A new computational method for solving weakly singular Fredholm integral equations of the first kind

E. S. Shoukralla, Faculty of engineering and technology, Future University, Cairo, Egypt.

emil.shokralah@fue.edu.eg

M. Kamel, Faculty of Electronic Engineering, Menoufia University, Cairo, Egypt.

dr_magdi_kamel@yahoo.com

M. A. Markos, Faculty of Electronic Engineering, Menoufia University, Cairo, Egypt.

menamarkos90@gmail.com

Abstract **— A new computational technique is given for the numerical solution of Fredholm integral equation of the first kind with a singular density function and a weakly singular logarithmic kernel. The singular density function and the given data function are approximated by using monic Chebyshev polynomials. The singularity of the kernel is treated by expanding the kernel into Maclaurin polynomial of the first degree about the singular parameter. Moreover, an adaptive Gauss-Legendre formula is applied. It turns out from the illustrated example that the presented method clearly simplifies the computations, saves time, and ensures a superior accuracy of the solution.**

Keywords— Monic Chebyshev polynomials; weakly singular; Fredholm integral equations; potential-type equations; numerical solutions.

I. INTRODUCTION

Fredholm boundary integral equations with Cauchy-type or logarithmic-type singular kernels have very important practical applications in many fields; particularly in the fields of potential theory, electron-optics, acoustic wave scattering, electromagnetism, and radiation. This class of equations are actually found to be equivalent to the solution of Dirichlet or Newman boundary value problems for Laplace or Helmholtz equations [1-9]. In fact, for the Dirichlet or Neumann boundary problems for open boundaries, the equivalent boundary integral equations have singular unknown functions in addition to the singularity of the kernels.

We have found many articles [10-15] which solve the aforementioned equation due to the complication caused by the singularity of both the unknown function and the kernel. In paper [14], the author presented a method based on the treatment of the singularity of the unknown function by changing the variables of the singular part and approximated the regular part by Chebyshev polynomials, while the kernel was approximated twice by Chebyshev polynomials and by the change of variables, which complicated the procedure strikingly.

Our goal is to establish an easy technique for the numerical solution, which at the same time treats the singularities of a certain potential-type Fredholm integral equation of the first kind whose unknown function is singular at the end-points of the integration domain, and has a weakly singular logarithmic kernel. The presented method is based on the monic Chebyshev polynomials approximation with an analytical treatment of the singularities. We begin by replacing the unknown function with a product of two functions; the first is a well-behaved unknown function, while the second is a badly-behaved known function. The badly-behaved function represents explicitly the singular behavior of the original unknown function at the end-points of the integration domain. The badly-behaved function itself is factorized into two badly-behaved known functions. Hence, the three previously mentioned functions in addition to the given potential data function are all approximated by monic Chebyshev polynomials of the same degree.

Furthermore, it was necessary to redefine the obtained improper integrals on two intervals so that we can substitute the approximate first badly-behaved function into the first integral, and substitute the approximate second badly-behaved function into the second integral. Thus, we found that the singularity of the unknown function of the considered integral equation is completely disappeared. For the analytical treatment of the singularity of the kernel, we have found method [15], but we provide here another technique without approximating it by using the monic Chebyshev polynomials. We investigated an approach based on the parameterization of the kernel, and by creating two appropriate asymptote formulas, that were obtained by expanding the two parametric functions of the kernel about the singular parameter. In addition to the above, a set of collocation points are applied and an adaptive Gauss-Legendre formula in matrix form is used for the computation of the obtained proper integrals. Thus, the singularity of the kernel is entirely isolated, the complexity of the calculations is minimized and the required numerical solution is reduced to the solution of a system of linear algebraic equations. The computed results are compared with the exact solution to show the advantages of the proposed method.

