Bayes-rule

Bayes' Theorem is a way of finding a probability when we know certain other probabilities.

The formula is:

$$P(A|B) = \frac{P(A)\,P(B|A)}{P(B)}$$

Which tells us: how often A happens *given that B happens*, written **P(A|B)**,
When we know: how often B happens *given that A happens*, written **P(B|A)**
and how likely A is on its own, written **P(A)**
and how likely B is on its own, written **P(B)**

Examples:

1)

Let us say P(Fire) means how often there is fire, and P(Smoke) means how often we see smoke, then:

P(Fire|Smoke) means how often there is fire when we can see smoke
P(Smoke|Fire) means how often we can see smoke when there is fire

So the formula kind of tells us "forwards" P(Fire|Smoke) when we know "backwards" P(Smoke|Fire)

Example:

- dangerous fires are rare (1%)
- but smoke is fairly common (10%) due to barbecues,
- and 90% of dangerous fires make smoke

We can then discover the **probability of dangerous Fire when there is Smoke**:

$$P(\text{Fire}|\text{Smoke}) = \frac{P(\text{Fire})\,P(\text{Smoke}|\text{Fire})}{P(\text{Smoke})} = \frac{1\% \times 90\%}{10\%} = 9\%$$

So it is still worth checking out any smoke to be sure.

2)

Being General

Why does it work?

Let us replace the numbers with letters:

	B	notB	
A	s	t	s+t
notA	u	v	u+v
	s+u	t+v	s+t+u+v

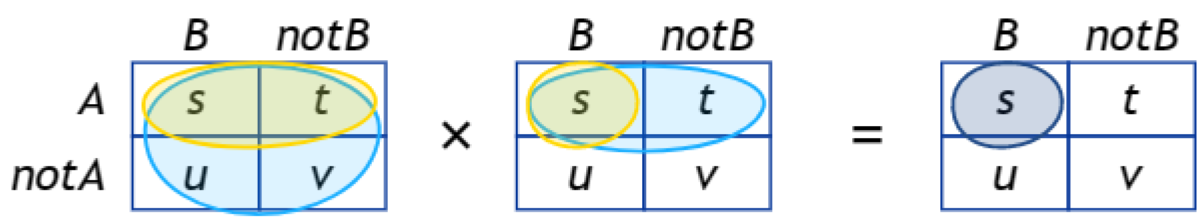
Now let us look at **probabilities**. So we take some ratios:

- the overall probability of "A" is P(A) = $\frac{s+t}{s+t+u+v}$
- the probability of "B given A" is P(B|A) = $\frac{s}{s+t}$

And then multiply them together like this:

$$P(A) \times P(B|A) = P(A)\,P(B|A)$$

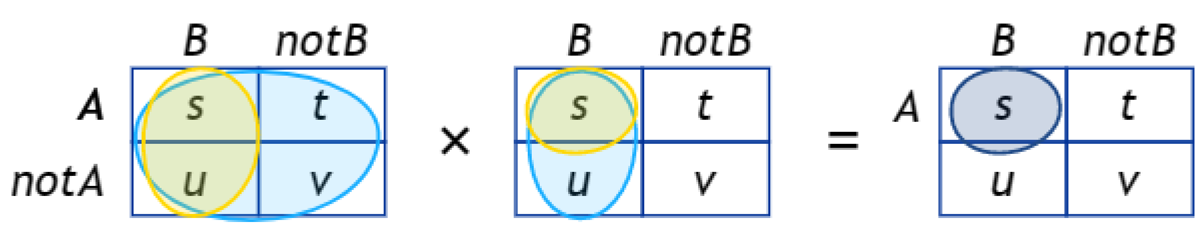
$$\frac{s+t}{s+t+u+v} \times \frac{s}{s+t} = \frac{s}{s+t+u+v}$$



Now let us do that again but use **P(B)** and **P(A|B)**:

$$P(B) \times P(A|B) = P(B)\,P(A|B)$$

$$\frac{s+u}{s+t+u+v} \times \frac{s}{s+u} = \frac{s}{s+t+u+v}$$



Both ways get the **same result** of $\frac{s}{s+t+u+v}$

So we can see that:

$$P(B)\ P(A|B) = P(A)\ P(B|A)$$

Nice and symmetrical isn't it?

