t) \quad \text{— ②}$$

from ① & ②

$$G(z) = Z\, \mathcal{L}^{-1}\, G(s)$$

Ex: find the digital Equivalent Transfer function for following system: $G(s) = \frac{1}{s+1}$

Solution

$$G(z) = \mathcal{Z}\,\mathcal{L}^{-1}\,G(s)$$

$$= \mathcal{Z}\,\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$= \mathcal{Z}\,e^{-t}$$

$$\boxed{G(z) = \frac{z}{z-e^{-T}}}$$

Ex: find the Digital Equivalent T.F for: $G(s) = \dfrac{1}{s(s+2)}$

Sol:

$$G(z) = \mathcal{Z}\,\mathcal{L}^{-1}\,G(s)$$

$$= \mathcal{Z}\,\mathcal{L}^{-1}\,\frac{1}{s(s+2)}$$

$$= \mathcal{Z}\,\mathcal{L}^{-1}\,\frac{A}{s} + \frac{B}{s+2}$$

$$A = \lim_{s \to 0} \frac{1}{s+2} = \frac{1}{2}$$

$$B = \lim_{s \to -2} \frac{1}{s} = -\frac{1}{2}$$

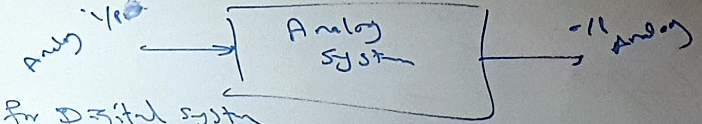
$$= \mathcal{Z}\,\mathcal{L}^{-1}\left(\frac{1}{2}\,\frac{1}{s} - \frac{1}{2}\,\frac{1}{s+2}\right)$$

$$= \mathcal{Z}\left[\frac{1}{2}\,u(t) - \frac{1}{2}\,e^{-2t}\right]$$

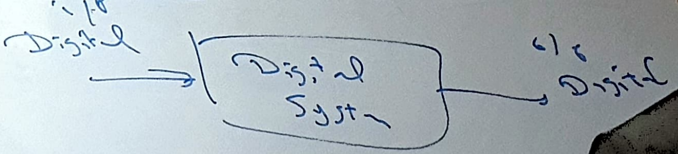
$$G(z) = \left(\frac{1}{2}\,\frac{z}{z-1} - \frac{1}{2}\,\frac{z}{z-e^{-2T}}\right)$$