II. COMPUTATIONAL ALGORITHM

Consider the weakly singular Fredholm integral equation of the first kind

$$\int_{\Gamma} k(x,y) f(x) ds_x = g(y)\ ;\ y \in \Gamma \tag{1}$$

Where $f(x)$ is the unknown function that has a singular behavior at the neighborhood of the end- points of the smooth, simple, and open arc $\Gamma \in R^2$, $g(y)$ is the given potential data function, and $k(x,y)$ is the weakly singular kernel such that $k(x,y) = ln\frac{1}{|d(x,y)|}$; $d(x,y)$ is the distance between the two-points $x(x_1,x_2), y(y_1,y_2)$ on Γ . However, the parameterization of the open arc Γ yields two-parametric equations that represent each of the two points $x(x_1,x_2), y(y_1,y_2)$ respectively as follows

$$\begin{gathered} x_1 = x_1(\alpha), x_2 = x_2(\alpha);\ a \le \alpha \le b; \\ x_1'(\alpha) \ne 0,\ x_2'(\alpha) \ne 0;\ a < \alpha < b, \\ y_1 = y_1(\beta),\ y_2 = y_2(\beta);\ \ a \le \beta \le b; \\ y_1'(\beta) \ne 0,\ y_2'(\beta) \ne 0;\ a < \beta < b, \end{gathered}$$

Hence, (1) becomes

$$\int_a^b F(\alpha) J(\alpha) K(\alpha,\beta) d\alpha = \phi(\beta)\ ;\ a \le \beta \le b \tag{2}$$

where α,β are the two parameters corresponding to the two-points $x(x_1,x_2)$ and $y(y_1,y_2)$ respectively, $J(\alpha) = \left|\sqrt{(x_1'(\alpha))^2 + (x_2'(\alpha))^2}\right|$ is the Jacobian due to the parameterization of the open arc Γ, the kernel takes the form

$$K(\alpha,\beta) = \frac{-1}{2} ln\left|(x_1(\alpha) - y_1(\beta))^2 + (x_2(\alpha) - y_2(\beta))^2\right| \tag{3}$$

By using the linear transformation $\xi = \frac{(\alpha - a)}{b-a} - \frac{(\alpha - b)}{b-a}$,

(2) becomes

$$\int_{-1}^{1} F(\xi) N(\xi) K(\xi,\beta) d\xi = \phi(\beta);\ -1 \le \xi,\beta \le 1 \tag{4}$$

where $N(\xi) = \frac{b-a}{2} J(\alpha)$ is a new Jacobian, $F(\xi)$ denotes the unknown singular function, $\phi(\beta)$ denotes the given potential data function, and $K(\xi,\beta)$ denotes the kernel in the parameterized form

$$K(\xi,\beta) = \frac{-1}{2} ln\left|(x_1(\xi) - y_1(\beta))^2 + (x_2(\xi) - y_2(\beta))^2\right| \tag{5}$$

The unknown singular function $F(\xi)$ of (4) is now replaced by the product of the two functions $\gamma(\xi)\theta(\xi)$ such that

$$F(\xi) = \gamma(\xi)\theta(\xi);\ \gamma(\xi) = \frac{1}{\sqrt{1-\xi^2}} \tag{6}$$

In fact, $\gamma(\xi)$ demonstrates the singular behavior of the unknown function $F(\xi)$ at the end-points $\xi = \pm 1$ and it is axiomatically equal to the weight function of the monic Chebyshev polynomials, while $\theta(\xi)$ is a regular (well-behaved) continuous unknown function to be approximated by using the orthogonal set of the monic Chebyshev polynomials, $\{\tilde{T}_i(x)\}_{i=0}^{n}$, defined on the interval $[-1,\ 1]$. Let

$$\theta(\xi) = \sum_{i=0}^{n} c_i \tilde{T}_i(\xi) \tag{7}$$

where

$$\begin{gathered} \tilde{T}_0(\xi) = 1,\ \tilde{T}_i(\xi) = \frac{1}{2^{i-1}} T_i(\xi); \\ i \ge 1\ ;\ \left\|\tilde{T}_i(\xi)\right\| = \frac{1}{2^{i-1}};\ |\xi| \le 1 \end{gathered} \tag{8}$$

Here $\theta(\xi)$ satisfies the following condition

$$\underset{\text{degree of } \theta(\xi) \le i\ ;\ |\xi| \le 1}{\text{Mini Max}} \left\|\theta(\xi) - \sum_{i=0}^{n} c_i \tilde{T}_i(\xi)\right\| \to 0 \tag{9}$$

where Chebyshev polynomials are defined by

$$T_{i+1}(\xi) = 2\xi T_i(\xi) - T_{i-1}(\xi);\ i \ge 1;\ T_0(\xi) = 1 \tag{10}$$

