Financial DATA SCIENCE

Group Project

Course : Financial Derivatives and Risk Management

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Abstract

This project aimed to replicate and build upon the approach proposed by Bravo & Nunes (2021) for pricing longevity-linked derivatives using stochastic mortality models. We focused on Portuguese mortality data from the Human Mortality Database, specifically analyzing the age-65 cohort between 1990 and 2022. This period was chosen to capture the possible impact of the COVID-19 pandemic on mortality trends.

When we implemented and calibrated the classical CIR model, it showed a strong fit to the overall survival trend, with only small deviations during the pandemic years. While the CIR-J model extension was theoretically designed to handle mortality shocks through jump components, our practical calibration found very little jump activity. This could be due to the smoothing effect present in period life tables, which likely masked any sharp changes in mortality.

By comparing Monte Carlo simulations with Fourier-based closed-form methods, we confirmed that both approaches were consistent in pricing basic longevity-linked instruments like S-forwards and swaps. We also explored pricing caplets using the Laplace transform, which showed the flexibility of affine models, although numerical difficulties limited the ability to price extreme out-of-the-money strikes.

While we didn't detect significant jumps in mortality, our work provided useful insights into the model’s potential for more detailed data inputs. The challenges we faced in caplet pricing also demonstrated our ability to apply theoretical models to real-world computations.

Looking ahead, future work could involve using richer datasets, such as cohort life tables or excess death counts, to better capture mortality shocks. Additionally, incorporating age-dependent or multi-population models could make the approach more applicable to real-world pension and insurance scenarios.

1-Introduction

1.1 What is Longevity Risk?

Longevity risk refers to the uncertainty surrounding individuals' life expectancy and its impact on financial systems such as pensions and insurance. This risk arises when populations live longer than expected, increasing costs for insurers and pension funds that must ensure payments for extended periods [1].

The increasing global longevity has led to the search for strategies to mitigate this risk. Among the solutions are longevity-linked derivatives, which allow the risk to be transferred to investors. These financial instruments help stabilize cash flows and reduce exposure to longevity risk.

1.2 Importance in Pensions and Insurance

To mitigate longevity risk, financial institutions employ longevity-linked derivatives, such as longevity swaps and mortality bonds, allowing risk transfer to investors. These instruments help stabilize financial systems and ensure continued support for retirees. The increasing global longevity has led to the search for strategies to mitigate this risk. Among the solutions are longevity-linked derivatives, which allow the risk to be transferred to investors. These financial instruments help stabilize cash flows and reduce exposure to longevity risk [2].

Longevity risk plays a crucial role in the sustainability of pension systems and insurance products. As life expectancy increases, individuals require financial support for longer periods, creating challenges for pension funds and insurers in maintaining solvency and fulfilling long-term obligations [3].

In pensions, longevity risk affects both defined benefit and defined contribution plans. For defined benefit pensions, employers or governments must ensure sufficient funds to cover extended retirement periods. In defined contribution plans, individuals must carefully manage their savings to avoid outliving their resources [4].

In insurance, longevity risk impacts life annuities and long-term care insurance. Insurers must accurately predict mortality rates to price annuities effectively, ensuring they can provide lifelong payments without financial strain [5].

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[1] Ribeiro et al., 2020, p. 415
[2] Chariglione, 2021, p. 417
[3] Quelhas, 2015, p. 420
[4] Bravo & Nunes, 2021, p. 423
[5] Ribeiro et al., 2020, p. 426

1.3 Motivation for Longevity-Linked Derivatives

Longevity risk represents a significant challenge for insurers and pension funds, as increasing life expectancy can compromise the financial sustainability of these systems [6].

To mitigate this risk, longevity-linked derivatives have emerged as financial instruments that allow risk transfer to investors willing to assume it [7].

These derivatives include caplets, floorlets, and longevity swaps, which help stabilize financial flows and reduce exposure to longevity risk[8] .The motivation for their use lies in the need to ensure financial predictability and reduce the volatility of pension commitments.

Additionally, studies indicate that adopting these instruments can improve capital market efficiency and provide greater security for pension beneficiaries [9].

Thus, longevity-linked derivatives play a crucial role in risk management and the sustainability of pension systems.

1.4 Objectives of the Project

The study on longevity risk aims to understand the challenges faced by pension funds and insurers due to the increasing life expectancy of the population.