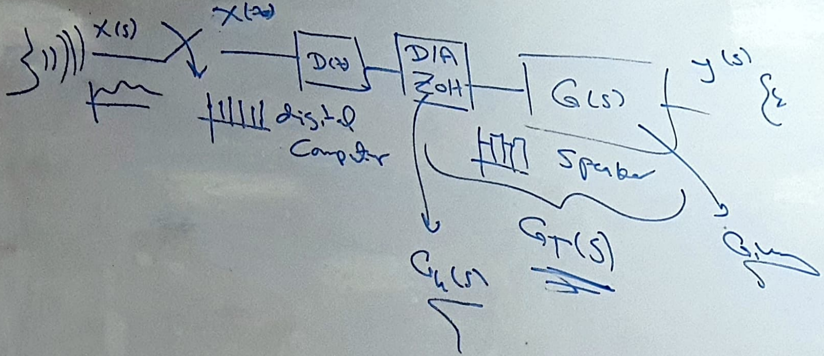
Note that

for Analog system

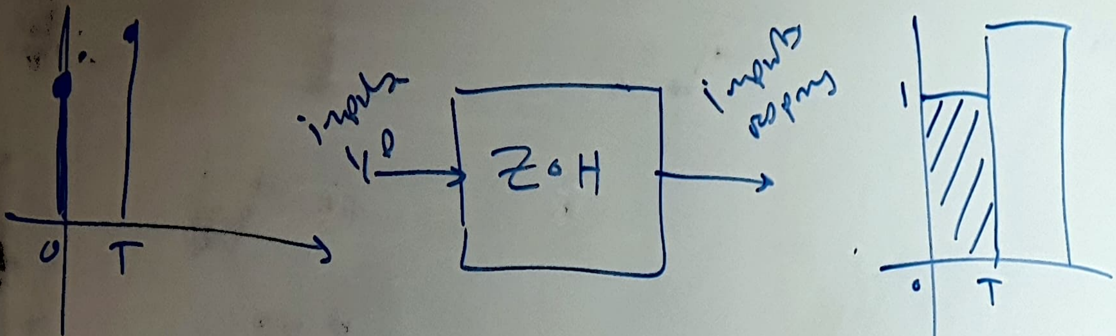


for Digital system





* Zero order Hold (ZOH)



