Diffrence equation

Function	Z-transform
$x(n+2)$	$z^2 x(z) - z^2 x(0) - z x(1)$
$x(n+1)$	$z x(z) - z x(0)$
$x(n)$	$x(z)$
$x(n-1)$	$z^{-1} x(z)$
$x(n-2)$	$z^{-2} x(z)$

Ex. Solve diffrace equation by Z-transform

$$y(n) + y(n-1) = u(n)$$

$$y(z) + z^{-1} y(z) = \frac{z}{z-1}$$

$$y(z)\left[1 + z^{-1}\right] = \frac{z}{z-1}$$

$$y(z) = \frac{z}{z-1} \div \left[1 + z^{-1}\right]$$

$$y(z) = \left[\frac{z}{z-1} \div \frac{z+1}{z}\right] = \frac{z}{(z-1)(z+1)}$$

Solve

$$\frac{y(z)}{z} = \frac{A}{z-1} + \frac{B}{z+1}$$

$$A = \lim_{z \to 1} \frac{z}{z+1} = \frac{1}{2}$$

$$B = \lim_{z \to -1} \frac{z}{z-1} = \frac{1}{2}$$

$$y(z) = \frac{1}{2}\left(\frac{z}{z-1}\right) + \frac{1}{2}\left(\frac{z}{z+1}\right)$$

$$= \frac{1}{2}\left(\frac{z}{z-1}\right) + \frac{1}{2}\left(\frac{z}{z-(-1)}\right)$$