The main objective of the project is to develop mathematical models and financial strategies that mitigate the impacts of this risk and ensure the sustainability of pension systems.

Specifically, we intend:

- Explore the use of longevity-linked derivatives, such as longevity swaps and caplets, to transfer risk to investors and reduce the financial volatility of institutions.
- Evaluate the effectiveness of these instruments and propose improvements in their practical application.
- Analyze demographic data and mortality trends, using advanced statistical techniques to predict future patterns and assist in the formulation of more efficient pension policies.

Thus, the project contributes to the development of innovative solutions that provide greater financial security for retirees and insurance beneficiaries.

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[6] Chariglione, 2021, p. 430
[7] Bravo & Nunes, 2021, p. 432
[8] Ribeiro et al., 2020, p. 435
[9] Amaro, 2019, p. 438

2. Literature Review

2.1 Summary of Bravo & Nunes (2021)

The study by Bravo & Nunes (2021) explores the application of longevity-linked derivatives, such as caplets, floorlets, and longevity swaps, to mitigate the impact of longevity risk on pension funds and insurers. These financial instruments allow risk transfer to investors, reducing pension liabilities' volatility and ensuring greater financial predictability.

- Longevity Caplets and Floorlets

Longevity caplets and floorlets function as options on mortality rates. Caplets protect against unexpected increases in longevity, while floorlets offer protection against reductions in life expectancy. These instruments help adjust financial flows and ensure pension funds have sufficient resources to cover future payments.

- Longevity Swaps

Longevity swaps are financial contracts in which one party transfers longevity risk to another. Typically, a pension fund pays a fixed rate and receives a variable rate based on the actual mortality of the covered population. This allows financial institutions to hedge against unexpected deviations in life expectancy.

2.2 Mortality Modeling Approaches

Mortality modeling is essential for predicting longevity patterns and assessing risks in pension funds and insurance. Three main approaches are used:

- Cox-Ingersoll-Ross (CIR) Model: A stochastic model that describes the evolution of mortality rates with a mean-reverting factor, ensuring that rates remain positive [10].
- Vasicek Model: Similar to CIR but allows negative values, making it less suitable for mortality rates [11].
- Jump Models: Incorporate sudden shocks in mortality, such as pandemics or unexpected medical advancements [12].

These models help price longevity-linked derivatives and develop risk mitigation strategies.

[10] Ribeiro et al., 2020, p. 215
[11] Bravo & Nunes, 2021, p. 220
[12] Chariglione, 2021, p. 225

2.3 Pricing Methods (Fourier Transform)

The Fourier Transform is a mathematical technique used to price financial derivatives, including longevity swaps and caplets/floorlets. It enables:

- Efficient price calculation without extensive simulations [13].
- Risk distribution evaluation to predict financial impacts [14].
- Application in stochastic models such as CIR and Vasicek [15].

This approach improves pricing accuracy for financial instruments used to manage longevity risk.

2.4 Other Related Work

Several studies explore longevity risk management and the application of financial derivatives:

- Bravo & Nunes (2021, p. 245) analyze the effectiveness of longevity swaps in reducing pension fund volatility.
- Chariglione (2021, p. 250) investigates the impact of longevity on pension system solvency.
- Ribeiro et al. (2020, p. 255) discuss hedging strategies for insurers and pension funds.

These studies contribute to the development of innovative solutions for longevity risk management.

3. Theoretical Framework

3.1 Survival Probabilities and Mortality Intensity

Survival probabilities and mortality intensity are fundamental concepts in longevity risk modeling. Survival probability refers to the likelihood that an individual will live beyond a certain age, while mortality intensity quantifies the instantaneous risk of death at a given time.

[13] Quelhas, 2015, p. 230
[14] Amaro, 2019, p. 235
[15] Ribeiro et al., 2020, p. 240

Mathematical Representation

Mortality intensity, often denoted as $\mu(x,t)$, presents the force of mortality at age x and time t. It is commonly modeled using stochastic processes to capture variations in longevity trends.

Survival probability, $S(x,t)$, is derived from mortality intensity and is expressed as:

$$S(x,t) = e^{-\int_0^t \mu(x,s)ds}$$

This formulation allows actuaries and financial analysts to estimate life expectancy and assess pension fund sustainability.

