Root Locus

By

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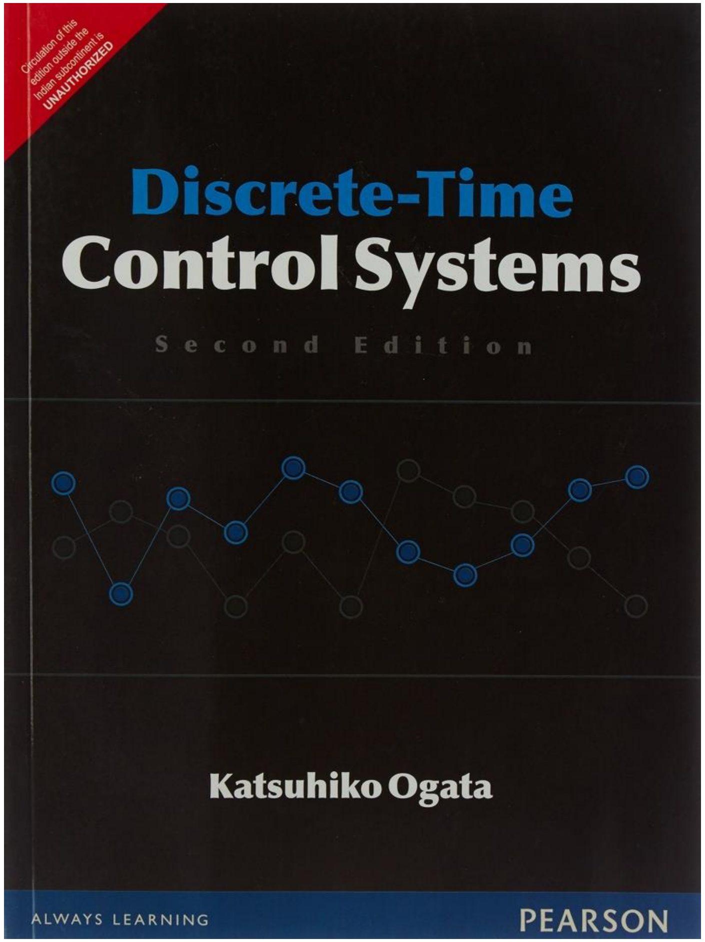


Relation between Laplace and Z - transform

- **Mapping from s-plane to z-plane:**

Since $z = e^{STs}$ And $S = \sigma + jw$ Then $z = e^{(\sigma + j\cdot w)Ts} = e^{\sigma Ts} \cdot e^{j\cdot w\cdot Ts}$

where $z = |z| \angle z$ Then $|z| = e^{\sigma Ts}$ And $\angle z = wTs$

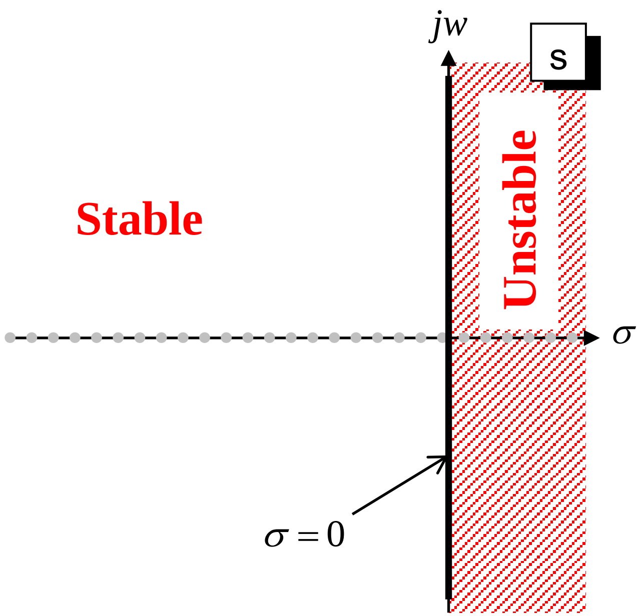
As per sampling theorem, $|wTs| = \pi$ Then $\angle z$ will vary from $-\pi$ to π (0 to 2π)