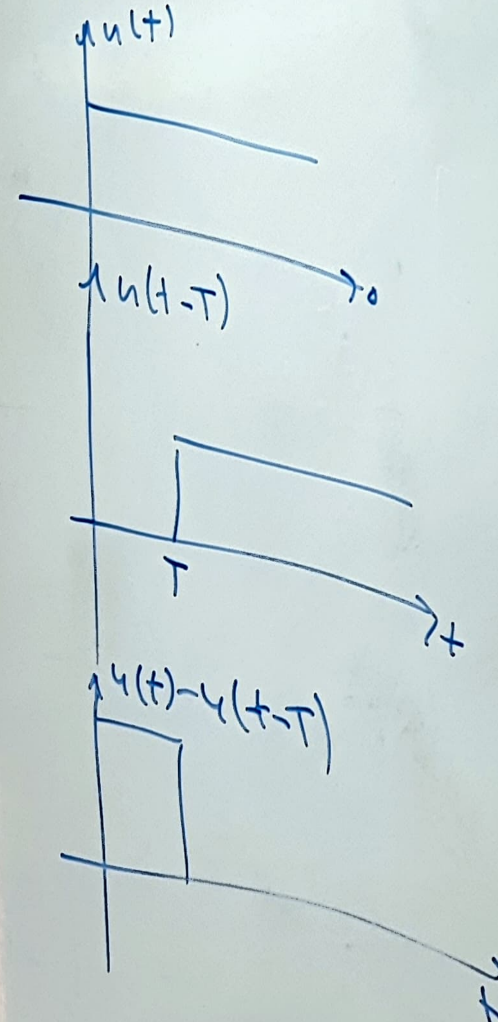
$$g_h(t) = u(t) - u(t-T)$$

$$G_h(s) = \frac{1}{s} - \frac{e^{-sT}}{s}$$

$$\boxed{G_h(s) = \frac{1 - e^{-sT}}{s}}$$

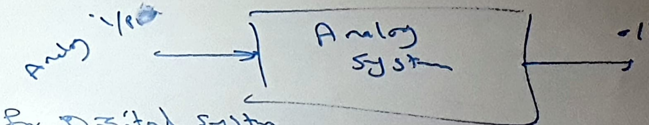
$$G_T(s) = G_h(s) \cdot G(s)$$

$$G_T(s) = \frac{1 - e^{-sT}}{s} G(s)$$

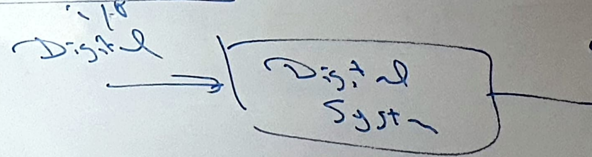


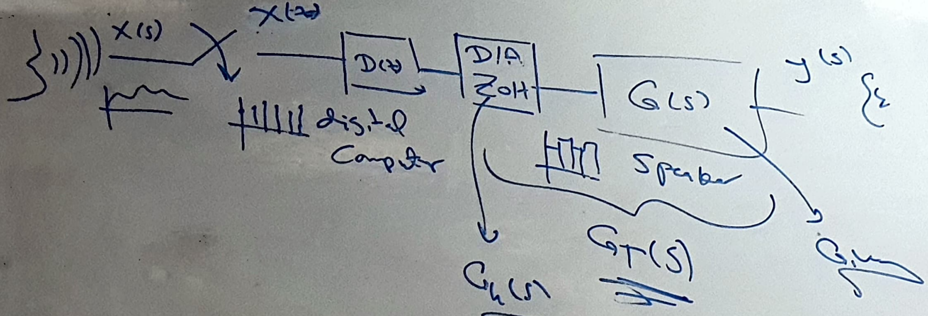
Note that

for Analog system



for Digital system

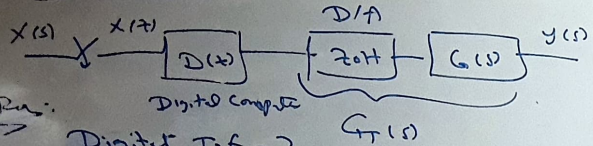




$$\boxed{G_T(s) = G_h(s) \cdot G(s)}$$

Pulse Transfer function:

① Open Loop system:



Rem.: →

Digital T.f = ?

$$T.f = \frac{Y(z)}{X(z)} = D(z) \cdot G_T(z)$$

but $G_T(z) = \mathcal{Z}\, G_T(s)$

$$= \mathcal{Z}\, G_h(s) \cdot G(s)$$

$$= \mathcal{Z}\, \frac{1 - e^{-sT}}{s} G(s)$$

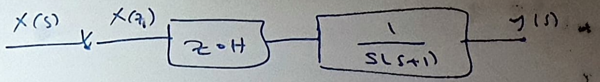
but $z = e^{sT}$

$$= \mathcal{Z}\, \frac{1 - z^{-1}}{s} G(s)$$

$$G_T(z) = 1 - z^{-1}\, \mathcal{Z}\, \frac{G(s)}{s}$$

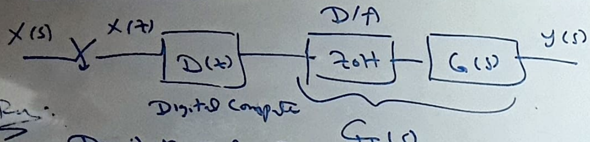
$$G_T(z) = \frac{z-1}{z}\, \mathcal{Z}\, \frac{G(s)}{s}$$

Ex: Find the digital T.f, assume $T_s = 1$ sec.



Pulse Transfer function:

① Open Loop system:



Req.: Digital T.f = ?

$$T.f = \frac{Y(z)}{X(z)} = D(z) \cdot G_T(z)$$

but $G_T(z) = \mathcal{Z}\, G_T(s)$

$$= \mathcal{Z}\, G_h(s) \cdot G(s)$$

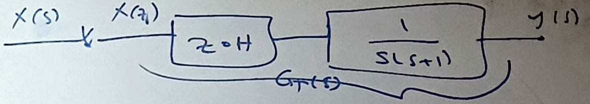
$$= \mathcal{Z}\, \frac{1-e^{-sT}}{s} G(s)$$

but $z = e^{sT}$ $\quad = \mathcal{Z}\, \frac{1-z^{-1}}{s} G(s)$

$$G_T(z) = 1 - z^{-1}\, \mathcal{Z}\, \frac{G(s)}{s}$$

$$G_T(z) = \frac{z-1}{z}\, \mathcal{Z}\, \frac{G(s)}{s}$$

Ex: Find the digital T.f, assume $T_s = 1$ sec.



Sol: $T.f = \frac{Y(z)}{X(z)} = G_T(z)$

but: $G_T(z) = \frac{z-1}{z}\, \mathcal{Z}\, \frac{G(s)}{s}$

$$= \frac{z-1}{z}\, \mathcal{Z}\, \frac{1}{s^2(s+1)}$$

$$= \frac{z-1}{z}\, \mathcal{Z}\, \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}$$

$$A = \lim_{s \to 0} \frac{1}{s+1} = 1$$

$$B = \lim_{s \to 0} \frac{d}{ds}\left(\frac{1}{s+1}\right) = \lim_{s \to 0} \frac{-1}{(s+1)^2} = -1$$

$$C = \lim_{s \to -1} \frac{1}{s^2} = 1$$

$$= \frac{z-1}{z}\, \mathcal{Z} \left\{ \overset{t}{\frac{1}{s^2}} - \overset{u}{\frac{1}{s}} + \overset{e^{-t}}{\frac{1}{s+1}} \right\}$$