In matrix form, (7) becomes

$$[\theta(\xi)] = \mathrm{C}\tilde{\mathrm{T}}(\xi);\ -1 \le \xi \le 1 \tag{11}$$

Where $\mathrm{C} = [c_i];\ i = \overline{0,n}$ is the unknown row coefficients matrix of order $1 \times (n+1)$ and $\tilde{\mathrm{T}}(\xi) = [\tilde{T}_i(\xi)];\ i = \overline{0,n}$ is the monic Chebyshev polynomials column matrix of order $(n+1) \times 1$, whose entries $\tilde{T}_i(\xi)$ are defined by (8). Hence, the unknown singular function $F(\xi)$ is transformed into the matrix form

$$[F(\xi)] = \mathrm{CM}(\xi) \tag{12}$$

where $\mathrm{M}(\xi)$ is the column matrix of order $(n+1) \times 1$ such that

$$\mathrm{M}(\xi) = [m_i(\xi)];\ i = \overline{0,n};\ m_i(\xi) = \tilde{T}_i(\xi)\gamma(\xi) \tag{13}$$

Similarly, we approximate the given potential data function $\phi(\beta)$ that was given by Eq. (4) in the matrix form

$$[\phi(\beta)] = \mathrm{H}\tilde{\mathrm{T}}(\beta);\ -1 \le \beta \le 1 \tag{14}$$

Here $\mathrm{H} = [h_i]_{i=0}^{n}$ is the known monic Chebyshev coefficients row matrix of order $1 \times (n+1)$ whose

$$\begin{gathered} h_0 = \frac{1}{\pi}\int_{-1}^{1} \phi(\beta)\gamma(\beta)\tilde{T}_0(\beta) d\beta; \\ h_i = \frac{2^{2i-1}}{\pi}\int_{-1}^{1} \phi(\beta)\gamma(\beta)\tilde{T}_i(\beta) d\beta;\ i \ge 1 \end{gathered} \tag{15}$$

Substituting (12) and (14) into (4), yields

$$\mathrm{CQ}(\xi,\beta) = \mathrm{H}\tilde{\mathrm{T}}(\beta) \tag{16}$$

where $\mathrm{Q}(\xi,\beta) = [q_i(\xi,\beta)]_{i=0}^{n}$ is the column matrix of order $(n+1) \times 1$ whose entries $q_i(\xi,\beta)$ can be computed by

$$q_i(\xi,\beta) = \int_{-1}^{1} \tilde{T}_i(\xi)\gamma(\xi) N(\xi) K(\xi,\beta) d\xi;\ i = \overline{0,n} \tag{17}$$

Now, if the matrix (16) is satisfied at the n-collocation points $\beta_j; j=\overline{0,n}; -1 \le \beta_j \le 1$, then the following algebraic linear system is obtained

$$\tilde{\mathrm{Q}}(\xi,\beta_j)\mathrm{C}^T = \Phi^T(\beta_j)\mathrm{H}^T \tag{18}$$

where the square matrix $\Phi^T(\beta_j) = \left[\tilde{T}_i(\beta_j)\right]_{i,j=0}^n$ is of order $(n+1)\times(n+1)$, and the square matrix $\tilde{\mathrm{Q}}(\xi,\beta_j) = \left[\tilde{q}_i(\xi,\beta_j)\right]_{i,j=0}^n$ is of order $(n+1)\times(n+1)$ whose entries can be computed by

$$\tilde{q}_i(\xi,\beta_j) = \int_{-1}^{1} \gamma(\xi)\tilde{T}_i(\xi)N(\xi)K(\xi,\beta_j)d\xi;\ i,j=\overline{0,n} \tag{19}$$