It actually *has* to be symmetrical as we can swap rows and columns and get the same top-left corner.

And it is also **Bayes Formula** ... just divide both sides by P(B):

$$P(A|B) = \frac{P(A)\ P(B|A)}{P(B)}$$

Remembering

First think "AB AB AB" then remember to group it like: "AB = A BA / B"

$$P(A|B) = \frac{P(A)\ P(B|A)}{P(B)}$$

Prior: $P(C) = 0.01 = 1\%$ $\quad P(\neg C) = 0.99$

$P(Pos|C) = 0.9 = 90\%$ $\quad P(Pos|\neg C) = 0.1$

$P(Neg|\neg C) = 0.9$

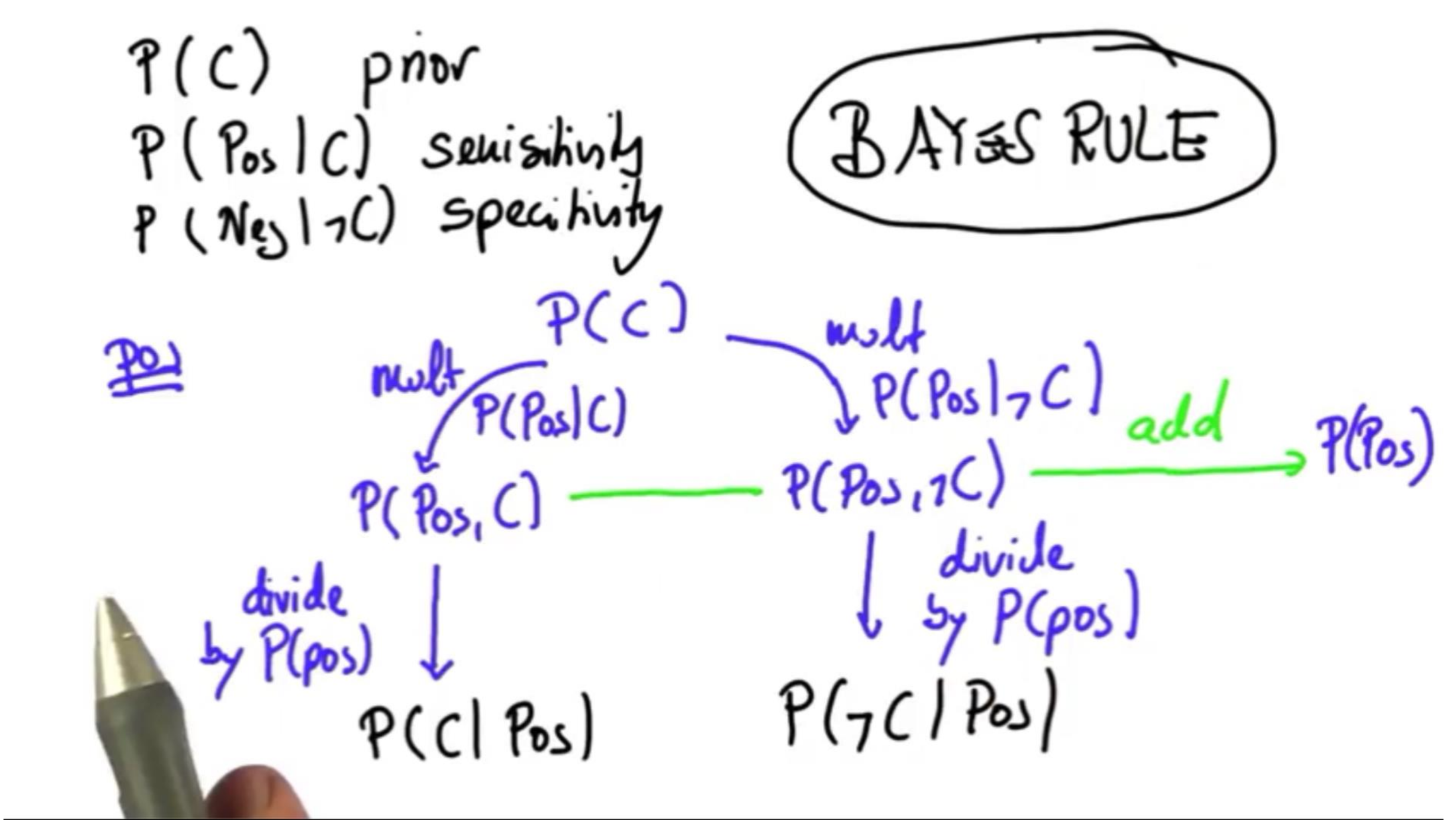
~~posterior~~ joint: $P(C, Pos) = P(C) \cdot P(Pos|C) = 0.009$