$$y(z) = \frac{1}{2} u(n) + \frac{1}{2} (-1)^n$$

Ex: Solve diffrence equation $y(n+1) + 2y(n) = 0$ where $y(0) = 0.5$

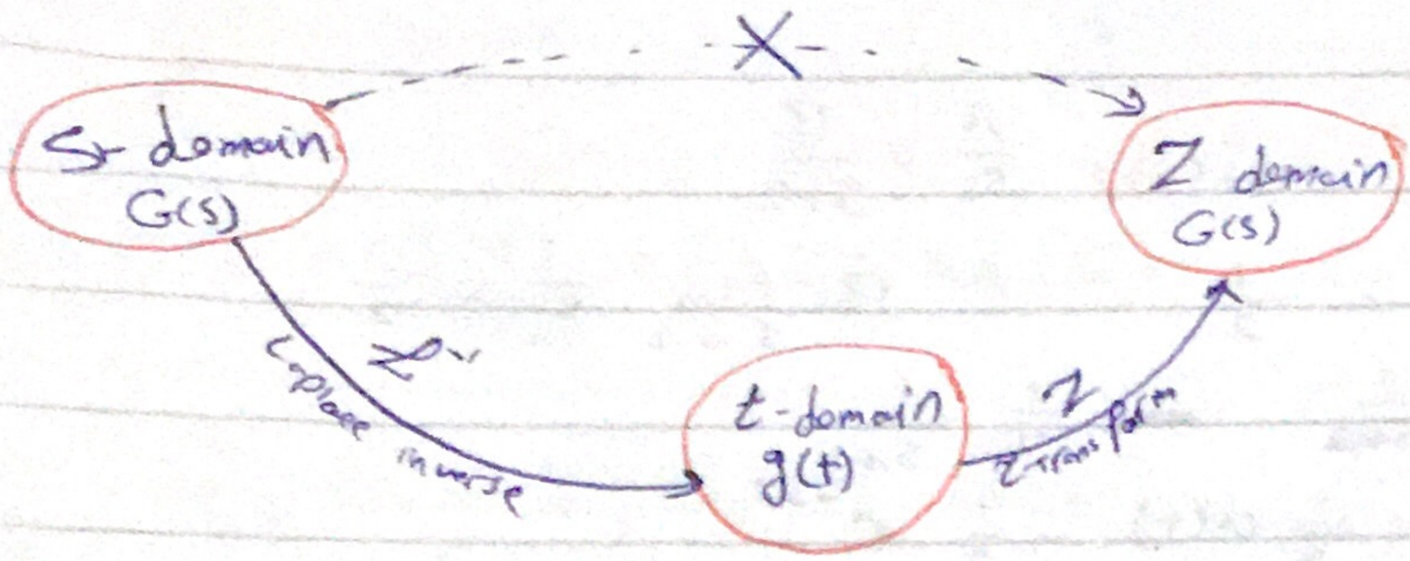
$$z\,y(z) - z\,y(0) + 2\,y(z) = 0$$

$$y(z)\left[z + 2\right] = \frac{1}{2} z = 0$$

$$y(z) = \frac{1}{2} \frac{z}{z+2} = \frac{1}{2} \frac{z}{z-(-2)}$$

$$y(n) = \frac{1}{2} (-2)^n$$

Note
• غالبا بنا خذ $y(z)$ عامل مشترك



$G(z) = Z\,\mathcal{L}^{-1} G(s)$

Table of z transform

x(t)	X(s)	X(z)
$\delta(t)$	1	1
$u(t)$	$\frac{1}{s}$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
$e^{\pm at}$	$\frac{1}{s \mp a}$	$\frac{z}{z - e^{\pm aT}}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$	$\frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$	$\frac{z(z - \cos \omega T)}{z^2 - 2z \cos \omega T + 1}$

Ex: Find impulse response due to transfer function $G(s) = \frac{1}{s+1}$

$\frac{1}{s+1}(s) \longrightarrow g(t)$ $\quad e^{-t}$

$g(t) \longrightarrow G(z)$ $\quad \frac{z}{z - e^{-t}}$

1) ope Ex: $G(s) = \frac{1}{s(s+2)}$

$g(t) = \mathcal{L}(G(s))$ $\quad G(s) = \frac{A}{s} + \frac{B}{s+2}$

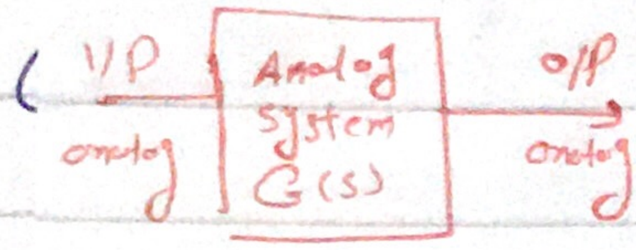
Re $\quad A = \lim_{s\to 0} \frac{1}{s+2} = \frac{1}{2}$ $\quad B = \lim_{s\to -2} \frac{1}{s} = \frac{-1}{2}$

$G(s) = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2}$

$g(t) = \mathcal{L}^{-1} G(s) = \frac{1}{2}u(t) - \frac{1}{2}e^{-2t}$

$G(z) = Z\, g(t)$

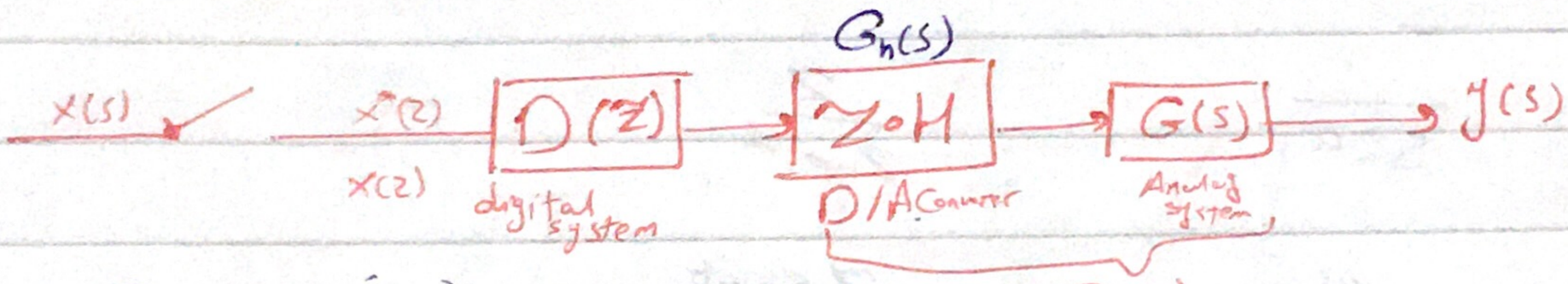
$G(z) = \frac{1}{2}\frac{z}{z-1} - \frac{1}{2}\frac{z}{z-e^{-2t}}$