The integrals $\tilde{q}_i(\xi,\beta_j)$ are improper due to the singularity of the badly-behaved function $\gamma(\xi)$ that was given by (6) when $\xi \to \pm 1$, and because of the singularity of the kernel $K(\xi,\beta_j)$ that was given by (5) when $\xi \to \beta_j$. In order to remedy these singularities for both functions $\gamma(\xi)$, $K(\xi,\beta_j)$, we define the improper integrals $\tilde{q}_i(\xi,\beta_j)$ given by (19) over the interval $[-1, 1]$ as a sum of the two integrals $\tilde{q}_i^1(\xi,\beta_j)$, $\tilde{q}_i^2(\xi,\beta_j)$ the first defined over the interval $[-1, 0]$ and the second over $[0, 1]$. Thus, we have

$$\tilde{q}_i(\xi,\beta_j) = \tilde{q}_i^1(\xi,\beta_j) + \tilde{q}_i^2(\xi,\beta_j);\ i,j=\overline{0,n} \tag{20}$$

where

$$\left.\begin{aligned} \tilde{q}_i^1(\xi,\beta_j) &= \int_{-1}^{0} \gamma(\xi)\tilde{T}_i(\xi)N(\xi)K(\xi,\beta_j)d\xi, \\ \tilde{q}_i^2(\xi,\beta_j) &= \int_{0}^{1} \gamma(\xi)\tilde{T}_i(\xi)N(\xi)K(\xi,\beta_j)d\xi \end{aligned}\right\} \tag{21}$$

Moreover, by factorization of the function $\gamma(\xi) = \frac{1}{\sqrt{1-\xi^2}}$ into the two functions $\omega(\xi)$, $\psi(\xi)$ such that $\omega(\xi) = \frac{1}{\sqrt{1-\xi}}$, $\psi(\xi) = \frac{1}{\sqrt{1+\xi}}$, and approximating both of them by using the monic Chebyshev polynomials $\left\{\tilde{T}_i(t)\right\}_{i=0}^n$, we get

$$\psi(\xi) = \sum_{i=0}^{n} a_i\tilde{T}_i(\xi),\ \omega(\xi) = \sum_{i=0}^{n} b_i\tilde{T}_i(\xi) \tag{22}$$

where the known coefficients a_i, b_i are computed by

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-1}^{1} \gamma(\xi)\psi(\xi)\tilde{T}_0(\xi)\,d\xi, \\ a_i &= \frac{2^{2i-1}}{\pi}\int_{-1}^{1} \gamma(\xi)\psi(\xi)\tilde{T}_i(\xi)d\xi;\ i \ge 1 \\ b_0 &= \frac{1}{\pi}\int_{-1}^{1} \gamma(\xi)\psi(\xi)\tilde{T}_0(\xi)\,d\xi, \\ b_i &= \frac{2^{2i-1}}{\pi}\int_{-1}^{1} \gamma(\xi)\psi(\xi)\tilde{T}_i(\xi)d\xi;\ i \ge 1 \end{aligned}\right\} \tag{23}$$

In view of (22), and by replacing $\gamma(\xi)$ by the product $\omega(\xi)\sum_{i=0}^{n} a_i\tilde{T}_i(\xi)$ in $\tilde{q}_i^1(\xi,\beta_j)$, and by replacing $\gamma(\xi)$ by the product $\psi(\xi)\sum_{i=0}^{n} b_i\tilde{T}_i(\xi)$ in $\tilde{q}_i^2(\xi,\beta_j)$, the integrals $\tilde{q}_i(\xi,\beta_j)$ that was given by (21) are transformed to $\tilde{q}_{sj}(\xi,\beta_j)$ as follows

$$\tilde{q}_{sj}(\xi,\beta_j) = \tilde{q}_{sj}^1(\xi,\beta_j) + \tilde{q}_{sj}^2(\xi,\beta_j);\ s,j=\overline{0,n} \tag{24}$$

where

$$\left.\begin{aligned} \tilde{q}_{sj}^1(\xi,\beta_j) &= \sum_{i=0}^{n}(a_i)\int_{-1}^{0} \omega(\xi)N(\xi)\tilde{T}_s(\xi)\tilde{T}_i(\xi)K(\xi,\beta_j)d\xi \\ \tilde{q}_{sj}^2(\xi,\beta_j) &= \sum_{i=0}^{n}(b_i)\int_{0}^{1} \psi(\xi)N(\xi)\tilde{T}_s(\xi)\tilde{T}_i(\xi)K(\xi,\beta_j)d\xi \end{aligned}\right\} \tag{25}$$