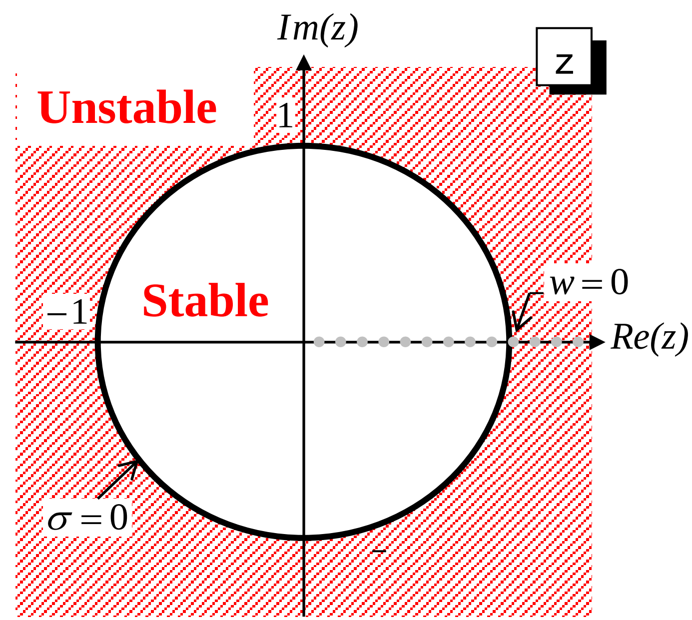
- The boundary between the stability and instability region can be defined as follows:

  $\sigma = 0$ Then $z = |1| \angle z$

- For stable :

  $\sigma < 0$ Then $|z| = e^{\sigma Ts} < 1$ so, system is stable inside the unit circle.





Design based on root locus method

❑ Objective:

- The main objective of a control system is to design a controller either in forward or in feedback path so that the closed loop system is stable with some desired performance.

❑ Two most popular design techniques for continuous time LTI systems are using:

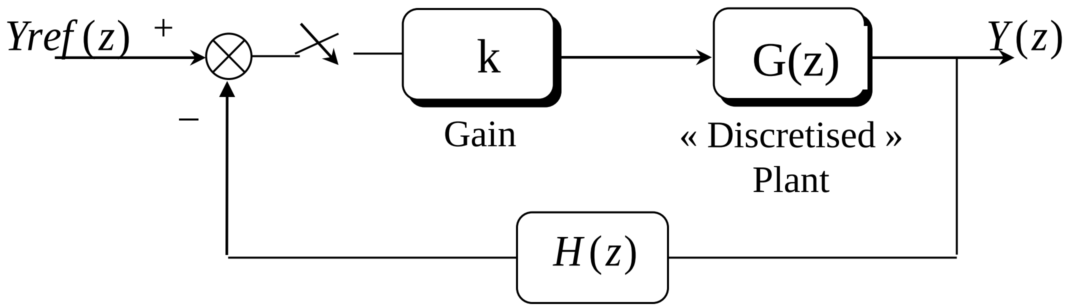
- **Root locus** and
- **Frequency domain methods.**

1. Design based on root locus method

- It is a necessary to investigate the effect of system **gain** and/or **sampling period** on the absolute and relative stability of the closed loop system should be investigated in addition to the transient response characteristics.
- For this purpose the **root locus** method prove to **be very useful**.
- The root locus method for continuous time systems can be extended to discrete time systems without much modifications.

Construction rules for root locus

- Root locus construction rules for discrete control systems are the same as that of continuous time control systems.



- In discrete time control system, the characteristics equation have the following forms:

$$1+k\mathbf{GH}\ (z)=0$$

- **<u>The general rules for constructing "Root Locus" are the following:</u>**

<u>Step1</u>: Locate the poles and zeros of the open loop transfer function $GH\ (z)$ where:

$$GH\ (z)=\frac{K\ (z-z1)(z-z2)\ \ldots(z-zm)}{(z-p1)(z-p2)\ \ldots(z-pn)}$$

- **<u>Zeros:</u>** z1, z2, …, zm
- **<u>Poles:</u>** p1, p2,…, pn

Construction rules for root locus

Step 2: **Draw Parts of root locus on real axis.**

Step 3: **Calculate the asymptotes lines .**

- **Number of asymptotes lines = $n_p - n_z$**
  - **Case1:** np = nz **then** number of asymptotic line = zero.
  - **Case2:** np – nz = 1 **then** single asymptotic line which is the real axis.
  - **Case3:** np – nz > 1 **then**:
    1. **Center of asymptotic** $\sigma_A = \dfrac{\sum poles - \sum zeros}{np - nz}$
    2. **Angle of asymptotic** $= \dfrac{180*(2q+1)}{np - nz}$, q = 0,1,2…
       - If np – nz = 2 **then angles of asymptotic** = ±90
       - If np – nz = 3 **then angles of asymptotic** = ±60, 180
       - If np – nz = 4 **then angles of asymptotic** = ±45, ± 135