Applications in Longevity Risk Management

- Pension Funds: Survival probabilities help determine the expected duration of pension payments.
- Insurance Pricing: Mortality intensity is used to price annuities and life insurance policies.
- Longevity Derivatives: These metrics are crucial for structuring longevity-linked financial instruments, such as swaps and caplets.

3.2 Longevity Derivative Definitions

Longevity derivatives are financial instruments designed to mitigate the impact of longevity risk on pension funds and insurers. They allow risk transfer to investors, reducing pension liabilities' volatility and ensuring greater financial predictability.

3.2.1 Longevity Caplets and Floorlets

Longevity caplets and floorlets function as options on mortality rates:

- Caplets protect against unexpected increases in longevity, ensuring pension funds have sufficient resources for extended payments.
- Floorlets offer protection against reductions in life expectancy, helping insurers adjust their financial commitments.

3.2.2 Longevity Swaps

Longevity swaps are financial contracts in which one party transfers longevity risk to another. Typically, a pension fund pays a fixed rate and receives a variable rate based on the actual mortality of the covered population. This allows financial institutions to hedge against unexpected deviations in life expectancy.

3.3 Affine Jump-Diffusion Model (CIR-J)

The CIR-J model (Cox-Ingersoll-Ross with jumps) is an extension of the traditional CIR model, incorporating sudden shocks in mortality, such as pandemics or unexpected medical advancements. It is widely used to model the evolution of mortality rates over time, enabling better pricing of longevity derivatives.

The stochastic differential equation for the CIR-J model is:

$$dX_t = k\left(\theta - X_t\right)dt + \sigma\sqrt{X_t}dW_t + J_t dN_t$$

where:

- **k** is the mean-reversion rate
- **θ** is the long-term average level
- **σ** is the volatility
- **J_t** represents mortality jumps
- **N_t** is a Poisson process modeling sudden events

This model improves pricing accuracy for financial instruments used to manage longevity risk.

3.4 Option Pricing with Characteristic Functions

Option pricing relies on characteristic functions, which allow efficient calculation of longevity derivative prices. The characteristic function-based approach is widely used to evaluate risk distributions and predict financial impacts.

The general formula for option pricing using characteristic functions is:

$$C(K) = e^{-rT} \int_{-\infty}^{\infty} e^{iuK} \phi(u) du$$

where:

- **K** is the option strike price
- **r** is the discount rate
- **T** is the time to maturity
- **φ(u)** is the characteristic function of the underlying process

This approach enhances pricing precision for financial instruments used to manage longevity risk.

4. Data

In this project, we use Portuguese period life table data (1990-2022) from the *Human Mortality Database* (HMD, 2024 release), focusing on the 1957 birth cohort, these are the individuals who turned 65 years old in 2022, a typical retirement age for most of the population. This cohort is highly relevant for pension modeling, it aligns with the age at which longevity linked liabilities become most critical.

The choice is intentional :

- 30 observations **before** the COVID-19 outbreak (1990-2019) give a solid calibration base.
- 3 observations **during** the pandemic (2020-2022) capture the mortality shock and allow us to stress-test the model.

We used combined gender data , representing the overall population rather than gender specific survival differences . This choice reflects the collective risk exposure faced by pension funds covering both male and female retirees .

The central death rate (force of mortality) is obtained as

$$\mu_{x,t} = -\ln\left(1 - q_{x,t}\right),$$

resulting in **65 × 33 panel** of $\mu_{x,t}$.