Thus, the system that was given by (18) is transformed to the new one

$$\tilde{\tilde{\mathrm{Q}}}(\xi,\beta_j)\mathrm{C}^T = \Phi^T(\beta_j)\mathrm{H}^T \tag{26}$$

where $\tilde{\tilde{\mathrm{Q}}}(\xi,\beta_j) = \left[\tilde{q}_{sj}(\xi,\beta_j)\right]_{s,j=0}^n$ is of order $(n+1)\times(n+1)$ matrix whose entries $\tilde{q}_{sj}(\xi,\beta_j)$ were given by (24) and (25). From this mathematical treatment, it turns out that, the singularity of the unknown function $F(\xi)$ has completely disappeared. The singularity of the kernel

$$K(\xi,\beta_j) = \frac{-1}{2}ln\left|\left(x_1(\xi)-y_1(\beta_j)\right)^2 + \left(x_2(\xi)-y_2(\beta_j)\right)^2\right| \tag{27}$$

when $\xi \to \beta_j$ is now treated by expanding each of the two-parametric functions $x_1(\xi)$, $x_2(\xi)$ into Maclaurin polynomial of the first degree about the singular parameter (collocation point), β_j, to get

$$\left.\begin{aligned} x_1(\xi) &= x_1(\beta_j) + (\xi-\beta_j)x_1'(\beta_j); \\ x_2(\xi) &= x_2(\beta_j) + (\xi-\beta_j)x_2'(\beta_j) \end{aligned}\right\} \tag{28}$$

Substituting (28) into (27), the kernel $K(\xi,\beta_j)$ can be replaced by the approximated kernel $\tilde{k}(\xi,\beta_j)$ such that

$$\tilde{k}\left(\xi,\beta_j\right) = ln\frac{1}{R\left|\xi-\beta_j\right|};\ a \le \xi \le b,$$
$$R = \left|\sqrt{\left(x_1'\left(\beta_j\right)\right)^2 + \left(x_2'\left(\beta_j\right)\right)^2}\right| \tag{29}$$

Now, by adding and subtracting to the integrals $\tilde{q}^1_{sj}\left(\xi,\beta_j\right)$ the asymptote expression $\omega(\xi)N(\xi)\tilde{T}_s(\xi)\tilde{T}_i(\xi)\tilde{K}\left(\xi,\beta_j\right)$, and to the integrals $\tilde{q}^2_{sj}\left(\xi,\beta_j\right)$ the asymptote expression $\psi(\xi)N(\xi)\tilde{T}_s(\xi)\tilde{T}_i(\xi)\tilde{K}\left(\xi,\beta_j\right)$, and by taking into account the following two limits

$$\left.\begin{aligned}&\lim_{\xi\to\beta_j}\left[\omega(\xi)N(\xi)\tilde{T}_s(\xi)\tilde{T}_i(\xi)\left[K\left(\xi,\beta_j\right)-\tilde{K}\left(\xi,\beta_j\right)\right]\right]\to 0\\&\lim_{\xi\to\beta_j}\left[\psi(\xi)N(\xi)\tilde{T}_s(\xi)\tilde{T}_i(\xi)\left[K\left(\xi,t_j\right)-\tilde{K}\left(\xi,\beta_j\right)\right]\right]\to 0\end{aligned}\right\} \tag{30}$$

we find that

$$\left.\begin{aligned}&\tilde{q}^1_{sj}\left(\xi,\beta_j\right)=\\&\sum_{i=0}^{n}(a_i)\int_{-1}^{0}\omega(\xi)N(\xi)\tilde{T}_s(\xi)\tilde{T}_i(\xi)ln\frac{1}{R\left|\xi-\beta_j\right|}d\xi\\&\tilde{q}^2_{sj}\left(\xi,\beta_j\right)=\\&\sum_{i=0}^{n}(b_i)\int_{0}^{1}\psi(\xi)N(\xi)\tilde{T}_s(\xi)\tilde{T}_i(\xi)ln\frac{1}{R\left|\xi-\beta_j\right|}d\xi\end{aligned}\right\} \tag{31}$$