Step 4: **Find the "break-away" and "break-in" points.**

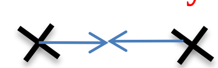
1. $P(z) = 1 + GH(z) = 0 \rightarrow GH(z) = -1 \rightarrow -k = \dfrac{1}{GH(z)}$

2. $$\frac{-dk}{dz} = \frac{d}{dz}\left(\frac{1}{GH(z)}\right) = 0.$$

break-in

z = z1

break-away



z = z2

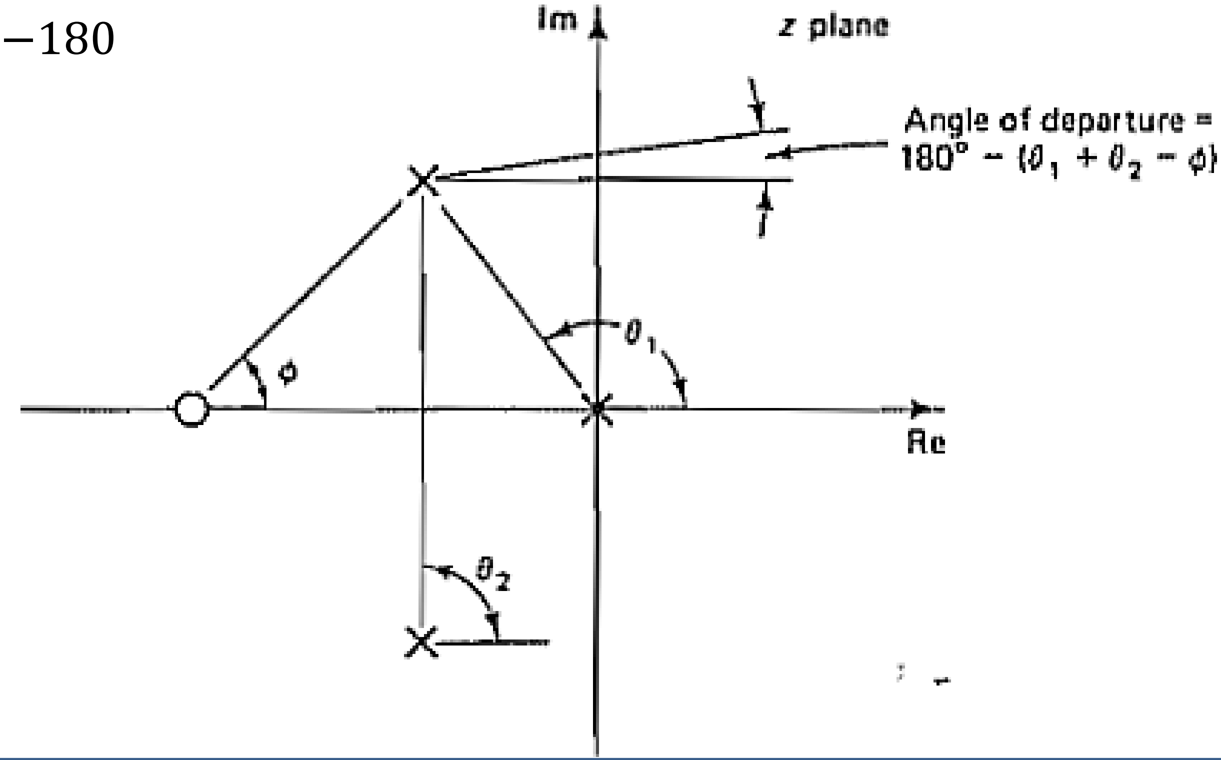
Construction rules for root locus

<u>Step 5:</u> Determine the "angle of departure" or "angle of arrival", connect the pole or zero that need to find its "angle of departure" or "angle of arrival", by all other poles and zeros of the system then:

- **"angle of departure" or "angle of arrival"** = $180 - (\theta 1 + \theta 1 - \emptyset)$