S-domain

Z-Domain

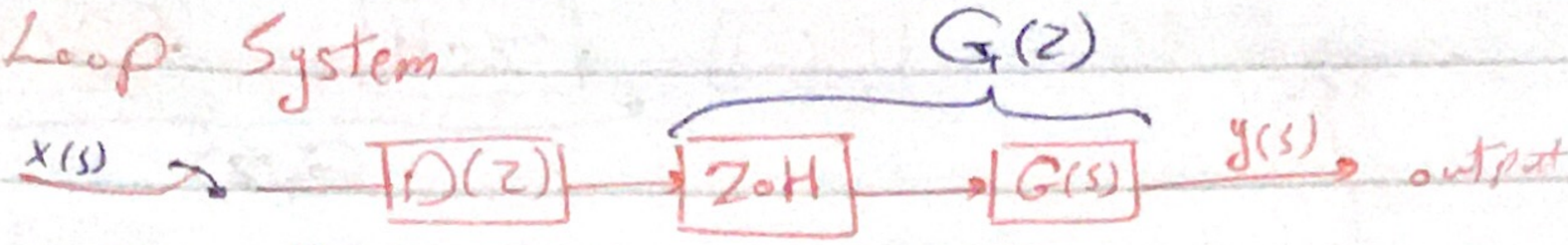
Ex



$G_T(s) = G_h(s)\, G(s)$

$\rightarrow G_h(s) = \frac{1 - e^{-sT}}{s}$

[1] Open Loop System



$$\text{Req: } T.F = \frac{y(z)}{x(z)} = D(z)\,G_t(z)$$

$$G_t(z) = Z[G_h(s)\,.\,G(s)]$$

$$= Z\ \frac{1-e^{-sT}}{s}\,.\,G(s)$$

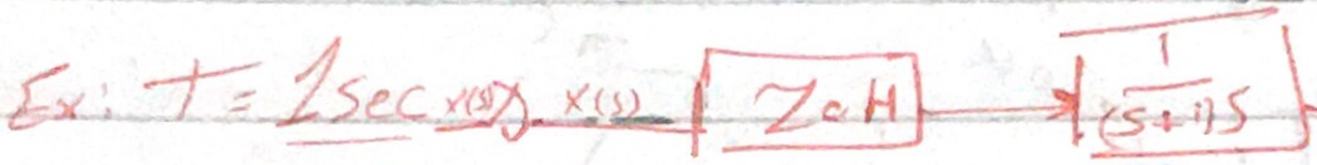
$$e^{sT} = z^{-1}$$

$$= Z\ \frac{1-z^{-1}}{s}\ G(s)$$

$$G_t(z) = (1-z^{-1})\,Z\ \frac{G(s)}{s}$$

$$= \frac{z-1}{z}\ Z\left(\frac{G(s)}{s}\right)$$

Ex: $T = 1\,sec$



$$T.f. \quad \frac{Y(z)}{X(z)} = Z\ \frac{1-e^{-sT}}{s}\cdot\frac{1}{s(s+1)}$$

$$= \frac{z-1}{z}\ Z\left[\frac{1}{s^2(s+1)}\right] = \frac{z-1}{z}\ Z\left[\frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}\right]$$

$$A = \lim_{s\to 0}\ \frac{1}{s+1} = 1\ ,\quad B = \lim_{s\to 0}\ \frac{d}{ds}\left(\frac{1}{s+1}\right) = \lim_{s\to 0}\ \frac{-1}{(s+1)^2} = -1$$

$$C = \lim_{s\to -1}\ \frac{1}{s^2} = 1$$

$$G_t(z) = \frac{z-1}{z}\,Z\left[\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right]$$