Thus, the singularity of the kernel is entirely isolated and the proper integralas $\tilde{q}^1_{sj}\left(\xi,\beta_j\right)$, $\tilde{q}^2_{sj}\left(\xi,\beta_j\right)$ can be computed using the adaptive $m-$nodes Gauss-Legendre formula

$$\int_a^b f(x)dx = \sum_{s=1}^{m}\delta_s f(\omega_s) \tag{32}$$

Thereby, by solving the algebraic linear system (26), where the integrals $\tilde{q}^1_{sj}\left(\xi,\beta_j\right)$, $\tilde{q}^2_{sj}\left(\xi,\beta_j\right)$ were given by equation (31) we can find the unknown singular function $F(\xi)$ of (4).

III. COMPUTATIONAL RESULTS

In this section, we will consider a numerical example to demonstrate the efficiency of the proposed scheme, while the comparative results with the exact solution are given to justify the superior behavior of the proposed method. Tables of the obtained numerical solutions and the graphs are illustrated with the absolute error estimation.

Example: Consider the Fredholm integral equation of the first kind

$$\int_{-1}^{1} f(x)\,ln\frac{1}{|x-t|}dx = 1;\ -1\le t\le 1 \tag{33}$$

whose exact solution [14] is given by

$$f(x) = \frac{1}{\left(\pi ln(2)\right)\sqrt{1-x^2}};\ -1<x<1 \tag{34}$$

In table 1, the obtained numerical solutions by using the monic Chebyshev method for $n=4,\ 6,\ 8$ respectively are compared with the exact solution $E_i(x_i)$. In table 2, the absolute errors estimation for the monic Chebyshev solutions for $n=4,\ 6,\ 8$ respectively. In Figures 1, 2 and 3, plotted the exact solution and the monic Chebyshev numerical solutions for $n=4,\ 6,\ 8$ respectively.

TABLE I. A comparison between the exact solution denoted by $E_i(x_i)$ with the monic Chebyshev numerical solutions for $n=4,\ 6,\ 8$ respectively.

i	x_i	$E_i(x_i)$	$n=4$	$n=6$	$n=8$
0	-0.9000	1.0535	1.0533	1.0536	1.0536
1	-0.8000	0.7654	0.7652	0.7654	0.7654
2	-0.7000	0.6430	0.6430	0.6431	0.6430
3	-0.6000	0.5740	0.5740	0.5740	0.5739
4	-0.5000	0.5303	0.5303	0.5302	0.5301
5	-0.4000	0.5011	0.5011	0.5010	0.5009
6	-0.3000	0.4814	0.4814	0.4814	0.4813
7	-0.2000	0.4687	0.4687	0.4687	0.4687
8	-0.1000	0.4615	0.4616	0.4616	0.4616
9	0.0000	0.4592	0.4593	0.4593	0.4593
10	0.1000	0.4615	0.4616	0.4616	0.4616
11	0.2000	0.4687	0.4687	0.4687	0.4687
12	0.3000	0.4814	0.4814	0.4814	0.4813
13	0.4000	0.5011	0.5011	0.5010	0.5009
14	0.5000	0.5303	0.5303	0.5302	0.5301
15	0.6000	0.5740	0.5740	0.5740	0.5739
16	0.7000	0.6430	0.6430	0.6431	0.6430
17	0.8000	0.7654	0.7652	0.7654	0.7654
18	0.9000	1.0535	1.0533	1.0536	1.0536

TABLE II. Absolute errors estimation, where E_1, E_2 and E_3 denote the absolute errors for $n=4,\ 6,\ 8$ respectively.

i	x_i	E_1	E_2	E_3
0	0.0000	4.2855E-05	3.7528E-05	1.1309E-04
1	0.1000	4.1177E-05	3.2515E-05	8.9655E-05
2	0.2000	3.6046E-05	1.8908E-05	2.7514E-05
3	0.3000	2.7145E-05	8.3262E-07	4.8951E-05
4	0.4000	1.3860E-05	1.5417E-05	1.0913E-04
5	0.5000	4.9160E-06	2.2353E-05	1.2924E-04
6	0.6000	3.1234E-05	1.2920E-05	1.0556E-04
7	0.7000	6.9294E-05	1.6709E-05	6.0162E-05
8	0.8000	1.2959E-04	6.1565E-05	2.0481E-05
9	0.9000	2.5188E-04	9.1791E-05	1.0513E-04