$P(\neg C, Pos) = P(\neg C) \cdot P(Pos|\neg C) = 0.099$

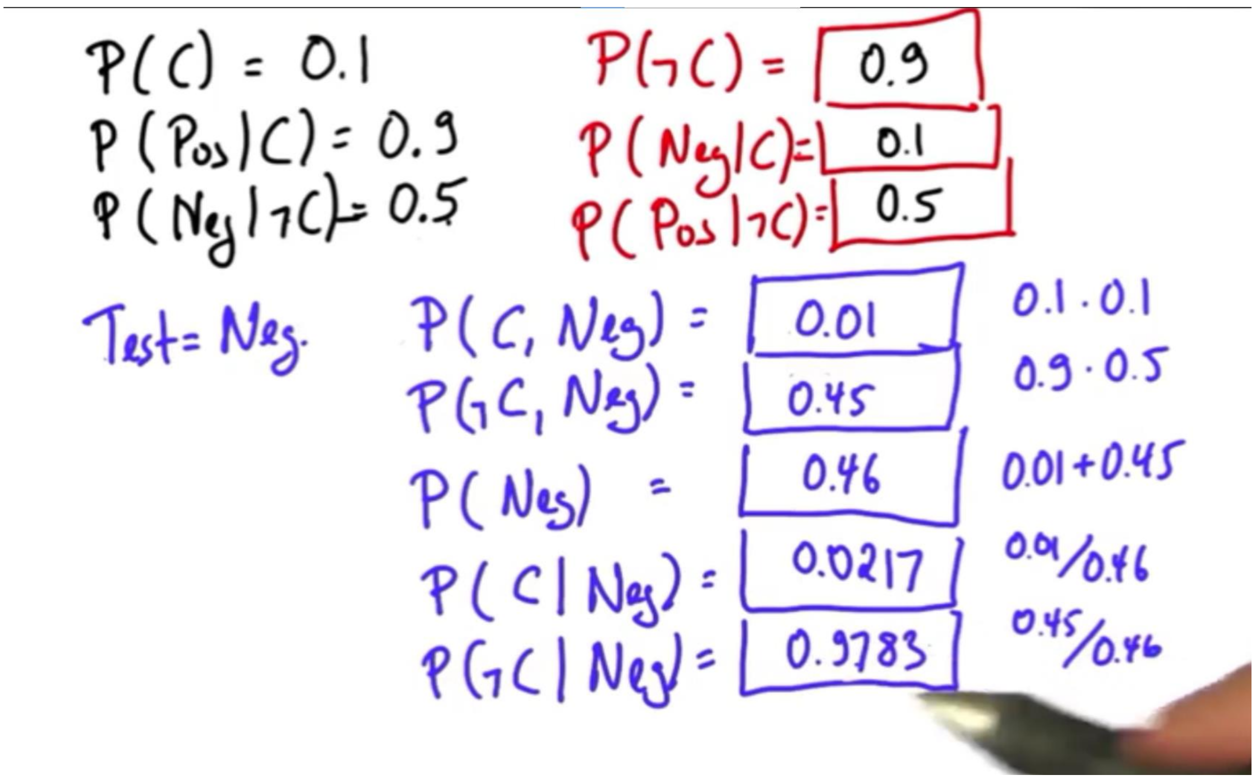
normalizer: $P(Pos) = P(C, Pos) + P(\neg C, Pos) = 0.108$

posterior:

$$\left.\begin{aligned} P(C|Pos) &= 0.0833 \\ P(\neg C|Pos) &= 0.9167 \end{aligned}\right\} = \boxed{1}$$

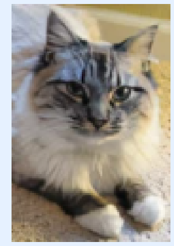


3) P(c) = cancer --- p(pos) = positive ---- p(neg) = negative --- p(<C) == no cancer



4)

Example: Allergy or Not?