- **Generally:**

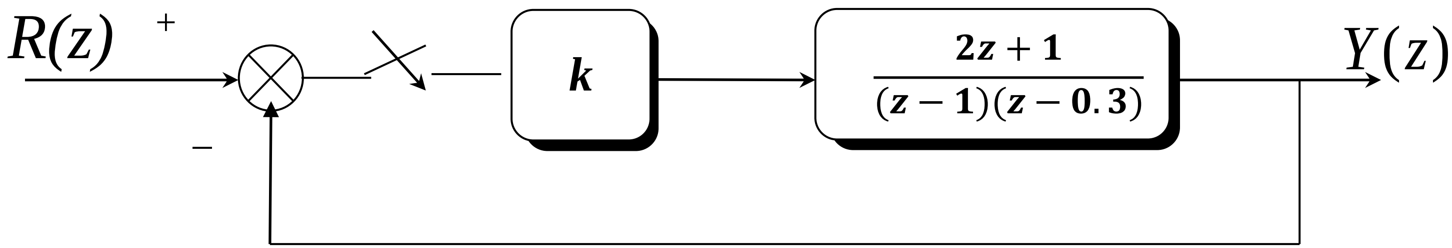
$$\sum \theta z - \sum \theta p = -180$$



Construction rules for root locus

❑ Example:

- **Obtain the root locus plot and the critical gain for the following system.**



❑ Solution:

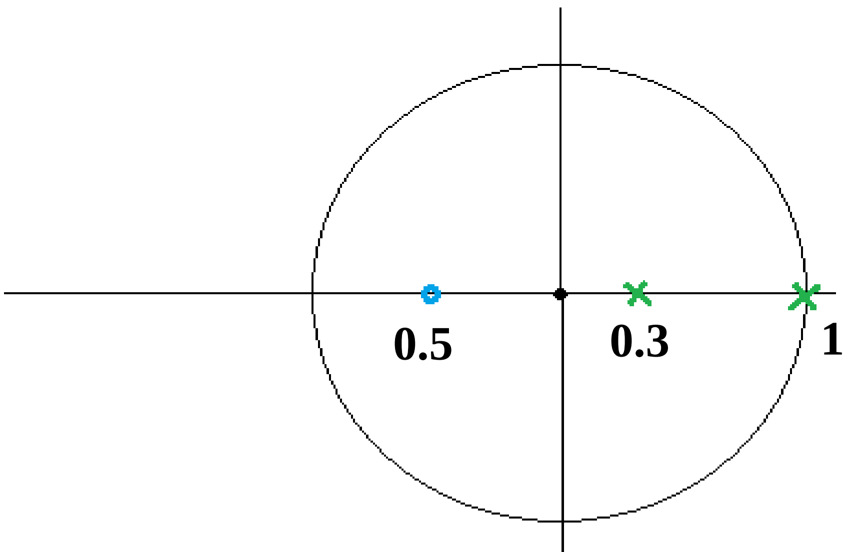
- **Step1:**

  Locate the poles and zeros of the open loop transfer function $GH\ (z)$ where:

  ○ $GH\ (z) = \frac{2k(z+0.5)}{(z-1)(z-0.3)}$

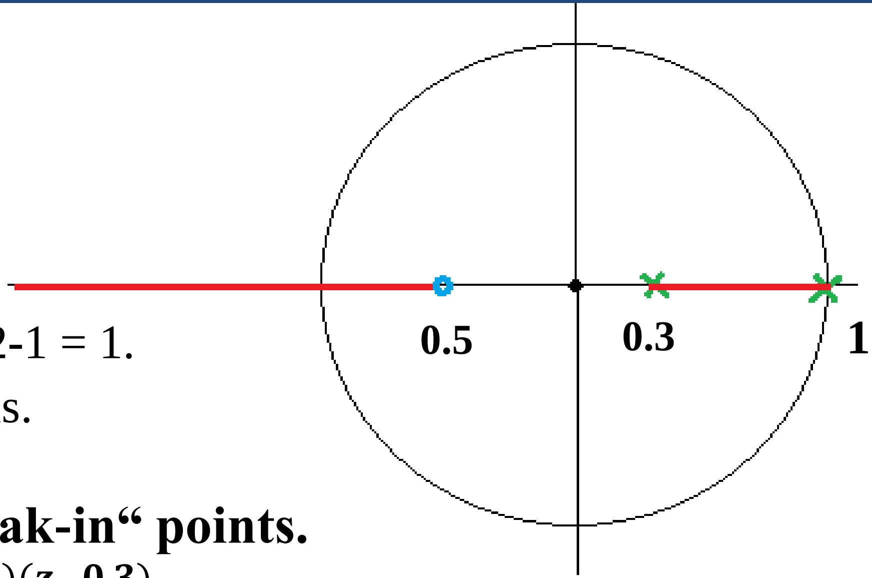
  – **Poles:** 0.3, 1

  – **Zeros:** -0.5



Construction rules for root locus

- **Step 2:**
  Draw Parts of root locus on real axis.
- **Step 3:**
  Calculate the asymptotes lines .
  - Number of asymptotes lines = $n_p - n_z$ = 2-1 = 1.
  - Single asymptotic line which is the real axis.



- **Step 4: Find the "break-away“ and "break-in“ points.**

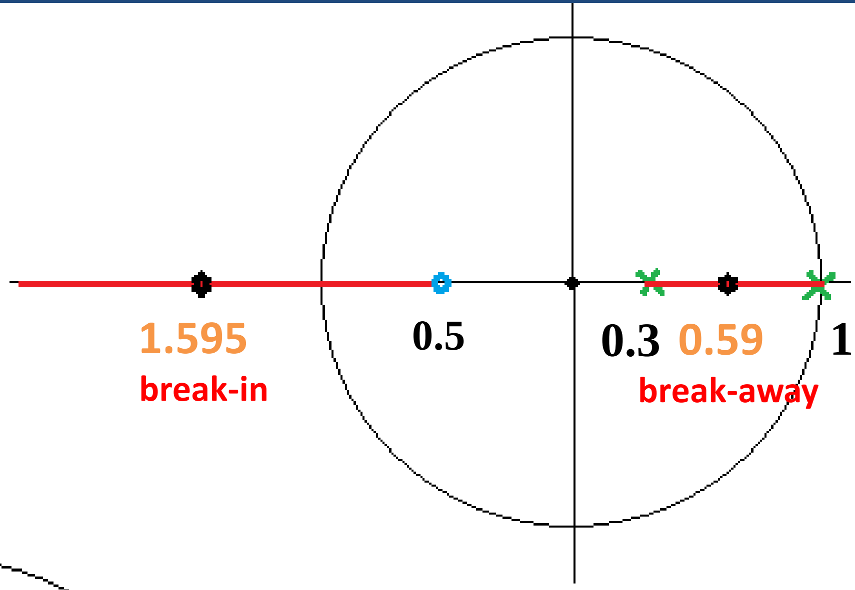
1. $P(z) = 1 + \frac{2k(z+0.5)}{(z-1)(z-0.3)} = 0 \rightarrow -k = \frac{(z-1)(z-0.3)}{2(z+0.5)}$
2. $\frac{-dk}{dz} = \frac{d}{dz}\left(\frac{(z-1)(z-0.3)}{2(z+0.5)}\right) = 0.$