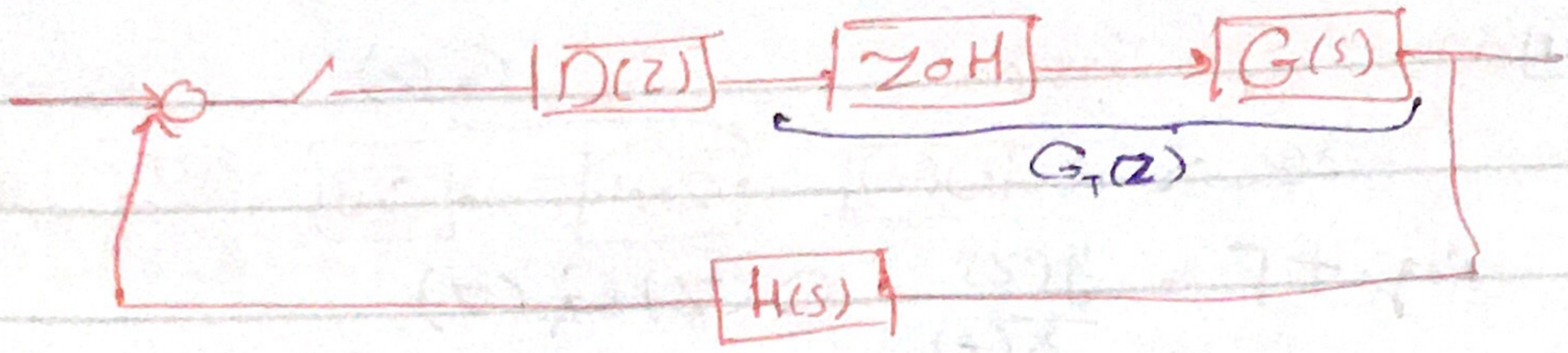
$$= \frac{z-1}{z}\,\mathcal{L}\left[t - u(t) + e^{-t}\right]$$

$$= \frac{z-1}{z}\left[\frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-T}}\right]$$

$$= \frac{1}{z-1} - 1 + \frac{z^{-1}}{z-e^{-T}} \qquad e^{-T} = 0.36$$

$$T.f = G(z) = \frac{0.36z + 0.28}{z^2 - 1.36z + 0.36}$$

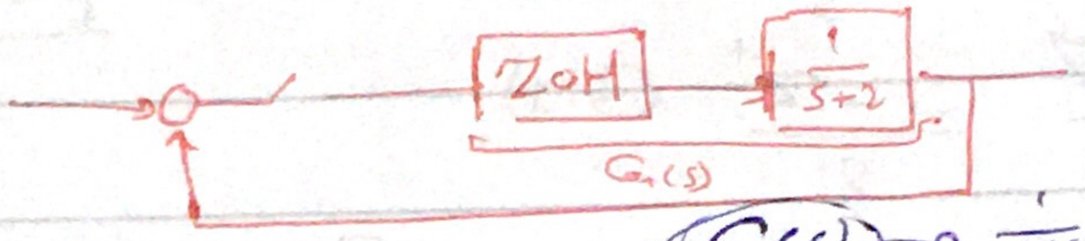
② Closed Loop



$$T.F = \frac{y(z)}{x(z)} = \frac{D(z)\,G_T(z)}{1 + D(z)\,G_T(z)\,H(z)}$$

$$G_T(z) = (1-z^{-1})\,Z\left(\frac{G(s)}{s}\right), \quad G_T H(s) = (1-z^{-1})\,Z\left[\frac{GH(s)}{s}\right]$$