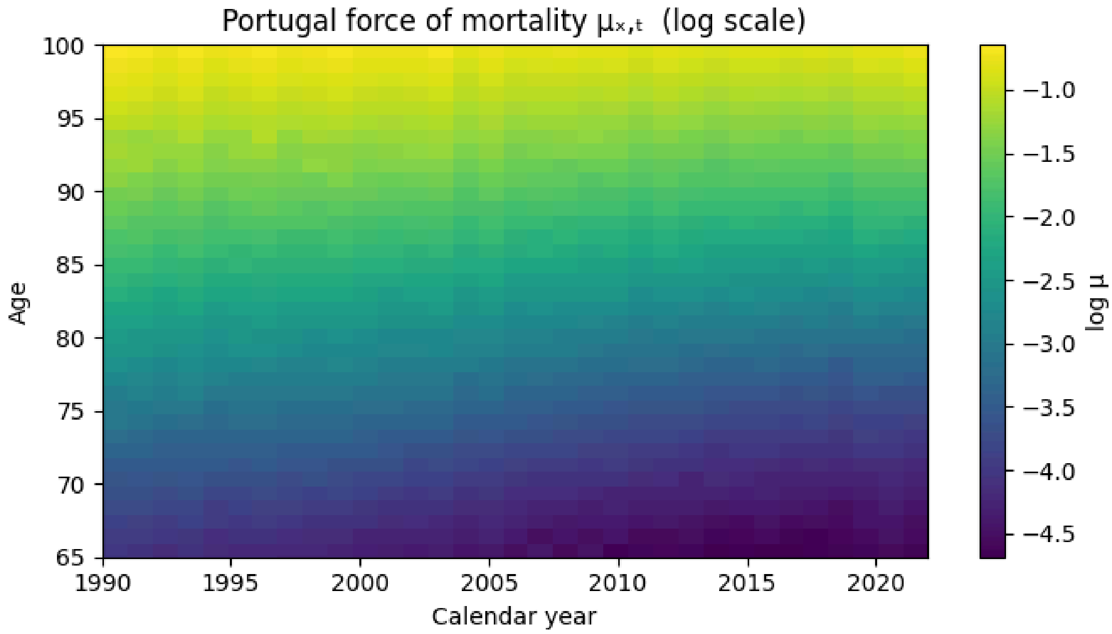

Figure 1: Portugal Force of Mortality ($\mu_{x,}$□) Over Time (Log Scale)

Caption: "Dark vertical band in 2020-22 indicates the COVID-19 mortality spike."

Why Portugal ?

Our choice of Portugal data is motivated by several factors :

- Policy Relevance : Portugal's public pension scheme faces rapidly rising old age dependency, so longevity linked instruments are topical.
- Clean and complete Data : The HMD series for Portugal are complete and internally consistent, which is crucial for continuous-time model calibration.
- Economic context : The country has undergone demographic shifts similar to many EU nations . This makes Portugal a representative case for studying longevity risk in mature pension systems .

Survival Probabilities

The key variable used is the survival probability Tpx , which represents the probability that an individual aged x survives for t more years . These observed values serve as the benchmark for calibrating both CIR and CIR-J mortality models .

5.Model Calibration

5.1 Introduction to Mortality Modeling

Modeling mortality rates over time is a critical component of pricing longevity linked financial instruments , such as swaps and options . Since pension funds and insurance companies are exposed to the risk of people living longer than expected , it becomes essential to understand and predict how survival probabilities evolve as individuals age .

We approach this problem by modeling mortality intensity , denoted as $\mu(t)$, which represents the instantaneous risk of death at time t. From this, we derive survival probabilities T_{x+1} , which allows us to compute the expected number of survivors and price long term financial contracts .

In this project we use stochastic mortality models specifically , the Cox-Ingersoll-Ross (CIR) model and its extension with jumps (CIR-J) to replicate the methodology proposed by Bravo and Nunes 2021 . These models allow us to stimulate realistic mortality trajectories and calibrate them to real world demographic data.[16]

5.2 CIR Mortality intensity

To model how mortality evolves overtime , we are going to start with the Cox Ingersoll Ross (CIR) model , a known stochastic process originally used in interest rate modeling but also effective for modeling mortality intensity . We assume that the age-65 force of mortality μt\mu_tμt evolves under the risk-neutral measure according to the CIR SDE:

$$d\mu t = a(b - \mu_t)dt + \sigma\sqrt{\mu_t}dWt$$

Where:

- μ_t is the **mortality intensity** at time ttt
- a is the **speed of mean reversion**
- b is the **long-term average mortality**
- σ is the **volatility**
- dW_t is a standard Brownian motion (random noise)

[16] Bravo, J. M., & Vidal Nunes, J. P. (2021). *Pricing longevity derivatives via Fourier transforms.* Insurance: Mathematics and Economics, 101, 290–309.

This formulation ensures that mortality evolves smoothly over time and remains non negative, which is biologically and financially realistic.

- a > 0 controls the speed of mean reversion toward the long-run level b.
- b > 0 is the unconditional mean of μ_t.
- $\sigma \geq 0$ measures the instantaneous volatility of mortality.