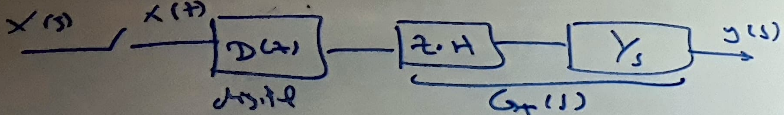
$$= \frac{z-1}{\cancel{z}}\left[\frac{T\cancel{z}}{(z-1)^2} - \frac{\cancel{z}}{z-1} + \frac{\cancel{z}}{z-e^{-T}}\right]$$

$$= z-1\left[\frac{1}{(z-1)^2} - \frac{1}{z-1} + \frac{1}{z-0.36}\right]$$

$$= \left(\frac{\cancel{z}\,1}{z-1} - 1 + \frac{z-1}{z-0.36}\right)$$

$$\boxed{T.F = \frac{0.36z + 0.28}{z^2 - 1.36z + 0.36}}$$

Ex: find digital T.f, $D(z) = \frac{z-1}{z}$, $T_s = 1$ sec
then find the step response.



Sol:

$$T.f = D(z)\, G_T(z)$$

but $G_T(z) = \frac{z-1}{z} \mathcal{Z}\left\{\frac{G(s)}{s}\right\}$

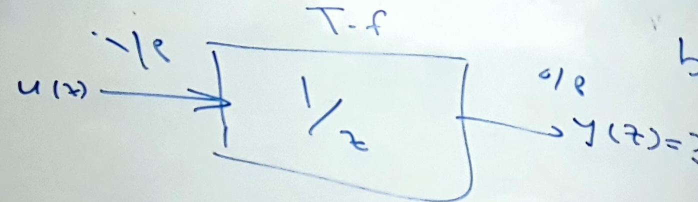
$= \frac{z-1}{z} \mathcal{Z}\left\{\frac{1}{s^2}\right\}$ t

$= \frac{\cancel{z-1}}{\cancel{z}} \cdot \frac{T\cancel{z}}{(z-1)^2}$

$$G_T(z) = \frac{1}{z-1}$$

$$T.f = \frac{\cancel{z-1}}{z} \cdot \frac{1}{\cancel{z-1}}$$

$$T.f = \frac{1}{z}$$



$Y(z) = \frac{1}{z} \cdot u(z)$

$= \frac{1}{\cancel{z}} \cdot \frac{\cancel{z}}{z-1}$

$$Y(z) = \frac{1}{z-1}$$

Ex: find the digital T.f

X(s)

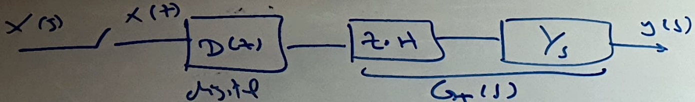
X(z)

Sol: T

but: G

Ex: find digital T.f, $D(z) = \frac{z-1}{z}$, $T_s = 1$ sec

then find the step response.



Sol:

$$T.f = D(z)\, G_T(z)$$

but

$$G_T(z) = \frac{z-1}{z}\, \mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

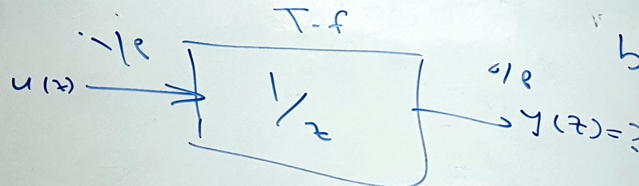
$$= \frac{z-1}{z}\, \mathcal{Z}\left\{\frac{1}{s^2}\right\}$$

$$= \frac{\cancel{z-1}}{\cancel{z}} \cdot \frac{T\cancel{z}}{(z-1)^{\cancel{2}}}$$