Ex $T_s = 0.5\,sec$



$$T.F\ \frac{G_T(z)}{1+G_T(z)}, \quad G_T(z) = (1-z^{-1})\,Z\left(\frac{G(s)}{s}\right), \quad \frac{G(s)}{s} \to \frac{1}{s(s+2)}$$

$$= \frac{z-1}{z}\,Z\left(\frac{A}{s} + \frac{B}{(s+2)}\right)$$

$$A = \lim_{s\to 0} \frac{1}{s+2} = \frac{1}{2} \qquad B = \frac{1}{s} = \frac{-1}{2}$$

$$G_T(z) = \frac{z-1}{z}\,Z\left[\frac{1}{2}\,\frac{1}{s} - \frac{1}{2}\,\frac{1}{s+2}\right] \quad (u \to \frac{1}{s}, \ e^{-2t} \to \frac{1}{s+2})$$

$$= \frac{z-1}{z}\left(\frac{1}{2}\,\frac{z}{z-1} - \frac{1}{2}\,\frac{z}{z-e^{-2t}}\right) \qquad t_s = 0.5$$

$$= \frac{1}{2}\left(1 - \frac{z^{-1}}{z-0.36}\right)$$

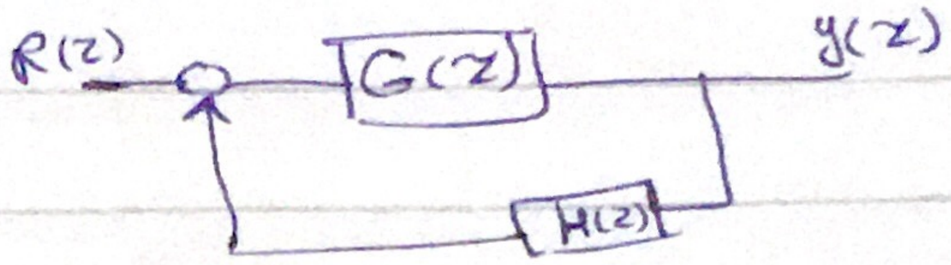
$$= \frac{1}{2}\left[\frac{z - 0.36 - z + 1}{z-0.36}\right]$$

$$G_T(z) = \frac{0.32}{z-0.36}, \quad T.F = \frac{G_T(z)}{1+G_T(z)} = \frac{\frac{0.32}{z-0.36}}{1 + \frac{0.32}{z-0.36}} = \frac{0.32}{z-0.36+0.32}$$

$$T.F = \frac{0.32}{z-0.04}$$

Stability

$$T.F = \frac{G(Z)}{1+GH(Z)} = \frac{q(Z)}{P(Z)} \quad \begin{matrix} \rightarrow \text{zeros} \\ \rightarrow \text{Poles} \end{matrix}$$