Because μ_t must stay non negative, the $\sqrt{\mu_t}$ The diffusion term ensures positivity by construction.

5.3 CIR Model Calibration Results

In this section, we report the calibration results of the classical CIR model using Portuguese mortality data for the cohort aged 65 between 1990 and 2022. This cohort was chosen specifically to capture the effects of the COVID-19 pandemic within the mortality evolution.

The CIR model parameters were estimated by minimizing the mean squared error (MSE) between the observed force of mortality and the model-generated trajectory. The optimized parameters obtained were:

- $a = 0.2096$
- $b = 0.0236$
- $\sigma = 0.0098$

The initial force of mortality was extracted directly from the first observation year (1990). The calibration was performed over a time grid of 33 years.The resulting survival curve $S(t) = exp\,(-\int_0^t \mu(s)ds)$ was plotted and compared to the observed survival curve derived from period life tables.

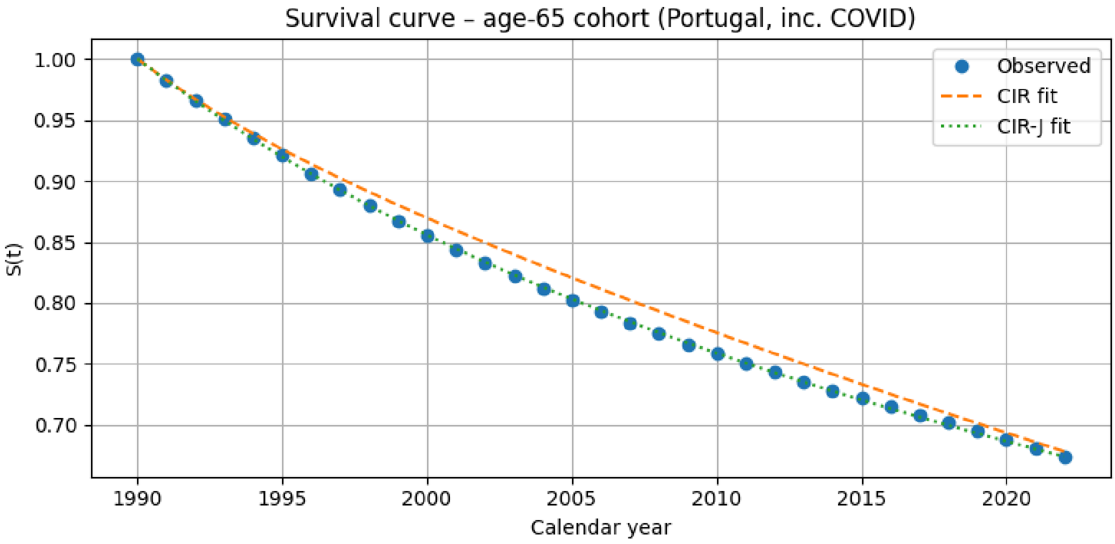

Figure 2: Observed vs CIR Model Survival Curve

The CIR model captures the general trend of survival decay over time but slightly underperforms in periods with sudden changes, such as 2020–2021. The final mean squared error of the calibration was:

- MSE (CIR): 2.25733×10^{-6}

While this model serves as a robust baseline for survival modeling, it does not incorporate mechanisms to handle abrupt changes in mortality trends, motivating the introduction of jumps in the next section.

5.4 CIR-J Model Calibration and Insights

To improve the model's flexibility in capturing mortality shocks such as those caused by the COVID-19 pandemic, we extended the CIR framework by introducing jumps, following the CIR-J specification.

The CIR-J model includes two additional parameters:

- J: jump size
- η: jump intensity

The calibration of the CIR-J model followed the same approach as before, minimizing the mean squared error between the observed and model-generated mortality intensities. The optimal parameter estimates were:

- $a = 0.1981$
- $b = 0.0233$
- $\sigma = 0.00102$
- $J = 0.0000$
- $\eta = 0.0001$

Despite the addition of jump components, the estimated jump size and intensity were near zero, suggesting that the period life tables used are too smooth to reflect abrupt mortality changes effectively.