$$G_T(z) = \frac{1}{z-1}$$

$$T.f = \frac{\cancel{z-1}}{z} \cdot \frac{1}{\cancel{z-1}}$$

$$T.f = \frac{1}{z}$$



$$Y(z) = \frac{1}{z} \cdot u(z)$$

$$= \frac{1}{\cancel{z}} \cdot \frac{\cancel{z}}{z-1}$$

$$Y(z) = \frac{1}{z-1}$$

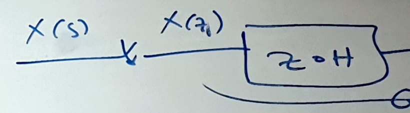
Take Z-inverse

$$Y(z) = z^{-1}\left(\frac{z}{z-1}\right)$$

$$y(t) = u(t-T)$$

$$y(n) = u(n-1)$$

Ex: find the digital T.f, assume



Sol: T f =

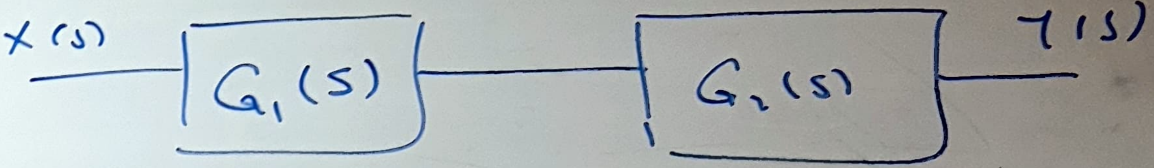
but: $G_T(z) =$

A =

B =

C =

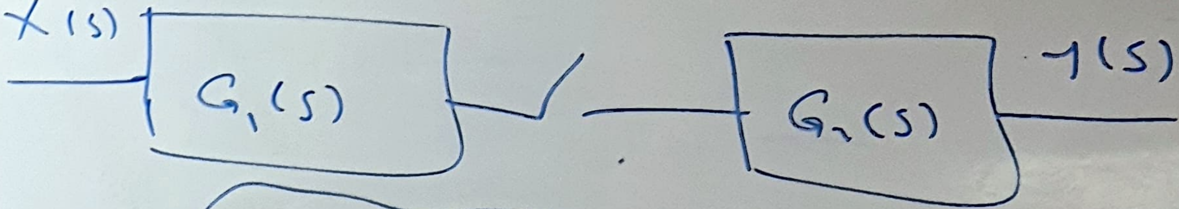
Note: Case (1)



$$T.f = \frac{Y(z)}{X(z)} = Z\, G_1(s)\cdot G_2(s)$$

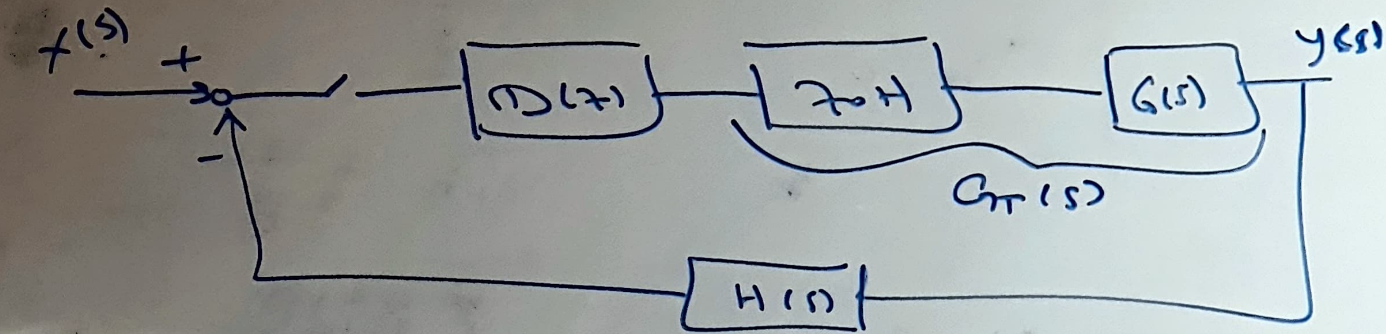
Case (2):

(z)=?