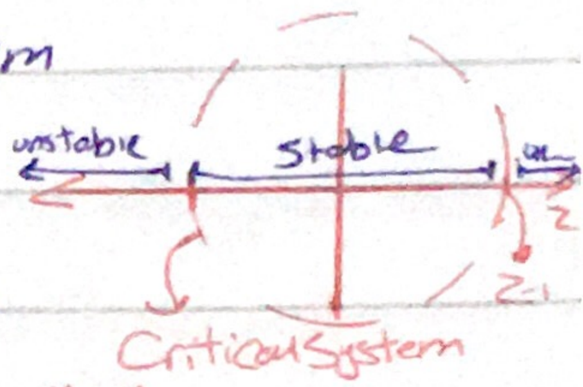


$P(Z) = 1+GH(Z)$ $\quad 1+GH(z)=0 \Rightarrow$ check stability

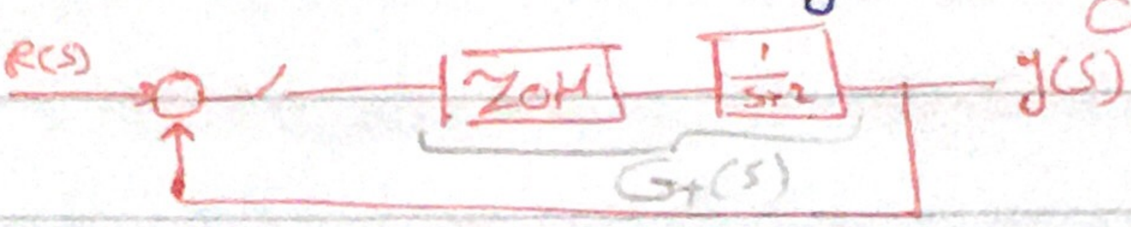
if Poles → all in unit circle → Stable system

→ any point outside unit circle → unstable system

→ Simple poles at $Z=1$ → critical system



Check Stability $T=0.5s$



① find transfer fraction

$$T.F = \frac{y(z)}{R(z)} = \frac{G_+(z)}{1+G_+(z)}$$

$$G_+(z) = \frac{z-1}{z} Z^{-1}\left[\frac{G(s)}{s}\right] = \frac{z-1}{z} Z^{-1}\left[\frac{1}{s(s+2)}\right]$$

$$G(s) = \frac{A}{s} + \frac{B}{s+2} \qquad A = \lim_{s\to 0} \frac{1}{s+2} = \frac{1}{2}, \quad B = \lim_{s\to -2} \frac{1}{s} = \frac{-1}{2}$$

$$\frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{1}{s+2}\right) \quad G_{\text{ZE}}(t) = \frac{1}{2}u(t) - \frac{1}{2}e^{-2t}$$

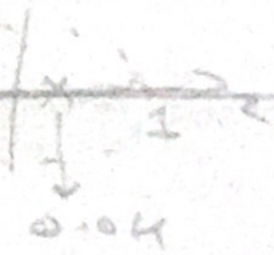
$$G(z) = \frac{1}{2}\left[\frac{z}{z-1}\right] - \frac{1}{2}\left[\frac{z}{z-e^{-2t}}\right]$$

$$G_+(z) = \frac{z-1}{z}\left[\frac{1}{2}\left(\frac{z}{z-1}\right) - \frac{1}{2}\left(\frac{z-1}{z-e^{-2t}}\right)\right]$$

$$G_+(z) = \frac{1}{2}\left[1 - \frac{z-1}{z-e^{-2t}}\right] = \frac{1}{2}\left[\frac{z-e^{-2t}+z-1}{z-e^{-2t}}\right] \quad \Big| \; t=0.5, \; e^{-2t}=0.36$$

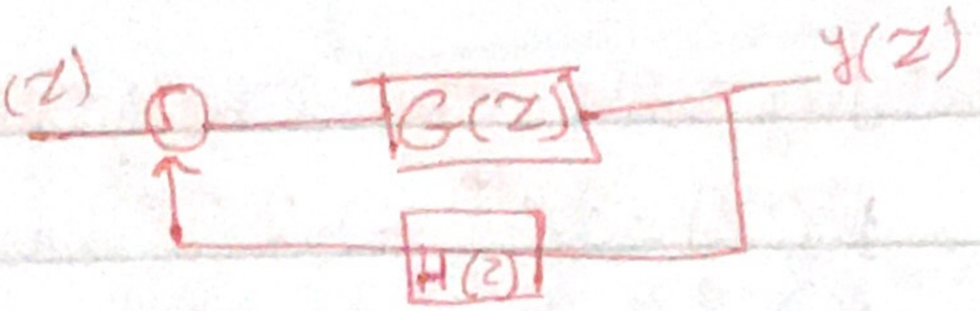
$$= \frac{1}{2}\left[\frac{-0.36-1}{z-0.36}\right]$$

$$T.F = \frac{\frac{0.32}{z-0.36}}{1+\frac{0.32}{z-0.36}} = \frac{0.32}{z-0.68} = \frac{q(z)}{P(z)}$$



Stable System.