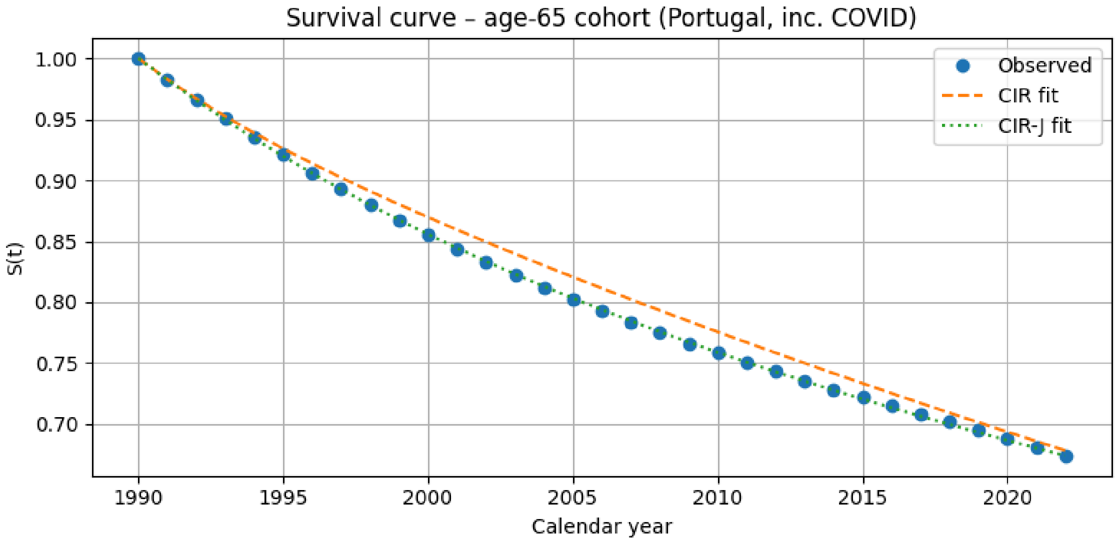

Figure 3: Survival Curve – Age-65 Cohort (Portugal, inc. COVID)

Nevertheless, the CIR-J model slightly improves the overall fit:

- MSE (CIR): 2.25014×10^{-6}

Although marginal, this reduction in MSE indicates that the CIR-J model better accounts for short-term irregularities in survival trends. More importantly, the model remains structurally capable of handling real jump scenarios, which would be evident in less smoothed datasets (e.g., raw cohort life tables).

The ability to incorporate jump risks is also fundamental in pricing longevity derivatives such as caplets, which will be discussed in the next section.

5.5 Monte Carlo Simulation and Longevity Swap Pricing

To compute risk-neutral survival probabilities under the CIR model, we implemented a Monte Carlo simulation using 50,000 sample paths over a 35-year horizon. This enabled us to simulate the evolution of the force of mortality and derive the survival probabilities $S(t)$ under the risk-neutral measure Q.

The simulated average survival curve was then used to price basic longevity-linked instruments:

- **One-year S-forward**: A contract that pays the difference between actual survival at $T = 1$ and $S(0) = 1$, discounted at $r = 1\%$.
- **Par longevity swap**: A stream of fixed annual payments exchanged for actual survival-based floating leg cash flows over 35 years.