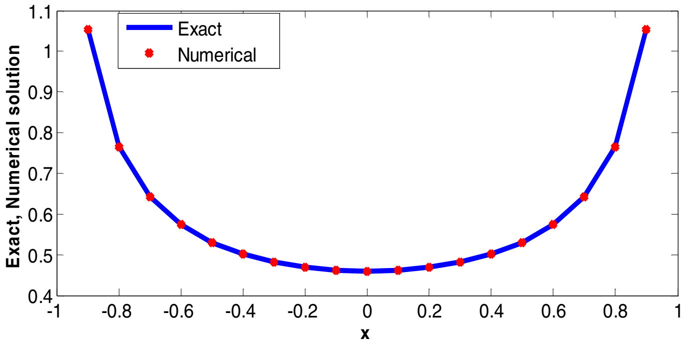

Fig. 1. The exact solution and the monic Chebyshev numerical solution for $n = 4$.

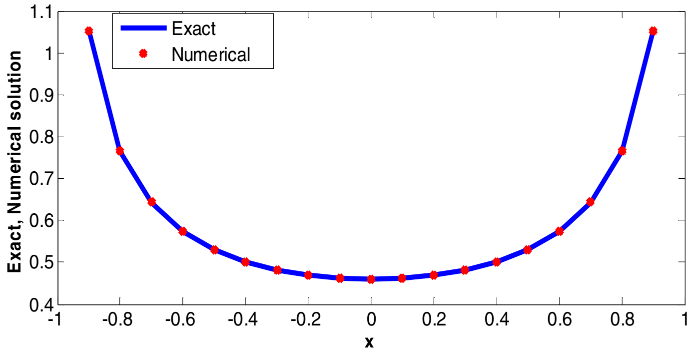

Fig. 2. The exact solution and the monic Chebyshev numerical solution for $n = 6$.

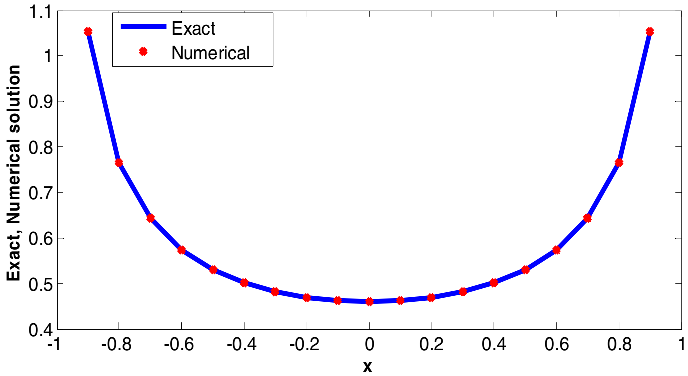

Fig. 3. The exact solution and the monic Chebyshev numerical solution for $n = 8$.

IV. CONCLUSION

The presented computational method solves a certain class of a potential-type Fredholm integral equation of the first kind whose unknown function is singular at the end-points of the integration domain, and has a weakly singular logarithmic kernel. We considered the unknown function as a product of three functions; the first is a well-behaved unknown function, while the two others are badly-behaved functions. These three functions and the given potential data function are all approximated by using the monic Chebyshev polynomials of the same degree. Moreover, we improved the approach based on the parameterization of the kernel and the four above-defined approximated functions by redefining the domain of the improper integral of the integral equation into two equal sub-domains in such a manner that the first badly-behaved function is inserted in the first integral, while the second is inserted in the second integral. Thus, the singularity of the unknown function was entirely isolated. Furthermore, we adopted two appropriate asymptote formulas that were obtained by expanding each of the two parametric functions of the kernel about the singular parameter so that the singularity of the kernel is perfectly eliminated. Additionally, the perfect choice of a set of collocation points reduces the solution of the considered integral equation to the solution of a linear system of algebraic equations, where an adaptive Gauss-Legendre formula was used for the computation of the obtained convergent integrals. The obtained numerical solutions underscore high accuracy compared with the exact solutions.

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