$$\frac{-dk}{dz} = z^2 + z - 0.95 = 0.$$

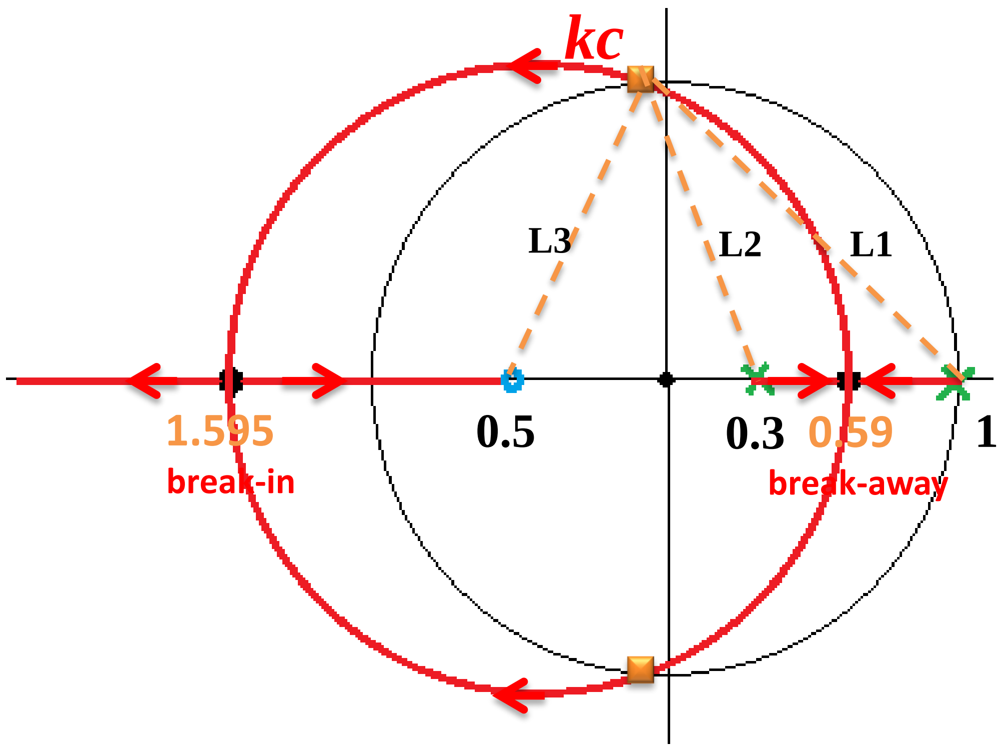
Z1=0.59,
Z2= 1.595

Construction rules for root locus

- $z1=0.59$ "break-away",
- $z2=-1.595$ "break-in".