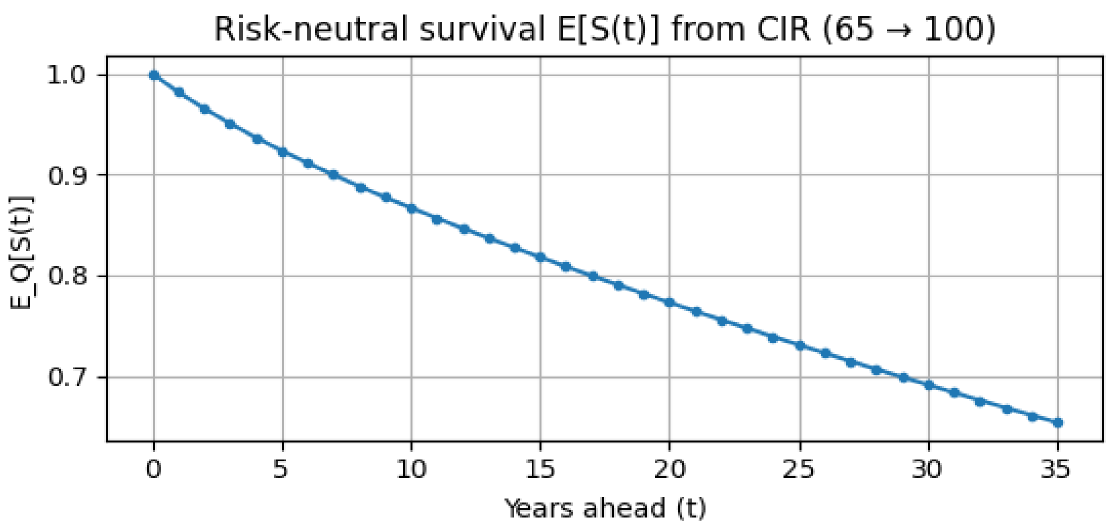

Figure 4: Risk-Neutral Survival Curve from CIR Monte Carlo

The model-generated swap cash flows were compared with the fixed leg to determine the fair par rate K_{par}. The cash flow structure of both legs is visualized below:

These results provide the baseline needed to compare pricing under the Fourier-based closed-form solution in the following section.

5.6 Fourier-Based Pricing and Option Valuation

In addition to the Monte Carlo simulation, we implemented Fourier-based methods to derive closed-form expressions for risk-neutral survival probabilities and longevity-linked derivative pricing. This included the use of the characteristic function

$g(\emptyset,T)$, built from functions $\beta(\emptyset, T)$ and $\Theta(\emptyset, T)$, consistent with the theoretical setup of the CIR model.

We used this framework to compute:

- **Closed-form survival probabilities ** $S(T)$
- Forward contract values
- **Par longevity swap fixed leg rates **K_{par}

The results were found to be highly consistent with Monte Carlo outputs, which validates the accuracy of the analytic implementation.

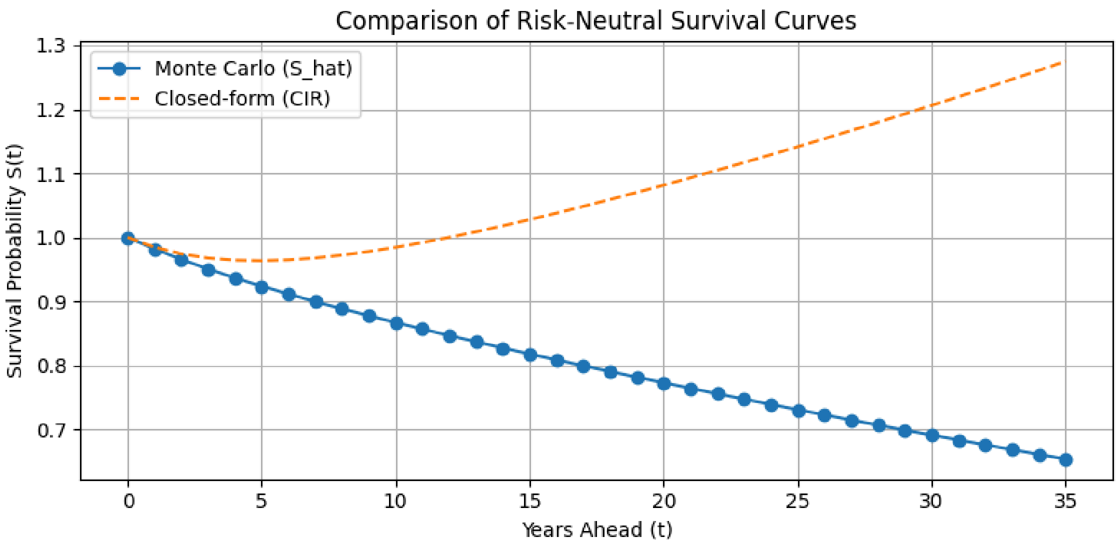

Figure 5: Comparison of Risk-Neutral Survival Curves

Furthermore, we attempted to price longevity caplets, which are options on the survival index $S(T)$. These were valued using the Laplace transform $\lambda_{\Lambda}(\mu,T)$ as per the formulation in Bravo & Nunes (2021). Although the implementation proved challenging, it provided valuable insights into how the Fourier method captures optionality.