$$T.f = \frac{Y(z)}{X(z)} = Z\, G_1(s)\cdot Z\, G_2(s)$$

* Closed Loop system:



By Digital T-f

$$T\text{-}f = \frac{Y(z)}{X(z)} = \frac{D(z)\, G_T(z)}{1 + D(z)\, G_T(z)\, H(z)}$$

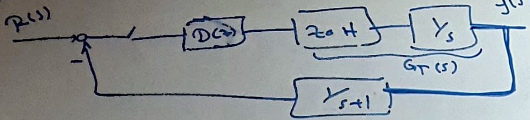
$$\boxed{G_T(z) = \quad \mathcal{Z}\left\{\frac{G(s)}{s}\right\}}$$

$$\frac{z-1}{z}$$

$$\boxed{G_T(z)\, H_T(z) = \quad \mathcal{Z}\left\{\frac{G(s)\cdot H(s)}{s}\right\}}$$

$$\frac{z-1}{z}$$

Ex: find digital T.f, $D(z) = \frac{z-1}{z}$, $T_s = 1$ sec



Sol: $T.f = \frac{D(z)\,G_T(z)}{1 + D(z)\,G_T(z)\,H(z)}$

$G(z) = \frac{z-1}{z} \mathcal{Z}\left\{\frac{G(s)}{s}\right\}$

$(1-z^{-1}) \quad = \frac{z-1}{z} \mathcal{Z}\left\{\frac{1}{s^2}\right\} \rightarrow t$

$= \frac{\cancel{z-1}}{\cancel{z}} \left[\frac{\cancel{z}T}{(z-1)^{\cancel{2}}}\right]$

$G_T(z) = \frac{1}{z-1}$

$D(z)\,G_T(z) = \frac{\cancel{z-1}}{z} \cdot \frac{1}{\cancel{z-1}}$

$D(z)\,G_T(z) = \frac{1}{z}$

$G_T(z)\,H(z) = \frac{z-1}{z} \mathcal{Z}\left\{\frac{1}{s^2(s+1)}\right\}$

$= \frac{z-1}{z} \mathcal{Z}\left\{\frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}\right\}$

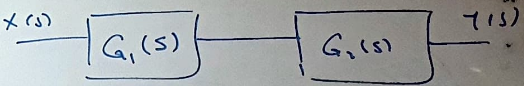
$A = \lim_{s \to 0} \frac{1}{s+1} = 1$

$B = \lim_{s \to 0} \frac{d}{ds} \frac{1}{s+1} = \lim_{s \to 0} \frac{-1}{(s+1)^2} = -1$

$C = \lim_{s \to -1} \frac{1}{s^2} = 1$

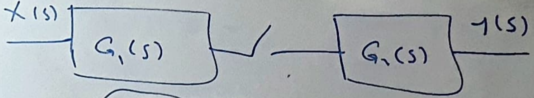
$= \frac{z-1}{z} \mathcal{Z}\left\{\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right\}$ (t, 1, e^{-t})

Note: Case (1)



$T.f = \frac{Y(z)}{X(z)} = \mathcal{Z}\, G_1(s) \cdot G_2(s)$

Case (2):



$T.f = \frac{Y(z)}{X(z)} = \mathcal{Z}\, G_1(s) \cdot \mathcal{Z}\, G_2(s)$

$$= \frac{z-1}{z}\left[\frac{zT}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-T}}\right]$$

$$= \left[\frac{1}{z-1} - 1 + \frac{z-1}{z-0.36}\right]$$

$$D(z)\,G_T H(z) = \frac{\cancel{z-1}}{z}\left(\frac{1}{\cancel{z-1}} - 1 + \frac{z-1}{z-0.36}\right]$$

$$= \left(\frac{1}{z} - \frac{z-1}{z} + \frac{(z-1)^2}{z(z-0.36)}\right]$$

$$\boxed{D(z)\,G_T H(z) = \frac{0.36z+0.38}{z(z-0.36)}}$$

$$T.f = \frac{1/z}{1+\frac{0.36z+0.38}{z(z-0.36)}}$$

$$\boxed{T.f = \frac{z-0.36}{z^2+0.28}}$$