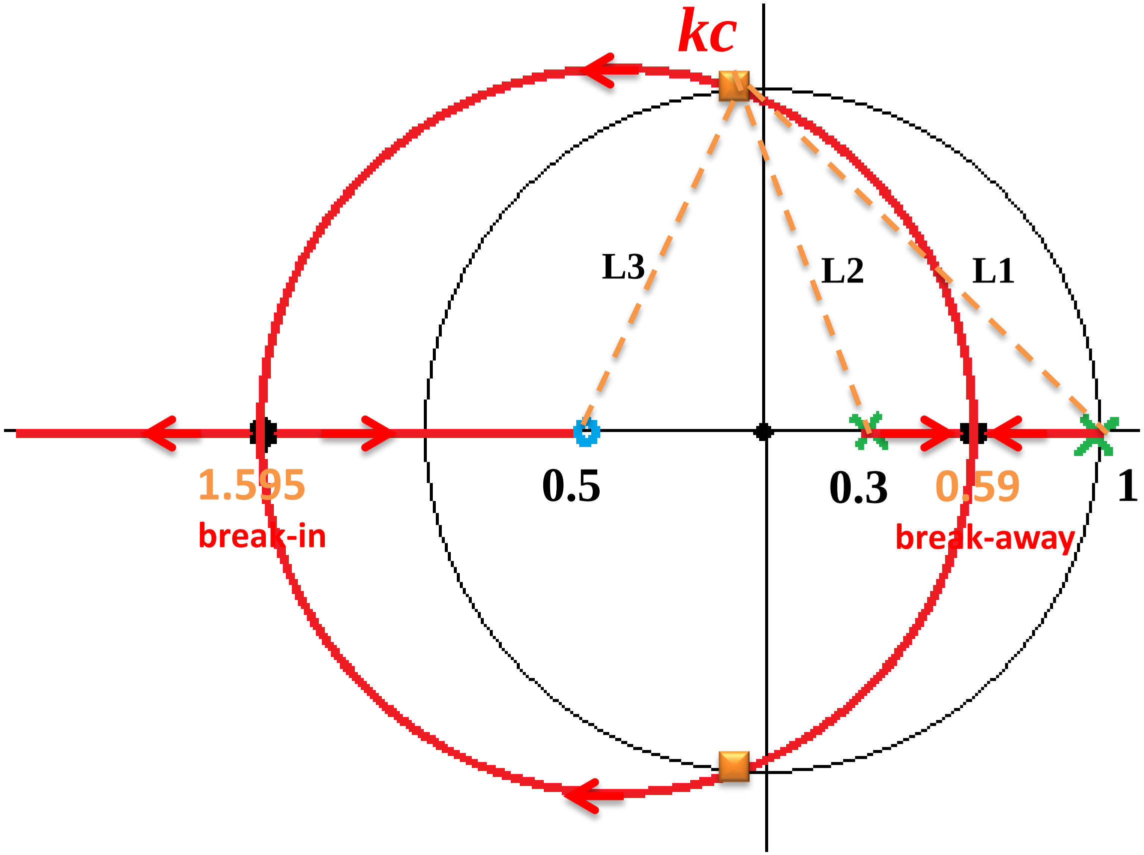


- **Root locus plot :**



Construction rules for root locus

- **To obtain kc:**
- $K = 2kc = \frac{\prod lp}{\prod lz} = \frac{L1*L2}{L3} = \frac{1.5*1.1}{1.1} = 1.5$
- $\rightarrow kc = 0.75$



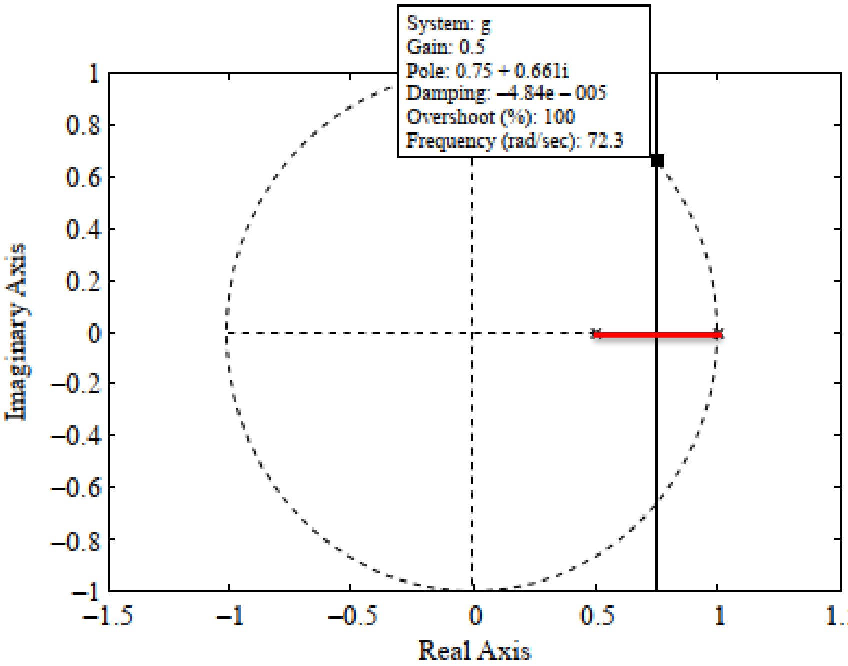
Construction rules for root locus

Example: Obtain the root locus plot and the critical gain for the first-order type 1 system with loop gain:

$$L(z) = \frac{1}{(z-1)(z-0.5)}$$

Solution:

- **Step1: Locate the poles and zeros of the open loop transfer function $L\ (z)$ where:**
  - **Poles:** 0.5, 1
  - **Zeros:** --
- **Step 2: Draw Parts of root locus on real axis.**



- **Step 3: Calculate the asymptotes lines .**

○ Number of asymptotes lines = $n_p - n_z = 2$

1. **Center of asymptotic** $\sigma_A = \frac{\sum poles - \sum zeros}{\text{np} - \text{nz}}$

   $= (1+0.5)/(2) = 0.75.$

1. **Angle of asymptotic =** ±90

Construction rules for root locus

Step 4: **Find the "break-away“ and "break-in“ points.**

- the closed-loop characteristic equation:
  - $1 + kGH\,(z) = 0.$
  - $1 + \frac{k}{(z-1)(z-0.5)} = 0.$
  - $(z-1)(z-0.5) + K = z^2 - 1.5z + K + 0.5 = 0.$
  - $-K = z^2 - 1.5z + 0.5$
  - $\frac{-dk}{dz} = 0.$
  - → 2z-1.5=0 → z = 0.75.

- The breakaway point is midway between the two open-loop poles at zb = 0.75.
- The critical gain now occurs at the intersection of the root locus with the unit circle.
- **To obtain kc:**
- $K = kc = \frac{\prod lp}{\prod lz} = \frac{L1*L2}{1} = \frac{0.7*0.7}{1}$ =**0.5**
- →kc= **0.5**

Construction rules for root locus