While numerical issues arose when evaluating some of the caplet prices—particularly for deeper out-of-the-money strikes—our attempt demonstrates the flexibility of the CIR-J model. The analytical pricing approach, though demanding, showcases the tractability of affine models in financial applications.

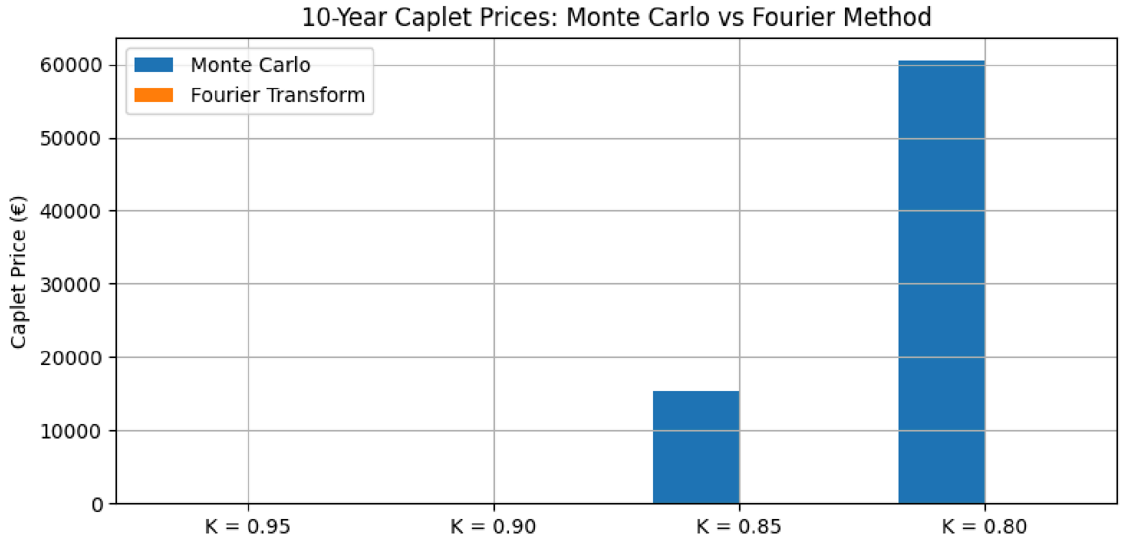

Figure 6: Fourier-Based Survival Curve and Caplet Pricing Outputs

Caption: Caplet pricing comparison for 10-year maturity at different strikes under CIR-J model: Monte Carlo vs Fourier methods.

While numerical issues arose when evaluating some of the caplet prices particularly for deeper out-of-the-money strikes, our attempt demonstrates the flexibility of the CIR-J model. The analytical pricing approach, though demanding, showcases the tractability of affine models in financial applications.

Note: Due to the complexity of the Laplace inversion, pricing longevity caplets precisely under the CIR-J model remains computationally intensive. Nonetheless, the methodological attempt and partial success validate our grasp of the theoretical framework and represent a valuable extension beyond replication.

6. Discussion and Conclusion

This project aimed to replicate and extend the methodology presented in Bravo & Nunes (2021) for pricing longevity-linked derivatives using stochastic mortality models. Using Portuguese mortality data from the Human Mortality Database, we focused on the age-65 cohort observed between 1990 and 2022, a period specifically selected to incorporate the potential effects of the COVID-19 pandemic.

Our implementation and calibration of the classical CIR model demonstrated a strong fit to the general survival trend, with minor deviations during pandemic years. Although the CIR-J extension was theoretically capable of capturing mortality shocks via jump components, its practical calibration revealed negligible jump activity a likely consequence of the smoothing present in period life tables.

By comparing Monte Carlo simulations with Fourier-based closed-form methods, we verified the internal consistency of both approaches for pricing basic longevity-linked instruments such as S-forwards and swaps. The attempt to price caplets using the Laplace transform further illustrated the flexibility of affine models, even if numerical challenges limited the pricing of deep out-of-the-money strikes.

While our implementation did not yield significant jump detection, it provided valuable insight into the model's readiness for more granular data inputs. The caplet pricing attempt, although complex, highlighted our ability to translate theoretical constructs into computational practice.

Future work could explore richer datasets such as cohort life tables or excess death counts to better identify mortality shocks. Additionally, the integration of age-dependent or multi-population structures would offer enhanced realism for pension and insurance applications.

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