Jury test



$$T.F = \frac{G(z)}{1+GH(z)} = \frac{q(z)}{p(z)} \quad , \quad p(z) = 1 + GH(z) = 0$$

Jury equation:

$$p(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n$$

Conditions for Stability

1. $|a_n| < a_0$ — معامل الصدارة أكبر من معامل ذيل أي
2. $p(z)|_{z=1} > 0$ — أما تعوض عن $z=1$ لازم الناتج يبقى أكبر من الصفر
3. $p(z)|_{-1}$ — تعوض عن $z=-1$
   - n even $\rightarrow p(-1) > 0$
   - n odd $\rightarrow p(-1) < 0$

   ولو أعلى أس عدد زوجي ← لازم الناتج أكبر من الصفر
   لو فردي ← لازم الناتج أصغر من الصفر

"لو أحد شرط من دول متحققش اقف ومتكملش لأنك كده"

unstable

4. Construct Jury table

z^0	z^1	z^2	z^3	.....	z^{n-1}	z^n
a_n	a_{n-1}	a_{n-2}	a_{n-3}	.....	a_1	a_0
a_0	a_1	a_2	a_3	.....	a_{n-1}	a_n
b_{n-1}	b_{n-2}	b_{n-3}	.....		b_1	b_0
b_0	b_1	b_2	.....		b_{n-2}	b_{n-1}
c_{n-2}	c_{n-3}	.....			c_1	c_0
c_0	c_1	.....			c_{n-3}	c_{n-2}
q_2	q_1	q_0				

أول عمود معاملات الـ z^n

لازم $a_n > a_0$ لو لا متحققتش يبقى unstable

بنقارن a_n, a_0 بنبدل Swapping

بنقارن b_{n-1}, b_0 بنبدل

بنقارن c_{n-2}, c_0 بنبدل

if $q_2 > q_0 \Rightarrow$ stable system

$$b_{n-1} = \begin{vmatrix} a_n & a_0 \\ a_0 & a_n \end{vmatrix} \quad b_{n-2} = \begin{vmatrix} a_n & a_1 \\ a_0 & a_{n-1} \end{vmatrix} \quad \cdots \quad c_{n-2} = \begin{vmatrix} b_{n-1} & b_0 \\ b_0 & b_{n-1} \end{vmatrix} \quad c_0 = \begin{vmatrix} b_{n-1} & b_{n-2} \\ b_0 & b_1 \end{vmatrix}$$

① هيديلك معادلة ويطلب منك تشوف هل Stable ولا لأ.

1 Construct Jury table Then Check Stability

Solu $P(Z) = Z^4 - 1.2Z^3 + 0.07Z^2 + 0.3Z - 0.08$

Stability Conditions ① $|a_n| < a_0$ $|-0.08| < 1$ ✓

② $P(Z)|_{Z=1} > 0$

$P(Z)|_{Z=1} = 1 - 1.2 + 0.07 + 0.3 - 0.08 = 0.09 > 0$ ✓

③ $P(Z)|_{Z=-1} > 0$ معاملات موجبة

$P(Z)|_{Z=-1} = 1 + 1.2 + 0.07 - 0.3 - 0.08 = 1.89 > 0$ ✓

④ Construct Jury table

Z^0	Z^1	Z^2	Z^3	Z^4
-0.08	0.3	0.07	-1.2	1
1	-1.2	0.07	0.3	-0.08

-0.994	1.176	-0.756	-0.204	$\|-0.994\| > \|-0.204\|$ ✓
-0.204	-0.756	1.176	-0.994	

0.946	-1.184	0.315	$\|0.946\| > \|0.315\|$ ✓

Coeffient equal 3

Stop ⚠

Since all Condition Satisfied then System is Stable.