Hunter says she is itchy. There is a test for Allergy to Cats, but this test is not always right:

- For people that **really do** have the allergy, the test says "Yes" **80%** of the time
- For people that **do not** have the allergy, the test says "Yes" **10%** of the time ("false positive")

If 1% of the population have the allergy, and **Hunter's test says "Yes"**, what are the chances that Hunter really has the allergy?

We want to know the chance of having the allergy when test says "Yes", written **P(Allergy|Yes)**

Let's get our formula:

$$P(\text{Allergy}|\text{Yes}) = \frac{P(\text{Allergy})\,P(\text{Yes}|\text{Allergy})}{P(\text{Yes})}$$

- P(Allergy) is Probability of Allergy = 1%
- P(Yes|Allergy) is Probability of test saying "Yes" for people with allergy = 80%
- P(Yes) is Probability of test saying "Yes" (to anyone) = ??%

Oh no! We **don't know** what the **general** chance of the test saying "Yes" is ...

... but we can calculate it by adding up those **with**, and those **without** the allergy:

- 1% have the allergy, and the test says "Yes" to 80% of them
- 99% do **not** have the allergy and the test says "Yes" to 10% of them

Let's add that up:

$$P(\text{Yes}) = 1\% \times 80\% + 99\% \times 10\% = 10.7\%$$

Which means that about 10.7% of the population will get a "Yes" result.

So now we can complete our formula:

$$P(\text{Allergy}|\text{Yes}) = \frac{1\% \times 80\%}{10.7\%} = 7.48\%$$

$$P(\text{Allergy}|\text{Yes}) = \text{about } \mathbf{7\%}$$

This is the same result we got on False Positives and False Negatives.

In fact we can write a special version of the Bayes' formula just for things like this:

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(\text{not }A)P(B|\text{not }A)}$$

5)

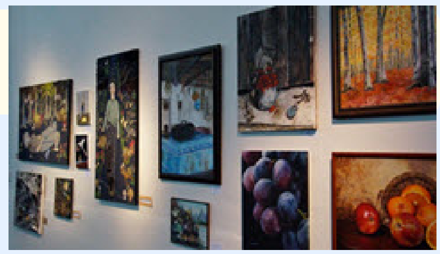
"A" With Three (or more) Cases

We just saw "A" with two cases (A and not A), which we took care of in the bottom line.

When "A" has 3 or more cases we include them all in the bottom line:

$$P(A1|B) = \frac{P(A1)P(B|A1)}{P(A1)P(B|A1) + P(A2)P(B|A2) + P(A3)P(B|A3) + \ldots\text{etc}}$$

Example: The Art Competition has entries from three painters: Pam, Pia and Pablo



- Pam put in 15 paintings, 4% of her works have won First Prize.
- Pia put in 5 paintings, 6% of her works have won First Prize.
- Pablo put in 10 paintings, 3% of his works have won First Prize.

What is the chance that Pam will win First Prize?

$$P(Pam|First) = \frac{P(Pam)P(First|Pam)}{P(Pam)P(First|Pam) + P(Pia)P(First|Pia) + P(Pablo)P(First|Pablo)}$$

Put in the values:

$$P(Pam|First) = \frac{(15/30) \times 4\%}{(15/30) \times 4\% + (5/30) \times 6\% + (10/30) \times 3\%}$$

Multiply all by 30 (makes calculation easier):

$$P(Pam|First) = \frac{15 \times 4\%}{15 \times 4\% + 5 \times 6\% + 10 \times 3\%}$$

$$= \frac{0.6}{0.6 + 0.3 + 0.3}$$

$$= 50\%$$

A good chance!

Pam isn't the most successful artist, but she did put in lots of entries.